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ORIGINAL ARTICLE

The function concept at the transition to upper secondary school level: tasks for a situation of change

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Abstract This paper is about the development of a task sequence to help overcome the fragmented understanding of the 'function' concept that students often bring with them into the initial stage of upper secondary school level. Our aim is to make the students' use of functions more flexible in certain respects, for example when functions are required to be used as tools for modelling. The core idea of our task design is to interpret formulas as functional relations. The tasks are developed along the lines of a Design-Based Research approach in which several theoretical approaches are employed in a complementary way. The paper will show how this complementarity frames and informs the design as well as the analysis of data about how a student solves the tasks. Mediated by the task sequence, students' development in terms of how their use of functions becomes more flexible is reconstructed. In the analysis, a key constraint for developing a flexible use of functions is identified: their poor understanding of the coordinate system and its scaling and utilization as a reference space for graphical representations of functions.

1 Introduction

A typical situation of change that students experience in the school system in Germany is the transition phase from lower to higher secondary school level when they reach grade 11. This transition is particularly challenging with regard to learning mathematics because the understanding of the function concept requires a change of view towards calculus. In addition, several other problems are likely to occur.

The heterogeneity of the pupil body of this initial stage is remarkably high due to students' social and educational background. Especially the fact that the students come from different feeder schools with a wide variety of school concepts makes the development of suitable tasks for mathematics classrooms particularly challenging. An additional problem is the curriculum structure with regard to the function concept. At lower secondary level in Germany, students normally do not experience functions as an overall comprehensive concept (see for example KMK 2004). Functions rather appear in a clustered or isolated way, for example as graphs, verbal descriptions and tables of functional relations or as proportional relations of magnitudes, as linear or quadratic functions. At grade 11, the initial stage of high school level, these function concepts are merged to build one comprehensive concept that enables students to use familiar and new functions flexibly in calculus and beyond. However, students at this level and beyond often carry with them only a kind of *fragmented view of functions* leading to various problems, some of them already documented in the field (Kösters 1996; Nitsch 2015; Zaslavsky 1997; Vinner and Dreyfus 1989; Malle 1993). For example, students often do not clearly distinguish between different types of functions and their features (e.g. the constant slope of a linear function contrasted with the varying slope of a quadratic function), between the function concept and its representations (Stölting 2008), between variables and parameters, or they regard variables as being tied to their symbols *x* or *y* (Kösters 1996; Bikner-Ahsbahs et al. 2015). In order to build a more comprehensive function concept, students have to overcome their fragmented views on functions. The question regarding how this can be achieved and what kinds of tasks are suitable

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are relevant but largely under-researched. In our project, these two questions are addressed by a task design process that is aligned with the aim of initiating and exploring the described transformation, namely, the change of conceptual understanding going from a fragmented towards a more flexible and merged concept of functions.

To overcome fragmentation, we assume flexibilization (for a definition see Sect. 3) to be a building brick in the transition towards a more comprehensive function concept enabling “individuals to solve problems quickly and accurately” (Heinze et al. 2009, p. 535). However, raising flexibility in the use of functions alone does not lead to a more comprehensive function concept; but it does have the potential to overcome the separation of function concepts, their parts and representations, hence, their fragmentation.

This paper presents the design process of a task sequence developed in an ongoing study.[1] The design goal is to raise the students’ flexibility in the use of functions at the initial phase of grade 11. The paper outlines the theoretical approaches utilized, their purposes in the design process, a partial data set which illustrates the manner in which these theoretical approaches are used, and the gains achieved. Particularly, the students’ written task solutions are investigated according to two more specific research questions: How do students use functions flexibly when solving the tasks, and which conditions hinder or support raising flexibility? Since the notion of flexibility in the use of functions is an idea inherent in various research results but not explicitly worked out in reports, we aim at elaborating it empirically in the course of the study. Methodologically, we suggest using a multi-theoretical approach to design tasks for students in a specific but typical context, and we study the solving of the tasks empirically following two directions: to explore (1) how the tasks are used and (2) which theory bricks are suitable for design and why they are useful.

We begin with elaborating the theoretical framework and the mathematical content of the task design, and define then what we call *flexible use of functions*. After that, the paper is structured along the steps of the methodological approach in terms of Design-Based Research. Finally, typical results of the empirical analysis of one student’s task solution are presented, and conclusions are drawn, looking back to the theoretical framework.

[1] This research project is part of the interdisciplinary project FaBiT conducted within a new research format called “Creative Unit” at the University of Bremen [FaBiT: “Fachbezogene Bildungsprozesse in Transformation”, (changing of domain specific educational processes, own translation)], (http://www.uni-bremen.de/cu-fabit). FaBiT investigates educational change in domain-specific teaching and learning with the aim of providing knowledge about conditions and constraints of innovation and change in domain-specific instruction (Doff et al. 2014).

2 Theoretical framework

We use four theoretical approaches, amplified in the next paragraph, for our task design and the subsequent analysis of students’ task solutions: (1) The Anthropological Theory of the Didactics (ATD), two epistemic actions models—(2) the GCSt-model, (3) the RBC + C-model—and (4) conceptual blending. We begin with a brief description of the theoretical approaches, highlight their added value in the design process, and describe how they are used together for design purposes.

The ATD allows signifying mathematical and didactical activities in an institutional setting. The core construct of ATD describes the institutional way of teaching in class through praxeologies. A praxeology consists of four interrelated parts: *tasks* students work on, *techniques* to solve a task, *technologies* to reason and the manner in which techniques are talked about, and a *theory* made of assumptions and background knowledge on the nature of the mathematics to be learned. Teachers employ a didactic praxeology (with didactical tasks, techniques, technology, and theory) to set up a mathematical praxeology (Bosch and Gacón 2014).

A technique is everything that is done to solve a given task. For example, a technique to construct a graph from a functional equation could be characterized by the following steps. First, a table could be produced, then the values are transferred to a coordinate system and finally, the points are connected to the right shape. A corresponding technology comprises the words which are used to describe the technique, for example: lookup table, equidistant marks, abscissa, and so forth. These aspects can be justified by the technology, for example, the known slope of a type of function (Barbé et al. 2005).

The GCSt-model consists of three collective epistemic actions, it can be used to analyse epistemic processes (Bikner-Ahsbahs and Halverscheid 2014) as well as to design tasks (Bikner-Ahsbahs 2014). Epistemic processes begin with gathering (G) examples and mathematical ideas, which are connected (C) and lead to seeing a mathematical structure (St). The GCSt-model assumes that adequate experience of gathering and connecting mathematical meanings shape the conditions for enabling structure-seeing in class. For example, different examples of tables about calculating the area *A(b)* of rectangles are gathered:

A(b)=1b		A(b)=5b		A(b)=0,5b		A(b)=7b	
b	A	b	A	b	A	b	A
1	1	1	5	1	0,5	1	7
2	2	2	10	2	1	2	14
4	4	4	20	4	2	4	28
8	8	8	40	8	4	8	56

If the students say something like "All these tables are representations for the function of an area" or "In all these tables, the widths are iteratively doubled", they connect the different examples with each other. If they say "It is always the same: If we double the width, the area is doubled, too. No matter how long the rectangle is", they see a structure that is a pattern generalized beyond the given examples.

