ZDM (2018) 50:91–102
https://doi.org/10.1007/s11858-017-0900-6

ORIGINAL ARTICLE

Open word problems: taking the additive or the multiplicative road?

Tine Degrande[1] · Jo Van Hoof[1] · Lieven Verschaffel[1] · Wim Van Dooren[1]

Accepted: 1 November 2017 / Published online: 7 December 2017

Abstract

Previous studies have repeatedly shown that children often incorrectly use an additive model for multiplicative word problems, and a multiplicative model for additive word problems. The present study aimed to investigate which model upper primary school children tend to choose in word problems that are open to *both* ways of reasoning. In particular, a non-symbolic variant of the snake task of Lamon (Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers, Taylor & Francis Group, New York, NY, 2008) was administered to 279 children in fifth and sixth grade of primary education. Children were asked to indicate which of two snakes had grown the most, and to verbally explain the reasoning behind their answer. Results revealed that additive reasoning (i.e., absolute growth) was more frequently used than multiplicative reasoning (i.e., relative growth), although it appeared to be harder to verbalize. Second, both trends were more prominent for fifth than sixth graders. Third, contrary to previous studies with younger children, we did not find any differences between answers on discrete and continuous variants of the task. Nevertheless, children's answers were more often explicitly verbalized in discrete than continuous items. Theoretical, methodological, and educational implications for solving word problems, and more generally for modelling in the domain of additive and multiplicative reasoning, are discussed.

Keywords Modelling · Word problem solving · Additive reasoning · Multiplicative reasoning · Open word problems

1 Introduction

Within the domain of mathematics education research, much attention has been devoted to the connection between mathematics and the real world. This process of applying the appropriate mathematical operations in order to make sense of everyday-life situations and to solve real-life problems, is called mathematical modelling (e.g., English and Lesh 2003; Van Dooren et al. 2006; Verschaffel et al. 2000, 2007). Mathematical modelling is conceived of as a complex process, consisting of a number of phases (e.g., see Blum and Niss 1991; Verschaffel et al. 2000). First, one needs to understand the problem situation and to build a situation model. Second, the situation model has to be translated into a mathematical model, and third, the mathematical model has to be worked out by means of calculations in order to derive results. Fourth, those results have to be interpreted, and lastly, evaluated and communicated (Verschaffel et al. 2000).

One major way in which the teaching of mathematical modelling is initialized is through word problems (Verschaffel et al. 2000, 2007). Although learning to model mathematically of course requires a range of modelling tasks that go far beyond classical school word problems (Verschaffel et al. 2000, 2007), there is little doubt that, when appropriately formulated and used, word problems are a valuable tool as "exercises in mathematical modelling" (Verschaffel et al. 2010, p. 9). Much research on mathematical modelling in primary school has focused on word problems. Initially, most of the research attention was devoted to additive word problems, in particular one-step addition and subtraction word problems. Since the late eighties, scholars have increasingly turned their attention to the field of multiplicative word problems. Subsequently, authors argued for a simultaneous investigation of the development of students' solutions of additive and multiplicative word problems (for reviews of this research domain, see Greer 1992; Nunes and Bryant 2010; Verschaffel et al. 2007). In this article, we

✉ Tine Degrande
Tine.Degrande@kuleuven.be

[1] KU Leuven, Centre for Instructional Psychology and Technology, Dekenstraat 2, Box 3773, 3000 Leuven, Belgium

focus on the models being chosen in word problems that are open to both additive and multiplicative reasoning. In what follows, we first introduce the field of multiplicative word problems and the additive errors that primary school children make in multiplicative word problem solving. Afterwards, we analyze the multiplicative errors that primary school children make in additive word problem solving. Moreover, we report the way in which both types of errors have been traditionally interpreted (i.e., in terms of abilities), and challenge this traditional interpretation by proposing an additional explanation for children's errors, that is a preference for additive or multiplicative reasoning. This additional explanation provides the rationale for looking at word problems that are open to both ways of reasoning.

2 Theoretical background

Multiplicative reasoning lays the foundation for many mathematical ideas, such as fractions, ratios, and linear functions (Vergnaud 1988). Hence, it plays a pivotal role in primary mathematics education. In Flanders (Belgium), the standards for elementary school mathematics are the same for all schools (Ministerie van de Vlaamse Gemeenschap 1997), so the (order of) contents and instructional approaches are quite similar. Children typically learn to solve basic additions and multiplications in first and second grade, and from third grade on, attention is paid to solving simple multiplicative word problems (e.g., of the type '1 kg of apples costs x EUR. How much would 5 kg of apples cost?'). However, the typical multiplicative missing-value problems are only introduced at the end of the fourth grade, and further thoroughly rehearsed and intensively practiced in fifth and sixth grade. In this problem type, three numbers are given and a fourth one has to be found, such as: "A car of the future will be able to travel 6 miles in 2 min. How far will it travel in 4 min?" (Kaput and West 1994, p. 267). Based on the assumption that the car is driving at a constant velocity, this problem situation needs to be modelled multiplicatively, i.e., by looking at the multiplicative relation between two of the given numbers, and comparing or applying this relation to a third given number in order to find the missing one (e.g., 2 min × 2 = 4 min, so 6 miles × 2 = 12 miles). Also other approaches to this problem correctly model the situation in a multiplicative way, and in this sense, fall under the heading of multiplicative reasoning. These are, for instance, the unit factor approach (e.g., 3 miles in 1 min, so 12 miles in 4 min) and the building-up approach based on repeated addition (e.g., in 2 + 2 min, 6 + 6 miles).

However, the literature on multiplicative reasoning has repeatedly pointed out that children often model multiplicative situations incorrectly. More specifically, they model those multiplicative situations in an additive way, also referred to as the "constant difference approach" (e.g., Hart 1988; Kaput and West 1994; Karplus et al. 1983; Noelting 1980; Van Dooren et al. 2010; Vergnaud 1983, 1988). Children who erroneously apply this approach consider the *differences* instead of the *ratios* between given numbers. For instance, in the above missing-value problem, they answer 8 miles instead of 12 miles (e.g., the difference between 2 and 4 is 2, so 6 + 2 = 8). This kind of erroneous additive modelling especially occurs in middle primary school children (Kaput and West 1994; Van Dooren et al. 2010; Vergnaud 1988). Therefore, it has been interpreted as evidence for an additive phase in the development of quantitative relational reasoning abilities that precedes a multiplicative one. In this additive phase, children would be able to reason only additively, and thus have to make an important transition to multiplicative reasoning later on. This transition has even been characterized as "one of the major barriers to learning mathematics" (Siemon et al. 2005, p. 1). More recently, however, researchers have started to question this interpretation of erroneous additive modelling in multiplicative word problems exclusively in terms of lacking abilities, as several authors provided evidence that 5- to 6- year old children are already able to correctly reason about multiplicative relations (e.g., Boyer et al. 2008; Jeong et al. 2007), particularly in non-symbolic tasks containing continuous quantities. More specifically, several studies showed that young children had less difficulties with multiplicative reasoning in problems containing proportions represented in continuous amounts (e.g., an unsliced pie or a glass of water) than in problems containing discrete sets (e.g., a pie sliced into units or a glass of water containing marks indicating water units), in which they often resorted to additive reasoning (e.g., Boyer et al. 2008; Jeong et al. 2007; Spinillo and Bryant 1999). Young children's difficulty with multiplicative reasoning in discrete tasks may be due to the overextension of numerical counting routines. They may engage in counting, and the whole numbers obtained by those counting routines may interfere with the proportions needed for multiplicative reasoning (Mix et al. 1999; Obersteiner et al. 2015; Wynn 1997).

