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ORIGINAL ARTICLE

Characteristics of teaching and learning single-digit whole number multiplication in china: the case of the nine-times table

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Abstract
This study investigates the teaching and learning of single-digit whole number multiplication in China. Analysis of data from documents, classroom teaching, and semi-structured interviews revealed three salient characteristics of emphasizing *oral calculation, calculation speed*, and *understanding* across standards, textbook and classroom practices. It also showed how mathematics teachers enact these features in their teaching practice to help students develop their computational skills. The study particularly elaborates the role played by the nine-times table, or *Chengfa Kou Jue Table* (CKJ Table) in teaching practices, as well as how teachers treat memorization of CKJ and how understanding of operations contributes to their better understanding the relationship between the two in the teaching process.

Keywords Multiplication · Mathematics curriculum · Teaching · Nine-times table

1 Introduction

The ability to do basic arithmetic is considered a critical skill for people in daily life (Nunes et al. 2016) and crucial for laying a solid foundation for the study of more advanced mathematics. Thus, the development of basic arithmetic abilities is essential. Research from various perspectives such as cognitive science (Campbell and Xue 2001), neuroscience (Zhou et al. 2007), and education and schooling (Lampert 1986; Steel and Funnell 2001) has already investigated students' ability to perform multiplication—an important part of basic arithmetic. The Programme for International Student Assessment (PISA) 2012 assessment (OECD 2014) showed that Chinese students have a highly developed ability to perform arithmetic operations, in the sense that they can calculate rapidly and accurately. Additionally, other studies also revealed that Chinese students also perform well on the tasks that require arithmetic computations (Ho and Fuson 1998; Silver and Kenney 2000; Cai 2002), as they have a high ability to retrieve whole number single-digit multiplication facts (LeFevre and Liu 1997). Existing research on how to teach single-digit multiplication focused on specific teaching methods that help students recall multiplication facts in meaningful ways, such as using tools, scenarios, or other materials (Kahn 2005; Steel and Funnell 2001). Some studies on teaching the nine-times table tended to identify patterns within the table and extend findings based on the table, but the retrieval/memorization of basic facts of the nine-times table is ignored (Woodward 2006). However, there has been a lack of research reported in the literature into how the teaching of single-digit multiplication through the nine-times table helps students to understand, memorize, and achieve fluent calculation (Woodward 2006). Research indicated that the nine-times table has historically been an elementary part of Chinese mathematics curricula for basic arithmetic (Cao et al. 2015), which may help explain Chinese students' proficiency in multiplication calculations. Therefore, the aim of this study is to explore the teaching and learning of single-digit multiplication facts with whole numbers, with a focus on the application of the nine-times table in China. This study may further contribute to uncovering Chinese students' outstanding performance

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in calculation and provide insights into teaching and learning of single-digit whole number multiplication in other contexts.

2 A description of the current state of the art

The teaching and learning of basic arithmetic have always been core contents in elementary mathematics education. Whole number arithmetic, along with its related concepts, lays the foundation for further learning in mathematics (Bussi and Sun 2018); therefore, there is wide agreement, concerning most studies and curricula, that acquiring both mathematical facts and computational skills in one's early years is of seminal importance (Campbell and Robert 2008).

It is widely accepted that computational skills include speed and accuracy of operations. Yet, due to the recent trend towards developing flexibility of arithmetic operations, various calculation strategies have been utilized, including comparative memorization, i.e., memorizing some facts and comparing the unknown problem with these memorized facts to solve problems, and retrieval (Lin and Kubina 2005; Nunes et al. 2016). While retrieval has long been considered the most effective calculation strategy for coping with basic arithmetic problems, Campbell and Xue (2001) explained that direct retrieval of basic arithmetic facts reduces responding time when compared to other strategies, for example, calculation through counting, etc., because no further operation is needed except for recalling facts. Research on students' calculation performance shows that Chinese students tend to use retrieval strategies more frequently and proficiently than their Western counterparts (He 2015, cited in Arzarello et al. 2018), resulting in relatively better calculation speed and accuracy (Cai 2002; Sun et al. 2015).

Effective retrieval of arithmetic facts requires students' memorization of those facts. Thus, the teaching of memorization and practicing of basic facts has been the focus in teaching arithmetic. Historically, direct instruction and drill have been deemed the best ways to teach arithmetic (Baroody and Dowker 2013). Geary (1994) claimed that conceptual understanding is not sufficient for developing computational proficiency; drill and practice are also necessary. Artigue et al. (2006) suggested that drill and practice help students form a sense for numbers and later learn more complex mathematics. Fluent memorization also contributes to people's mind functions 'running in the background.'

Teaching through drill, practice, and memorization is crucial for developing children's arithmetic fluency and accuracy (Artigue et al. 2006). However, some researchers argued that it may impede students' ability to understand underlying concepts, thereby inhibiting the development of conceptual understanding (e.g., James 2008). In particular, in 1989, the National Council of Teachers of Mathematics (NCTM) developed a new curriculum standard that deemphasized rote memorization in the teaching of basic arithmetic while emphasizing teaching through problem solving.

The debate about whether arithmetic should be taught through rote memorization (e.g., emphasizing computational skills) or in more meaningful ways (e.g., emphasizing understanding) has a long history (Resnick and Ford 1981). Some researchers argued that the development of computational skills predates the development of understanding, while others insisted on the opposite (Rittle-Johnson and Siegler 1998). However, recently, there is a perspective that regards computational skills and conceptual understanding as having an iterative or simultaneous relationship, resulting in students developing both types of knowledge in a more complicated and dynamic way (Baroody and Ginsburg 1986; Rittle-Johnson and Siegler 1998). Research has also suggested that developing computational skill and conceptual understanding are not contradictory; rather, they are mutually supportive (Ma 1999), and there is no fixed way to teach and learn arithmetic expertise.

2.1 Research background

The nine-times table is quite popular and has sparked interest all over the world, albeit often under different names (e.g., multiplication table, pithy table) (Bussi et al. 2018). Teaching the nine-times table gives students a solid foundation from which to perform calculations proficiently and has historically been a stable part of the Chinese mathematics curriculum (Cao et al. 2015). Fluent memorization of nine-times table facts plays an important role in helping students calculate multiplication rapidly and accurately (Bussi et al. 2018).