Abstraction in Context (AiC) (Dreyfus et al. 2015) is a theoretical approach that describes individual epistemic processes within a context. The starting point of such a process is an epistemic need, the need for a new construct which drives the construction of knowledge. Given this need, the epistemic process can be reconstructed with the help of a nested epistemic actions model, briefly named as RBC + C-model. Through recognizing (R) students take up previous constructs as relevant for solving the task; these constructs are linked by building-with-actions (B) which shape the basis for constructing-actions (C) resulting in a new mathematical construct. Consolidating-actions (+C) stabilize the new construct making it more flexible and accessible for further learning.

Before we describe conceptual blending and its purpose in the study, we show how the three other theoretical approaches are used together in a complementary way to inform the design and research process and justify the inclusion of conceptual blending. First we consider institutions. They frame the settings in which the project aims at initiating transformation processes of the students by allowing or restricting an up-take of their experience of learning functions in feeder schools. The ATD is used to gain insight into how praxeologies may constrain epistemic processes. Didactical and mathematical praxeologies experienced in feeder schools are indicated by the manner students cope with tasks about functions, specifically when task requirements appear as constraints for the student. Hence, previous experience with praxeologies may constrain current learning. We assume that these constraints can be identified as traces expressed in the students' contributions. The current praxeology in class may or may not build on previous praxeologies, hence, it may provide conditions that respectively foster or hinder transforming the students' functional understanding towards a more comprehensive one (Bikner-Ahsbahs and Best 2016).

The students' transformation processes are shaped by individual epistemic processes towards constructing knowledge. The RBC + C-model is a tool to reconstruct these individual epistemic processes related to contextual conditions, e.g., providing insight into the way previous knowledge is used and incorporated to transform fragmented understanding of functions by their flexible use. It is not a design tool: epistemic processes in the study are rather initiated by a sequence of tasks being solved individually and collectively in class. These tasks are structured by the GCSt-model used as heuristics. Mathematical structures the students collectively perceive during the solving of the tasks should open up a renewed and more flexible understanding of functions.

However, the GCSt-model only allows organising collective epistemic processes; it does not inform the task design cognitively in terms of how to shape application contexts. For that, we use conceptual blending. Conceptual blending is "a basic mental operation that leads to new meaning, global insight and conceptual compression" (Fauconnier and Turner 2003, p. 56). To describe this operation, a network model is used. It contains in its simplest form two partially matched input spaces, a mental generic space constituted by a familiar structure common to the inputs, and the blended space. The blended space is the mental space constructed by the students through selective projections from the inputs (ibid.). An example of this operation is requiring the students to interpret the formula of the cone volume as a function of its height h or its base radius r (for further details, see Sect. 6); to accomplish this result, the students must blend geometric and functional aspects of the cone volume. The generic space is made of familiar algebraic expressions in which geometric as well as functional aspects are already addressed together. The two input spaces consist of the geometric aspects of the cone on the one hand and the functional aspects of the cone volume on the other. When students perceive the formula of the cone volume as a functional expression depending on the height of the cone, they are blending geometric and functional aspects of the two input spaces. This is possible if they have already built a generic space, for example, for calculating an area of a rectangle.

3 Clarifying the mathematical content of the task design

The task sequence is about interpreting formulas as functions to evoke flexibilization. For the purpose of untangling the content, we start with a literature review about research on functions and add some mathematical aspects. We then define flexibility in the use of functions (the Design-Subject, see Fig. 2) and add some prior findings about interpreting formulas as functions.

Already in 1991, Dreyfus showed that switching among representations, and translating the same problem into different contexts, are relevant for conceptualizing functions but difficult to achieve (pp. 32–34). Given the many problems students have with the function concept (Bikner-Ahsbahs et al. 2015; Nitsch 2015; Beckmann 2007), switching between representations and translating between concepts is not enough. The different concepts of variables have to be considered, too (see Malle 1993, pp. 46, 80). Thus, the function concept is a highly complex mathematical notion.

Its complexity is related to the contexts addressed, the mathematical content that functions encompass, and their representations (Beckmann 2007; Nitsch 2015). Therefore, learning about functions requires the integration of several aspects, e.g., functions as correspondence, covariation, and object (see Vollrath 1989, pp. 8–16). Students need to be familiar with all these aspects if they want to interpret functional representations. Covariation is crucial, but specifically difficult to apprehend (Johnson 2015). In our study, covariation means how changes in one magnitude affect a change of the depending magnitude. For example, covariation is addressed by sentences like “the more x increases, the more y increases, too” (Vollrath 1989, p. 12). For example, the covariation of a linear function is determined by its slope and can be described by a constant increase of the dependent variable whenever the independent variable increases by the same size. Regarding a function as a correspondence means to focus more on a relationship between variables where each independent variable of a set is assigned to exactly one dependent variable of another set.

In the project, a task sequence is developed through Design-Based Research (see Sect. 4). This methodological approach provides an empirical frame for elaborating the phenomenon of flexibility in the use of functions. Ellis (2011) has investigated students’ flexible understanding of functions focusing on quantitative relationships. The main focus is as follows: to let students investigate “functions from multiple perspectives” and “support their abilities to shift flexibly across different perspectives”. Star and Newton propose a more definite concept of “flexibility as knowledge of multiple solutions as well as the ability and tendency to selectively choose the most appropriate ones for a given problem and a particular problem-solving goal” (Star and Newton 2009, p. 5) in the context of solving equations. Translating these descriptions of flexibility to *using functions* positions it close to the notion of semiotic-theoretical control on functions (Bikner-Ahsbahs et al. 2014; Arzarello and Sabena 2011). Semiotic control is addressed “when the decisions concern mainly the selection and implementation of semiotic resources” (Arzarello and Sabena 2011, p. 191), and theoretical control is “when the decisions concern mainly the selection and implementation of a more or less explicit theory or parts of it” (ibid.). Taken together for using functions, the notion of flexibility concerning functions encompasses three aspects:

- flexible semiotic usage: *local* semiotic-theoretical control *within* a conceptual mathematical frame (e.g. algebra, geometry, arithmetic, functions) and *global* semiotic-theoretical control *between* different conceptual mathematical frames;
- flexible switching: ability and willingness to change between different perspectives in the interpretation of functional relations; and
- robustness: flexible semiotic usage and flexible switching are robust against small changes.

Flexibilization then means raising flexibility if flexibility is only partly developed.

We now briefly describe the aspects of this definition addressing functions. Performing semiotic-theoretical control means choosing signs of different registers (table, graph, equation) and different areas (algebra, analysis, geometry) based on theoretical knowledge according to the task requirements given. Moreover, it means dealing with them correctly, in other words, satisfying conventions or being at least consistent. Acting robustly against small changes implies that small changes in the formulation or the choice of notations do not confuse the students or prevent them from understanding. For example, if the names of the variables are not common or unfamiliar, students can still understand the concept and solve the given task. Different perspectives often refer to different meanings of variables, correspondence and covariation, different registers, different functions or types of functions.

The core idea for task design (see Sect. 6) is interpreting a formula as a function. For instance, the equation of the volume for a cone is an algebraic expression of magnitudes to calculate another magnitude; the volume of a cone. It can be interpreted considering different functional relations (see Fig. 1), for example as V(h) or V(r), that is the volume is regarded as a variable dependent on the height h or radius r. Switching flexibly between $V(h)$ and $V(r)$ requires changing the notion of the variables as parameters or as independent variables.

First investigations of solving tasks on the formula of the cone volume (Bikner-Ahsbahs et al. 2015) have uncovered problems of flexible switching, such as the following: translating functional relationships into graphical representations collides with the geometric form of the cone, for the *conceptual frame* (see Bikner-Ahsbahs et al. 2014) of the cone as a geometric object and the *conceptual frame* of functional relations were strictly separated for the students. That is, when the students reasoned on geometric objects there was no room for functional relationships and vice versa.