Besides the erroneous additive modelling of multiplicative problem situations, the inverse mistake has been repeatedly reported as well. In this line of research, missing-value word problems that have an additive underlying mathematical model, as originally developed by Cramer et al. (1993), take a central place. For instance, word problems such as "Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 4 laps, Kim has run 8 laps. When Ellen has run 12 laps, how many has Kim run?" (e.g., Van Dooren et al. 2010, p. 364), are frequently erroneously modelled in a multiplicative way (e.g., "4 × 3 = 12, so 8 × 3 = 24 laps"). This mistake has been especially found in children in upper primary education (Van

Dooren et al. 2005, 2008, 2010). However, just as in the case of erroneous additive modelling in multiplicative missing-value word problems, one can assume that those upper elementary school children are able to reason correctly about such additive missing-value word problems. Hence, it seems unlikely that this erroneous multiplicative modelling in additive missing-value word problems could be fully explained by children's lacking additive modelling ability.

It seems that neither children's erroneous additive modelling nor their erroneous multiplicative modelling exclusively depends on their *ability* to reason multiplicatively or additively. A complementary explanation was raised by Resnick and Singer (1993), who interpreted children's erroneous additive reasoning in multiplicative word problems as an indication of their *preference* for additive relations. Also in the broader mathematics education literature, the term "preference" refers to the way of reasoning that "has precedence over" the other (Pellegrino and Glaser 1982, p. 310; also see; Bailey et al. 2012; Resnick and Singer 1993). Likewise, children's multiplicative reasoning in additive word problems may be based on a preference for multiplicative relations.

Although it has been suggested that a preference is at play in children's missing-value word problem solving (Resnick and Singer 1993), classical word problems, including the missing-value word problems mentioned above, may not be best suited to capture this preference. Because those word problems *do* contain an underlying additive or multiplicative mathematical model, children's choice, for instance, of a multiplicative model to solve a multiplicative problem may be due to their preference for the multiplicative model as such, or to a conscious consideration of its applicability in the given problem situation. To measure children's preference validly, it may be more useful to resort to "open" word problems that do not contain any intrinsic indication for additive or multiplicative reasoning, and, thus, for which both an additive and a multiplicative model are equally correct.

Several authors have suggested the usefulness of open word problems for both research and teaching purposes (e.g., see Schukajlow and Krug 2014; Schukajlow et al. 2015; Star and Rittle-Johnson 2008). In the domain of additive and multiplicative reasoning in particular, the usefulness of open word problems has been suggested too (e.g., Lamon 2008; Lamon and Lesh 1992; Pellegrino and Glaser 1982), but little systematic research using such open tasks has been conducted. Degrande et al. (2014) developed open word problems by posing word problems in Greek to Flemish pupils. Since those word problems were posed in the Greek language, these pupils could absolutely not understand the problem situation, except for the given numbers which were represented in the usual Arabic format. This way, neither the multiplicative nor the additive solution method could be considered as *the* correct one in those word problems. Degrande et al. (2014) found that a substantial percentage of middle primary school children (3rd and 4th graders) chose to solve those problems using additive relations, while a substantial percentage of upper primary school children (5th and 6th graders) solved those problems multiplicatively, despite the incomprehensible context. No children at all reacted by stating that both answers were possible. These open word problems, hence, exposed children's preferred type of relations between numbers (i.e., additive or multiplicative). Moreover, those results closely resembled previous findings (e.g., Van Dooren et al. 2005, 2009, 2010) obtained by means of classical word problems that do have a clear additive or multiplicative underlying model. This suggests that children's preferred way of reasoning may not only be at play in open word problems, but in *closed* classical word problems as well. Thus, rather than relying on the mathematical model that is underlying these latter problems, children tend to fall back on their preferred way of reasoning, which may be triggered by certain superficial cues in the word problem, such as the presence of key words (such as "many times" or "per" for multiplicative reasoning, or "in total" for additive reasoning) or of specific numbers or number combinations (as originally suggested by Sowder 1988).

3 The present study

The present study aims to get a view on children's preference for additive or multiplicative reasoning by means of a novel task, namely a non-symbolic variant of the snake task that was originally suggested by Lamon and colleagues (Lamon 2008; Lamon and Lesh 1992) but, to the best of our knowledge, which had not been empirically tested yet. In our non-symbolic variant of the snake task, children are presented with a picture of two snakes that differ in length at a given time point. A second picture shows the same snakes at a later moment. Children are then asked which of the two snakes has grown the most. Additive reasoners would consider absolute growth, i.e., compare how much length was added to each snake, whereas multiplicative reasoners would consider relative growth, i.e., compare the ratios between the present length of the snake and its original length. These two distinct understandings of the same task can be considered as two different, but equally plausible, mathematical *models* of the situation depicted in that task (in the sense of e.g., Gravemeijer 2004; Greer et al. 2007; Usiskin 2007). Hence, the way in which children modelled the problem situation in the snake task indicated whether or not they had a preference, and if so, which preference. This task was open with respect to the underlying mathematical model, and therefore we call it an *open* word problem.

The research literature contains several other mathematical modelling tasks that are open to multiple solutions (see classification by, e.g., Schukajlow et al. 2015). Some tasks are open to multiple solution methods that still lead to the same mathematical result, while in other tasks different results can be obtained based on different assumptions about missing data but using the same solution methods. A third type of tasks can be solved using different solution methods (based on different assumptions about missing data) leading to different results. The snake task belongs to this third type of multiple solution tasks. It does not contain any information about the *nature* of the growth (since it is *open*), hence children should make assumptions about this missing information in order to interpret the problem situation. Different assumptions lead to different solution methods (i.e., additive or multiplicative) and to different results (i.e., the snake that had grown the most from an additive perspective or the snake that had grown the most from a multiplicative perspective).