In China, the nine-times table (Table 1) has a total of 45 cells; in each line, only the case of $a < b$ for $a \times b$ has been written, because of the commutative property. The table has also been called the *Chengfa Kou Jue Biao* (*CKJ table*), where *Chengfa* means "multiplication," *Kou* is "mouth" (i.e., oral), *Jue* refers to a rhyming and/or easily-memorized sentence based on a particular rule and/or fact, and *Biao* means "table". So the CKJ table is actually a combination of the written and oral nine-times table. That is, the CKJ table contains not only mathematical expressions of multiplication facts, but also linguistic properties that make them easily readable and easy to memorize. In the following table, two lines have been written in each cell, the top being CKJ and the lower one the corresponding mathematics expression.

As the following textbook excerpt illustrates, the presentation of multiplication information is always coupled with the appearance of a corresponding CKJ—that is, the learning of multiplication facts is based on CKJ. Further,

Table 1 The nine-times table

一一得一 (1×1=1)								
一二得二 (1×2=2)	二二得四 (2×2=4)							
一三得三 (1×3=3)	二三得六 (2×3=6)	三三得九 (3×3=9)						
一四得四 (1×4=4)	二四得八 (2×4=8)	三四十二 (3×4=12)	四四十六 (4×4=16)					
一五得五 (1×5=5)	二五一十 (2×5=10)	三五十五 (3×5=15)	四五二十 (4×5=20)	五五二十五 (5×5=25)				
一六得六 (1×6=6)	二六十二 (2×6=12)	三六十八 (3×6=18)	四六二十四 (4×6=24)	五六三十 (5×6=30)	六六三十六 (6×6=36)			
一七得七 (1×7=7)	二七十四 (2×7=14)	三七二十一 (3×7=21)	四七二十八 (4×7=28)	五七三十五 (5×7=35)	六七四十二 (6×7=42)	七七四十九 (7×7=49)		
一八得八 (1×8=8)	二八十六 (2×8=16)	三八二十四 (3×8=24)	四八三十二 (4×8=32)	五八四十 (5×8=40)	六八四十八 (6×8=48)	七八五十六 (7×8=56)	八八六十四 (8×8=64)	
一九得九 (1×9=9)	二九十八 (2×9=18)	三九二十七 (3×9=27)	四九三十六 (4×9=36)	五九四十五 (5×9=45)	六九五十四 (6×9=54)	七九六十三 (7×9=63)	八九七十二 (8×9=72)	九九八十一 (9×9=81)

CKJ of 5 & math expressions of 5

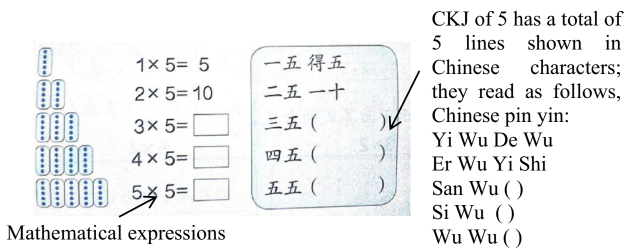

Fig. 1 Text excerpts presentation of mathematics expressions and CKJ

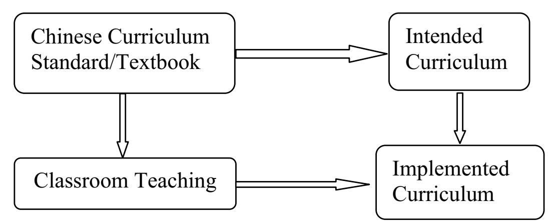

Fig. 2 Theoretical framework of the study

in textbooks, every multiplication lesson has been named a *Chengfa Kou Jue* of a specific number; for example, *Chengfa Kou Jue of 5* would be a lesson that introduces the multiplication facts for the number five. Calling every multiplication topic CKJ is also widely accepted in educational contexts (Fig. 1).

2.2 Theoretical framework

The nine-times table has kept its place in the Chinese mathematics curriculum over decades although there were many rounds of curriculum reforms. Ni (2015) argued that the Chinese school curriculum, textbooks, and classroom teaching related to the study of mathematics all work to promote the proficiency of Chinese pupils. The present study has the aim of examining how the nine-times table is presented at different curriculum levels. Thus, the analytical framework for this study is based on part of the Trends in International Mathematics and Science Study (TIMSS) curriculum model—i.e., the intended curriculum and implemented curriculum (Mullis and Martin 2013). According to the TIMSS curriculum model, the intended curriculum explains the national, social, and educational context, while the implemented curriculum represents the school, teacher, and classroom contexts. Therefore, the Chinese national Curriculum Standard and textbook were chosen as documents representing the intended curriculum, while actual classroom teaching describes the implemented curriculum. Figure 2 illustrates the operational conceptual framework of this study:

With regard to the research purpose and theoretical framework, in this study we address the following research questions:

1. What are the characteristics of single-digit whole number multiplication as presented in the intended curriculum?
2. How do Chinese teachers teach single-digit whole number multiplication?
3. Are the curriculum standards, textbooks, and classroom teaching consistent, and if so, how strongly?

3 Methods

3.1 Data collection

In line with the research questions and the corresponding conceptual framework, Table 2 (below) outlines the data used in this study.

In China, the national mathematics curriculum for compulsory education is known as the *Full-time Compulsory*

Table 2 Data sources addressing research questions

Conceptual framework	Participants	Main data
Intended curriculum	NA	National Curriculum Standard Textbook (People's Education Press, PEP)
Implemented curriculum	Two teachers (T1, T2)	Audio recordings of two taught lessons with T1 Post-lesson interview with T1 and T2 together
	Two teachers (T3, T4)	Audio recordings of semi-structured interviews with T3 and T4

Education Mathematics Curriculum Standard (hereafter the *Curriculum Standard*) (Ministry of Education 2011). It is a national-level document depicting social expectations of, and initial standards for, what students are expected to learn in the field of mathematics. Therefore, the national *Curriculum Standard* is the main source representing what the curriculum intends to achieve in the realm of mathematics education.

