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ORIGINAL ARTICLE

Student understanding of linear combinations of eigenvectors

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Abstract

To contribute to the sparse educational research on student understanding of eigenspace, we investigated how students reason about linear combinations of eigenvectors. We present results from student reasoning on two written multiple-choice questions with open-ended justifications involving linear combinations of eigenvectors in which the resultant vector is or is not an eigenvector of the matrix. We detail seven themes that analysis of our data revealed regarding student responses. These themes include: determining if a linear combination of eigenvectors satisfies the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$; reasoning about a linear combination of eigenvectors belonging to a set of eigenvectors; conflating scalars in a linear combination with eigenvalues; thinking eigenvectors must be linearly independent; and reasoning about the number of eigenspace dimensions for a matrix. In the discussion, we explore how themes sometimes cut across questions and how looking across questions gives insight into individuals' conceptions of eigenspace. Implications for teaching and future research are also offered.

Keywords Linear algebra · Student reasoning · Eigenspace · Linear combination

1 Introduction

Linear algebra is particularly useful to science, technology, engineering and mathematics (STEM) fields and has received increased attention by undergraduate mathematics education researchers in the past few decades (Artigue, Batanero, & Kent, 2007; Dorier, 2000; Rasmussen & Wawro, 2017). A useful group of concepts in linear algebra is eigentheory, or the study of eigenvectors, eigenvalues, and eigenspaces.[1] We focus on eigentheory because it is a conceptually complex set of ideas that build from and rely upon student understanding of multiple key ideas in mathematics, and its application is widespread in mathematics and beyond. First, investigating student understanding of eigentheory contributes to what is known about how students conceptualize related key ideas, such as linear transformation, linear (in)dependence, solution sets, subspace, and geometric interpretations of these ideas when applicable. For instance, eigenvectors and eigenvalues have geometric interpretations in two and three dimensions; however, those may or may not connect to students' understanding of the algebraic representations of these ideas (Hillel, 2000). Second, students also encounter eigentheory in a wide spectrum of mathematics courses beyond linear algebra—such as differential equations, probability, graph theory, cryptography, matrix theory—as well as courses in chemistry, physics, economics, and engineering. For instance, within mathematics the uses of eigentheory include stochastic processes, predator–prey models, and connectivity of graphs and digraphs; in quantum mechanics, the uses of eigentheory include determining the possible measurements of observables in spin or energy systems. Thus, investigating student understanding of a complex topic that exists in a variety of areas in the undergraduate STEM curriculum has the potential to impact not only the field of mathematics education but various discipline-based educational research fields as well.

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[1] "An eigenvector of an *nxn* matrix A is a nonzero vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\boldsymbol{x}$ of $A\boldsymbol{x} = \lambda\boldsymbol{x}$; such a $\boldsymbol{x}$ is called an *eigenvector corresponding to* λ" (Lay et al., 2016, p. 269). The eigenspace of A corresponding to λ is "the set of all solutions of $A\boldsymbol{x} = \lambda\boldsymbol{x}$, where λ is an eigenvalue of A. [It c]onsists of the zero vector and all eigenvectors corresponding to λ" (p. A9).

One aspect of eigentheory that seems particularly understudied in educational research is eigenspace, including how students understand linear combinations of eigenvectors. If $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are eigenvectors of A both with eigenvalue λ, then all vectors that are a linear combination of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ (i.e., $span\{\boldsymbol{x}_1, \boldsymbol{x}_2\} = k_1\boldsymbol{x}_1 + k_2\boldsymbol{x}_2$ for scalars k_1, k_2) are eigenvectors of A associated with λ. Some research on student understanding of eigentheory has included eigenspaces, but usually not as the focus of the study (Beltrán-Meneu, Murillo-Arcila, & Albarracín, 2016; Gol Tabaghi & Sinclair, 2013; Salgado & Trigueros, 2015; Thomas & Stewart, 2011). In addition to the importance to eigentheory specifically, studying student understanding of linear combinations of vectors is relevant to knowing how they understand vector spaces, span, and linear independence. By focusing on linear combinations of eigenvectors, this paper considers how students combine knowledge of the meaning of the eigenequation (both algebraically and geometrically) with their understanding of linear combinations of vectors and subspaces.

To explore students' understanding of eigenspaces further, the research question for this paper[2] is: How do students make sense of and reason about linear combinations of eigenvectors?

2 Theoretical framework and literature review

The research reported in this paper is part of a larger research program in which we endeavor to investigate how students reason about and symbolize eigentheory in linear algebra and in quantum physics (*Project LinAl-P*, NSF-DUE 1452889). Herein we focus on the variety of mathematical conceptions that individual students bring to bear in their mathematical work, choosing to immerse ourselves in the multi-faceted nuances of students' constructed conceptions (von Glasersfeld, 1995) about a particular set of mathematical ideas, namely those related to eigentheory and eigenspace. We note that these conceptions are not constructed in isolation; rather, the norms and practices of classroom activity give "shape and purpose to individuals' goal-directed activities" (Saxe, 2002, p. 277). As such, we recognize the importance of an overarching theoretical approach towards learning called the Emergent Perspective (Cobb & Yackel, 1996), which is based on the assumption that mathematical development is a process of individual cognition through constructivism (von Glasersfeld, 1995) and mathematical enculturation through symbolic interactionism (Blumer, 1969). This paper is a first step toward understanding how students learn the concept of eigenspace by specifically focusing on the individual conceptions they hold about linear combinations of eigenvectors.

In order to understand nuances in students' conceptions of eigentheory and eigenspaces, it is important to situate this study within the literature on the teaching and learning of eigenvectors and eigenvalues, which points to several aspects of eigentheory that are important as students build their understanding. In drawing upon these various studies, we recognize that each has a chosen theoretical framework that led to or illuminated particular results. Our focus here is to provide a broad overview of the educational research that has focused on eigentheory, tying together results that come from various theoretical framings into a cohesive whole.

Thomas and Stewart (2011) found that some students struggle to coordinate the two different mathematical processes (*matrix* multiplication versus *scalar* multiplication) captured in the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to make sense of equality as "yielding the same result" between mathematical entities (i.e., two equivalent vectors), an interpretation that is nontrivial or even novel to students (Henderson, Rasmussen, Sweeney, Wawro, & Zandieh, 2010). This importance of understanding $A\boldsymbol{x} = \lambda\boldsymbol{x}$ was further explored by Bouhjar et al. (2018). They found that a robust conceptual understanding (Hiebert & Lafevre, 1986) that leads to productive solutions in solving eigentheory problems includes the ability to interpret the matrix–vector multiplication in $A\boldsymbol{x} = \lambda\boldsymbol{x}$ as a transformation (see Larson & Zandieh, 2013) which "yields a vector that is a scalar multiple of x, or that lies on the same line as x, or that points in the same (or opposite) direction as x" (p. 212). Furthermore, in navigating eigentheory problems, students have to keep track of multiple mathematical entities (matrices, vectors, and scalars), all of which can be symbolized similarly. For instance, the zero in $(A - \lambda I)\boldsymbol{x} = 0$ refers to the zero vector, whereas the zero in $\det(A - \lambda I) = 0$ is the number zero. This complexity of coordinating mathematical entities, operations, and their symbolizations is something students have to grapple with when making sense of eigentheory concepts and solving eigentheory problems.