A specification of the formula by fixing $h = 1$ (length unit) was meant to be helpful for the students, but irritated them: it led to a change of the formula into

$$V = \frac{1}{3}\pi r^2,$$

which was interpreted as an area and not as a volume anymore. In addition, missing letters x and y basically hindered the students from seeing functional relations; their semiotic

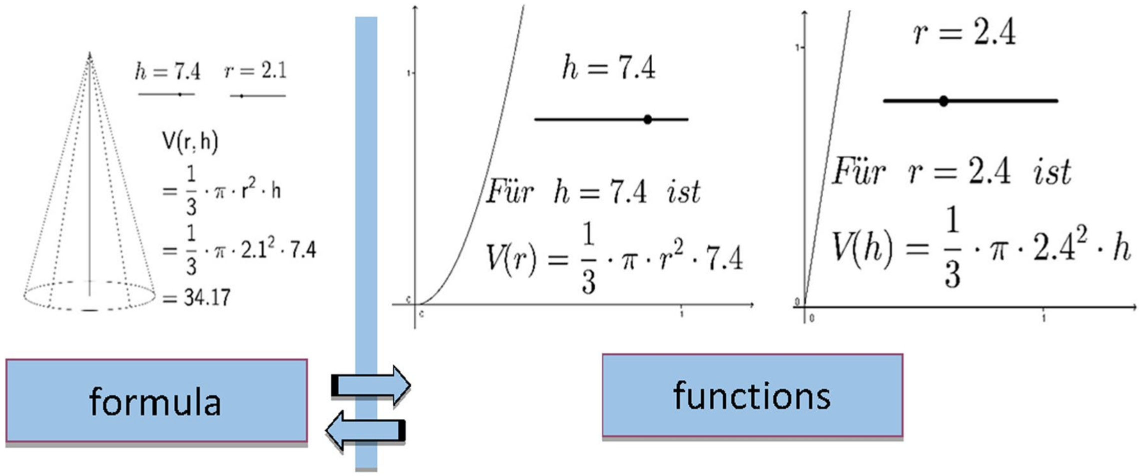
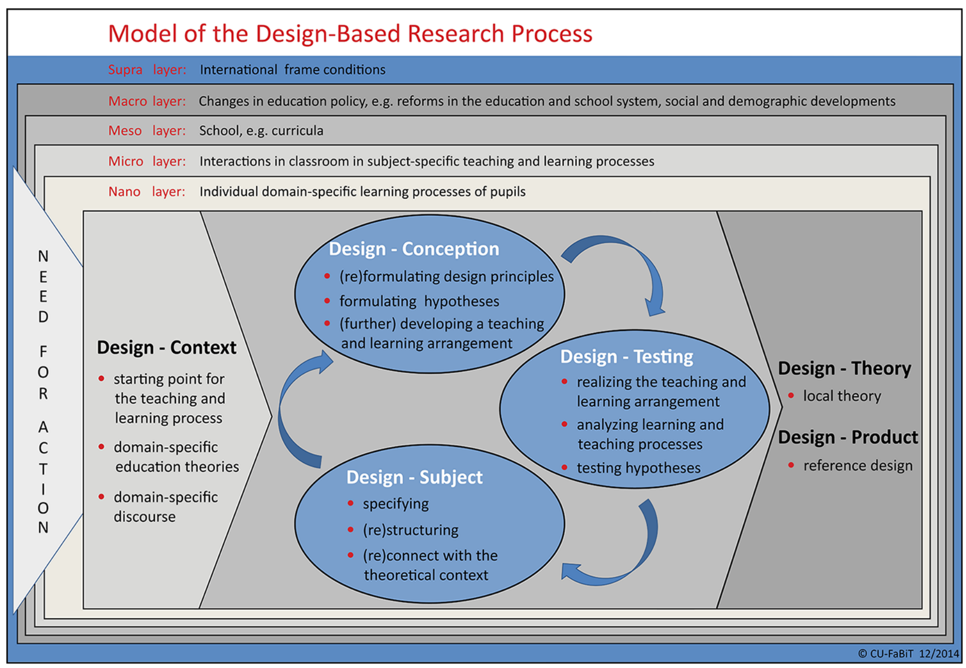


Fig. 1 Functions in the formula of the cone volume

Fig. 2 FaBiT design process (Peters and Róviro 2017, p. 32, ©CU FaBiT 2014, translated)



control was not robust. Instead, the students placed the cone into the 3D-coordinate system, such that the cone's height was on the y-axis and the radius r was marked on the x-axis. This way, a more familiar representation was produced, hiding functional relationships between volume and height or radius. If conceptual frames are strongly separated, as in this case, the students' (local) semiotic-theoretical control within one conceptual frame (e.g. geometry) may still be possible while the (global) semiotic-theoretical control across these frames cannot be achieved (see Bikner-Ahsbahs et al. 2014). Hence, if we use formulas of geometric objects for exploring functional relationships, we also have to globalize semiotic-theoretical control across conceptual frames.

4 Design-Based Research as a methodological approach for task design

In this study, we use an adapted form of the methodology of Design-Based Research presented by Prediger et al. (2012). The core of our methodology consists of three steps (Fig. 2): clarifying the mathematical subject matter of the design, describing the Design-Conception, and the Design-Testing.

These three steps are part of an iterative cyclical design process starting from a need for action, based on the analysis of the Design-Context. This process is regarded as being conducted within five nested institutional layers (one of which is the school). It finally leads to (1) theoretical knowledge about how fragmented understanding of functions may be overcome at the initial stage of grade 11 at high school and (2) a reference design of tasks and the teaching-learning arrangement, showing how this can be achieved.

The following sections are structured along these steps. We start by clarifying the Design-Context. As we have already specified the Design-Subject, we then describe the Design-Conception. Afterwards we report how we have developed the final tasks. Finally, a data set is analysed, results are presented, and conclusions are drawn.

5 Design-Context

The didactical praxeologies on teaching functions of the feeder schools are disclosed by interviews with teachers addressing their teaching of function at lower secondary school level, their use of books, typical tasks and techniques they implement in their classes, their reasoning about teaching functions, and their understanding of the nature of functions.

First analyses show a preliminary picture (see Bikner-Ahsbahs and Best 2016). Teachers at the *Gymnasium* (a type of school that includes high school level up to grade 12/13) seem to think about preparing conceptual understanding as it is needed right from the beginning, from grade five towards grade 12/13. This preparation is not done by teachers of lower secondary schools, if they do not have any experience at upper secondary school level. The latter is often the case. Mathematical issues of lessons and grades with which teachers have not yet had experience do not seem to be within the scope of their behaviour in class; these teachers are more concerned with current conditions of instruction. Two extreme poles of functional understanding addressed by teachers have been identified so far. On the one side, some teachers stress functional understanding whenever it appears, including reasoning activities with functions throughout all grades of secondary level. On the other side, other teachers teach how to use the different representations from grade five on without being explicit about functions: algebraic representations of functions shape the starting point of talking explicitly about functions in terms of linear functions. In addition, we could distinguish between a holistic view and a fragmented view on teaching functions. In a holistic view, a correspondence is as much used as covariation is addressed, while specific features like the slope or the y-intercept are included in the considerations as early as possible. The fragmented view considers teaching functions to consist in predominantly teaching its parts.