As part of the centralized education system, school textbooks are also developed within the framework set by the *Curriculum Standard*. Textbooks are considered important guidelines for teaching (Cao and Wu 2016), and the selection of and information provided by the textbook are crucial factors that influence teaching. This study selected the textbook version published by People's Education Press, which is used in approximately 70–80% of all Grade 2 classrooms in mainland China.

Regarding the implemented curriculum, one specific case study was conducted with the participation of an in-service mathematics teacher (T1). Two of her normal lessons on the topic were individually observed and audio-recorded. The lessons were taught in consecutive weeks—the first lesson was on the Friday of the first week and the second on the Tuesday of the following week. Both lessons took place during the second period of the morning session (each morning is comprised of four periods).

There were also two after-class semi-structured interviews with T1 and T2 together on the topic of teaching practices, lasting between 1.5 and 2 h each. T2 was the headmaster of the school and responsible for managing everyday mathematics teaching affairs; thus, the lesson plans and teaching designs used by T1 were all outcomes of discussions with T2. T1 was selected because she had been teaching for 6 years, had become a mature teacher, and her lessons, as did most teachers', following the teaching guidelines suggested by National Curriculum and textbook, were representative of typically taught lessons.

Interviews with another two teachers facilitated the data triangulation, which helps us better to understand teaching style in terms of the topic (Denzin 1970). T3 and T4 were approached separately and individually in October and November. Both teachers were experienced in teaching multiplication. At the time of the study, T3 was teaching Grade 2; T4 was teaching Grade 4 but had taught Grade 2 on four or five previous occasions. T4 was also a teaching research specialist whose opinions were very important to other teachers. Their interviews were conducted at their school and lasted around one hour each. Later, during the coding and analyzing process, some questions and concepts required the researchers to seek clarifications from the interviewees; this was normally done over the phone (Table 3).

3.2 Data analysis

3.2.1 Curriculum standard

The *Curriculum Standard* sets the basic standards for compulsory education in China, with criteria specifying what is to be learned at each of the different learning stages. The teaching of single-digit whole number multiplication takes place in Grade 2, which is part of the first learning stage (Grades 1–3). Taking *multiplication inside of the table* (*biao nei cheng fa*), and *single-digit multiplication* as keywords, all related sentences in the national *Curriculum Standard* about the topic were classified and major characteristics were identified.

Table 3 Teachers' demographic information

Teachers	Teaching grade	Teaching experience (years)	Year of completing bachelor's degree	City/county	Position
T1	Grade 2	6	2011	B	Teacher
T2	Grade 2	26	2005	B	Headmaster
T3	Grade 2	7	2010	P	Teacher
T4	Grade 4	25	2009	B	Teaching Research Specialist

3.2.2 Textbook

The content structure presented in the textbook was analyzed. The entire nine-times table takes up two chapters. Table 4 (below) summarizes how the teaching of all CKJs is organized over the two chapters and how much time should be devoted to covering the material. In every normal CKJ lesson, the material is organized in a similar manner. The "CKJ of 7" (at the middle position) was chosen to be specifically analyzed in this study, focusing on how particular content is presented and organized.

3.2.3 Audio-taped lesson and classroom observation

As the structure of the two observed lessons—CKJ of 6 and CKJ of 7—are similar, we chose to analyze one of the lessons, in addition to referring to the teacher's teaching plan. Main activities were selected from the lesson and are presented in the Sect. 4; this study also addresses how students learned multiplication through the activities in the lesson.

3.2.4 Semi-structured interview

Teachers' interviews were analyzed mainly to examine how teachers perceived the teaching and learning of single-digit whole number multiplication. Specifically, teachers were asked what they thought of and how they planned such teaching, their opinions about CKJ tables, and their experiences in teaching them. The constant comparing coding process (Flick 2007) was adopted in the study to analyze all transcripts. First, the researchers read the texts and highlighted important related intercepts; at the same time, memos were written for some texts to document related information and possible ideas based on the literature and the researchers' prior knowledge. Second, open coding was conducted, wherein specific incidents and behaviors were recognized and coded. The third step consisted of constructing theme categories (CKJ, context) based on the connections identified among facts in the open coding process. We then collected all the texts belonging to a specific category and further sorted them into different sub-categories. Finally, the resulting sub-categories became our main unit of analysis for further study. The process is presented below (Table 5).

4 Results

4.1 National curriculum standard

Two passages in the document were found that specifically addressed multiplication and the nine-times table:

1. "Students should be able to orally calculate addition and subtraction of whole numbers less than 20, as well as multiplication and division inside the nine-times table proficiently" (Ministry of Education [MOE] 2011, p. 17).
2. The assessment of calculation skills acquired in the first learning stage (Table 6).

Broadening our view to include the teaching of whole numbers and main operations, we found that the *Curriculum*

Table 5 The coding themes and categories

Theme	Subcategory
CKJ	The format of CKJ
	Distinguish between CKJs and mathematical expressions
	The rationale of CKJ table
	Memorization and recitation of the CKJs
Context	Count specific items to calculate
	Understand the meaning behind the operation
	Motivating the interests of the students

Table 4 Content organization and arrangement of teaching time

Grade	Chapter	Content	Recommended lessons
The first semester of grade 2	The fourth chapter multiplication Table 1	A brief introduction of multiplication	3
		CKJ of five	1
		CKJ of two, three, four	3
		CKJ of six	4
		Reflection and review	1
	The sixth chapter multiplication Table 2	CKJ of seven	2
		CKJ of eight	3
		CKJ of nine	4
		Reflection and review	2

Each lesson lasts 40 min

Table 6 Calculation speed requirements as listed in the *Curriculum Standard* (MOE 2011, p. 53)

Skill	Speed requirements
Addition/subtraction of numbers less than 20 and multiplication/division table	8–10 items/min
Addition/subtraction of numbers less than 100 and mental calculation between one-digit number and two-digit number	3–4 items/min

Standard also proposes that students should be able to do the following:

3. "…combine specific contexts in order to understand and feel the meaning of addition, subtraction, multiplication and division" (MOE 2011, p. 17).

Therefore, with regard to the *Curriculum Standard*, the three main attributes students are required to demonstrate are: *oral calculation, calculation speed*, and *understanding*.