So, like the Greek word problems used by Degrande et al. (2014), the snake task is open with respect to the underlying mathematical model, as an additive and a multiplicative model are equally valid for making a decision on which snake has grown the most. However, contrary to the Greek word problems, our variant of the snake task is non-symbolic in nature: There are no concrete numbers provided in the task, so children cannot fall back on their preferred type of relations between *numbers*. Another difference is that children can understand the problem situation, which is not the case for the Greek word problem (due to its Greek language).

The present study aimed to answer three questions. First, how do children in upper primary education answer these open word problems: Do they prefer additive relations, multiplicative relations, or do they indicate that both types of relations are possible (RQ 1)? A subquestion to this first question is as follows: How do these children verbalize their reasoning? Several authors argued that young children have some implicit knowledge about additive and multiplicative relations, long before they are able to verbalize their reasoning (Nunes and Bryant 2010; Sophian 2000). In contrast, others did not find a discrepancy between children's quantitative relational reasoning (as demonstrated by their answers) and the verbalizations accompanying their answers (McMullen et al. 2011).

Previous research focused mainly on children in upper primary education, and showed that children tend to focus increasingly on multiplicative relations and decreasingly on additive relations across (upper) primary education (for a review, see Van Dooren et al. 2008). Therefore, our second research question was whether children from fifth and sixth grade differ in terms of their answers to those open word problems and their accompanying verbalizations (RQ 2)?

A third question related to the way in which the problem is presented: Does the presentation of the snake task in terms of discrete vs. continuous quantities have an impact on children's answers and verbalizations (RQ 3)? Previous studies showed that young children have more difficulties with multiplicative reasoning in discrete than in continuous problems, as in the former they are more inclined to reason additively (e.g., see studies with 5- to 6-year olds by Boyer et al. 2008; Jeong et al. 2007; Spinillo and Bryant 1999). However, little or no research addressed the impact of this task characteristic in upper primary school children.

4 Method

4.1 Participants, instruments and procedure

Participants were 279 children (157 fifth and 122 sixth graders) from four Flemish primary schools. Children's preference was measured by means of a non-symbolic paper-and-pencil variant of the snake task of Lamon (2008), which was administered as part of a larger data collection. In this task, two snakes were presented at two distinct moments in time (see Fig. 1). Children were asked to indicate which snake had grown the most between the two moments. Additive reasoners were expected to consider how much length was added to both snakes (i.e., absolute growth) and, thus, to answer that the snake named "Sleng" had grown the most, whereas multiplicative reasoners were expected to look at the ratio between the present length of each snake and its original length (i.e. relative growth) and thus conclude that the snake named "Sling" had grown the most.

Each child solved four items. The ratios between the snakes' lengths differed amongst these four items, but the ratios between the lengths of each snake at the two moments were nevertheless integer, in order to allow straightforward multiplicative comparisons. Two out of four items contained continuous quantities (i.e., snakes of different lengths, see Fig. 1a), while two other items contained discrete quantities (i.e., dotted snakes, see Fig. 1b). The order of the items was counterbalanced in different versions of test booklets.

The number of items per task per child was limited to four, to avoid the development of response tendencies across items. The snake tasks were presented in a paper-and-pencil form and were collectively administered in the children's usual classrooms. Children were explicitly asked to indicate which snake had grown the most, and to verbalize the reasoning behind their answer in written form.

4.2 Analyses

Answers to each of the four items of the snake task were classified as additive (if "Sleng", the snake that had grown the most from an absolute perspective was chosen), as multiplicative (if "Sling", the snake that had grown the most from a multiplicative perspective was chosen), or belonging

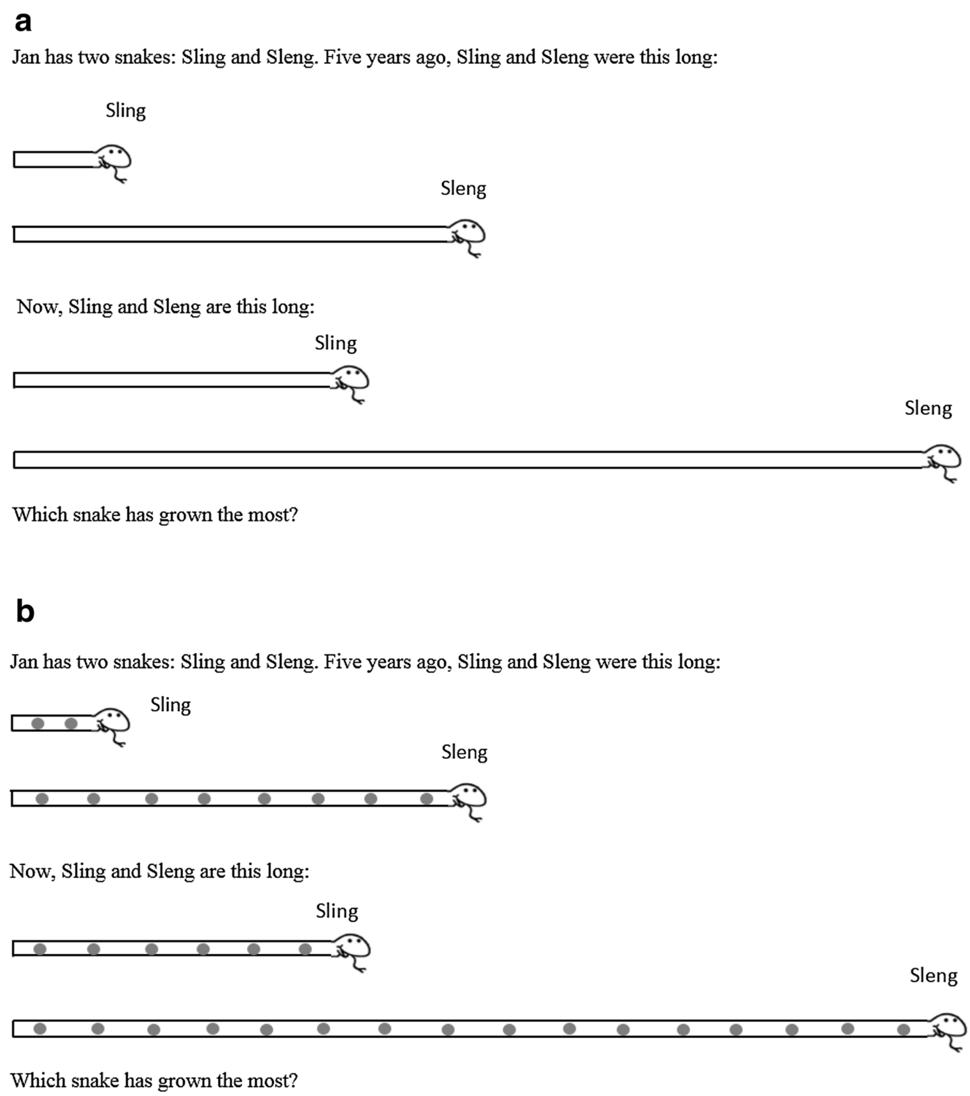

Fig. 1 **a** One item of a continuous and **b** discrete variant of the snake task of Lamon (2008)

to the rest category (if the child did not make an explicit choice between both snakes, or if the item was left unanswered). Children's *verbalizations* to each of the four items were classified as:

- Additive, if children explicitly referred to growth in absolute terms[1] (i.e., difference between the snakes' lengths); e.g., "Sling only grew 4 dots, and Sleng grew 8 dots", "the distance in between both snakes is now larger than five years ago".
- Multiplicative, if children explicitly referred to growth in relative terms (i.e., ratio between the snakes' lengths); e.g., "because you look at the body, and Sling fits four times in itself and Sleng only three times", "Sling has become 4 times as large and Sleng only 3 times".
- Additive-and-multiplicative, if children explicitly referred to both absolute and relative growth; e.g., "(1) Sleng is now twice as long as before, (2) Sling has only grown a few cm".
- Rest, if children referred neither to absolute nor to relative growth, by
    - Referring to growth in general terms, e.g., "because this snake has grown larger", "because this one was the largest one and had most dots, and now it is still the largest one with most dots", or
    - Referring to a comparison of the lengths of one snake at two moments or the lengths of two snakes

[1] Also verbalizations that involved only a comparison of the lengths of one snake at two moments, or the lengths of two snakes at one moment, (or verbalizations that left implicit whether they were comparing the former or the latter) but still clearly referred to *differences* in lengths, *ratios* between lengths, or both, fell under the categories of resp. additive, multiplicative, or additive-and-multiplicative verbalizations.

at one moment in general terms, e.g., “because Sleng is now larger than Sling”, or leaving implicit which lengths are compared, e.g. “there are more dots” or “Slong is bigger”
- Referred neither to growth, nor to the lengths of snakes, e.g., “because I think so” or “you simple see that”.

After coding, children’s answers and verbalizations were analyzed by means of generalized estimating equations (GEE) analyses, which allow one to conduct repeated measures logistic regression analyses on repeated (and thus possibly correlated) categorical data. GEE analyses with grade (between-subject) and type of presentation (within-subject) as independent variables and the likelihood of occurrence of additive answers and multiplicative answers as dependent variables were conducted, as well as GEE analyses with grade and type of presentation as independent variables on the likelihood of occurrence of additive verbalizations of additive answers and multiplicative verbalizations of multiplicative answers. Besides children’s answers to all individual items, their answer profiles were analyzed, using logistic regressions with grade as independent variable and each profile type as dependent variable.

5 Results

In what follows, we first give an overview of children’s *answers* to the four items of the snake task, i.e., whether they mentioned the snake that had grown the most from an additive or multiplicative perspective. After that, we report results regarding the *verbalizations* that accompanied children’s answers.

5.1 Answers

First, results showed that most children (67.9%) chose the snake that had grown the most from an additive perspective, considering how much length was added to both snakes (i.e., absolute growth, see Table 1). A smaller percentage (29.9%) chose the snake that had grown the most according to a multiplicative perspective, considering the ratio between the original and present length of each snake (i.e., relative growth). A small group of children (2.2%), belonging to the rest group, did not make an explicit choice between the two snakes, or left the problem unanswered.

Second, both in fifth and in sixth grade there were many more additive (respectively 72.6 and 61.9%) than multiplicative answers (respectively. 25.2 and 36.1%, see Table 2). The discrepancy was larger in fifth than in sixth grade, which was due to a decrease of additive (*Wald* $\chi^2(1) = 5.526$; $p = .019$) and an increase of multiplicative answers between the two grades (*Wald* $\chi^2(1) = 6.101$; $p = .014$), as was expected based on previous research.

Table 1 Children’s answers and verbalizations in the snake task

Answers	%	Verbalizations	%
A	67.9	A	35.5
		M	7.1
		A & M	0.1
		Rest	57.3
M	29.9	A	1.8
		M	76.0
		A & M	1.5
		Rest	20.7
Rest	2.2	A	12.5
		M	29.2
		A & M	4.2
		Rest	54.2

Third, contrary to our expectations based on research with younger children (i.e., 5- to 6 year olds, see Boyer et al. 2008; Jeong et al. 2007; Spinillo and Bryant 1999), we did not find a difference between discrete and continuous presentations (see Table 3). In particular, both discrete and continuous tasks mainly evoked additive answers (respectively 68.6 and 67.2%, *Wald* $\chi^2(1) = .716$; $p = .398$.), and far fewer multiplicative answers (respectively 29.6 and 30.3% *Wald* $\chi^2(1) = .165$; $p = .684$). Neither did we find an interaction between grade and presentation format (i.e., discrete vs. continuous) (respectively *Wald* $\chi^2(1) = .001$; $p = .972$ and *Wald* $\chi^2(1) = .165$; $p = .684$ for additive and multiplicative answers).

Fourth, analyses of the consistency of the above results revealed that there is consistency in children’s answers, as shown in their answer profiles across all four items. Most children consistently modelled the situation additively in at least three out of four items (i.e., 62.7%). Another substantial group of children consistently modelled the situation multiplicatively in at least three out of four items (i.e., 22.9%). In addition, 1.8% of all children answered format-sensitively: 0.7% of them answering multiplicatively in continuous items and additively in discrete items, and 1.1% vice versa, i.e., answering additively in continuous items and multiplicatively in discrete items. The remaining 12.5% of children combined additive and multiplicative answers, and mixed them irrespective of presentation characteristics. Only this latter group did not show a consistent answering behavior. In line with the answers per item, the percentage of additive profiles decreased from 68.2% in fifth to 55.7% in sixth grade (*Wald* $\chi^2(1) = 4.494$; $p = .034$) while the number of multiplicative profiles increased from 15.9 to 32.0% (*Wald* $\chi^2(1) = 9.683$; $p = .002$) between fifth and sixth grade.