6 Design-Conception

The aim of Design-Based Research is to construct and explore tasks for a teaching and learning arrangement for about four lessons (of eight hours): that is, two weeks of instruction on geometric formulas with the aim to make functional understanding more flexible. For developing the Design-Conception, we use design principles.

A principle of design is a prescriptive theoretical element that contains the aims and the means to achieve the instructional goal. This principle is meant to be worked out during the experimental phases resulting in a local theory. As regards the design, typical questions are as follows: "What do I have to do to achieve a certain goal? Which conditions must be considered?" (Prediger 2015, p. 652, own translation) What kinds of procedures can be addressed? Are there theories that substantiate the decisions?

The following learning goals are the focus of our study: The students are to …

- …learn to talk about and work with the same functions in different representations, and switch between different representations;
- …recognize similarities and differences between linear and quadratic functions (concerning the functional aspects and visible properties of representations) while they switch between different types of functions.

To achieve these goals, we use the following **Principle of Design**:

"To construct new knowledge about functions to achieve the above mentioned goals,

- a task that requires conceptual blending and a flexible use of representations could be used;
- taking up praxeologies of feeder schools as traces the students bring with them into class;
- initiating a change by starting with a generic model on formulas as functions and enlarging it" (Best 2016, p. 38);
- using the GCSt-model as heuristics for task design.

Connecting the four parts in our principle is demanding for our design study. The first two are linked to our definition of flexible use of functions, the last two are theoretically informed. Thereby a generic model models the generic space to be activated mathematically. We now show how we developed the task sequence methodically.

The design of tasks for the lesson is supposed to align with the fragmented understanding of functions students bring with them into class and to transform it by raising its flexibility. This design takes advantage of the following aspects: formulas are interesting because they predominantly are meant to be used and built for specific concrete objects and cases. Normally, they are not related to functions. Therefore, different frames could be addressed within the same task. And it can make the students aware that a function can also be described by variables different from x and y. This way, the students need to have a robust semiotic usage to solve the developed tasks. In $A(a)=ab$, the side length b is regarded as fixed, while the side length a varies. Or it could be considered the other way around (a fixed and b varies). This could be used to initiate flexible switching. On these grounds, a task developed on the idea "formulas as functions" could flexibilize semiotic usage and switching and foster robustness. Thus, we assume that flexibility (see Sect. 3) can be achieved if students investigate functional relationships in formulas, such as the formula of the cone volume.

Furthermore, considering functional relations, formulas are considerably rich resources for mathematical exploration. As an example, we take a look at different aspects of the volume of a cone for describing the interpretation of its formula as a function. Based on this view, the advantages for achieving flexibility are illustrated more precisely. Consider a cone and double its height, then geometric and physical representations indicate that the volume is also doubled (see Fig. 3).

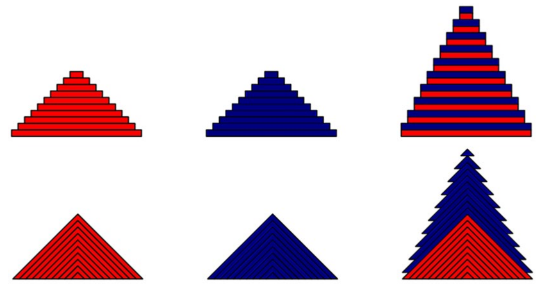


Fig. 3 Doubling the height of a cone, geometrical view (Walser 2015)

An analogical inference can be drawn if we make the height h three times as high as before. This makes proportionality between the height and the volume of the cone visually and practically plausible. Once the formula of the cone volume $V(r,h)$ (Fig. 2) is given, this can enrich insight into how the volume of the cone depends on h. It allows the students to explore and uncover the size of the volume in different cases, thus flexibly focusing on different aspects of the ways variables may be considered.

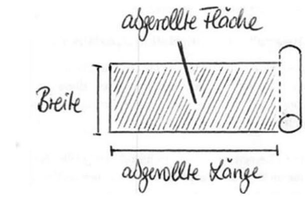
Considering the proportional relationship $V(h) \sim h$ while keeping r constant, simplifies and generalizes geometrical argumentation: If the height becomes n times as large, the cone volume will become n times as large, too. We now fix the height $h=1$ (length unit). If we ask how the cone volume will change if the radius r is doubled, considering the formula will lead to the result that the volume will be four times as big, but geometrical considerations only will not lead immediately to the correct answer. In this case the roles of the variables have changed: the height now is a fixed parameter and the radius is varying. This shows that the cone volume increases quadratically with the radius no matter how high the cones are, as long as they are regarded as fixed. However, the formula of the cone volume encompasses many more ways of using functions. For example, if the height h and the radius r have the same size but increase together, the volume increases cubically.

To summarize, the nature of the functional relationship in the formula of the cone volume is not visually obvious because the change of the height or the radius alters the cone's shape. This lack of similarity makes it more or less likely that students will have to understand the underlying relationships between the variables. Moreover, formulas may allow students to build on proportional, linear, or quadratic functions and their ordinary representations, algebraic, graphical, by tables, in speech and through the objects for which they are built. And they provide a frame for recognizing previous constructs and functional relationships (see the RBC + C-model, Sect. 2): formulas may—according to Malle—"be described more precisely as dependencies with the help of the functions concept" (own translation of "Abhängigkeiten […] mit Hilfe des Funktionsbegriffes präziser beschrieben werden.") (Malle 1993, p. 79). Hence, formulas provide concrete and meaningful contexts for reasoning with functions, an aspect that is new to the students at this stage. Flexibility can partly be achieved by a change of view on different functional relationships within the same formula and their representations. This change also asks for a change of view on variables and their specific meanings and covariations. Thus, the students could learn to switch between different perspectives flexibly within the same object and connect this to building semiotic–theoretical control.

Since a formula can represent one single algebraic expression on features of geometric objects including different functional relationships, a change of view on functional relationships can easily be interpreted geometrically, but also point to the distinction between the function concept, its algebraic expression and the geometrical object about which they inform some feature. The goal is that through a change of view students become able to *see* specific functions as *structures of growth* in a formula (according to the GCSt-Model, see Sect. 2). This goal

Fig. 4 A task activating a generic model, part 1 (own translation)

Task 1: Chose a roll.



Scrolled length of the scrolled area

Its width is ___________ .

a) State the size of the scrolled area of the roll if its length is 1 cm, 2 cm, 3 cm, 4 cm, 5 cm, 9 cm or 10 cm.

b) There is a correlation between the length of the unrolled area and the size of the area. Represent this correlation. Think of the representations we have written on the white board.

could be reached while regarding some variables as independent or dependent, exploring their relationships in the same expression, and representing them in different ways.

7 Developing the final task design sequence

In a pilot design study, three design cycles were conducted to explore how students build a flexible understanding of functions via formulas, starting with a generic model for building the generic space: the formula for calculating the area of a rectangle. The results of these design cycles are complemented by a series of interviews with teachers from feeder schools to identify typical (didactical) praxeologies that indicate ways through which students may have access to understanding functions (see Sect. 2, ATD).

The teachers' interviews show that "*Oberschulen*" (our feeder schools) implement calculating the area of a rectangle at grade five or six. In these lessons, often functional relations are considered by questions such as the following:

> How does the area of a (square) rectangle change if we change one side length by making it two times, three times, … as long as the original one?