Thomas and Stewart (2011) also posit that this complexity may prevent students from making the symbolic progression from $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to $(A - \lambda I)\boldsymbol{x} = 0$ through the introduction of the identity matrix, which is often an important step in solving for the eigenvalues and eigenvectors of a matrix A. In their genetic decomposition[3] describing how a

[2] This paper builds from and is an extension of a conference presentation given at the 2018 Research in Undergraduate Mathematics Education Conference (Wawro, Watson, & Zandieh, 2018).

[3] "A genetic decomposition is a hypothetical model that describes the mental structures and mechanisms that a student might need to construct in order to learn a specific mathematical concept" (Arnon et al., 2014, p. 27).

student might construct eigentheory concepts, Salgado and Trigueros (2015) also point out the importance of understanding the equivalence of the two equations through coordinating solutions to $A\boldsymbol{x} = \lambda\boldsymbol{x}$, solutions to homogeneous systems of equations resulting from $(A - \lambda I)\boldsymbol{x} = 0$, and the null space of the matrix $A - \lambda I$. Adding further complexity to understanding these equations is the fact that the interpretation of "solution" in this setting, the (infinite) set of all vectors $\boldsymbol{x}$ that make the equation true, is much different from solving equations such as $cx = d$, where c, x, and d are real numbers (Harel, 2000). Furthermore, our own work indicates students' preference for using either $A\boldsymbol{x} = \lambda\boldsymbol{x}$ or $(A - \lambda I)\boldsymbol{x} = 0$ can influence their reasoning when solving eigentheory problems (Watson, Wawro, Zandieh, & Kerrigan, 2017).

Not only must students navigate complexities involved in understanding the equations $A\boldsymbol{x} = \lambda\boldsymbol{x}$ or $(A - \lambda I)\boldsymbol{x} = 0$, they must also make sense of different representations and levels of abstraction involved in eigentheory. Hillel (2000) found that instructors often move between geometric, algebraic, and abstract modes of description without explicitly alerting students; although the various ways to think about and symbolize linear algebra ideas are second nature to experts, they often are not within the cognitive reach of students. In fact, Thomas and Stewart (2011) mentioned that students in their study primarily thought of eigenvectors and eigenvalues symbolically and were confident in matrix-oriented algebraic procedures, but "the vast majority had no geometric, embodied world view of eigenvectors or eigenvalues … losing out on the geometric notion of invariance of direction" (p. 294).[4] In contrast, other researchers have shown how exploration of eigentheory through dynamic geometry software (Çağlayan 2015; Gol Tabaghi & Sinclair, 2013; Nyman, Lapp, St John, & Berry, 2010) or reinvention-oriented task sequences (Plaxco et al. 2018; Zandieh, Wawro, & Rasmussen, 2017) can help students develop a robust geometric understanding of eigenvectors and eigenvalues. Another potentially fruitful way to help students develop an understanding of eigentheory is using real-world contexts (Beltrán-Meneu et al., 2016; Salgado & Trigueros, 2015). For example, using a Models and Modeling approach (Lesh, Hoover, Hole, Kelly, & Post, 2000), Salgado and Trigueros (2015) showed how a model-eliciting activity supplemented with additional activities explicitly designed from their genetic decomposition helped many students in the class develop at least a process conception[5] of eigenvectors and eigenvalues, and several were able to demonstrate an object conception.

Regarding eigenspaces in particular, some researchers have suggested specific ideas about eigenspaces that a student needs to develop as they build their understanding of eigentheory. Thomas and Stewart (2011) pointed out that students need to understand (a) there are infinitely many eigenvectors associated with an eigenvalue, (b) every scalar multiple of an eigenvector is also an eigenvector, and (c) "a linear transformation represented by an $n \times n$ matrix has at most n distinct, non-parallel eigenvectors" (p. 279). In Salgado and Trigueros (2015), their genetic decomposition culminates with: "The need to compare spaces spanned by different eigenvectors allows students to encapsulate the spanned space process into an object, defined as the eigenspace corresponding to a given eigenvalue of a matrix" (p. 107). They found that through their model-eliciting activity, students were able to construct a process view of eigenspace by coordinating the concepts of span and the null space of $(A - \lambda I)$ to find the solution set to the system of equations. However, in interviewing students three weeks later, they found many students struggled to construct an object conception of eigenspace and to coordinate the number of eigenvectors corresponding to a given eigenvalue with the dimension of the space spanned by the eigenvectors of that eigenvalue.

Although researchers have made some progress in delineating what is involved in understanding eigentheory as well as ways to teach it, there is a need for research that explicitly examines the various conceptions students have about eigenspaces and linear combinations of eigenvectors. Some researchers (Beltrán-Meneu et al., 2016; Thomas & Stewart, 2011) have asked questions related to reasoning about eigenspaces and briefly shared results of student thinking on those problems in the context of their larger investigations. In our results section, we tie in these previous findings to help create a more comprehensive understanding of the various ways that students reason about eigenspaces and linear combinations of eigenvectors.

3 Methods

The data for this study come from student written responses to the 6-question Eigentheory Multiple-Choice Extended (MCE) Assessment Instrument (Watson et al., 2017), which aims to capture nuances of students' conceptual understanding of eigentheory. At the time of data collection, there existed multiple versions[6] that varied by question format.

[4] See Tall (2004) for a more detailed discussion of the Three Worlds of Mathematics (geometric/embodied, symbolic, formal) used as part of the framing in Thomas and Stewart (2011).

[5] "Process conception" and "object conception" are constructs from APOS Theory; see, for example, Dubinsky and McDonald (2001) for more information.

[6] The main MCE version prompts students to justify their answer to the multiple-choice stem by selecting all pre-made justification statements that support their choice (Watson et al., 2017; Zandieh, Plaxco, Wawro, Rasmussen,Milbourne, & Czeranko 2015).

3. Suppose A is a $n \times n$ matrix, and $\boldsymbol{y}$ and $\boldsymbol{z}$ are linearly independent eigenvectors of A with corresponding eigenvalue 2. Let $\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$. Is $\boldsymbol{v}$ an eigenvector of A?
   (a) Yes, $\boldsymbol{v}$ is an eigenvector of A with eigenvalue 2.
   (b) Yes, $\boldsymbol{v}$ is an eigenvector of A with eigenvalue 5.
   (c) No, $\boldsymbol{v}$ is not an eigenvector of A.
   Because … (Please write a thorough justification for your choice)

5. Suppose a 3x3 matrix B has two real eigenvalues: for eigenvalue 2 its eigenspace E_2 is one-dimensional, and for eigenvalue 4 its eigenspace E_4 is two-dimensional. Also suppose that vector $\boldsymbol{x} \in \mathbb{R}^3$ lies on the plane created by the eigenspace E_4 and $\boldsymbol{y} \in \mathbb{R}^3$ lies on the line created by the eigenspace E_2, as illustrated in the graph below.
   If $\boldsymbol{z} = \boldsymbol{y} + 0.5\boldsymbol{x}$, which of the following is true?
   (a) The vector $\boldsymbol{z}$ is an eigenvector of B with an eigenvalue of _____ [fill in the blank]
   (b) The vector $\boldsymbol{z}$ is not an eigenvector of B.
   Because … (Please write a thorough justification for your choice)

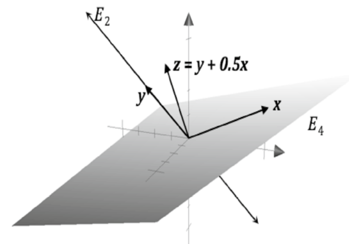


Fig. 1 Questions 3 and 5 of the Eigentheory MCE Assessment Instrument

All versions have questions that begin with a multiple-choice element; the variation exists in how the students are asked to justify their conclusions in the multiple-choice stem. The two populations whose responses we analyze in this paper were administered an open-ended version of the Eigentheory MCE, in which they were asked to select an answer to the multiple-choice stem and then respond to the open-ended justification prompt: "Because…(Please write a thorough justification for your choice)." In particular, we focus on student responses to Questions 3 and 5 (Q3 and Q5), which are about linear combinations of eigenvectors (Fig. 1).

