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ORIGINAL ARTICLE

Impact of the course teaching and learning of mathematics on preservice grades 7 and 8 mathematics teachers in Singapore

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Abstract This paper explores the impact of the course "Teaching and Learning of Mathematics (Grades 7 and 8)" for the preparation of pre-service secondary mathematics teachers. It examines how pre-service teachers perceived mathematical problem solving and communicating mathematical knowledge before and after the course. The main objective of the course is to equip them with a working knowledge of basic teaching principles, an understanding of the theories of learning that inform instruction, and a thorough knowledge of the curriculum. The participants of the study are professor X, who teaches the course, and her five pre-service teachers (PTs). The PTs wrote a journal entry after every topic explored during the tutorials, detailing their learning during the course of study. For their journals, specific prompts were provided for them to reflect on their learning and experiences. At the end of the course the PTs were also interviewed. The interview sought PTs perceptions about their communication of mathematical knowledge during the microteaching sessions. The data collected were subjected to qualitative analysis. The findings of the study show that the course deepened PT's knowledge of mathematical problem solving and also shaped their ideas about how to communicate mathematical knowledge in ways that address the why and how of it.

Keywords Pedagogy course · Pre-service mathematics teachers · Grades 7 and 8 · Communication of mathematical knowledge · Reflective journals · Interviews · Narratives

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1 Introduction

Learning to teach mathematics calls for the reversal of roles by pre-service teachers, from being a student to being a teacher. The transition from being a student to being a teacher, if it is to be successful, must for many involve a considerable degree of "unlearning" and discarding of mathematical baggage (Brown and McNamara 2011, p. 35). Often pre-service teachers have fixed views of what mathematics teaching is and seek to replicate their views that are based on their experiences as students (Zaslavsky and Sullivan 2011, p. 13). Lack of attention to past experiences of pre-service teachers "may help to account for why teacher education is often such a weak intervention—why teachers, in spite of courses and workshops, are most likely to teach math just as they were taught" (Ball 1988, p. 40). Studies have shown that courses for pre-service teachers that include features that engage them to understand and reconstruct what they know with more depth and meaning, particularly learning or re-learning mathematics they will teach in school, have led to positive outcomes for them (Crespo and Sinclair 2008; Prediger 2010; Steele et al. 2013).

In Singapore, pre-service teachers are recruited by the Ministry of Education from the top third of respective local university cohorts (Darling-Hammond and Rothman 2011). Following recruitment they are enrolled at the National Institute of Education (NIE) for their pre-service teacher education. Pre-service secondary school teachers generally are trained to teach two curriculum subjects. As part of their teacher education that leads to the award of a Postgraduate Diploma in Education, pre-service secondary mathematics teachers take the course Teaching and Learning of Mathematics that is 108 h in duration spanning a period of 21 weeks.

Teachers who have completed a major course in mathematics at the university are trained to teach grades 7 to 10, while those who have completed only a minor in mathematics or passed mathematics at grade 12 are trained to teach grades 7 to 8. The main objective of the course is to equip them with working knowledge of basic teaching principles, an understanding of the theories of learning that informs mathematics instruction, and a sound knowledge of the school mathematics curriculum. The appropriate combination of these components, when utilized by motivated individuals, will provide the background required for the development of expertise in the teaching of mathematics (Ponte and Brunheira 2001).

The primary goal of the school mathematics curriculum is mathematical problem solving and the intended curriculum places emphasis on engaging learners through meaningful learning experiences (Ministry of Education 2012). Therefore during the course, the instructor places emphasis on mathematical problem solving, making sense of the sequence and content of the curriculum, and communicating mathematical knowledge. This paper examines the impact of the course Teaching and Learning of Mathematics that was conducted for a group of pre-service teachers who were preparing to teach mathematics in grades 7 and 8. It explores how the pre-service teachers developed an understanding of mathematical problem solving and inquired into problem solving and instructional explanations for topics that are part of the grades 7 and 8 mathematics curriculum in Singapore schools. The two specific questions explored in this paper are as follows:

1. How did pre-service teachers perceive mathematical problem solving before and after an introduction to problem solving in the curriculum studies course?
2. How did pre-service teachers perceive the communication of mathematical knowledge before and after introduction to topics such as Arithmetic and Mensuration in the curriculum studies course?