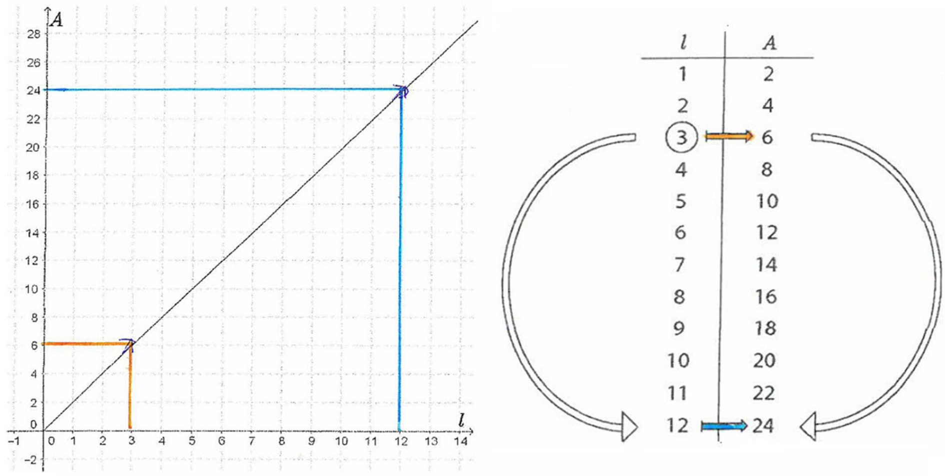
Since students at the initial stage of high school seem to share this experience about rectangles, this was used as a generic model for our design experiment. Since representational fluency (i.e., its flexible usage) seems to be underdeveloped (see Shu and Moyer 2007), several representations of functions such as tables, graphs, equations were taken up in the design. This way, the first and second aspect of the Principle of Design are incorporated into the task.

The generic model consists of the formula of the area of a rectangle: $A=ab$ with the side lengths a and b (see Figs. 4, 5). This equation can be interpreted considering functional relations (see Sect. 6). It is complemented by the formulas of the areas of triangles and circles. In the second step, this generic model is transformed to the formula of the volume of a quadratic pillar and the volume of a right circular cylinder. The third step addresses the formula of the cone volume. This kind of enlarging of the generic model manifests the third aspect of the Principle of Design.

The final task design for the lessons was prepared together with two teachers in order to build a common praxeology (Arzarello et al. 2014). The teachers and the researchers discussed the tasks before their implementation. These discussions were audio taped. They served as bases for the final revisions of the tasks, executed by the researchers and adopted by the teachers. Further, the researchers provided suggestions as to how the lessons could be given but left the final decisions to the teachers, who made changes and adaptations during the process if needed. This way, we expected to adapt the tasks to the learners' situation and to the teachers' praxeologies in class. In particular, the unrolling tape task (Fig. 4) provides a context in which the students could experience a continuous change of the length and simultaneously a continuous change of the corresponding area visually and practically. This feature of continuity can be substantiated visually, if the different values calculated for the table were illustrated by roll-ups.

In order to make the use of representations more flexible, translations between representations are initiated

Fig. 5 A task activating a generic space, part 2 (own translation)



Arisha has drawn a table for a roll of 2 cm width. She is looking for the starting length and demonstrates the change when unrolling by arrows.

c) Interpret the arrows: What does Arisha want to show us?

Arisha transferred the arrows onto the graph. How did she manage to do that?

d) Draw the corresponding arrows with the same color.

(Remark: Colour the arrows in the table and use the same colours in the graph. In other words, a red arrow [see: from 3 to 6] in the table means the same as a red arrow in the graph.)

and investigated. In the first task (Figs. 4, 5), this aim is addressed by letting the students transfer corresponding arrows from one representation to another (first aspect of the Principle of Design).

Besides this translation aspect, the students gather examples about the correlation between the area and the length of a rectangle. The gathered examples consist of corresponding values, different representations and examples of the covariation property (Figs. 6, 7, 14). In addition, the students are instructed to connect different representations by transferring arrows (Figs. 8, 9, 15). This way, the fourth aspect of the Principle of Design is addressed.

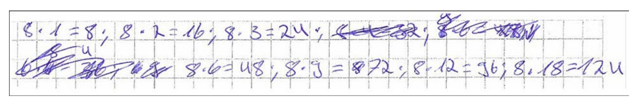


Fig. 6 Solution of task 1 (a) by student S1

expressions of the change of functional understanding are expected to occur in their acting. Expected results will answer the following questions:

- Which constructs are recognized and built-with?
- How does this step influence improving flexibility in the use of functions?

8 Investigating students' transformation processes in the Design-Testing

The final sample consists of four classes, one of which was a focus class being videotaped with three cameras. One camera videotaped the whole class, specifically during class discussions. The other two videotaped two pairs of students over the duration of the whole data collection in class. Additional field notes were made and the solutions of the students of all the four classes were collected. The data analysis aims at reconstructing traces of the praxeologies of the feeder schools and their interaction with the tasks. The RBC + C-model is used to reconstruct individual epistemic processes since in these processes

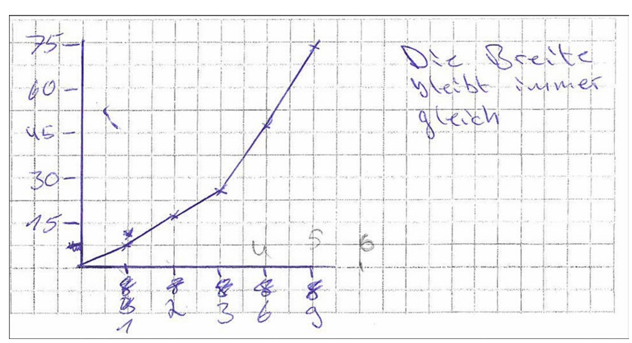


Fig. 7 The solution of task 1 (b) by S1 (see Bikner-Ahsbahs and Best 2016)

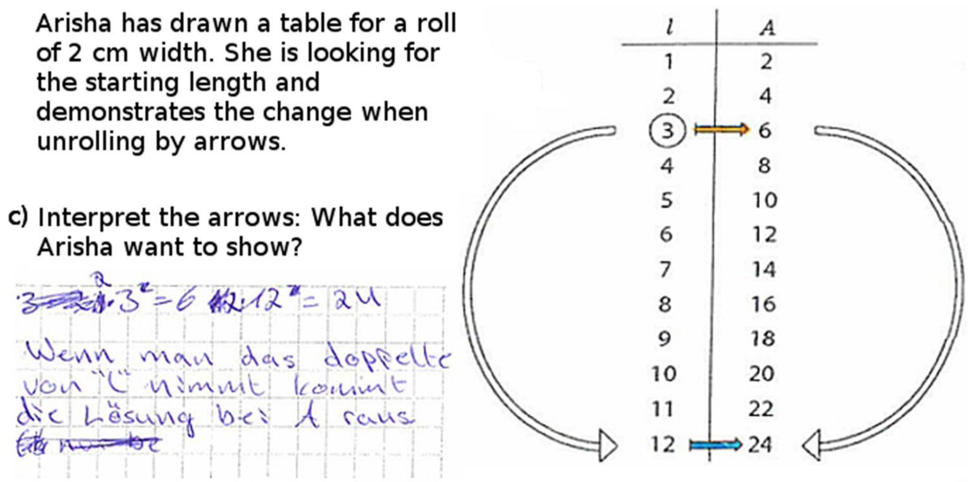


Fig. 8 Task 1 (c) and the student S1's solution: "If one doubles "*l*" you get the solution of *A*" (own translation)