4.2 Textbook presentation

For each specific topic, the CKJ of 7 for example, the textbook allocates three pages; one page regarding the main presentation of the new material, followed by two pages of practice drills. On the first page, there is a realistic game (tangram, a classic Chinese puzzle) that sets up a context that is used to introduce the idea behind the CKJ of 7. This context is interesting and visually presents the model of multiplication as repeated addition. At the end of the page, there is a small practice exercise that lists some expressions without their answers (Fig. 3).

On the next page, the textbook presents another 10 exercises with different features. Among the exercises, five (numbered 1, 3, 5, 7, 9) are word problems based on some realistic scenarios or visual representations, in which students are required to use the CKJ directly. Three (numbered 2, 6, 10) are pure arithmetic problems where students are required to solve 7, 16, and 9 arithmetic expressions, respectively. Purposely, the difficulty of arithmetic expression increases from only multiplication (item 2), to including multiplication and addition (item 6), finally including equivalent expressions with multiplication, addition and subtractions (for example, 6 × 7=, 5 × 7 + 7=, 6 × 7–7=_, item 10). The remaining two exercises (4 and 8) have students orally recite the CKJ of 7 (item 4) and recite the CKJ up to 7 (e.g. 1 × 1 = to 7 × 7=). At the end of this topic is a small mathematics game that depicts two students playing a game; their conversation goes something like this (Fig. 4):

> Five seven thirty-five.
> You calculate faster, the card belongs to you.

Every unit has a similar structure of presenting the teaching content: exploring the CKJ using context situations, and introducing the CKJ by incorporating arithmetic equations, CKJ and the multiplicative communicative property, and practicing the CKJ using reciting and variation practicing. Yet, there are differences regarding the context used for the setting and the ratios between word problems and pure mathematics problems in the practice part of the unit. However, every unit, including the two Reflection and Review sections, contains two reciting drills in each set of practice exercises. Furthermore, in the final Reflection and Review

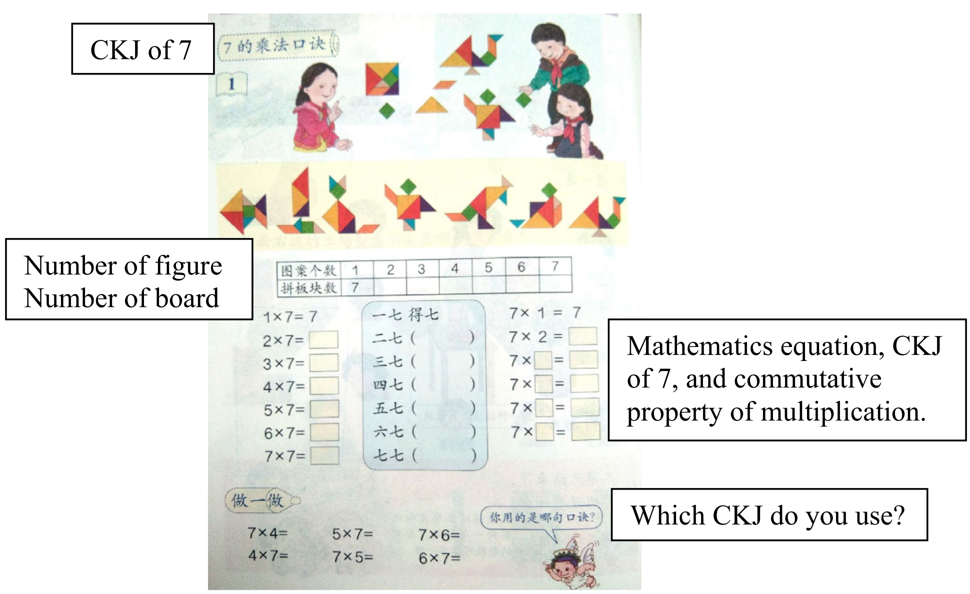

Fig. 3 Textbook presentation (first page in CKJ of 7)