Table 2 Fifth and sixth graders' answers and verbalizations in the snake task

Grade 5				Grade 6			
Answers	%	Verbalizations	%	Answers	%	Verbalizations	%
A	72.6	A	29.8	A	61.9	A	44.0
		M	7.9			M	6.0
		A&M	0.2			A&M	0.0
		Rest	62.1			Rest	50.0
M	25.2	A	2.5	M	36.1	A	1.1
		M	72.8			M	79.0
		A&M	2.5			A&M	0.6
		Rest	22.2			Rest	19.3
Rest	2.2	A	14.3	Rest	2.0	A	10.0
		M	28.6			M	30.0
		A&M	7.1			A&M	0.0
		Rest	50.0			Rest	60.0

Table 3 Children's answers and verbalizations in discrete and continuous items in the snake task

Discrete				Continuous			
Answers	%	Verbalizations	%	Answers	%	Verbalizations	%
A	68.6	A	40.7	A	67.2	A	30.1
		M	6.0			M	8.3
		A & M	0.3			A & M	0.0
		Rest	53.0			Rest	61.1
M	29.6	A	3.0	M	30.3	A	0.6
		M	80.6			M	71.6
		A & M	1.8			A & M	1.2
		Rest	14.5			Rest	26.0
Rest	1.8	A	10.0	Rest	2.5	A	14.3
		M	20.0			M	35.7
		A & M	10.0			A & M	0.0
		Rest	60.0			Rest	50.0

5.2 Verbalizations

We also analyzed the way in which the different types of answers were verbalized. First, the large majority of multiplicative answers (76.0%) was verbalized multiplicatively, and only very small percentages of multiplicative answers were verbalized additively (1.8%), or additively-and-multiplicatively (1.5%). This results in a small portion of multiplicative answers that did not explicitly refer to additive or multiplicative relations (20.7%). This large percentage of multiplicative verbalizations accompanying multiplicative answers, and small percentage of additive verbalizations accompanying those multiplicative answers, confirms the validity of the task.

Contrary to the multiplicative verbalizations accompanying multiplicative answers, a smaller percentage of the additive answers (35.5%) was indeed also verbalized additively. Most verbalizations of those additive answers (57.3%) did not explicitly refer to a specific type of relation, although almost all of those referred to changes in length of both snakes at both moments (without being specific about the additive or multiplicative nature of the growth of both snakes), and hence, were relational in nature. Hence, although the task was more frequently answered in an additive way, it appeared to be harder to explicitly verbalize than multiplicative reasoning. Nevertheless, only small percentages of additive answers were verbalized multiplicatively (7.1%, see Table 1), and hardly any children used both additive and multiplicative justifications for their additive answer (0.1%). Those results indicate that only in rare cases the additive alternative was chosen while the underlying reasoning was clearly multiplicative, confirming the validity of our task.

Second, especially in fifth grade additive reasoning seemed to be hard to verbalize (see Table 2). This was confirmed by the smaller percentage of additive answers that were verbalized additively in fifth than in sixth grade (respectively 29.8 vs. 44.0%, *Wald* $\chi^2(1) = 6.447$, $p = .011$).

This was not the case for multiplicative answers, where the difference in multiplicative verbalizations between grades was not significant (respectively 72.8 vs. 79.0%, *Wald* $\chi^2(1) = 1.189$, $p = .275$). Both additive verbalizations in multiplicatively answered tasks and multiplicative verbalizations in additively answered tasks occurred slightly more often in fifth than sixth grade (2.5 and 1.1% for additive verbalizations in multiplicatively answered tasks, and 7.9% and 6.0% for multiplicative verbalizations in additively answered tasks, respectively).

Third, with regard to the presentation format, more additive answers were verbalized additively in discrete (40.7%) than in continuous items (30.1%, *Wald* $\chi^2(1) = 15.392$, $p \leq 0.001$, see Table 3). The same presentation effect was found amongst the multiplicative answers, i.e., more multiplicative answers were verbalized multiplicatively in discrete (80.6%) than in continuous items (71.6%, *Wald* $\chi^2(1) = 5.279$, $p = .022$). The interaction between grade and presentation format (i.e., discrete vs. continuous) was not significant, neither for the additive answers that were verbalized additively (*Wald* $\chi^2(1) = 5.222$, $p = .470$), nor for the multiplicative answers that were verbalized multiplicatively (*Wald* $\chi^2(1) = 1.368$, $p = .242$).

6 Conclusion and discussion

Within the research domain of mathematics education, much attention has been devoted to the connection between mathematics and the real world. This complex process of applying the appropriate mathematical operations in order to make sense of everyday-life situations and to solve real-life problems, is called mathematical modelling (e.g., English and Lesh 2003; Van Dooren et al. 2006; Verschaffel et al. 2000). The literature on mathematical modelling is vast, and word problems are only one of the ways of teaching it (Verschaffel et al. 2000, 2007). Still, word problem solving constitutes an important part of the mathematics curriculum in primary education, and is one of the major ways in which modelling is introduced in the elementary math classroom (Verschaffel et al. 2000, 2007).

Previous studies in the domain of additive and multiplicative word problem solving, however, have repeatedly shown that middle primary school children often incorrectly model multiplicative word problems in an additive way (Hart 1988; Kaput and West 1994; Karplus et al. 1983; Noelting 1980; Van Dooren et al. 2010; Vergnaud 1983, 1988), and on the other hand, upper primary school children erroneously apply a multiplicative model to additive word problems (Van Dooren et al. 2005, 2008, 2010). Underlying the present study is the idea that the ability to reason additively or multiplicatively seems not to be the only reason for this incorrect application of additive and multiplicative models in word problems. Hence, the present study aimed to get a view of upper primary school children's preference for additive or multiplicative reasoning. We used a non-symbolic variant of the snake task of Lamon and colleagues (Lamon 2008; Lamon and Lesh 1992) to get a view of children's preference. Contrary to classical word problems, our task was open to the underlying mathematical model, in the sense that additive and multiplicative reasoning were equally valid to compare the growth of both snakes. This implies that the way in which children modelled the problem situation indicated whether they had a preference for additive or multiplicative reasoning. This task was presented to fifth and sixth graders, both in a continuous and a discrete presentation format.

Results, first, revealed that additive reasoning occurred more frequently than multiplicative reasoning, but was harder to verbalize (RQ 1). Second, both findings were more prominent for fifth than sixth graders (RQ 2). Third, while we did not find any differences between children's answers on discrete and continuous items, their answers were more often explicitly verbalized in discrete than continuous items (RQ 3). In what follows, we discuss the theoretical, methodological and educational implications of those results.