2 Review of literature

To provide a context for the study reported in this paper relevant literature is reviewed. Effective approaches to developing pedagogy of pre-service mathematics teachers, pre-service teachers' knowledge and experience of mathematical problem solving and communication of mathematical knowledge provide the theoretical basis of the course. Journal writing and narratives provide tools for pre-service teachers to engage in reflection and articulation of their learning which provides data for the study.

2.1 Effective approaches to develop pedagogy of pre-service mathematics teachers

Ponte and Chapman (2008) noted that studies of pre-service teachers' knowledge of teaching mathematics are much less represented in the research literature when compared with studies of their knowledge of mathematics. Nevertheless the studies available do suggest that there are several challenges that mathematics educators face when developing desired pedagogies for pre-service mathematics teachers. The challenges arise from the experiences of pre-service teachers when they were students (Jaworski and Gellert 2003; Brown and McNamara 2011; Zaslavsky and Sullivan 2011) and their resistance to change (Brown and Borko 1992; Lampert and Ball 1998).

The pre-service teachers in the study are university graduates who have done their respective content courses at the university prior to making a decision to become a teacher. Therefore, as suggested by Ponte and Chapman (2008), a good approach for mathematics teacher education may be an integration of the content and pedagogy courses such that experiences of the pre-service teachers learning mathematics will directly impact their preparation to teach mathematics in the immediate future. This is possible through the "isomorphism" principle (Ponte and Chapman p. 238) as they can be taught during their pedagogy course the same way they are expected to teach later as teachers.

Liljedahl et al. (2007) argue that the learning of content knowledge, pedagogy knowledge, and pedagogical content knowledge (Shulman 1986) will not ensure that novice teachers do not "revert to a method of teaching that is more reflective of their own experiences as students" (p. 319) if their beliefs about teaching and learning mathematics are not addressed. They note that robust beliefs may be difficult to change (Op't Eynde et al. 2001) but beliefs implicitly constructed from personal experiences as learners of mathematics (Green 1971) can be challenged. This is done when pre-service teachers learn mathematics and mathematics pedagogy, in a constructivist environment (Ball 1988; Feiman-Nemser and Featherstone 1992). This will result in change through two distinct but related processes. The first is that it involves pre-service teachers as learners in mathematical experiences that may be completely absent from their prior learning and the second is it models for them teaching strategies that are more conducive to facilitating the aforementioned mathematical experiences.

Wasserman and Ham (2013), in their work with beginning teachers of secondary mathematics, noted that learning instructional strategies during pre-service education programmes was formative for developing pre-service teachers' pedagogy about how mathematics education should be in practice, particularly engaging students as active learners. They found this finding uncommon

as many teachers would simply "teach how they were taught" (p. 82) which they attributed to the high quality of beginning teachers who were their participants in their study. They also found that the beginning teachers' experiences during their respective teacher education programmes were the most influential for them in developing an array of tools, such as technology, manipulatives, questioning strategies, heuristics and adapting problems to students, to use in the classroom. Contextual experience was also highlighted by the pre-service teachers as a must for being able to adapt classroom activities for particular sets of students. This is made possible during teaching experiences that often vary across teacher education programmes.

2.2 Mathematical problem solving knowledge for teaching

Based on a rigorous review of literature on mathematical problem solving, Chapman (2015) has outlined seven key components of Mathematical Problem-Solving Knowledge for Teaching (MPSKT). The components are as follows: knowledge of (1) mathematical problem solving proficiency, (2) mathematical problems, (3) mathematical problem solving, (4) problem posing, (5) students as mathematical problem solvers, (6) Instructional practices for problem solving, and (7) affective factors and beliefs (p. 31). Chapman has cautioned that having knowledge of the components alone is not helpful, rather knowing how to use the knowledge effectively, and recognizing the interdependence of the components is what teachers need for developing problem solving proficiency in their teaching.