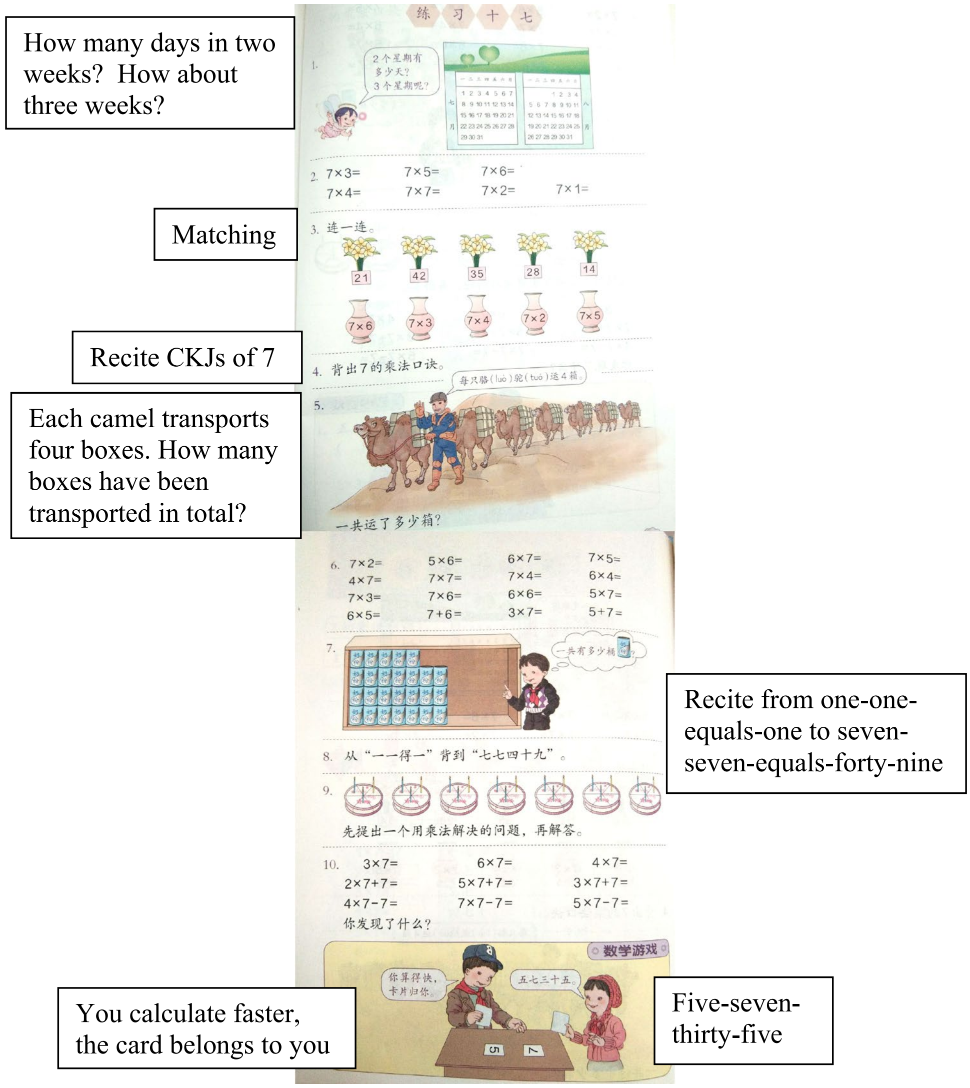

Fig. 4 Textbook presentation (second and third page in CKJ of 7)

section, the review of multiplication facts and the CKJs are combined to cover the whole CKJ table. Thus, in total, 45 pieces of CKJ in the multiplication table should have been learned, step-by-step. In a single lesson, as shown in the previously described text, the introduction is only about the facts as they pertain to a single number, but all previously learned CKJ sentences are incorporated in practice exercise. For instance, in the lesson on the CKJ of 5, the learned multiplication facts include only the five equations in the fifth line of the nine-times table; however, in the final lesson, students need to review the whole table. They are required to fill the whole table, where some facts are listed while others are not.

In short, the textbook emphasizes drill and practice based on understanding and connection. In addition, a variety of contexts are embedded in both the introduction and word problems of the practice section, forming another feature. Furthermore, a specific emphasis on calculation speed can be seen in the mathematics games at the end of each unit.

Given the main characteristics found in the *Curriculum Standard*, we attempted to identify the characteristics of the textbook and whether they corresponded to the *Curriculum Standard*. First, in line with the *Curriculum Standard*, the learning progression of CKJs provided appropriate scaffolding for students to develop their ability to perform oral calculations. The specific features of the CKJs extensively may contribute to fluent oral calculations because it is far easier for students to retrieve multiplication facts.

Second, the large number of pure arithmetic exercises presented in every single lesson to some extent demonstrates the intention of developing calculation speed, in that the large number of exercises shows how variation practice helps students increase their speed as memorization is consolidated. This routine echoes a common Chinese belief that “practice makes perfect” and reflects a Chinese tradition of learning through variation practice. As the discourse between the two students in the final mathematics game in the CKJ of 7 unit demonstrates, calculation speed was valued and rewarded.

Third, the embedding of specific contexts in the lessons is a salient textbook feature. Each CKJ is introduced using a specific context, while the word problems in the practice sections often employ realistic contexts or situations. According to the *Curriculum Standard*, this combining of contexts is designed to help students better understand the unit content. While it may be difficult to argue that use of these situations helps students understand multiplication, they at least help students understand the material in terms of the contexts and understand what they are trying to achieve through mathematics. Accordingly, the design of the textbook follows the three characteristics of multiplication teaching contained in the *Curriculum Standard*.

4.3 How the teaching and learning of single-digit whole number multiplication is enacted in the classroom

How T1 organized her lesson to teach the CKJ of 6 is portrayed in detailed descriptions of the main activities below.

Set context for review At the beginning of the lesson, the teacher set up a scenario to review the CKJ of 5. She used a zombie shooting game named *Plants vs Zombies* as the context to introduce the lesson and prepared a square chart as an aid to display the results. The final question asked: "To know how many bullets we need to shoot the zombies, we would need to use the CKJ of what number?" Students then reviewed the CKJ of 5 together.

Introduce new knowledge Turning to the second part of the lesson, which was to explore new knowledge and material, the teacher posed the following question: "If a more powerful zombie is coming, how many bullets do we need to shoot it down?" The accompanying slides revealed a cartoon scene that illustrated six bullets would be needed. The students were then asked to form the first CKJ, which was: One-six-is-six (*yi-liu-de-liu*). The teacher then presented a new chart that contained one more vertical line than the previous one, enabling the students to see the difference between the CKJ of 5 and the CKJ of 6. Here, the students were also asked to share their ideas on the meaning of mathematics expressions, such as, "What does the expression $2\times 6=12$ mean?" Students would deduce that it means to add two sixes or add six twos (Fig. 5).

Learning and presenting. The third part of the lesson was to complete the entire CKJ of 6. Accordingly, the students were asked to complete the chart displayed below (Table 7).

Here, students were encouraged to present their explanation and the corresponding mathematics expressions for the CKJ. For instance, a student may follow the teacher's question about how many bullets one needs to shoot five zombies,

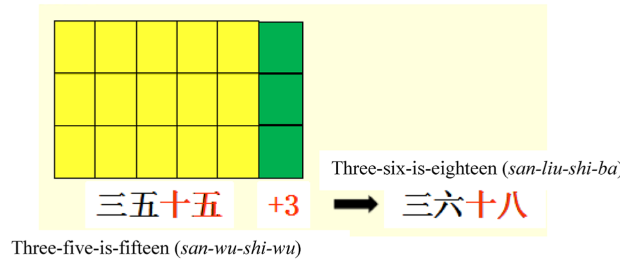

Fig. 5 Connections between CKJs shown through square chart

with the expected answer, the mathematics expression $5\times 6=30$—i.e., the CKJ *wu-liu-san-shi* (five-six-is-thirty).

Memorization. The fourth step was to review and strengthen the memorization process of the CKJs. The teacher asked students to memorize the CKJs through oral drills, both in pairs and as a whole class.

> T1: "Well, now let's try to remember the short formulas through clapping your hands with your desk mates, ready? Go!"

The students then paired off and recited the short formulas, clapping their hands to keep a rhythm. The next step in the lesson was to have the students observe the relationships between the CKJs. If the CKJs were displayed together, except for the number 6, the students could see that increasing the multiplier by one would increase the product by six.