Q3 asked if a linear combination of two eigenvectors of the same eigenvalue is an eigenvector (it is), and Q5 asked if a linear combination of two eigenvectors with different eigenvalues is an eigenvector (it is not). Because MCE questions were created to elicit student thinking about eigentheory within and across different settings and interpretations (see Wawro, Zandieh, & Watson, 2018), we note that Q3 had no geometric wording and was for a more abstract, *nxn* matrix, whereas Q5 was worded in terms of a line and plane of eigenspaces for a 3×3 matrix. In both cases these questions ask students to coordinate their knowledge of linear combinations of vectors with their understanding of eigenvectors and eigenvalues. If students have experience with properties of eigenspaces, they may apply that knowledge. On the other hand, students who only have minimal exposure to eigentheory can still coordinate their knowledge of linear combinations of vectors from other parts of linear algebra with their knowledge of eigenvectors and eigenvalues.

As this work is part of a larger study of student understanding of eigentheory in mathematics and physics, data sources for this paper come from a junior-level quantum mechanics class at a northwestern US university (which we refer to as "Class A") and a sophomore-level introductory linear algebra class at a university in the eastern United States (referred to as "Class B"). Both universities are large, public, research-active doctoral universities. The relevant prerequisite for students in Class A was either a combined introductory linear algebra and differential equations course or two full introductory courses in both linear algebra and differential equations; we have no data regarding the instructional methods nor curriculum in these courses. Class B was a student-centered, active learning course that utilized the Inquiry-Oriented Linear Algebra curricular materials (Wawro, Zandieh, Rasmussen, & Andrews-Larson, 2013) with a supplementary textbook (Lay et al. 2016). Eigenvectors, eigenvalues, and matrix diagonalization were taught in Class B and in Class A's linear algebra prerequisites. Note that we do not intend this study as a comparison of these student groups; rather we draw on students from two different settings to allow for a variety of types of student reasoning to emerge.

Of the 32 students in Class A, 23 answered Q3 and 16 answered Q5. Of the 28 students in Class B, 27 answered Q3 and 23 answered Q5. Class A was asked to work on the MCE for no more than 20 min as homework, without outside consultation, during the first week of their Quantum Mechanics course; Students in Class B worked on the MCE individually for approximately 20 min in-class during the last day of their linear algebra course. The version of Q5 given to Class A had a different wording in which the vector under consideration was on the plane of the two-dimensional eigenspace, and students were asked to decide what happened geometrically to the vector when acted on by the matrix. We determined, however, that this question under-emphasized the aspect of linear combination, and it

Fig. 2 The 24 codes developed from the data to describe student reasoning on Q3 and Q5

<table>
<tr><td colspan="12">Algebraic/Symbolic</td></tr>
<tr><td colspan="3">$\boldsymbol{Ax = \lambda x}$</td><td colspan="3">$(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$</td><td colspan="3">$\det(A - \lambda I) = 0$</td><td colspan="3">Calculation-based</td></tr>
<tr><td colspan="12">Geometric</td></tr>
<tr><td colspan="3">Geometric -Vector</td><td colspan="3">Geometric -Transformation</td><td colspan="3">Geometric - Eigenspace</td><td colspan="3">Geometric –Span</td></tr>
<tr><td colspan="12">Global</td></tr>
<tr><td colspan="4">LC of eigenvectors IS an eigenvector</td><td colspan="4">LC of eigenvectors is not necessarily an eigenvector</td><td colspan="4">LC of eigenvectors is NOT an eigenvector</td></tr>
<tr><td colspan="6">Only scalar multiples of eigenvectors are eigenvectors</td><td colspan="6">Vector in e-space is also e-vector</td></tr>
<tr><td colspan="4">Combination</td><td colspan="4">Vectors in LC are LI</td><td colspan="4">Vectors in LC have same eigenvalue</td></tr>
<tr><td colspan="3">Scale k</td><td colspan="3">Scale λ</td><td colspan="3">Eigenvectors are LI</td><td colspan="3">Unique</td></tr>
<tr><td colspan="3">Dimension</td><td colspan="3"># of eigenvectors</td><td colspan="3">Size of the matrix</td><td colspan="3">Miscellaneous</td></tr>
</table>

did not seem to differentiate or lend insight into students' understanding of eigentheory. Thus, the results regarding Q5 in this paper (see Sect. 3.4) only report on Class B, who was given the version of Q5 in Fig. 1.

To analyze the data in such a way that would allow us to characterize the concepts students brought to bear as they justified their answers to Q3 and Q5, we engaged in qualitative analysis consisting of two levels of coding (Miles, Huberman, & Saldaña, 2014):

Codes are labels that assign symbolic meaning to the descriptive or inferential information compiled during a study. Codes usually are attached to data 'chunks' of varying size and can take the form of a straightforward, descriptive label or a more complex one (e.g., a metaphor) (p. 71–72).

In the first level of analysis, we engaged in a cyclical process to code all students' justifications and develop a coding book of the codes and their descriptions. This cyclical process included: (a) each author coding students' justifications individually (open or inductive coding initially, and then using the developed codes with each subsequent pass); (b) group discussions with all three authors about each student to work towards a coding consensus; and (c) development and refinement of codes and their descriptions within the coding book. The codes developed and given to students' justifications in this first level of coding were generally descriptive or in vivo codes. Descriptive codes are short words or phrases assigned to data as a label that briefly describes the ideas contained therein, and in vivo codes use "words or short phrases from the participant's own language in the data record as codes" (Miles et al. 2014, p. 74). In the second level of analysis, we identified how the first-level codes were loading together in the various student justifications, which assisted us in inferring underlying themes among the conceptions that students brought to bear in their justifications for particular multiple-choice answers to Q3 and Q5. This second-level of analysis is similar to axial coding in Grounded Theory (Glaser & Strauss, 1967). In the results that follow, we share the codes developed and themes discovered, with explicating the themes being the major focus of our results.

4 Results

Our analysis of the data revealed 24 codes and seven themes that characterized the concepts and reasoning that students brought to bear as they justified their answers to Q3 and Q5. We list all 24 codes in Fig. 2; to facilitate a brief explanation of a sampling of codes, we organize some of them into three clusters—Algebraic/symbolic, geometric, and global. Although all 24 codes grew out of our analysis, not all are discussed in detail in this paper. Rather, we provide additional detail on specific codes throughout the subsequent Results subsections as needed when the seven themes are delineated.