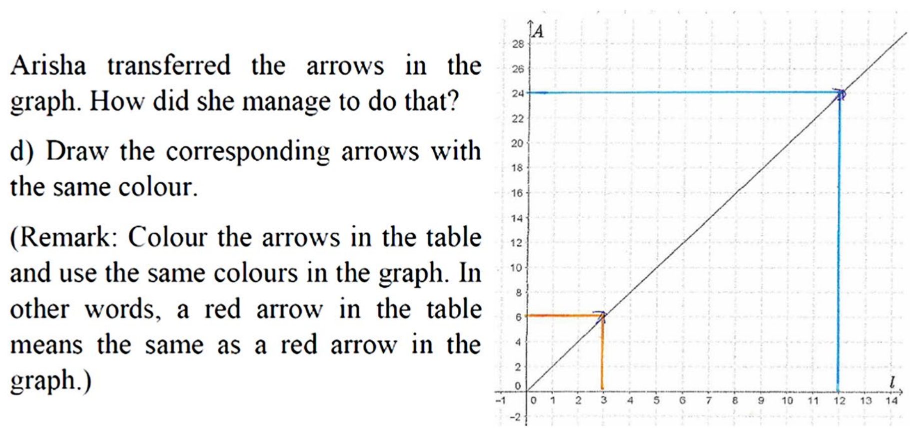


Fig. 9 Task 1 (d) addresses representation fluency (own translation)

- Which conditions foster or hinder students to be more flexible in using functions?
- What traces of praxeologies of the feeder schools are identified and how do they constrain the students' transformation processes?
- How can the teacher meet the students' difficulties?

9 A student case: preliminary empirical results

We now analyse written solutions of the student S1 to answer the questions above. The results will serve as hypotheses being investigated by the analysis of further video data and further written solutions of the students.

9.1 Tasks addressing the area of a rectangle as a generic model for blending

In Task 1 (Fig. 4), the blending of the arithmetic space and the rolls as real objects activates a generic space of S1 because S1 is able to calculate the areas for different lengths (Fig. 6). However, the graphical interpretation as a proportional functional relationship of the area dependent on the length of the unrolled part is not shown (Fig. 7) because S1 does not take up the conventions for graphical representations. For example, the equidistant marks on the abscissa do not correspond to the sizes of their numbers. This student even named all the different marks representing the same size (8 cm) at the beginning. Therefore, S1 does not yet seem to have achieved semiotic control, hence, semiotic flexibility, in the use of proportional functions.

The resulting graph (Fig. 7) makes it difficult for S1 to interpret the slope as being constant. As a consequence, it is difficult for the student to grasp covariation of the proportional function between the lengths and the area.

In the task sequence, Arisha is an imaginative girl who is used to offering the students ideas or solutions to be discussed. In Task 1c (Fig. 5), Arisha shows the change of the coordinates in the functional relation by using arrows. The students are supposed to interpret her arrows. In the table of Fig. 8, S1 refers only to the correspondence arrow. This reference begins arithmetically to be mixed up with the distinction between "times" and "the power of 2". But then

Arisha states now: "If and only if the side length, *l*, is getting three times as long as the original one the area, *A*, becomes 9 times as big".

f) Give reasons why this rule is valid.

g) Give at least one rule in addition to the one Arisha has formulated.

Fig. 10 Task 1f and g address functional understanding (own translation)

Fig. 11 The answer of S1 to Task 1f: "the more the length of one side increases the more the area increases. It proceeds proportionally" (own translation)

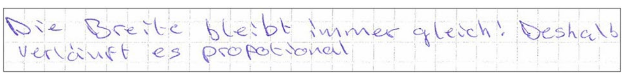


Fig. 12 The answer of S1 to Task 1g: "the width of the area is constant and therefore it [the process] proceeds proportionally" (own translation)

S1 transfers the calculations to a more general description, recognizing the correspondence between the length and the area only: "If one doubles "*l*" you get the solution of *A*" (Fig. 8). Covariation is not recognized in the second arrow going from 3 to 12 and from 6 to 24. Furthermore, in the next task (Fig. 9), the covariation goes unnoticed, too, since the corresponding arrows should be drawn and coloured. S1 coloured only the arrows that stand for the correspondence, and not the arrows that stand for covariation. In short, S1 shows only one view on the functional relation and thus was probably not able to flexibly switch between these two perspectives.

The answers to the two further tasks (f) and (g) (Fig. 10) show that S1 interprets (recognizes) the increase of the function value A(*b*) with A(*b*) = *ba* as proportionality justifying: "the more the length of one side increases the more the area increases. It proceeds proportionally" (Fig. 11). Then the student is asked to find another rule. The rule he describes is contextually bound, it says that "the width of the area is constant and therefore it [the process] proceeds proportionally" (Fig. 12). Hence, we do not find any indication towards the constant slope, which we would expect in a proportional function. Covariation is reduced to a simple rule: *the more … the more…* The student's graphical representation (Fig. 7), which is not a straight line, substantiates the interpretation that S1's view on covariation follows the simple rule that is viable with his calculations in Fig. 6.

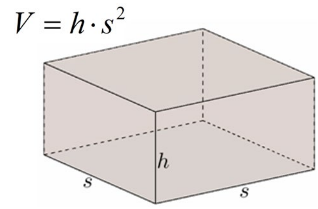


Fig. 13 The formula of a quadratic pillar

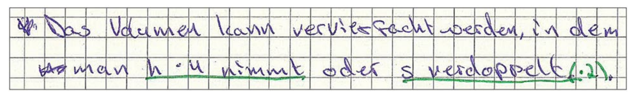


Fig. 14 The solution of the task solved by S1: The volume can be made four times as large if *h* is taken 4 times or *s* is doubled

Therefore, we may conclude that the generic space the student S1 activates is situated in the arithmetic plane of calculating the area and building a correspondence between numbers. The specific property of covariation of the proportional function is not activated beyond the simple rule (Fig. 11). The graphical representation (Fig. 7) is built by inserting the calculated numbers to build points, which the student connects using straight lines. This case is a first indication that students need to conceptualize covariation of a functional relation within every representation and also to link covariation between different representations. This is prerequisite for being able to switch flexibly among different representations. To achieve flexibility in the use of functions, these steps have to be considered in the design of tasks. Only then may we expect covariation to be used as a feature of the specific type of function in any representation.

9.2 Tasks addressing the formulas of the volume of pillars for blending geometric and algebraic spaces

Let us look at the solutions of S1 three lessons later.

In one task, the students are asked: How can you increase the volume of a box shaped by a quadratic pillar (Fig. 13) to get it four times as large as the original one?