Research has shown that pre-service teachers' beliefs and knowledge of problem solving are largely shaped by their experiences as students of mathematics (Mkomange and Ajagbe 2012; Ryve 2007). Therefore it necessary to engage them in realizing, through meaningful activities, components of MPSKT and their inter-connectedness. Certainly not all of the seven, but some, need to be realized, thus setting the stage for the development of the rest as they progress year by year in their teaching practices. As the primary goal of the school mathematics curriculum in Singapore is mathematical problem solving, it is pertinent that pre-service mathematics teachers are introduced to it in a manner that (1) facilitates examination of their beliefs and interpretations of problem solving, which are largely based on their experiences as students, (2) introduces them to knowledge of concepts and vocabulary such as problem, problem solving, processes of solving a problem, heuristics, etc (Kaur and Toh 2011), and (3) provides a context for further deliberations on instructional practices for problem solving throughout the pedagogy course.

2.3 Communication of mathematical knowledge

One key aspect of mathematics teaching is the communication of mathematical knowledge via instructional explanations by the teacher with her/his students. Leinhardt et al. (1991) defined the act of providing instructional explanations as "the activity in which teachers communicate subject-matter knowledge content to students" (p. 89). This may be done in varied ways, for example, merely through telling; but as elaborated further by Leinhardt et al. (1991), the activity includes not only literal explanations but also "systematic arrangement of experiences so that the student can construct a meaningful understanding of a concept or procedure" (p. 89). This is coherent with Martin's (1970) clarification that the primary function of instructional explanations is to support learners' understanding and not merely to transmit the content by telling. The work of Kinach (2002) has shown that it is not easy for pre-service teachers to shift from providing "telling-math" explanations (explanations merely describing procedures) to "teaching-for-understanding" explanations (explanations that build on past knowledge, linked to visual demonstrations, etc.) Kinach argues that in order to support pre-service teachers in providing "teaching-for-understanding" explanations, the mathematics educator must constantly and persistently negotiate two ideas with the pre-service teachers, namely, mathematical explanations must address both the *hows* and the *whys* of knowledge being explained.

Charalambous et al. (2011) found that making the practice of providing explanations part of a pre-service mathematics teacher education can be beneficial, as pre-service teachers can grow, although at varying degrees, in their practice. They also suggest that the practice is learnable, but caution that merely affording pre-service teachers opportunities to engage in and reflect on this practice cannot guarantee their growth and learning as recurrent opportunities to engage in providing explanations should be accompanied by structures that enable them actively and deliberately to reflect on their work.

In the pre-service mathematics teacher education course reported in this paper the mathematics educator placed much emphasis on explaining the how and why of the mathematics content always. In addition, mathematical tasks that were used for the explanation were also the focus as these tasks are the starting points of "teaching-for-understanding". Though it is not expected of pre-service teachers to design the learning tasks (Mok and Kaur 2006), nevertheless it is vital for them to source for appropriate tasks. Throughout the tutorials there were abundant opportunities for the pre-service teachers to practice their instructional explanations. During the micro-teaching sessions the pre-service teachers demonstrated the degree to which they had developed their instructional explanations. As a teaching development technique, microteaching

was first introduced at Stanford University for the preparation of secondary school mathematics teachers (Allen and Ryan 1969). It is organised practice teaching in which each pre-service teacher is given 15 min to develop a mathematical concept that they are given ahead of the teaching schedule. A review of the evidence for microteaching, undertaken by John Hattie as part of his Visible Learning project (Hattie 2008), found that it was an effective method for improving student outcomes.

2.4 Journal writing and narratives

Preservice teachers are students of mathematics themselves for a considerable period of time before they embark on the journey to 'become a math teacher'. They bring with them extensive knowledge about mathematics teaching and views about the nature of mathematics that are based mainly on their experience as students (Jaworski and Gellert 2003). As suggested by Jaworski and Gellert, scrutiny of this previous knowledge is essential for purposeful teacher education. Reflection is necessary for this scrutiny as it helps them to confront, clarify and extend their knowledge (Ponte and Chapman 2016). Chapman (2004), in her work with 28 pre-service secondary mathematics teachers, found that engaging the teachers in self-reflection whilst they were involved in problem-solving activities led them to make sense of problems, and also to confront the problem-solving process in ways consistent with the traditional classroom practices. Following the inquiry-reflective activities, the teachers thought of problems from the perspectives of the solver.