There are four algebraically/symbolically oriented codes; we coded if students wrote some version of any of the three equations $Ax = \lambda x$, $(A - \lambda I)x = 0$, and $\det(A - \lambda I) = 0$. We also gave the code "Calculation-based" if students seemed to utilize any of these equations to carry out computations and deduce a conclusion based on these calculations. There are four geometrically-oriented codes: "Geometric–vector," "Geometric–transformation," "Geometric–eigenspace," and "Geometric–span." The purpose of each was to describe which type of object the student was reasoning about geometrically. For example, "An eigenvector doesn't change direction under transformation" [A9] is coded as "Geometric–transformation" because the student focused on the graphical quality of the vector after a transformation was applied, whereas "*v* is lay [sic] on the plane formed by *y, z*" [B82] is coded as "Geometric–vector" because the student focused on the graphical quality of the vector implied by the linear combination. Five codes are of a more global quality, intended to capture when students said statements that had a universally true type of quality to them. These include: a linear combination of eigenvectors is (1) always an eigenvector, (2) not necessarily an eigenvector, or (3) never an eigenvector; (4) only scalar multiples of eigenvectors are also eigenvectors; and (5) vectors in an eigenspace are also eigenvectors of that eigenvalue.

In the remainder of the results, we focus on the seven themes we found among students' justifications for their answers to Q3 and Q5. For Q3, less than half of each class correctly chose (a) "Yes, $\boldsymbol{v}$ is an eigenvector of A with

Table 1 Students' multiple-choice answers on Q3 of the eigentheory MCE

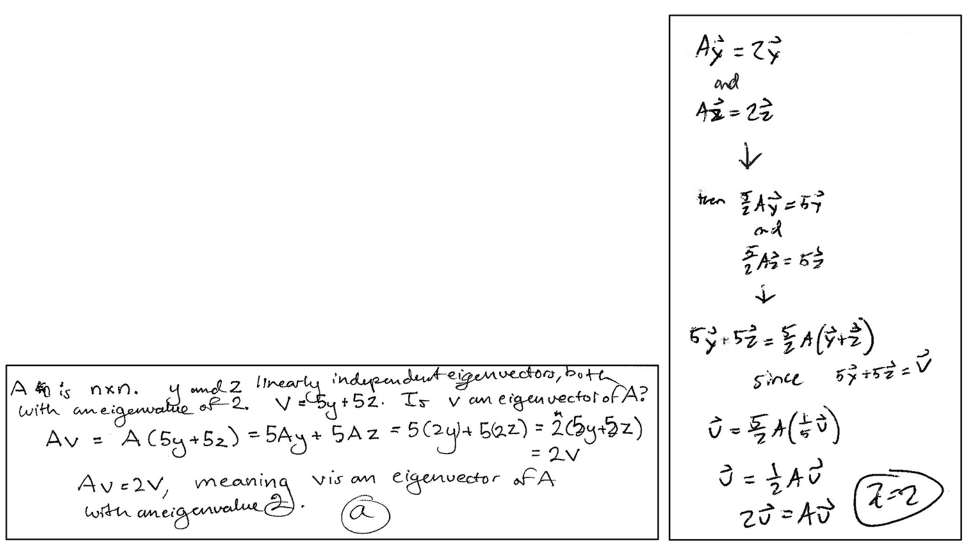
Class	Chose (a)	Chose (b)	Chose (c)	No answer	Yes eigenvector, but $\lambda \neq 2, 5$
Class A	13	5	3	9	2
Class B	6	5	16	1	0
Total	19	10	19	10	2

eigenvalue 2," indicating that Q3 was a particularly difficult question for students to answer on the MCE (see Table 1). Exploring the reasoning in students' justifications for each of the three answer choices for Q3 constitutes the majority of our themes in the following subsections. For Q5, most students were able to choose the correct answer (b) "The vector z is not an eigenvector of B," and many gave justifications related to a common theme but to varying degrees of mathematical sophistication. In the last section of our results, we explain the nuances among students' justifications for Q5.

4.1 Equation satisfaction and membership satisfaction: justification themes of students who chose (a) on Q3

Examining the justifications of students who correctly chose that $\boldsymbol{v}$ is an eigenvector of A with eigenvalue $\lambda = 2$ revealed two main themes: Equation Satisfaction and Membership Satisfaction. The Equation Satisfaction theme is typified by students who performed a series of calculations to verify that $A\boldsymbol{v} = 2\boldsymbol{v}$ within their justification; these seven students were coded with both "$A\boldsymbol{x} = \lambda\boldsymbol{x}$" and "Calculation-based" in our coding scheme. Here, we share two different examples of this approach (see Fig. 3). Notice both justifications use linearity of matrix multiplication and the fact that $\boldsymbol{y}$ and $\boldsymbol{z}$ are eigenvectors of A with eigenvalue $\lambda = 2$ to conclude that $\boldsymbol{v}$ satisfies the equation $A\boldsymbol{v} = 2\boldsymbol{v}$. Student A24 began with $A\boldsymbol{v}$ and substituted in $5\boldsymbol{y} + 5\boldsymbol{z}$ for $\boldsymbol{v}$. S/he then used linearity, determined the results of A acting on $\boldsymbol{y}$ and $\boldsymbol{z}$, and used commutativity of scalar multiplication and substitution to arrive at $2\boldsymbol{v}$ (Fig. 3a). Student B61 started differently, by taking the information about $\boldsymbol{y}$ and $\boldsymbol{z}$ and introducing a factor of 5 in order to have equations in terms of $5\boldsymbol{y}$ and $5\boldsymbol{z}$. Then combining this with the linear combination information for $\boldsymbol{v}$, the student arrived at $\boldsymbol{v} = \frac{5}{2}A\left(\frac{1}{5}\boldsymbol{v}\right)$, which led to $2\boldsymbol{v} = A\boldsymbol{v}$ after simplification (Fig. 3b). These rather powerful approaches used the given information to connect $\boldsymbol{v}$ to the definition of eigenvectors by determining that it satisfied the eigen-equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Second, the Membership Satisfaction theme is typified by students who, rather than looking at how $\boldsymbol{v}$ satisfies the eigen-equation, reasoned about $\boldsymbol{v}$ belonging to a set of eigenvectors. This was done in two ways: by pointing out that $\boldsymbol{v}$ is a linear combination of the eigenvectors $\boldsymbol{y}$ and $\boldsymbol{z}$ that have the same eigenvalue (coded with "Combination" and "Same Eigenvalue") or by stating that $\boldsymbol{v}$ would be a vector in the same eigenspace as $\boldsymbol{y}$ and $\boldsymbol{z}$ (coded with "Eigenspace"). First, five students that correctly chose (a) focused on the fact that $\boldsymbol{v}$ is a linear combination of the eigenvectors $\boldsymbol{y}$ and $\boldsymbol{z}$, with all except B62 pointing out in some way that $\boldsymbol{y}$ and $\boldsymbol{z}$ have the same eigenvalue. For example, B72 wrote, "$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$ which have same eigenvalue," and A13 wrote, "$\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$. To be an eigenvector: $\lambda A = \lambda\boldsymbol{v}$ [sic], $\boldsymbol{v}$ is composed of linearly independent eigenvectors of A with eigenvalues of 2." These students may have reached this