S1 no longer needs any calculation to find the answer to the pillar task. He suggests that the result could be achieved by making *h* exactly four times as large or by doubling *s* (Fig. 14). His graphical representations show that the correspondence as well as the covariation are represented by arrows adequately, hence, are well recognized (Fig. 15). In this task, the requirements to flexibly switch between the perspectives of correspondence and covariation are accomplished; hence, in the coordination both are built-with. However, the graph of the quadratic function *V* with

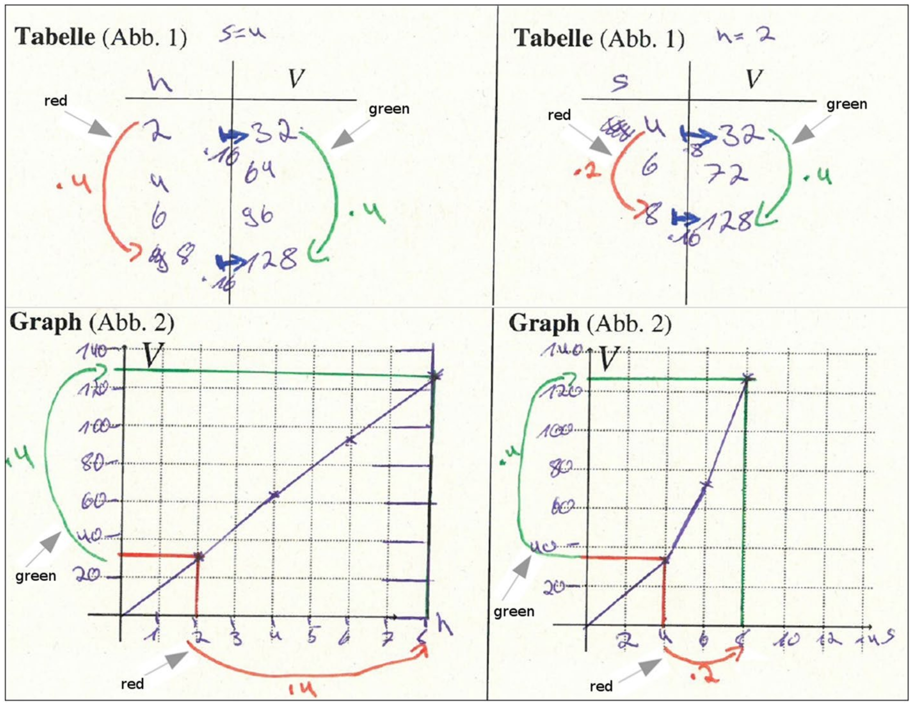


Fig. 15 S1 represents the solution of the pillar task in a table and in a graph

$V(s) = h \cdot s^2$

still does not meet mathematical conventions. It is built by drawing points based on the numbers in the table and connecting these points with a polygon chain. But using arrows, the student has represented a specific covariation. This indicates that the student has grasped a graphical construct of covariation.

Problems with the graphical representation also cause problems in the next step, i.e., in a similar task for the volume of a cylindrical box (Fig. 16).

The students were asked how the volume of a cylindrical box could become a quarter of the original volume. S1 answered correctly, that this can be done by dividing h by 4 or r by 2 (Fig. 17). Again, S1 was able to answer flexibly what has to be done by using words. So, S1 could flexibly switch between different perspectives of the notions of variables. But looking at the corresponding table and the graphical representation indicates difficulties with flexible semiotic control (Fig. 18).

The first difficulty of S1 (Fig. 18) is that making the volume smaller instead of bigger is a new kind of task. This task challenges S1 when asking how the table and the graph could be represented. While S1 is meanwhile able to manage working with the tables as well as the graphical representations of the proportional function, the graph of the quadratic function still shows difficulties that have already appeared before. S1 does not follow the conventions of using the coordinate system for representing a graph: Besides a missing π at the ordinate, the scaling is

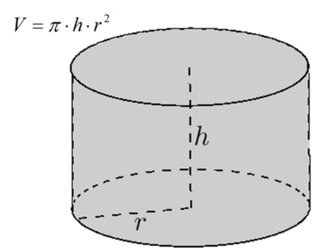


Fig. 16 The volume of a cylindrical box

wrong. The partition of the abscissa is equidistant—S1 adds "r in cm"—but the scaling does not seem to follow mathematical conventions. The values on the axis grow from right to left, instead of the other way round. Interestingly, the scaling of the ordinate is done in the conventional way. It seems as if it is not clear that the coordinate system needs to be built following clear mathematical conventions, and that the coordinate axis of the independent variable should increase from left to right. The concept of the coordinate system functioning as a reference system

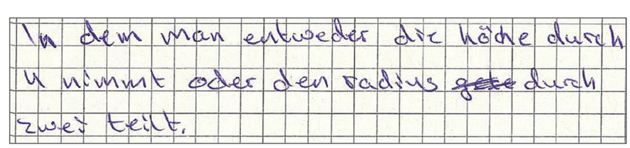


Fig. 17 S1 solves the cylindrical box task: "this can be done by dividing the height by 4 or the radius by 2"

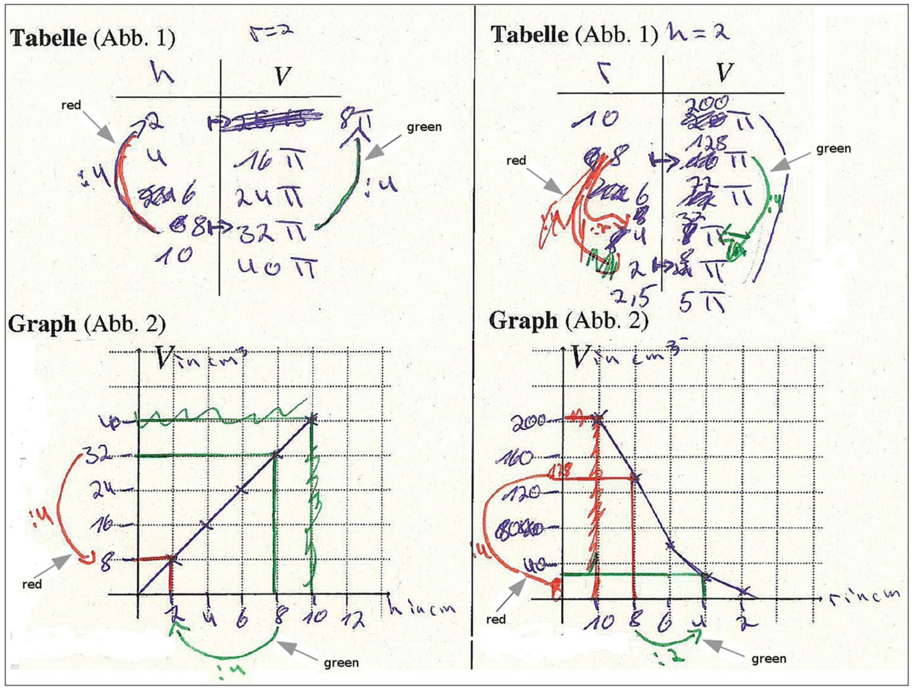


Fig. 18 Solution for the cylindrical box task

for representations does not seem to be stable for the student. S1 should take it as being fixed before drawing the graph, keeping the orientation of the axes and the measure units fixed, too. Instead, the student's starting point may be a familiarized or routine way of transferring the numbers from the table to the scaling beginning with the first mark on the left; that is 10.

To sum up, building a graphical representation is not robust against small changes. In addition, a fixed orientation of the covariation arrow for ":2" warrants orientation of the (wrong) coordinate system. Covariation describes change, which seems to be connected to orientation in space; hence, graphical representations of covariation need to consider orientation in space.

9.3 Preliminary empirical results

Based on the analysis above, we may answer our questions for the student S1 as follows:

- S1 recognized proportional functions but did not recognize suitable covariation for proportional functions. Therefore, building-with both, in terms of the covariation of the proportional function, was only partially correct and caused problems.
- At the beginning, S1 recognized the function concept as a correspondence, represented by arrows in a table, but did not recognize covariation in the table. This seems to hinder cultivating a flexible use of functions in both aspects.
- S1 did not build adequate graphical representations for covariations at the beginning. Later this was possible for proportional functions but still not completely possible for quadratic functions. Hence, covariation seems to be a difficult construct that may be there for one type of function but not for another.
- S1 did not recognize an adequate concept of the coordinate system, therefore S1 showed difficulties in using graphical representation. Specifically, when covariation is represented graphically with different or wrong orientations in space, this may confuse students and hinder their recognizing of covariation. To sum up, the concept of a coordinate system did not seem to have been built as a reference system for graphical representations. Since this occurred right from the beginning as an obstacle for representing covariations graphically, it seems to be a trace in the technique coming from the praxeologies of the feeder school. It constrained the development of flexibility from the beginning. We assume the concept of the coordinate system as a reference system for graphs to be a key prerequisite for any graphical representations of functions.