6.1 Theoretical implications

In this section, we elaborate on the main results related to the three research questions and their theoretical implications. With respect to the first research question, results revealed that *answers* based on additive reasoning occurred more often than answers based on multiplicative reasoning, and that those answers were mainly consistent within children. The finding that most children consistently chose the additive answer was in sharp contrast to previous research indicating that children in upper primary education tend to incorrectly model classical additive word problems in a multiplicative way (for a review, see Van Dooren et al. 2008). Also Degrande et al.'s (2014) study using open word problems indicated that children preferred multiplicative reasoning (Degrande et al. 2014). This discrepancy between the nature of children's preference (i.e., additive or multiplicative) in the present study and in previous research suggests that children's preference for additive or multiplicative reasoning depends on features that are inherent in the task that is used. Previous research has repeatedly suggested that children strongly rely on superficial cues, such as the given numbers, to determine the operations they need to perform, both in classical word problems (e.g., Verschaffel et al. 1994, 2000) and in open word problems (Degrande et al. 2014). Specifically for additive and multiplicative word problems, research has shown that the numbers in a word problem (and particularly whether numbers in the word problem form integer number ratios or not) have a larger impact on the model that is selected by children than whether the word problem actually is an additive or a multiplicative one (Van Dooren et al. 2009). Our variant of the snake task did not contain any numbers, and was thus non-symbolic. Children could not fall back on their preferred type of relations between numbers, so they had no alternative but to make sense of the situation described in the snake task itself. It seems that many children in upper primary education associate this specific problem context, namely comparing growths, with an additive rather than multiplicative model, while it may be that other contexts, for instance contexts related to time and distance, might have a privileged relation with the multiplicative model.

The finding that most children modelled the problem situation unilaterally additively while others modelled the situation unilaterally multiplicatively and only very few pointed out that both alternatives were valid also implies that the vast majority of the children did not notice the openness of the problem. This obviously indicates upper primary school children's preference for additive or multiplicative reasoning. Future studies could reveal whether children who modelled the problem situation multiplicatively would also consider the additive model, and children who modelled the problem situation additively would also consider the multiplicative model. The way in which the snake task was presented in the present study did not lend itself perfectly to this purpose, since children–based on their extensive experience with school mathematics tasks–may have thought that only one single answer was allowed, especially because it was left implicit in the "experimental contract" (Greer 1997) whether more than one answer could be given. Future studies could, for instance, ask children to verbalize why the snake named "Sleng" has grown the most, and afterwards, to find arguments for the idea that the other snake named "Sling" has grown the most (or vice versa).

As a subquestion of the first research question, we also looked at children's *verbalizations*. Even though additive reasoning was more frequently used than multiplicative reasoning, the former was harder to *verbalize*. The finding that a relatively small portion of all additive answers was also explicitly verbalized additively was in line with previous research indicating that children have some implicit knowledge about quantitative relations before they are able to verbalize their reasoning (e.g., Nunes and Bryant 2010; Sophian 2000). This result, however, did not hold for multiplicative relations, where the vast majority of the answers was verbalized multiplicatively in an explicit way. The latter finding confirms previous research by McMullen et al. (2011), who did not find a discrepancy between reasoning based on multiplicative relations and verbalizations of those relations. The finding that multiplicative reasoning was easier to verbalize than its additive counterpart may be due to its prominent role in the mathematics curriculum in Flemish primary education. From third or fourth grade on, Flemish children are confronted with multiplicative missing-value word problems, and the accompanying multiplicative verbalizations. In contrast, additive word problems in which children should consider the difference in a given number pair and apply this to a second number pair, hardly occur in the primary mathematics curriculum. Hence, verbalizing those additive relations is not taught as extensively as verbalizing multiplicative ones.

Concerning our second research question about the impact of grade, we found that, especially in fifth grade, additive *answers* were more prominent than multiplicative ones. Additive answers decreased and multiplicative answers increased between fifth and sixth grade. This trend was in line with previous studies using open word problems (see Degrande et al. 2014), well as previous research using classical additive or multiplicative word problems (for an overview, see Van Dooren et al. 2008). With respect to the *verbalizations*, we found that additive reasoning was especially hard to verbalize in fifth grade. Additive verbalizations of those additive answers tended to increase between fifth and sixth grade. The same trend was found for multiplicative verbalizations of multiplicative answers, but differences between grades failed to be significant there. While this result may be due to children's developing mathematical competencies in additive and multiplicative reasoning, it might as well indicate the simultaneous acquisition of other skills, such as the verbal skills to express the reasoning about quantitative relations in a given situation.

With respect to our third research question, we did not find an impact of the discrete or continuous presentation format on children's additive or multiplicative answers. This was in contrast to expectations based on previous studies revealing young children's difficulties with multiplicative reasoning in discrete rather than continuous tasks (i.e., 5- to 6-year olds, see Boyer et al. 2008; Jeong et al. 2007; Spinillo and Bryant 1999). Given that this presentation effect has been explained by young children's overextension of numerical counting routines to multiplicative situations (Mix et al. 1999; Obersteiner et al. 2015; Wynn 1997), it is not that surprising that upper primary school children did not experience those difficulties any more in discrete items. Regarding children's verbalizations, we even found an opposite trend: Multiplicative answers were more often verbalized multiplicatively in discrete than continuous items, and the same result was found for additive answers, which were more often verbalized additively in discrete than continuous items. Whereas the discreteness of tasks obstructed younger children from reason multiplicatively, it fostered upper primary school children to verbalize their additive or multiplicative reasoning.

6.2 Methodological implications

From a methodological perspective, the present study used a novel task to get a view of children's preference. In particular, we used a non-symbolic variant of the snake task that was originally suggested by Lamon and colleagues (Lamon 2008; Lamon and Lesh 1992), but not empirically tested yet. Contrary to classical word problems that can only be correctly modelled in one way, our task was open with respect to the underlying mathematical model. Hence, the model that children chose in the word problem indicated whether they had a preference, and if so, for which way of reasoning (i.e., additive or multiplicative reasoning).

The finding that our open snake task mainly evoked additive or multiplicative answers, and that the large majority of children showed consistent answer profiles, confirms its validity. The validity was further supported by substantial percentages of additive or multiplicative answers that were also verbalized additively or multiplicatively, respectively. Only very small percentages of additive answers were verbalized multiplicatively, and vice versa, only very small percentages of multiplicative answers were articulated additively. In order to further warrant the reliability and validity of the open word problems, future studies may opt for a larger number of items. While the relatively small number of open tasks in the present study was a deliberate methodological choice in order to preclude response tendencies across items, a larger number of items may be highly valuable, particularly in studies focusing on individual differences in children's preference (i.e., by means of answer profiles).

Furthermore, administering open tasks in one-on-one interview situations instead of by means of paper-and-pencil tests may provide more detailed information about the decisions children make when modelling the problem situation. This may also help in increasing the reliability and validity of our task. On the other hand, we need to be aware that the decisions that children make, and the line of reasoning they follow may be implicit, and therefore difficult to articulate (see e.g., Siegler 2000). Moreover, requesting children to verbalize how they solved a certain problem may affect their reasoning (e.g., see Boyer et al. 2008).