(a) (b)

Fig. 3 Justifications of A24 (**a**) and B61 (**b**) on Q3

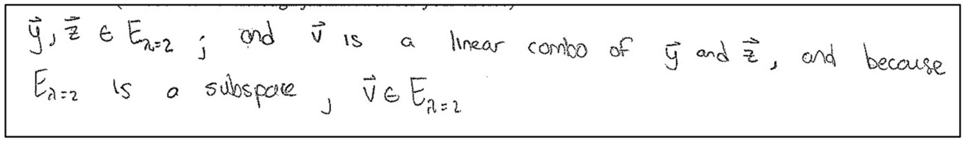


Fig. 4 Justification of A25 on Q3

conclusion if they were reasoning that linear combinations of eigenvectors are always eigenvectors; in this case, that would have allowed them to choose the correct answer. Because 4 of 5 students explicitly mentioned the shared eigenvalue 2, however, we conjecture more may have been involved in their reasoning. For example, consider A27's justification: "Combinations of eigenvectors will still be an eigenvector. Because eigenvectors are paired with an eigenvalue, the combination of its eigenvectors is another one (of infinite possible) corresponding to the same eigenvalue." This sophisticated response could also involve aspects of the second possible type of reasoning: that of $\boldsymbol{v}$ belonging to an eigenspace. Acknowledging that there are infinitely many eigenvectors for the same eigenvalue is consistent with reasoning about linear combinations of elements from the same eigenspace being members due to closure, but none of the 5 students in this group explicitly made this claim regarding subspace.

Examples of the second type of justification within the Membership Satisfaction theme include those that did explicitly state that $\boldsymbol{v}$ belongs to the eigenspace corresponding to the eigenvalue $\lambda = 2$. Five students included this as part of their justification; examples include B68 who simply wrote, "$\boldsymbol{v}$ is in the eigenspaces of both $\boldsymbol{y}$ and $\boldsymbol{z}$!" and A25 who added some reasoning about subspaces (see Fig. 4). As research on students' understanding of eigentheory has previously pointed out the difficulty for students to understand eigenspaces (Salgado & Trigueros, 2015), it is notable that these students were successful in this type of reasoning.

Connecting to previous research on student understanding of eigentheory, we find evidence of student reasoning consistent with our Equation Satisfaction and Membership Satisfaction themes in one aspect of the work by Beltrán-Meneu et al. (2016). In their study, students were given the following test question: "Consider the following matrix $A = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$ [and] consider the vectors $\boldsymbol{v}_1 = (2, 0)$, $\boldsymbol{v}_2 = (1, 1)$, and $\boldsymbol{v}_3 = (2, 2)$. Are the vectors $\boldsymbol{v}_1 + \boldsymbol{v}_2$ and $\boldsymbol{v}_2 + \boldsymbol{v}_3$ eigenvectors of A? Justify your answer." Note there are three ways in which this question differs from our Q3: it is framed in $\mathbb{R}^2$, specific vectors are given rather than abstract, arbitrary vectors, and the question does not specify if $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are eigenvectors. The authors categorized students' responses as either *symbolic* or *formal*. Students that checked if $\boldsymbol{v}_1 + \boldsymbol{v}_2$ and $\boldsymbol{v}_2 + \boldsymbol{v}_3$ satisfied the eigen-equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ were identified by the authors as using a symbolic approach; in terms of our analysis in this paper, students using a symbolic approach would be categorized under the Equation Satisfaction theme. Students were identified as using a formal approach by Beltrán-Meneu et al. when they "reasoned that the sum of eigenvectors is an eigenvector if and only if all vectors belong to the same subspace" (p. 7). Students using this formal approach would most likely be categorized through our analytical framing as falling under the Membership Satisfaction theme. We note that although this small finding was only one aspect in their larger study on student visualization in eigentheory, we find this coding consistency across studies to be encouraging. Furthermore, because Beltrán-Meneu et al. confined themselves to Tall's (2004) Three Worlds (embodied, symbolic, formal) as their categorizations of student approaches, our analytical approach allows us to uncover additional nuances in student reasoning, if they exist. We hope the results shared within this and subsequent sections better elaborate on those nuances in student reasoning.

4.2 Scalar Confusion: Justification theme of students who chose (b) on Q3

Question 3 was constructed with coefficients of 5 in the equation $\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$ and with a multiple choice option (b) $\lambda = 5$ to test whether students would conflate the scalar multiple of 5 with the eigenvalue of 2. In our experience, because the eigenvalue appears as a scalar of the vector in the eigenequation $A\boldsymbol{x} = \lambda\boldsymbol{x}$, students sometimes confuse other scalar multiples with the eigenvalue. This did in fact occur in our data, with 10 students choosing option (b) $\lambda = 5$. Six of the ten wrote statements that were true, primarily arguing that the linear combination should be an eigenvector without stating why they thought the eigenvalue should be 5 (illustrated with B71 and A9 below); the other four students were much less certain using phrases such as "not sure" or "I guessed."

We define the theme Scalar Confusion to describe this phenomenon. Students are labeled as exhibiting Scalar Confusion when they implicitly or explicitly referred to one type of scalar in a situation where it was mathematically correct to refer to the other type of scalar. For example, by choosing option (b), a student has referred to the scalar 5 instead of the correct answer of $\lambda = 2$. Therefore, all 10 students who chose (b) had their responses labeled with this theme whether or not their written justifications contained explicit statements conflating the scalars. Of students who did not answer (b), only two students provided justifications that fit the Scalar Confusion theme. Examples of each are provided after a description of the relationship of the codes to the theme.

The scalar in the linear combination $\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$ and the scalar in the eigenequation were explicitly referred to by students in both correct and incorrect ways. In our initial coding of the data we created codes "Scale k" and "Scale λ" for instances in which students explicitly referred to the two types of scalar multiples. "Scale k" was given if a student wrote about a vector being a scalar multiple of other vectors, which could include linear combinations of vectors; "Scale λ" was given if a student wrote about a vector being scaled by an eigenvalue or the eigenvalue being some type of scaling quantity. One might expect that students exemplifying the Scalar Confusion theme would be labeled with both of these because the codes refer to the two scalars that are conflated. However, these codes were not a good predictor for this theme because student responses that correctly discussed a scalar could also be coded with "Scale k" or "Scale λ." Also, many students who did exhibit Scalar Confusion did not explain their choice of (b) by referring to these scalars explicitly. First, students who chose (b) $\lambda = 5$ and had their explanations labeled with the Scalar Confusion theme wrote both true statements and false statements for their explanations. For example, B71 stated, "$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$. Both $5\boldsymbol{y}$ and $5\boldsymbol{z}$ are scalar multiples of their previous form so the resultant vector would be an eigenvector as well." This is a true statement, but the student wrote this as her/his explanation for (b) $\lambda = 5$. A9 wrote, "An eigenvector doesn't change direction under transformation, and since the vector $\boldsymbol{v}$ is composed of eigenvectors, it, too, would not change." In this case the student provided a geometric argument based on the idea that $\lambda\boldsymbol{v}$ is in the same direction as $\boldsymbol{v}$; this is a true statement (coded with the aforementioned "Geometric–Transformation"), but the student incorrectly chose (b) $\lambda = 5$. In general, only three students were coded with "Geometric –Vector" and two with "Geometric–Transformation" for Q3, and A9 is the only one of them who answered (b). Second, two students were labeled as exhibiting Scalar Confusion even though they did not choose $\lambda = 5$. B81 chose (c) not an eigenvector, stating.