Journal writing helps students to reflect on their learning (Mewborn 1999; Liljedahl et al. 2007). It also can provide qualitative researchers a means to gain insights into the students' thinking (Mewborn 1999; Miller 1992). Similarly, a narrative is a tool for collecting data on, teachers' experiences (Chapman 2008). It can also foster reflective thinking, an important skill for both teaching and learning. Researchers may use interviews to obtain the stories from participants about their teaching or learning experiences, as shown in the study by Kaasila (2007) on the development of prospective teachers' beliefs about, and emotions towards, mathematics (Drake 2006; Chapman 1997, 2002). In the pre-service course reported in this paper, journal writing and narratives were adopted to engage the pre-service teachers in reflection about their learning.

3 Method

3.1 Subjects

The subjects of the study reported in this paper are the course instructor and five pre-service teachers in the course—*Teaching and Learning of Mathematics* (Grades 7 and 8). The course instructor is Professor X, who has a PhD in Mathematics Education and has been involved in the education of mathematics teachers for the past 30 years. For more than 15 years the students of Professor X have rated her teaching as exemplary.

There were 19 students in the course and they were put into five groups. The first four groups had 4 members each while the fifth group had 3 members. One member from each group was selected randomly to participate in the study. The five pre-service teachers, PT1, PT2, PT3, PT4, and PT5, all have degrees from universities in Singapore or the United Kingdom and have at least passed Mathematics at grade 12. PT1, PT2, PT4 and PT5 are females, while PT3 is a male. Their ages range between 22 and 27 years.

3.2 Pre-service teacher education course for secondary mathematics teachers (grades 7 and 8)

3.2.1 Outline of the course

The course *Teaching and Learning of Mathematics* (Grades 7 and 8) at the National Institute of Education in Singapore, is comprised of lectures, tutorials, and micro-teaching sessions. The eight lectures, of an hour each in duration, provide the context of the course. The 42 tutorials, of 2 h each in duration, engage pre-service teachers in the application of pedagogical content knowledge and other relevant knowledge/skills in the teaching of mathematics topics for grades 7 and 8 in the school mathematics curriculum. The 6 micro-teaching sessions, each of 2 h in duration, provides opportunity for pre-service teachers to teach mathematics topics and to practise their teaching skills. During these sessions they are expected to put into practice the strategies and methods learnt during the course, and feedback is given by the instructor and fellow student teachers. Special attention is given to the communication of mathematical knowledge. Videos of authentic teaching and related materials are used in the preparation of the pre-service teachers for their micro-teaching sessions.

3.2.2 Implementation of the course

In this section, I describe only how the pre-service teachers were introduced to mathematical problem solving and the two selected topics Arithmetic and Mensuration. This focus in important because introduction to mathematical problem solving is critical as it is the primary goal of the school mathematics curriculum, and arithmetic is a core subject in the primary grades (1–6). Thus teachers in grades 7 and 8 need to develop conceptual understanding that extends knowledge of numbers, and also of their connectedness. Mensuration is a topic that often culminates in the use of

formulae, and therefore meaningful activities leading to conceptual understanding that goes beyond routine formulae are essential for pre-service teachers to experience. Furthermore all the five PTs participating in this study were assigned their micro-teaching sub-topics for Arithmetic and Mensuration. I also outline how the pre-service teachers prepared for their microteaching sessions and wrote reflective journals after every topic was completed.

3.2.3 Introduction to mathematical problem solving

The course begins with three tutorials on mathematical problem solving, these tutorials link theory and practice. The goal of the tutorials is to create the potential for pre-service teachers to commence their journeys as mathematics teachers, understanding that inquiry into teaching and learning is possible. As an introduction to mathematical problem solving, we engage our pre-service teachers in two tasks, *The Circular Flower Bed* and *Solve 2 Problems*, to jump start discussion on mathematical problem solving and to bridge theory into practice. The tasks, goals of the tasks and the purpose of the responses they illicit from the pre-service teachers, are reported elsewhere (Kaur and Toh 2011). The pre-service teachers are provided with a resource book on mathematical problem solving (Kaur 2008) which they refer to as the red book in their journals.