The results show that the flexible use of functions requires building-with flexibility for every type of function as correspondence and as covariation within and between all kinds of representations, and also between different functions. In addition, a specific constraint was identified. S1 has not yet constructed the concept of the coordinate system with its purpose to provide a reference system in respect of

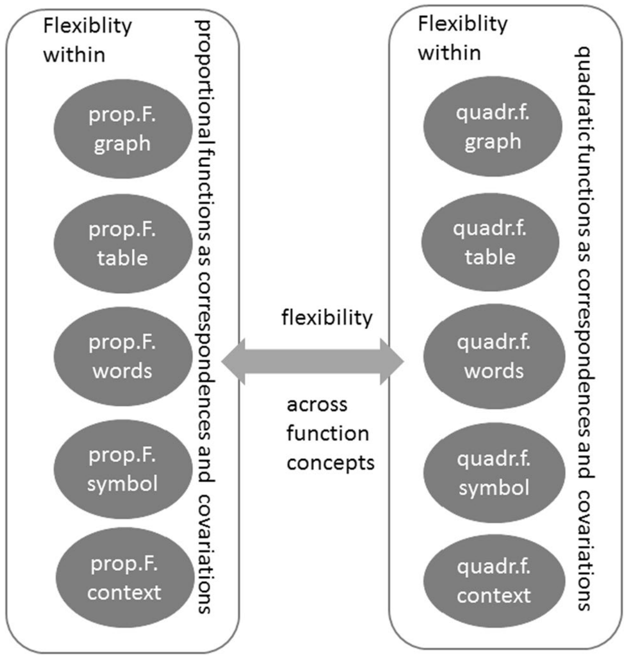


Fig. 19 Flexible use of functions also means being flexible among their parts and across different function concepts

its conventionalized rules for graphical representations of functions. Therefore, graphical representations often could not be built adequately, hindering the subsequent epistemic process.

Interpreting covariation in a graph does not happen automatically when graphs are just drawn by inserting points and connecting them with straight lines. It is an additional step to be addressed as early as possible by specifically designed tasks.

Figure 19 condenses the preliminary results: Flexible switching and flexible semiotic usage cannot be achieved all in one. They require to be addressed in tasks already within the context of learning specific types of functions, and relationships between them, for example, by the change of representations and contexts, and the change between correspondence and covariation within every representation.

9.4 Adapting to students' learning in class

Given the results above, how can the teacher support the student to improve flexibility? This question will better be answered after the analysis of the video data, which is still to be conducted. However, a key constraint, which may be a trace left from previous praxeologies of the feeder schools, is the lack of understanding of the coordinate system as a reference system for graphical representations. Since the student S1 has manifested interest in representing the function in a graphical way, he is about to learn more about graphical representations. Such an interest can be taken up by an emergent task, that is a task in which the teacher starts from the students' demonstrated interest directions and activates them cognitively to go beyond their boundaries or overcome their obstacles. In this case, the teacher could offer the following emergent tasks for adapting the student's needs to those of the tasks (Bikner-Ahsbahs and Janßen 2013; Ainley and Margolinas 2015, p. 133):

Emergent task 1 The teacher gives the student a slide with a fixed coordinate system. This way the student may use it as a fixed reference system without having constructed the concept himself. At the end of the lesson, the concept of the coordinate system should be discussed and explained. For example, the teacher could ask: "Why do you think it makes sense to first fix the coordinate system before drawing a graph?" or the other way round: "Imagine a situation in which an engineer adapts the coordinate system to get a straight line for a quadratic function. What risk would he run?"

A similar approach could be used if the student presents covariation in a table but is not able to translate it to another representation, for example to a graphical one. In this case the teacher could ask:

Emergent task 2 "How could you illustrate the changing of both variables shown in the table in a graphical way by the use of a coordinate system? Try it out and show us your ideas."

To make the reference feature of the coordinate system clearer, a third variation (specifically for high-achieving students) may be to transform the coordinates of the coordinate system after the graph of the quadratic function $f(x) = x^2$ has been sketched.

Emergent task 3 How does the shape of the parabola change after transforming the coordinates x and y into $\frac{1}{x}$ and $\frac{1}{y}$. The answer is, it does not change, except that the origin is not defined. Even in this case, the new coordinate system is a reference system for the new shape as soon as the system's coordinates are fixed.

Emergent tasks adapt the content to the students; they take up ideas a student produces or the regularities the student has found and transform them into tasks, e.g., an exploration task about graphical representation. This way, the teacher may raise mathematical awareness for the core idea of the coordinate system as a reference concept in an adaptive way to support the student in raising flexibility.

10 Conclusions

This paper has presented an overview of a Design-Based Research process on developing a task sequence for enabling students to use functions more flexibly at the transition phase to grade 11. Reflecting the design process, the relationship between the theoretical approaches, the tasks and the student in the context given, becomes clearer. Praxeologies experienced in class beforehand, influence the way students work with tasks, for example as strong constraints identified empirically. The paper describes two ways tasks may take constraints into account: in advance during the design process or in emergent tasks posed by the teacher to direct the student's learning into a fruitful direction. In class, tasks have to take into account learning paths, collectively and individually. The GCSt-model was used as a heuristic tool for task design that enacts a process of collectively gathering and connecting mathematical meanings to prepare structure-seeing for all students. However, both models fall short in addressing the individual-cognitive process. For blending two or more semiotic frames into the task, conceptual blending has informed task design in that a generic model may enact the students' generic space as a relevant step. To answer the question as to whether and how students have accomplished the goals individually, none of the three theoretical accounts would provide a suitable tool. For such an a posteriori analysis, we needed the RBC + C-model to analyse the students' epistemic processes of solving the tasks related to the contextual conditions.

The analysis of one single student's task solutions against the background of the theories above has shown how flexibility in the use of functions involves flexible switching across different function concepts as well as among the different perspectives of the same function, its partial concepts, its variables, representations and aspects such as correspondence and covariations. Further, activating a generic space and blending different frames such as geometric and algebraic or functional frames requires flexible semiotic control, that is, the learning of local as much as global semiotic–theoretical control including strengthening robustness according to small variations. The student's solutions show that beginning with a familiar generic model (the formula of the area of the rectangle) has indeed activated a suitable generic space. Solving the subsequent task led to a blended space, which in turn served as the next generic space for the following task. However, activating a generic space alone was just a first step in accomplishing the task. Developing flexibility was still necessary.

One specific key constraint for developing flexibility in the use of functions is the lack of understanding of the coordinate system as a reference concept. It may cause students' difficulties in representing functions graphically, specifically when covariation is considered. Wrong graphical representations may entice students onto a wrong track because of their ostensible *visual and physical evidence*. We conjecture that this deficiency is part of the praxeologies at the lower secondary school level, leaving traces in the students' experience. The data analysis shows that this deficiency may cause confusion even if other parts of the function concepts are well understood. This way, previous praxeologies may influence teaching and learning in the transition context as constraints (Bikner-Ahsbahs and Best 2016). However, such a constraint is not necessarily a bad thing; it can also be transformed into challenging emergent tasks for the individual student (Bikner-Ahsbahs and Janßen 2013).

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