No, because an eigenvector is defined as some linear combination defined by the eigenvalue, so that $A\boldsymbol{x} = \lambda\boldsymbol{x}$, where $\boldsymbol{x}$ is the eigenvector and λ is the eigenvalue. The vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ are being scaled by a factor of 5 and $\lambda = 2$ so they cannot be corresponding eigenvectors.

B81's explanation was the only one that received both "Scale k" and "Scale λ" codes. In this justification, B81 clearly and explicitly recognized the different roles that 5 and 2 play in the problem, but at the same time, did not realize that $\boldsymbol{v}$ could have an eigenvalue of 2 even though it is a linear combination of vectors scaled by 5. This student seems to conflate the scaling by 5 of the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ in the linear combination with the scaling by 2 of the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ when acted upon by the matrix A. In the former, $\boldsymbol{y}$ and $\boldsymbol{z}$ have not been acted upon by a transformation—the 5 is used to define the amount of each vector that is needed to create the vector $\boldsymbol{v}$. In the latter, the 2 is used to define that the result of multiplying each vector by A is twice the input vector. B81's reasoning seems to explain the role of the 5 in ways that would be more compatible with the role of the 2 and, because the scalars are different, concluded that "they" could not be eigenvectors. It is unclear what vectors "they" were—it could be some combination of $\boldsymbol{v}$, $5\boldsymbol{y}$ and/or $5\boldsymbol{z}$.

4.3 Linear dependence exclusion and linear combination exclusion: justification themes of students who chose (c) on Q3

There were two main themes for the justifications given by students who chose "(c) No, $\boldsymbol{v}$ is not an eigenvector of A" on Q3: Linear Dependence Exclusion and Linear Combination Exclusion. Linear Dependence Exclusion is characterized by students who conveyed an underlying rationale that a set of eigenvectors of a matrix must be linearly independent, including stating that $\{\boldsymbol{v}, \boldsymbol{y}, \boldsymbol{z}\}$ is linearly dependent, or that eigenvectors need to be "unique." We coded student justifications in this theme with either "Eigenvectors must be Linearly Independent" or "Unique." One example of this theme is B58's justification: "Eigenvectors must be LI from each other so if $\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$ then it cannot be an eigenvector." By assuming eigenvectors must be linearly independent from one another, B58 concluded $\boldsymbol{v}$ could not be an eigenvector because $\boldsymbol{v}$ is not linearly independent from the eigenvectors $\boldsymbol{y}$ and $\boldsymbol{z}$.

Although uniqueness and linear independence are different ideas, it is our experience that students often use "unique" as a synonym or substitute for "linearly independent." This is consistent with Zandieh, Adiredja, and Knapp (2018) that found some students using the word "unique" when asked to create "everyday examples" to describe the concept of basis (which involves linear independence). As an example, consider B79's justification: "Because they all correspond to the same eigenvalue they all must have unique eigenvectors and $\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$ and therefore not unique and not an eigenvector of A." We note B79 explained a need to have unique eigenvectors because all of the vectors would "correspond to the same eigenvalue," which, as pointed out previously, would indicate that the vectors all belong to the same eigenspace. We hypothesize students reasoning that $\boldsymbol{v}$ could not be an eigenvector because the set $\{\boldsymbol{v}, \boldsymbol{y}, \boldsymbol{z}\}$ is linearly dependent may have been conflating the need for eigenvectors in a basis for an

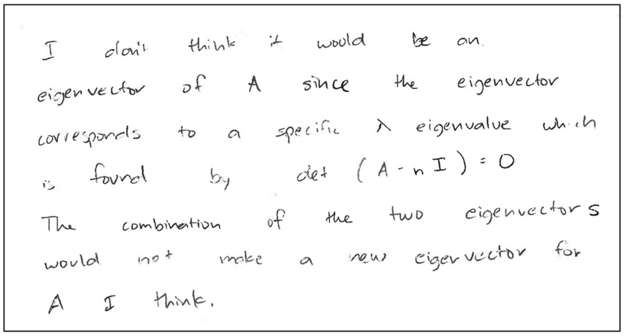


Fig. 5 Justification of B65 on Q3

eigenspace to be linearly independent with their decision about whether or not $\boldsymbol{v}$ is an eigenvector of the matrix.

Linear Combination Exclusion was typified by students who reasoned $\boldsymbol{v}$ would not be an eigenvector of A because it was a linear combination of eigenvectors. We coded these student justifications with "Linear Combinations of Eigenvectors are not Eigenvectors," or "Linear Combinations of Eigenvectors are not necessarily Eigenvectors." Two examples are "Having $\boldsymbol{v}$ as a sum of the two eigenvectors does not ensure that $\boldsymbol{v}$ is an eigenvector" [B75], and "The eigenvectors are not combinations of each other" [B64]. Though it is not completely clear which eigenvectors B64 was referring to, we assume s/he was making a global statement that eigenvectors in general are not linear combinations of other eigenvectors. We think there are two possible reasons why students concluded that $\boldsymbol{v}$ is not an eigenvector of A when giving these justifications. First, they may have thought similarly to students whose justifications were categorized with the aforementioned Linear Dependence Exclusion theme, assuming that a linear combination of eigenvectors could never be an eigenvector because the vectors involved would form a linearly dependent set. Second, students may have been recalling the definition of eigenvector and/or processes used to find eigenvectors and eigenvalues but were unable to determine how a linear combination of eigenvectors would satisfy that definition or process. For example, consider B65's justification shown in Fig. 5. Here, B65 noted that eigenvectors correspond to specific eigenvalues, found using the characteristic equation, but s/he was not able to see how a linear combination of eigenvectors related to this process.

An interesting blend of reasoning consistent with both Linear Dependence Exclusion and Linear Combination Exclusion was evident within A8's justification: "Eigenvectors are always linearly independent to other eigenvectors (assuming it is not a scalar multiple of the original) therefore $\boldsymbol{v}$ cannot be a unique eigenvector because the set of $\boldsymbol{v}, \boldsymbol{y}$, and $\boldsymbol{z}$ is linearly dependent." Note A8's parenthetical caveat, wherein s/he demonstrated an understanding that scalar multiples of eigenvectors are also eigenvectors; however, this seems to be an exception to eigenvectors needing to be linearly independent from each other. We find it commendable that A8 knew that scalar multiples of eigenvectors are also eigenvectors, but we note this as a potentially limiting approach to reasoning about eigenspace because it excludes the viability of linear combinations of vectors in an eigenspace to be an eigenvector, which is guaranteed by the closure of eigenspaces as subspaces (of $\mathbb{R}^n$ or other vector spaces).