3.2.4 Teaching of arithmetic and mensuration

Three tutorials were devoted for the teaching of Arithmetic. The tutorials focused on the content of the topics in grades 7 and 8 of school mathematics. The tutorials engaged the pre-service teachers in the following aspects: exploring the relationship of composite and prime numbers to arrive at the Fundamental Theorem of Arithmetic; uncovering methods for prime factorization; exploring the properties of rational numbers, properties of negative numbers, and the four operations using alge-discs; revisiting concepts of ratio, rate, speed, and percentages to solve problems; exploring proportions (direct and inverse) and uncovering the three tests (table, equation, graph) for proportions. Figure 1 shows an example of tasks used to contextualize pre-service teachers' learning of Arithmetic.

Two tutorials were devoted to the teaching of Mensuration. For both the tutorials emphasis was placed on using models of three-dimensional objects and paper cut-outs as manipulatives to verify known results for the surface areas and volumes. Essentially the manipulatives were tools for creating relationships between the object and mathematical relations and structures they represented (Nührenbörger and Steinberg 2008). Pre-service teachers worked in groups and used paper models to verify the following formulae: area of a parallelogram = length × height; area of a trapezium = ½ (sum of parallel sides) × height; area of a circle is πr^2; surface area of cone is πrl; surface area of cylinder is $2\pi rh$, the surface area of a sphere is $4\pi r^2$ (using orange peel and four circles). They also examined the properties of prisms and pyramids and explored the relationships between their volumes. The explorations of three-dimensional solids were limited to cylinders, cones and spheres. Figure 2 shows an uncommon approach, in most textbooks, to verify that the area of a circle is πr^2. This approach was explored during the tutorials. Figure 3 shows an example of tasks used to contextualize pre-service teachers' learning of Mensuration. For all the tutorials group-work and whole-class discussion were the predominant ways of facilitating discourse between the pre-service teachers and amongst the instructor and the teachers.

A teacher wrote the following question on the board for her grade 7 mathematics students:

A grocery store offered a store-wide 10% discount on every item during a moving out sale. Mrs Smith owned a credit card which entitled her to a cash rebate of 5% on the discounted bill. She paid $30 for her purchases at this store. How much more must Mrs Smith pay had she forgotten to present her credit card during the payment in this sale?

After reading the question, a student, Joyce, remarked that Mrs Smith had effectively enjoyed a 15% discount on her purchases.

If you were Joyce's Mathematics teacher, describe how you would guide her to realize her misconception.

Fig. 1 Arithmetic task for pre-service teachers

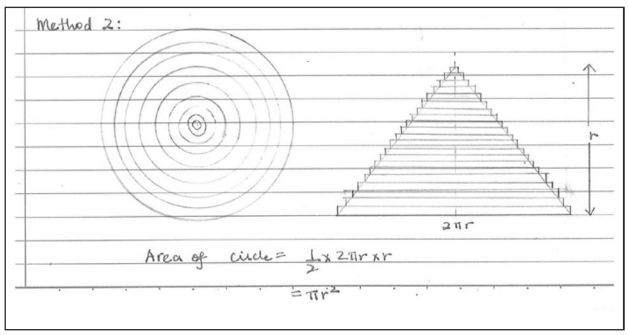


Fig. 2 Verifying that area of a circle is πr^2

A grade 8 mathematics teacher wants to use a prism and a pyramid to help students arrive at the formula for the volume of a pyramid.

State two conditions that the teacher must make on the dimensions of the prism and pyramid to ensure the correct derivation of the formula for the volume of a pyramid from the volume of a prism.

Describe how you would help students arrive at the formula for the volume of a pyramid.

Fig. 3 Mensuration task for pre-service teachers

3.2.5 Preparation for microteaching

The course is conducted across two semesters. The first semester is from July to November and the second is the following year from January to June. Microteaching is held during six tutorials in the second semester before pre-service teachers go for their 10 week practicum at schools. The pre-service teachers are given a topic, from the school mathematics curriculum for grades 7 or 8 (Ministry of Education 2012), some weeks before their scheduled microteaching. They plan a 40 min lesson and detail it in a lesson plan. But they teach only a 15 min segment of the lesson planned for the microteaching session. During the 15 min segment they are encouraged to enact the instructional explanation they have planned. They are encouraged to put into practice the strategies and methods learned during the course. Every pre-service teacher teaches only once but participates in all the microteaching sessions of their fellow pre-service teachers.