Strengthen memorization through understanding. The final part of the lesson was to strengthen the memorization process using a visual model that consisted of circles in squares (Fig. 6). The teacher showed an animation of the squares to help reinforce the CKJ of 6 in the minds of the students.

During the lesson, the teacher started exploring contextual problems to help students understand the meaning of multiplication expressions, with 6 as the multiplier. She then helped students memorize multiplication facts by reciting CKJs. Classroom activities were designed to help students drill and practice, strengthen their memorization process, and calculate quickly and accurately.

4.4 How teachers perceived the teaching and learning of single-digit whole number multiplication

Based on the analytic framework (Table 5), the two main themes that emerged from the teacher interview data—*CKJ* and *context*—underpinned the entire teaching structure. Teachers proposed a normal process of learning CKJ as well as the role of using context in teaching, presented in the following section.

Table 7 Working table displayed in students' worksheet

Number of zombies	1	2	3	4	5	6
Number of bullets						

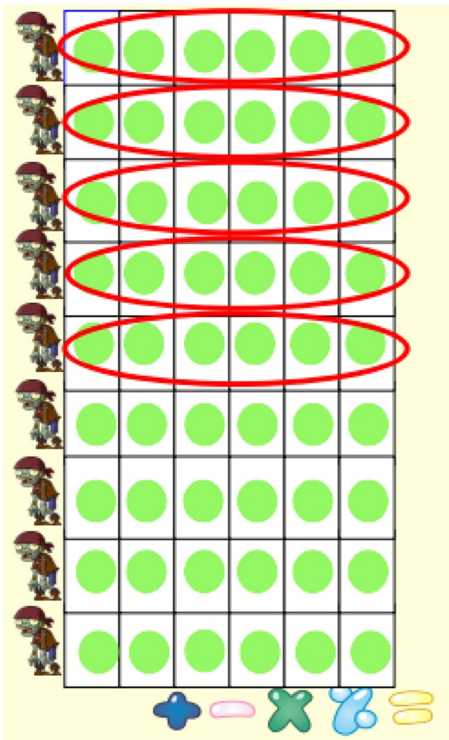

Fig. 6 Revision of CKJ through visual model

4.4.1 The process of learning CKJ

4.4.1.1 Start from the format of CKJ According to the teachers, teaching multiplication begins with introducing the way in which CKJs are composed, including understanding the rules for forming CKJs. At the start of the lesson, teachers taught how the CKJs were developed:

> T3: "...when I am teaching the CKJ of 5, firstly, I will let students use their fingers to assist. For on one hand, you have five fingers, then you have the one five, namely: Yi-Wu-De-Wu (one-five-equals-five). Since the product is a one-digit number, then you need to add De (equals) between the multiplication equation and the product so as to make the sentences rhyme and be more readable. Then, if your two hands have 10 fingers, which is Er-Wu-Yi-Shi (two-five-is-ten). At this moment, I will stress that the product, 10, is a two-digit number, so we do not need to add De (equals) inside [the expression], but just directly connect the product which follows..."

> T2: "…they then practice the formation process again and repeat this every time they learn a new CKJ. Students then get familiar with the rules and feel more comfortable forming similarly-structured CKJs..."

This first lesson, on how to form a CKJ, laid a foundation for students to develop more CKJs through imitation. To form a CKJ, students tried to find the rules between the numbers and the Chinese characters in the CKJ. Teachers tended to encourage students to form and develop the CKJs themselves, believing this to be an essential basic step. They argued that the process is not only important for students, enabling them to form more CKJs, but also helps them commit the CKJs to memory. Further, the development of the CKJs established the relationships between numbers, which directly corresponded to mathematics expressions; that is, when a student formed a CKJ, they were also identifying the mathematical relationship between the numbers. For example, *Er-Wu-Yi-Shi* (two-five-is-ten) can, of course, be written down as the mathematics expression, $2 \times 5 = 10$.

4.4.1.2 Distinguishing between CKJs and mathematical expressions In the lesson taught by T1, students were asked to answer the mathematical expression ($1 \times 5 = 5$) first, followed by the related CKJ (yi-wu-de-wu). However, some made mistakes in their order—e.g., providing the CKJ when they were supposed to answer with the mathematical expression. At the time, the teacher directly pointed out the problem, then repeated the instructions so students realized the two were different. The corresponding relationship between the mathematical symbol "×" and the Chinese character "*De*" holds different meanings for teachers. In their interviews, T1 and T2 shared that they would like students to say "×" (*Cheng Yi*) to represent multiplication and to reinforce that the term denoted an operation. However, in the CKJ sentences, "*De*" referred to the product of a multiplication.

4.4.1.3 The rationale of the CKJ table To help students memorize CKJs, teachers encouraged them to infer calculations from one expression to another. For example, students generally found *Er-Wu-Yi-Shi* (two-five-is-ten) quite a bit easier to remember than *Er-Liu-Shi-Er* (two-six-is-twelve). As the former was one of the first CKJs they learned, if students found the latter difficult to understand, they could first recall that two fives is ten, and then add two plus ten to make twelve—the final result of two sixes. In this sense, teachers treated CKJs as an organized structure, rather than individual sentences. As one teacher explained:

> T3: "Students can memorize the CKJ of 6 or 7 based on the CKJ of 5; they may also be able to memorize the CKJ of 9 based on the CKJ of 10..."

The ability to infer calculations is premised on students understanding the meaning of multiplication. In practice, when teaching CKJs, teachers focused more on the relationship between 'old knowledge' (i.e., what students had previously learned and understood) and 'newly-taught knowledge'. As the lesson on the CKJ of 6 illustrated, T1 spent an appreciable amount of time having students explore the relationship between the CKJ of 5 and the CKJ of 6 in an effort to help them understand the relationships between CKJs across the entire table, from both the horizontal and vertical axes.

4.4.1.4 Memorization and recitation of the CKJs Memorization of CKJs is always a long-term goal for students and may happen earlier than in formal instruction, due to their readability and popularity. As teachers explained:

> T4: "...children have no idea of some nursery rhymes, or poems, but they are still able to remember them and perform them at family parties, so, why not show off their knowledge of the CKJs...I usually require my students to memorize CKJs in the summer vacation before the semester when second grade starts ..."