Connecting to previous research, we find evidence of student reasoning consistent with our themes here in research by Thomas and Stewart (2011) on written exam questions. For instance, when asked how many different eigenvectors are associated with a given eigenvalue, several students in their study stated there is only one eigenvector associated with each eigenvalue. Thomas and Stewart posit that the textbook's focus on finding a single eigenvector for an eigenvalue rather than writing the set of eigenvectors parametrically may have led students to think in this way. In light of our results, an alternative explanation could be students conflating the process of finding a basis for the eigenspace (which requires one vector for a one-dimensional eigenspace) with the number of possible eigenvectors for a given eigenvalue. Another related question explored by Thomas and Stewart was "Can $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ both be eigenvectors of a given matrix?" One-third of the students were able to answer correctly, mostly arguing that one vector is a scalar multiple of the other; however, in light of our example A8 above, we caution that a knowledge of scalar multiples of eigenvectors also being eigenvectors does not necessarily correspond to a robust understanding of eigenspace. Additionally, seven of the 42 students said the two vectors could not both be eigenvectors because eigenvectors must be linearly independent. This is a clear example of Linear Dependence Exclusion in previous research, indicating this may be a common occurrence in student reasoning about eigenspaces and linear combinations (or scalar multiples) of eigenvectors.

4.4 Eigenvector total and dimension total: justification themes of students who chose (b) on Q5

This subsection details results related to Question 5 (see Fig. 1). Our last themes regarding student understanding of linear combinations of eigenvectors focus on reasoning about the total number of eigenvectors for a matrix versus the total eigenspace dimensions for that matrix.

In Q5, $\boldsymbol{z} = \boldsymbol{y} + 0.5\boldsymbol{x}$, where $\boldsymbol{y}$ is an eigenvector of B with eigenvalue 2 and $\boldsymbol{x}$ is an eigenvector of B with eigenvalue 4. So $B\boldsymbol{z} = B(\boldsymbol{y} + 0.5\boldsymbol{x}) = B\boldsymbol{y} + 0.5B\boldsymbol{x} = 2\boldsymbol{y} + 0.5(4\boldsymbol{x})$, leading to $B\boldsymbol{z} = 2\boldsymbol{y} + 2\boldsymbol{x}$. When the question was written, we conjectured that students might be tempted to think z was an eigenvector of B with eigenvalue 2; however, we did not see evidence of this in the data. Of the 21 students from Class B (from 28 total) that completed Q5, 19 correctly chose "(b) The vector z is not an eigenvector of B." Not all of their justifications, however, are ones that an expert would deem to be mathematically sound. One prominent undercurrent that we evidenced in our data (in both productive and unproductive student justifications) was that of students reasoning about a finite quantity from within the question situation, namely a number of eigenvectors, a size of a matrix, or a dimension of a vector space or eigenspace. In our coding, these justifications were frequently identified through codes such as "# e-vectors" (3 students), "dimension" (8 students), or "size of matrix" (5 students). These were parsed into two different themes, depending on how students used these numerical aspects in their justifications.

The Eigenvector Total theme describes student justifications in which reasoning about a finite number of eigenvectors for a matrix played a prominent role; this is a potentially problematic view because each eigenspace has an infinite number of eigenvectors. The Dimension Total theme describes student justifications in which reasoning about the numerical value of the dimensions of a matrix's eigenspaces played a prominent role. For the first theme, consider B59, who correctly chose (b) and wrote this was because "Matrix B already has 3 eigenvectors so there's no room for a 4th" (coded with "# e-vector" and "size of matrix"). We coded this with "# e-vector" because the student focused on there already being three eigenvectors of matrix B; we are uncertain, however, which "3 eigenvectors" the student meant. The question indirectly names vectors $\boldsymbol{y}$ (on the eigenline) and $\boldsymbol{x}$ (on the eigenplane), so possibly the student was imagining a second vector on the eigenplane such that this imagined second vector and $\boldsymbol{x}$ were linearly independent. We coded the response with an implicit "size of matrix" because the student said that matrix B had "no room for a 4th" eigenvector. One explanation for this response may be if the student was considering the fact that a nxn matrix can have at most n linearly independent eigenvectors (which often is considered with the concept of diagonalization). Student B66 gave a very similar justification: "$\boldsymbol{z}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{x}$, and there are already 3 eigenvectors for 3 dimensions, so z cannot be an eigenvector of B." This response, coded with "# e-vectors," "combination," and "dimension," focused on the existence of three eigenvectors (again, we cannot be sure what vectors the student was considering) for three dimensions. The student did not elaborate on whether s/he was considering the dimensionality of $\mathbb{R}^3$ or that of the given one-dimensional and two-dimensional eigenspaces (possibly the latter because of the question wording). One explanation for this response may be if the student was considering the fact that there must be exactly three linearly independent vectors to create the bases for the given one- and two-dimensional eigenspaces. In the case of both B59 and B66, the underlying rationale that possibly explain for their justifications are themselves correct; however, they do not invalidate that z could be an eigenvector and thus are incorrect justifications for the correct conclusion (b). The students' reasoning within the Eigenvector Total theme was inappropriate when paired with reasoning about linear independence because it seemed to block them from thinking that matrix B could have more than three eigenvectors.

Second, consider justifications from two student responses categorized according to the Dimension Total theme. Both students correctly chose (b) and seem to reason about finite number and dimension in productive ways. B58 stated, "In a 3×3 matrix there can only be 3 dimensions to the eigenspace. E_2 and E_4 together span the entire space of $\mathbb{R}^3$ so there cannot be another eigenvector of B besides E_2 and E_4." This was coded with "size of matrix," "# e-vectors," "dimension," "span," and "eigenspace." B68 stated, "B is nxn matrix where $n = 3$. The dimensions of $E_2$and E_4 add up to n, so there are no more eigenspaces. Because $\vec{z} \notin \{E_2, E_4\}$, $\vec{z}$ is not an eigenvector." This was coded with "size of matrix," "dimension," and "eigenspace." Both justifications focused on the fact that the dimensions of the eigenspaces of a nxn matrix can sum to at most n, and that the two given eigenspaces had dimensions that added up to three; because the vector z was an element of neither eigenspace and the allowable eigenspace dimension at already at the maximum possible, there could be no more eigenspaces to contain z. Although these students also used finite number reasoning, the focus here was on dimensions of eigenspaces, rather than on an amount of eigenvectors. We conjecture grasping the difference between finiteness of eigenspace dimensions (in $\mathbb{R}^n$) and infiniteness of eigenvectors may be particularly important for understanding eigenspaces.

Connecting to previous research on student understanding of eigentheory, we were intrigued by an exam question analyzed by Thomas and Stewart (2011): why the situation in Fig. 6 is impossible if A is a 2×2 matrix. This tested if students could see that three vectors that were geometrically described as eigenvectors could not all exist for a 2×2 matrix. The authors categorized only six of 42 students as correct. For instance, one student said, "Diagram shows 3 eigenvalues/ eigenvectors a 2×2 matrix should have only 2." This is sensible if the student meant only two eigenvalues; if the student meant only two eigenvectors, it could be an example of reasoning consistent with the Eigenvector Total theme (without more explanation about why those three particular vectors could not all be eigenvectors of the matrix).