3.2.6 Reflective journals

Journal writing was used as a tool to engage the pre-service teachers in reflecting on their learning during the course. It also provided the instructor a means of gaining insights into how the pre-service teachers perceived their learning during the course, in particular the tutorials. After every topic that was developed during the tutorials, they were asked to write an entry in their journals. The journals were guided by prompts. The first prompt sought pre-service teachers' perceptions about their introduction to mathematical problem solving and experience solving the circular flower bed problem and two problems that were part of their take home assignment. The next eight sought their perceptions about their learning for the specific topics that were the focus of the tutorials. Several pre-service teachers in the course remarked that "this journal is a good way of getting me to reflect on what I've learnt in class and thinking about how I could utilize and apply the skills and knowledge learnt in class in the future". Such entries affirmed that the purpose of journals was coherent.

3.2.7 Interview and narratives

Following the micro-teaching sessions the pre-service teachers were interviewed by the course instructor. The micro-teaching lesson plans were used to stimulate the interview, following which the preservice teachers wrote their narratives. Five prompts were used but the three relevant for the study reported in this paper were: (1) what guided you in planning your instructional explanation of the idea you were to teach? (2) Why did you do it the way you did? and (3) how would you have done it prior to your learning in the course?

3.3 Sources of data

The data reported in this paper are drawn from two sources. The first source is the reflective journals that the five PTs wrote, in particular for the topics mathematical problem solving, arithmetic and mensuration. The second source is the narratives written by the PTs.

3.4 Data analysis methods

The qualitative data were analysed using qualitative analysis tools. Both deductive and inductive approaches were used to identify the categories or themes in the narrative data collected. For the reflective journals of the pre-service teachers we adopted an inductive approach and carried out content analysis (Weber 1990). The responses were first scanned through for common themes, following which, codes were generated and the data coded. Inevitably "a progressive process of sorting and defining and defining and sorting" (Glesne 1999, p. 135) led to the establishment of the final list of codes for the themes. This task was carried out by the author and a colleague independently of each other. The outcome was unanimous agreement on the three themes that emerged from the content analysis of the reflective journals on the introduction to mathematical problem solving.

For the narrative data arising from the interviews, as is often the case (Chapman 2008), we identified themes and searched for characteristics that related to specific research questions. Again the author and a colleague independently scanned the narrative data and deduced similar findings.

4 Results and findings

In this section the results and findings from the two sources of data are presented.

4.1 PTs reflections on the introduction to mathematical problem solving

From the content analysis of the reflective journals, three common themes emerged. The following describes each of them with evidence from the journals.

4.1.1 *Pre and post lesson understanding about what is a problem*

From the following excerpts, it is apparent that there was a significant difference between their pre and post lesson understanding of the word "problem" in the context of mathematical problem solving.

PT1: Prior to the tutorials on mathematical problem solving, I did not realise the distinction between "problems" and "exercises"; to me, everything to be solved in a maths class was a maths problem.

PT2: Before attending any problem solving tutorials, my idea of a problem is the same as an exercise or a question. I never really paid much attention to the "problems". However, the tutorials made me more aware of what mathematical problems should actually be and I became more conscious when I use the word "problem".

PT3: The tutorials have made me aware of the words problem and exercise, routine and non-routine problems.

PT4: My teachers used the word "problem" for most of our math work, so every question to me was a problem. It never came across as they could be categorized as either exercises or problems. Only in NIE during tutorial, it was highlighted that we cannot and should not use the two terms interchangeably.

PT5: Before the tutorial sessions, I'd never thought about or knew the difference between a "problem", a mathematical "task" or an "exercise". These terms were used so interchangeably that I'd completely ignored the real meaning behind them.