Our snake task has the potential to be used with younger children as well. Its non-symbolic character makes it possible to offer the task to children who have been instructed neither on additive nor multiplicative reasoning, and even to pre-schoolers who have not been taught the number symbols yet. While a lot of recent studies documented young children's (additive and) multiplicative reasoning abilities (e.g., Boyer et al. 2008; Jeong et al. 2007), it remains an open question how those children would model those open tasks.

Lastly, compared to a classical word problem, which can be modelled and solved correctly in only one single way, our experimental snake task is open with respect to the underlying mathematical model. In that sense, problems such as our snake task, whereby "arithmetic operations should be mindfully evaluated as candidate models for a given situation" (Greer et al. 2007, p. 92), may be more valuable than classical word problems for the development in learners of a genuine disposition towards mathematical modelling. However, open word problems such as our snake task may still not be considered as a genuine mathematical modelling task as envisaged by authors such as Blum and Niss (1991) or Lesh and Doerr (2003), because both the way in which the task is formulated and the context in which it is presented do not truly invite children to activate and integrate their real-world knowledge about the problem situation into the modelling process (e.g., see Verschaffel et al. 2000, 2010). Still, we are convinced that the snake task has the potential to be transformed, in a more genuine mathematical modelling experience in which children can bring in all kinds of real-world knowledge and experiences to tackle the problem (including knowledge they gain from the domain of biology on growth processes of living organisms, for instance, or by doing measurements in real life situations). This may ultimately lead to a thorough and inspiring class discussion about different perspectives on growth.

6.3 Educational implications

The current results have implications for educational practice too. First, it seems advisable to scrutinize the word problems that occur in mathematics curricula carefully, to avoid a stereotyped offering of word problems. Therefore, superficial task characteristics that may shape children's preference should be identified (such as the presence of key words, nature of the numbers, etc.).

Second, pedagogical interventions that force children to model the actual problem situation–rather than relying on such superficial problem characteristics–are advisable. For instance, this genuine modelling can be required by using word problems that contain letters or very large numbers (e.g., see Greer 1987). Relatedly, explicit instructional attention to the similarities and differences between additive and multiplicative word problems–despite similar superficial task characteristics–is needed. In this respect, children may be invited to compare and solve word problems that differ with respect to the underlying mathematical model, but have similar other task characteristics. Vice versa, children may be invited to draw parallels between word problems that are similar with respect to the underlying mathematical model, but differ in terms of superficial task characteristics. This awareness of the invariance of operations when modelling a certain problem situation has been identified as "conservation of operations" (Greer 1987).

Third, besides solving of additive and multiplicative problems in mathematics classes, sufficient educational attention should be paid to verbalizing the accompanying reasoning. While this is already extensively practiced for multiplicative reasoning (due to its prominent role in primary mathematics education curricula), our results suggest that especially in the case of additive reasoning, this appears to be a stumbling block for children.

Fourth, besides classical word problems that typically contain only one underlying mathematical model, open word problems such as the snake task lend themselves to educational purposes too (e.g., as suggested by Lamon 2008). Teaching units in which children were prompted to find multiple solutions for the same mathematical task have been proven to be successful, since they increased the number of solutions that children came up with during and after the teaching unit (Schukajlow and Krug 2014). Those teaching units typically consist of activities such as collectively solving modelling problems that require multiple solutions, exploring different solutions during group work, discussing and summarizing similarities and differences between solutions, etc. In the snake task in particular, a discussion on the similarities and differences as well as applicability of an additive and a multiplicative model to describe the snakes' growths could lead to a better understanding of both models. Such a discussion, moreover, may increase the awareness that certain problem contexts can be viewed additively as well as multiplicatively. Such knowledge may help children to articulate the considerations they make when deciding on the appropriateness of a solution method, and to make conscious decisions when modelling problem situations, in open word problems and not only in classical word problems in which the one underlying mathematical model can be clearly and undoubtedly determined.

References

Bailey, D. H., Littlefield, A., & Geary, D. C. (2012). The codevelopment of skill at and preference for use of retrieval-based processes for solving addition problems: Individual and sex differences from first to sixth graders. *Journal of Experimental Child Psychology, 113*, 78–92. https://doi.org/10.1016/j.jecp.2012.04.014.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects: State, trends, and issues in mathematics instruction. *Educational Studies in Mathematics, 22*, 37–68. https://doi.org/10.1007/BF00302716.

Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: Where young children go wrong. *Developmental Psychology, 44*, 1478–1490. https://doi.org/10.1037/a0013110.

Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 159–178). New York: Macmillan.

Degrande, T., Verschaffel, L., & Van Dooren, W. (2014). How do Flemish children solve 'Greek' word problems? On children's quantitative analogical reasoning in mathematically neutral word problems. *Mediterranean Journal for Research in Mathematics Education, 13*(1–2), 57–74.

English, L. D., & Lesh, R. A. (2003). Ends-in-view problems. In R. A. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective on mathematics problem solving, learning, and teaching* (pp. 297–316). Mahwah: Lawrence Erlbaum.

Gravemeijer, K. (2004). Emergent modelling as a precursor to mathematical modelling. In H.-W. Henn & W. Blum (Eds.), *Applications and modelling in mathematics education (ICMI Study 14)* (pp. 97–102). Dortmund: Universität Dortmund.

Greer, B. (1987). Understanding of arithmetical operations as models of situations. In J. A. Sloboda & D. Rogers (Eds.), *Cognitive processes in mathematics* (pp. 60–80). Oxford: Clarendon.

Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of research on learning and teaching mathematics* (pp. 276–295). Reston: National Council of Teachers of Mathematics.

Greer, B. (1997). Modelling reality in mathematics classroom: The case of word problems. *Learning and Instruction, 7*, 293–307. https://doi.org/10.1016/S0959-4752(97)00006-6.

Greer, B., Verschaffel, L., & Mukhopadhyay, S. (2007). Modelling for life: Mathematics and children's experience. In W. Blum, P. L. Galbraith, H.-W. Henn & M. Niss (Eds.), *Modelling and applications in mathematics education (ICMI Study 14)* (pp. 89–98). New York: Springer.

Hart, K. (1988). Ratio and proportion. In M. Behr & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (pp. 198–219). Reston: National Council of Teachers of Mathematics.

Jeong, Y., Levine, S., & Huttenlocher, J. (2007). The development of proportional reasoning: Effect of continuous vs. discrete quantities. *Journal of Cognition and Development, 8*, 237–256. https://doi.org/10.1080/15248370701202471.