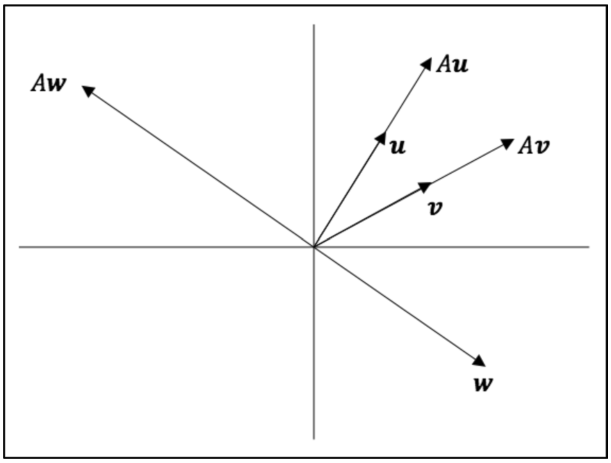


Fig. 6 Impossible Linear Transformation in $\mathbb{R}^2$. Reproduction of Picture in Fig. 2 of Thomas & Stewart (2011, p. 282)

5 Discussion

From analyzing student written responses to multiple-choice questions with open-ended justifications that aimed to help us characterize students' understanding of linear combinations of eigenvectors, we were able to delineate seven themes (see Fig. 7). In our data, students' reasoning within the first, second, and seventh of these themes led to productive solution strategies. On the other hand, categorizing students' reasoning as aligning with the third through sixth themes in our data allowed us to extract nuance about how students may have been reasoning in ways that were sensible to her/him yet did not align with either a correct answer or a correct justification.

These themes arose from our data by concentrating on students' responses to one question at a time. We also acknowledge that relating a particular student's responses to both questions may afford us additional insight into his/her understanding of eigentheory. In particular, comparing a student's responses across the two questions may provide triangulation regarding our analysis of their reasoning on the individual questions. For example, B69 answered Q3 correctly but gave a rather vague justification: "since it is a linear combination of the other eigenvectors, it would also be an eigenvector." On Q5, however, B69 explained that the vector would only be an eigenvector if the two vectors in the linear combination had the same eigenvalue. When considering B69's Q3 response in light of his/her Q5 response, we hypothesize that B69's vague response to Q3 was most likely based in a correct understanding of linear combinations of eigenvectors.

We also note that the themes may appear in categorizations of student thinking across the questions. For instance,

Fig. 7 Summary of the seven themes of student reasoning found in our data

Theme Name	*Theme Description*	*Section*
Equation Satisfaction	Students use the given information [$\boldsymbol{v} = 5\boldsymbol{y} + \mathbf{5}\boldsymbol{z}$ for eigenvectors $\boldsymbol{y}$ and $\boldsymbol{z}$ with eigenvalue 2 of matrix A] to perform a series of calculations to verify that $A\boldsymbol{v} = 2\boldsymbol{v}$.	3.1
Membership Satisfaction	Students use the given information [$\boldsymbol{v} = 5\boldsymbol{y} + \mathbf{5}\boldsymbol{z}$ for eigenvectors $\boldsymbol{y}$ and $\boldsymbol{z}$ with eigenvalue 2 of matrix A] to conclude that $\boldsymbol{v}$ belonged to the same set of eigenvectors as $\boldsymbol{y}$ and $\boldsymbol{z}$.	3.1
Scalar Confusion	Students implicitly or explicitly refer to one type of scalar [either the 5 that appeared in $\boldsymbol{v} = 5\boldsymbol{y} + \mathbf{5}\boldsymbol{z}$ or the 2 that appeared as the eigenvalue of $\boldsymbol{y}$ and $\boldsymbol{z}$] where it was mathematically correct to refer to the other type.	3.2
Linear Dependence Exclusion	Students convey an underlying rationale that $\boldsymbol{v}$ could not be an eigenvector of A because a set of eigenvectors of a matrix must always be linearly independent [but $\boldsymbol{v} = 5\boldsymbol{y} + \mathbf{5}\boldsymbol{z}$ implies $\{\boldsymbol{v}, \boldsymbol{y}, \boldsymbol{z}\}$ is dependent].	3.3
Linear Combination Exclusion	Students reason that $\boldsymbol{v}$ is not an eigenvector of A because it is a linear combination of eigenvectors [$\boldsymbol{v} = 5\boldsymbol{y} + \mathbf{5}\boldsymbol{z}$].	3.3
Eigenvector Total	Students reason that $\boldsymbol{z}$ could not be an eigenvector of the 3x3 matrix B because that would violate some maximum number of eigenvectors.	3.4
Dimension Total	Students use the given information [$\boldsymbol{z} = \boldsymbol{y} + 0.5\boldsymbol{x}$ and $\boldsymbol{z}$ does not graphically lie on either eigenspace in $\mathbb{R}^3$ (of different eigenvalues) that contain $\boldsymbol{y}$ or $\boldsymbol{x}$] to reason about the numerical value of the dimensions of a matrix's eigenspaces to conclude $\boldsymbol{z}$ could not be an eigenvector of the 3x3 matrix B.	3.4

the analysis referring to a finite number of eigenvectors was developed with respect to Q5, but there may also be evidence of this in Q3. For instance, B78 reasoned that the linear combination of eigenvectors could not be another eigenvector because "technically, you could multiply the eigenvectors by any number and if you did so and another eigenvector was achieved there would be a possibility for infinite eigenvectors which doesn't make sense." We note that the student was envisioning all possible scalar multiples of an eigenvector, realizing that would produce infinitely many vectors. The perturbation for the student occurred when trying to consider these infinitely many vectors as eigenvectors; we cannot know, unfortunately, why this was not sensible to the student. Further exploring and developing these two analytical connections across Q3 and Q5—comparing student responses and using the themes across questions—is a direction of future research for us. It would entail integrating analyses of student work on the other four MCE questions as relevant and exploring any seemingly contradictory reasoning offered by the same student in different tasks.

We conclude our paper with some possible implications for teaching. First, our results presented in this paper suggest consistency with research that points to challenges student have with solving systems that have infinitely many solutions (Zandieh & Andrews-Larson, 2018). We caution that an overemphasis on finding bases for eigenspaces before students have developed a strong understanding of eigenvectors and eigenvalues could lead students to think that there is only one eigenvector for each eigenvalue or that eigenvectors must always form a linearly independent set. For example, an overemphasis on diagonalization of matrices may encourage students to think that all sets of eigenvectors are linearly independent. Relatedly, we echo the sentiment of Salgado and Trigueros (2015) that the concepts of basis, span, spanning set, and subspace are particularly challenging for students to develop a rich conception of; hence, if students are only asked to find bases for eigenspaces, they may never fully grasp what an eigenspace is. A focus on eigenspaces as subspaces has the potential to mitigate these challenges and help students see connections across the linear algebra course.

We propose that Linear Algebra instructors may want to give their students more questions like the ones shared in this paper that provide students opportunities to wrestle with linear combinations of eigenvectors and explore elements of eigenspaces. Additionally, having students engage with a dynamic "eigen-sketch," such as one proposed by Gol Tabaghi and Sinclair (2013) in Geometer's Sketchpad, could help students understand the existence of multiple eigenvectors for a single eigenvalue as they explore dragging an input vector $\boldsymbol{x}$ along the line of an eigenspace. Having students engage with these types of questions and activities, and encouraging discussions about them, might help students develop a more sophisticated understanding of eigenspace.

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