4.1.2 *Pre and post lesson knowledge of Polya's framework*

The following excerpts show that PTs were not aware or did not know that they were adopting a framework similar to that of Polya (1973) when solving problems. Knowledge of Polya's framework was welcomed as it provided them with a tool to help their students in the near future.

PT 1: I think Polya's framework came as something that I was looking for, but was unsure of. I say this because I prefer to be systematic when solving any problems and I lacked the knowledge of such existing frameworks. It is also consoling that through the assignments I found that my train of thoughts (when forced to consciously adopt one) bear much resemblance with Polya. What I lacked as compared to him was his level of analysis, and in particular his level of meta-cognition. I found myself learning to be more reflective, and in particular one point that struck me was to look back and re-examine the method I have chosen and whether the solution could have been obtained in a simpler form or even by observation. A framework is extremely useful especially if one encounters an extremely tricky problem and does not know how to start tackling the questions.

PT 2: I had been following Polya's four step framework without even knowing it, although at times I did leave out the last step (checking). Now that I am aware of the framework, it allows me to better guide my students and I will emphasis a lot of the 4th step.

PT 3: Being exposed to Polya's strategies to problem solving helped me break up the process. This is important as it would help me identify where my students' weakness is and how to address it accurately.

PT 4: When we were solving problems before we were introduced to Polya's framework we were not very organized in our thinking but after we were introduced to the framework—we knew that there is a 4 step plan we can adopt when solving problems.

PT 5: The Polya's four steps of problem solving was particularly useful. We were never taught this when we were students. We just read the questions and attempted it. I'd definitely use it when teaching, for example getting all students to paste the four steps on the first page of their exercise books so they can always refer to it when doing their work.

4.1.3 *Solving the flower bed problem and two problems as part of a take home assignment*

From the excerpts shown below it is evident PTs understanding was deepened through the problems they solved and resulting classroom discourse during the lessons on mathematical problem solving. In particular the experience gained by the PTs in solving the circular flower bed and two other problems provided them with several insights about the following aspects: feelings such as "struggle", "shocked", "overwhelmed", "curiosity", "engagement", "motivation", "truly experiencing problem solving", and "understanding how students may think"; different strategies and heuristics leading to solving problems in varied ways; reflecting on the solution, and checking for possible errors.

PT 1: Even though I did struggle with many of the problem solving questions, I think it was a good experience and exposure to the many different types of problems. I appreciated the red book which listed out different strategies and worked solutions… In fact I am happy the curriculum is designed towards problem solving which is really, the crux of learning maths in school.

PT 2: Both the group and individual assignment made me reflect on the way I did my math problems when I was a student. Apart from having to come up with solutions, I felt that having to identify the heuristics used helped me to become more creative. It allowed me to solve the problem in many more ways. This is particularly useful when

teaching a class of 40 as every student learns differently. If I am able to provide 2–3 solutions via different heuristics, I am confident that the students will be more engaged and motivated to do math.

PT 3: Solving the "circular flower bed" and the other two problems were fun because it allowed me to explore and learn. This curiosity is something I hope I can inculcate in my students. With knowledge of Polya's framework I hope to be able to diagnose where the problem solving process breaks down for my weaker students and guide them in their problem solving process.

PT 4: When my classmate and I first read the circular flower bed question, we were pretty much shocked. We were thinking to ourselves if lower secondary questions could be this tough. We took some time to understand the question. Students will also probably go through what we felt at the initial stage; feeling a little overwhelmed with all the information presented, taking into consideration that this question isn't similar to typical questions that are done in school. Similarly, my experience solving the two problems was equally rich. I must say I have truly experienced problem solving!

PT 5: Solving the "circular flower bed" problem and the two problems provided me with opportunity to reflect on the process of problem solving. I particularly like the last step of looking back, which is the step that most of us/students forget when we're solving problems in school. Looking back enables us to reflect on what we've done, and check on the possible errors and other ways of looking at the same problem. It is definitely important as a teacher when we have to put ourselves in students' shoes to understand their ways of thinking, thereby helping them to ease their learning and correct their misconceptions.

4.2 PTs reflections on tutorials on arithmetic

The PTs were asked to reflect on their learning journeys for the topic Arithmetic in their reflective journals. The following are some excerpts from their journals.