Kaput, J. J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235–287). New York: State University of New York Press.

Karplus, R., Pulos, S., & Stage, E. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 45–89). New York: Academic Press.

Lamon, S. J. (2008). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (2nd edn.). New York: Taylor & Francis Group.

Lamon, S. J., & Lesh, R. (1992). Interpreting responses to problems with several levels and types of correct answers. In R. Lesh & S. J. Lamon (Eds.), *Assessment of authentic performance in school mathematics* (pp. 319–342). Washington, DC: American Association for the Advancement of Science Press.

Lesh, R., & Doerr, H. M. (2003). *Beyond constructivism. Models and modelling perspectives on mathematical problem solving, learning, and teaching*. Mawah: Lawrence Erlbaum.

McMullen, J. A., Hannula-Sormunen, M. M., & Lehtinen, E. (2011). Young children's spontaneous focusing on quantitative aspects and their verbalizations of their quantitative reasoning. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 217–224). Ankara, Turkey: PME.

Ministerie van de Vlaamse Gemeenschap. (1997). *Decreet van juli 1997 tot bekrachtiging van de ontwikkelingsdoelen en eindtermen van het gewoon basisonderwijs* [Decree of July 1997 to ratify the development goals and standards of primary education]. Brussels: Author.

Mix, K. S., Levine, S. C., & Huttenlocher, J. (1999). Early fraction calculation ability. *Developmental Psychology, 35*, 164–174. https://doi.org/10.1037/0012-1649.35.1.164.

Noelting, G. (1980). The development of proportional reasoning and the ratio concept: Part 1. Differentiation of stages. *Educational Studies in Mathematics, 11*, 217–253. https://doi.org/10.1007/BF00304357.

Nunes, T., & Bryant, P. (2010). *Understanding relations and their graphical representation*. Retrieved from http://www.nuffield-foundation.org/sites/defaukt/files/P4.pdf.

Obersteiner, A., Bernhard, M., & Reiss, K. (2015). Primary school children's strategies in solving contingency table problems: The role of intuition and inhibition. *ZDM, 47*, 825–836. https://doi.org/10.1007/s11858-015-0681-8.

Pellegrino, J. W., & Glaser, R. (1982). Analyzing aptitudes for learning: Inductive reasoning. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 2, pp. 269–345). Hillsdale: Lawrence Erlbaum.

Resnick, L. B., & Singer, J. A. (1993). Protoquantitative origins of ratio reasoning. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 107–130). Hillsdale: Lawrence Erlbaum.

Schukajlow, S., & Krug, A. (2014). Do multiple solutions matter? Prompting multiple solutions, interest, competence, and autonomy. *Journal for Research in Mathematics Education, 45*, 497–533.

Schukajlow, S., Krug, A., & Rakoczy, K. (2015). Effects of prompting multiple solutions for modelling problems on students' performance. *Educational Studies in Mathematics, 89*, 393–417. https://doi.org/10.1007/s10649-015-9608-0.

Siegler, R. S. (2000). Unconscious insights. *Current Directions in Psychological Science, 9*, 79–83.

Siemon, D., Breed, M., & Virgona, J. (2005). From additive to multiplicative thinking—The big challenge of the middle years. In J. Mousley, L. Bragg, & C. Campbell (Eds.), *Proceedings of the 42nd Conference of the Mathematical Association of Victoria*. Bundoora, Australia.

Sophian, C. (2000). Perceptions of proportionality in young children: Matching spatial ratios. *Cognition, 75*, 145–170. https://doi.org/10.1016/S0010-0277(00)00062-7.

Sowder, L. (1988). Children's solutions of story problems. *The Journal of Mathematical Behavior, 7*, 227–238.

Spinillo, A. G., & Bryant, P. (1999). Proportional reasoning in young children: Part–part comparisons about continuous and discontinuous quantity. *Mathematical Cognition, 5*, 181–197. https://doi.org/10.1080/135467999387298.

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem solving: The case of equation solving. *Learning and Instruction, 18*, 565–579. https://doi.org/10.1016/j.learninstruc.2007.09.018.

Usiskin, Z. (2007). The arithmetic operations as mathematical models. In W. Blum, P. L. Galbraith, H.-W. Henn & M. Niss (Eds.), *Modelling and applications in mathematics education (ICMI Study 14)* (pp. 257–264). New York: Springer.

Van Dooren, W., De Bock, D., Evers, M., & Verschaffel, L. (2009). Students' overuse of proportionality on missing-value problems: How numbers may change solutions. *Journal for Research in Mathematics Education, 40*, 187–211.

Van Dooren, W., De Bock, D., Hessels, A., Janssens, D., & Verschaffel, L. (2005). Not everything is proportional: Effects of age and problem type on propensities for overgeneralization. *Cognition and Instruction, 23*, 57–86. https://doi.org/10.1207/s1532690xci2301_3.

Van Dooren, W., De Bock, D., Janssens, D., & Verschaffel, L. (2008). The linear imperative: An inventory and conceptual analysis of students' overuse of linearity. *Journal for Research in Mathematics Education, 39*, 311–342. https://doi.org/10.2307/30034972.

Van Dooren, W., De Bock, D., & Verschaffel, L. (2010). From addition to multiplication … and back. The development of students' additive and multiplicative reasoning skills. *Cognition and Instruction, 28*, 360–381. https://doi.org/10.1080/07370008.2010.488306.

Van Dooren, W., Verschaffel, L., Greer, B., & De Bock, D. (2006). Modelling for life: Developing adaptive expertise in mathematical modelling from an early age. In L. Verschaffel, F. Dochy, M. Boekaerts & S. Vosniadou (Eds.), *Instructional psychology: Past, present and future trends. Sixteen essays in honour of Erik De Corte* (pp. 91–112). Oxford: Elsevier.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–174). New York: Academic Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141–161). Reston: Lawrence Erlbaum & National Council of Teachers of Mathematics.

Verschaffel, L., De Corte, E., & Lasure, S. (1994). Realistic considerations in mathematical modeling of school arithmetic word problems. *Learning and Instruction, 4*, 273–294. https://doi.org/10.1016/0959-4752(94)90002-7.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making sense of word problems*. Lisse: Swets & Zeitlinger.

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557–628). Charlotte: Information Age Publishing.

Verschaffel, L., Van Dooren, W., Greer, B., & Mukhopadhyah, S. (2010). Reconceptualising word problems as exercises in mathematical modeling. *Journal für Mathematik-Didaktik, 31*, 9–29.

Wynn, K. (1997). Competence models of numerical development. *Cognitive Development, 12*, 333–339. https://doi.org/10.1016/S0885-2014(97)90005-8.