> T3: "...some of my students told me that her/his mom helped them memorize CKJs when they were in the first grade...they will spend every morning for the whole semester on reciting CKJs till they realize fluent retrieval..."

The emphasis teachers placed on memorization reflects the traditional educational culture, wherein rote learning and memorization play vital roles. They insisted on pushing students to memorize the CKJs since they believed it was the beginning of learning and a very natural and embedded method. On the one hand, they strictly required students to recite and memorize multiplication facts; on the other, they tried to help students attain more comprehensive understanding.

4.4.2 The role of using context

4.4.2.1 Count specific items to calculate Based on the feedback from the teacher interviews, a teachers' first objective when creating a context was to help students count specific items. This was important because, when teachers used audio/visual slides or other teaching aids, students could directly see what items and how many items were there, instead of just looking at abstract numbers. Secondly, the teachers then normally posed a hint question to expand the problem, which may not be shown in visual form. Then the problem had to be extracted from the real context in order to become a pure mathematics problem. At this moment, students needed to make an inferential calculation, based on what they had counted before. The students needed to imagine what number had been added and what the final answer should be. In the teaching case of this study, T1 explained she used a square chart as an aid, as it was "in mathematical form, which is used to summarize the mathematical problems that students have abstracted from the context," and helped develop her students' ability to think in a mathematical way. During the lesson, she asked students to answer questions in a way that first described the story using specific items within the context, and then illustrated the answers using mathematical expressions, followed by the corresponding CKJ (Fig. 7).

4.4.2.2 Understanding the meaning behind the operation Teachers noted that the contexts were meaningful, helped students to understand mathematical relationships, and were good training for their oral presentation skills. The main idea was to let students understand that, for instance, if they had four fives, they could write it down using the mathematics expression: $4 \times 5 = 20$. From there, they could see the CKJ of this mathematical expression as *Si-Wu-Er-Shi* (four-five-is-twenty). In the lesson, teachers asked students to present the process and to work in pairs or groups to interact and repeat this process. As a result, students appeared to grasp the difference between mathematical expressions and CKJs, and the mathematical relationships between the items in a specific context.

Learning sheet for the CKJ of 6(2)

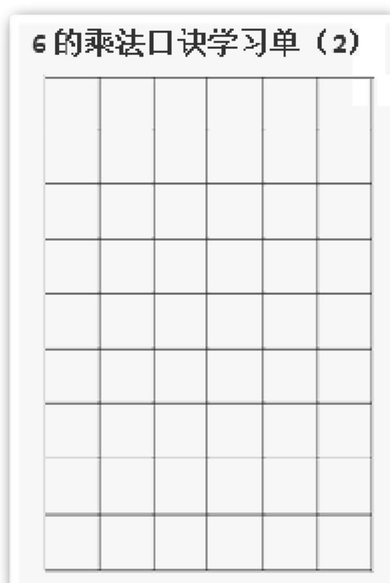

Fig. 7 Learning sheet for the CKJ of 6

This helped students feel and, to some extent, understand the meaning of multiplication.

> T2: "I always try different methods to let them understand what multiplication is, since to what extent they memorize it is difficult to control. What I can do is make an effort to have them understand more deeply...then when they have difficulty in retrieval, they can find other ways to solve problems..."

4.4.2.3 Motivating the interests of the students During the interviews conducted after the observed lessons, teachers reflected on the contexts they had planned for the taught lesson. The general criteria teachers used to assess whether a lesson was effective was how engaged the students were in the context used and what they got from the context. As one teacher explained:

> T1: "I am quite satisfied with the taught lesson, because I think students found the context or maybe the game interesting. They enjoyed the process and engaged in exploring the mathematics relationships."

Teachers tended to set various contexts in which students derived mathematical problems from real situations. In the two observed lessons, T1 used the game *Plants vs. Zombies* as a context to let students count the "bullets" needed and obtain their products. For example:

> T1: "...if shooting one zombie needs 6 bullets, how many bullets will be needed to shoot 2 zombies?"

These activities were lively and vivid, which deepened students' familiarity with mathematics expressions.

To summarize, from the teachers' perspective, the CKJ learning process played the most important role in students learning table facts. The role of context in teaching, on the other hand, was to help students understand the underlying meaning of operations, which also helped strengthen students' memorization. Teachers claimed that CKJs could serve as a shortcut for students memorizing the nine-times table. However, if students could not make use of CKJs, they would then think of the specific meaning of every mathematical expression they had encountered. What they had learned from the context would help students' retrieval. Therefore, the teaching observed during this study consistently helped students strengthen their CKJ memorization, because teachers believed fluent CKJ retrieval was key to calculating quickly and accurately.

> T2: "…in the calculation process, we do not need students to count... This is a pure retrieval process. Students already have powerful calculation ability. [The] CKJ provides a shortcut for them to realize single-digit whole number multiplication proficiently."

5 Conclusion and discussion

The following discussion illustrates how the results address our research questions. On the one hand, the 'how' question sought to explore teaching practices. As shown in Table 5, the two main themes—CKJ and context—help us to grasp the picture of actual teaching and illustrate the complementary roles of memorization and understanding. The study then provides a specific case illustrating that the debate on the teaching through memorization or for understanding (Baroody 2003) has been negotiated at a practical teaching level. On the other hand, the 'what' question regarding the characteristics of the intended curriculum is confirmed by our results from a tangential perspective. It means that those features of teaching we identified from the national curriculum drove our identification of how consistently the attributes of teaching multiplication were maintained among the national curriculum, textbook, and classroom teaching, which further contributes to understanding how teaching the development of fluent arithmetic could be enacted in the classroom.

5.1 Stress on calculation speed and accuracy

As noted in the literature, the commonly regarded values in computational skill are calculation speed and accuracy (Bussi and Sun 2018). This study detailed the role played by CKJs in teaching single-digit whole number multiplication in China and revealed that calculation speed and accuracy were stressed throughout the theoretical framework. Fluent memorization of CKJ helps students retrieve information quickly and precisely, which may contribute to students developing strong calculation abilities. The linguistic features of CKJs promote the fluent memorization of CKJ tables. In contrast to several Western languages, in Chinese, spoken whole numbers are always the same as the written numbers, which means "the written numeral directly reflects its pronunciation and thus has not diverged from the spoken language" (Sun and Bussi 2018, p. 60). This means that once students memorize CKJs, they are able to directly reflect on calculating the corresponding mathematical expressions. On the one hand, the cultural and linguistic characteristics of the CKJs can be an aid to teaching (Zhou et al. 2007). On the other, as CKJs offer direct support for oral computation, the use of the CKJs is consistent with the emphasis on oral calculation skills specified in the *Curriculum Standard*. As the discussion of the CKJ of 7 noted, the two recitation tasks in the textbook stress oral computation as well as memorization of the CKJs.

5.2 Emphasizing memorization along with understanding the meaning of operation

As the case study lessons and teacher interviews revealed, the memorization of the CKJs was a step-by-step process that began with a context showing where CKJs were first introduced and formed, which helped students understand how a CKJ is developed. Teachers were careful to distinguish between CKJs and mathematical expressions. This was in line with the textbook presentation, and it revealed teachers' attitudes toward the CKJs as a tool for memorization. This reflects a feature found by Sun and Bussi (2018), who noted that nine-times tables have long been used in the teaching of arithmetic in China; thus, from the teachers' standpoint, learning the rationale of the CKJ table and the different properties of specific CKJs were all aids to help students memorize. Therefore, memorization was the *de facto* main goal of their teaching and a consistent long-term goal of the learning process. The linguistic advantages of the CKJ decreased students' memory load and contributed to effective retrieval; the process calls for continuous memorization until students can fluently recall mathematical facts.

At the same time, the meaning of mathematical facts is never neglected throughout the process. The combining of contexts is a way to promote understanding, in line with previous research (Lampert 1986). In addition, as noted, setting a context helps students understand the meaning of mathematical relationships and the corresponding connections with CKJs. As teachers stated, a good context employs familiar situations to raise students' interest in counting and calculating. This is also in line with previous research concerning teachers' perceptions on understanding the meanings of and relationships among mathematical operations (Ma 1999). However, it seems that understanding mostly appeared during the taught lesson, in which teachers kept asking students to explain the meaning of each mathematical expression, indicating their commitment to ensuring students understanding thereof. After the taught lesson, the goal then turned to memorization. From the teachers' perspective, conceptual understanding's main role was helping students' stable memorization.

In line with the debates about rote memorization and more meaningful instruction, students' development of computational skills and conceptual understanding in this study can be seen as iterative or intertwined with each other in the teaching process (Baroody and Ginsburg 1986). This study reveals that, in the teaching of multiplication, teachers' views on these two processes may be different. From the teachers' perspective, the use of CKJs and the use of contexts, respectively, serve two different purposes—memorization and understanding. Fluent memorization is still the main goal in teaching, but from the textbook presentation to the delivery of the lesson, the two processes always appeared together and were intertwined with each other.

5.3 Keeping consistency throughout the intended and implemented curriculum

The study began by recognizing the main characteristics required for the teaching of multiplication, as listed in the national *Curriculum Standard*. The required three characteristics—oral calculations, calculation speed, and understanding of operations—are partly in line with previous findings claiming that the speed and accuracy of oral calculation are important requirements in the Chinese curriculum standards (Sun and Bussi 2018). Throughout the entire study—from the *Curriculum Standard*, to the textbook, to the classroom teaching—it is possible to see the consistency of these characteristics, although perhaps with different levels of performance in different areas.

The textbook plays a crucial part in setting the context. Moreover, the word problems in the textbook all have their own contexts, suggesting fulfillment of the third characteristic (the combining of contexts to aid understanding). This is consistent with the findings of previous studies, particularly the "Variation tradition in word problems" (Sun and Bussi 2018, p. 59). At the same time, the textbook also contains a sizable number of pure mathematics problems and two recitation tasks focused on training in the areas of oral calculations and calculation speed. With regard to mathematics teaching, as stated above, the application of the CKJ table helps students to memorize and recall facts more efficiently, which may improve students' oral calculation accuracy and speed (Campbell and Xue 2001). However, teachers never ignored the goal of having students understand the meaning of operations.

China is a country with a uniform standard curriculum that governs education for the entire country. Therefore, the *Curriculum Standard* plays an important role in guiding mathematics education, not only in terms of textbook content, but also classroom teaching, teacher education, assessment, etc. (Wang et al. 2018; Cao and Leung 2018). Accordingly, textbooks have been regarded as playing a dominant role in instruction (Robitaille and Travers 1992). Therefore, the consistency of the characteristics throughout the curriculum is reasonable. Furthermore, the study also illustrates how classroom teachers take the material in the textbook and expand upon it, suggesting (Huang et al. 2013), that in the process of implementing the curriculum, the "guidelines" in the *Curriculum Standard* have effectively been broadened.

6 Concluding remarks

This study followed the curriculum model in a specific way to see how single-digit whole number multiplication has been taught in China and has revealed the characteristics of the teaching of multiplication in detail. The three main characteristics revealed in the curriculum may offer some insight into the relatively high computation skills of Chinese students. The teaching practices detailed in this study touch on both memorization and understanding, which helps us better understand how this specific teaching with CKJ can happen. The application of the nine-times table has drawn a great deal of attention in the area of whole number arithmetic globally. This study therefore offers insight into how we can make use of the table to help students enhance their computational skills. Further, the results pertaining to how to teach students to memorize and/or understand content better also help us to understand teaching practices better.

The limitations of this study should also be taken into consideration. First, this study did not involve students' post-lesson interviews. In addition, the teacher interviews may not offer a holistic picture of classroom teaching, and since only four teachers participated in the study, it would be difficult to say these four teachers represented the whole spectrum of teacher opinions on the subject of single-digit whole number multiplication instruction. That said, the purpose of this study was only to open our eyes to a specific situation and to help us understand a particular teaching context. Further research could include student participation and more detailed classroom observation. Further investigation using the study's theoretical framework could include the attained curriculum, which assesses student achievements and would help us better to understand the quality of instruction.

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