PT 1: Interesting to uncover that prime numbers are the "DNA" of numbers….The idea of "zero pairs" is new to me and it is also a more visual representation… I never thought of negative signs as a "direction" prior to the tutorial.

PT 2: Before attending the tutorial I was very certain that I can teach Arithmetic effectively because it is quite simple….However, after attending the first tutorial, I realized that it isn't that simple … how do I explain $-2 \times -3 = 6$?

PT 3: I found that alge-discs [are] a good pedagogical tool to introduce positive and negative numbers and also do the number operations with negative numbers.

PT 4: I was stunned when asked to explain how to divide a fraction by a fraction? I only knew how to do it but could not explain it. The tutorial helped me develop an understanding of the algorithm I have always been using.

PT 5: I have always thought that when one quantity increases and so does the corresponding one, the two were in direct proportion. During the tutorial I discovered that my thinking was flawed…

From the content analysis of the reflective journals it was apparent that the tutorials on Arithmetic did provide them with insights of which they were not previously cognizant, related to the concepts of numbers and their relations. Furthermore, the approaches they were engaged in to uncover concepts, and the use of specific teaching tools like the alge-discs, provided them with much need knowledge concerning how to teach Arithmetic meaningfully. They also provided them with opportunities to examine flawed reasoning such as that two quantities are in direct proportion if both quantities are either increasing or decreasing, and to examine direct and inverse proportions robustly. Prior to the tutorials all of them thought that arithmetic was a simple topic to teach. This apparently was so, as they knew the 'how' to do it, but when confronted with the 'why' we do it, they had to engage with relational understanding (Skemp 1976).

4.3 PTs reflections on tutorials on mensuration

The PTs were asked to reflect on their learning journeys for the topic Mensuration in their reflective journals. The following are some excerpts from their journals.

PT 1: As a student, my teacher never verified any of the formulae I have been using for finding areas and volumes. I enjoyed the lessons and practical sessions on mensuration very much, I can't wait to try them out with my pupils.

PT 2: The tutorials for mensuration were exceptionally useful. I can now explain to my students where all that formulae came from!

PT 3: I did not know a lot of things before attending the tutorials. I learnt so many ways to teach mensuration through hands-on experiments and illustrations. These lessons really opened my eyes to how fun and enriching math lessons can be.

PT 4: I really enjoyed the activities we did in our groups to verify the formulae for areas of parallelograms, circles and surface areas and volumes of cylinder, cone and sphere.

PT 5: I was stunned when our instructor gave us a sheet of paper and asked us to use the paper and make a hollow cylinder with the largest possible volume. No calculators were allowed. As a student I have never come across such an activity. It was very thought provoking and I really enjoyed it.

From the content analysis of the reflective journals it was apparent that the tutorials on Mensuration provided the pre-service teachers with an opportunity to verify for

themselves the age-old generalizations they have known since their school days about areas and volumes of coplanar and three dimensional shapes. The hands-on work they did during the tutorials helped them make connections, for example between the area of a parallelogram and that of a rectangle. Further knowing how the formula for the area of a rectangle could help a student arrive at the area of a parallelogram using paper cut-outs could make the process visually impactful. It was also apparent that they appeared to have no recollection of "learning" mensuration the way it was explored during the tutorials.

4.4 The interviews and narratives

The interviews explored what guided the PTs in framing and enacting the instructional explanation of the mathematical concept they were assigned for micro-teaching and how they would have done it prior to their learning in the course.

4.4.1 What guided the PSTs in framing and enacting the instructional explanation of mathematical concepts for their microteaching and why they did so

Table 1 shows a summary of what guided the pre-service teachers in framing how to enact the instructional explanations for their micro-teaching and why they did so. From Table 1, it is apparent that their learning during the course along with other resources such as the textbook, curriculum document, and the internet, guided them in framing how to enact the instructional explanations. They did so because they believed visual models created using manipulatives, hands-on activities, real life contextual tasks, and tasks to engage students in higher order thinking were promoters of meaningful learning.

4.4.2 How the PSTs would have framed their instructional explanations prior to their learning in the course

The following are excerpts from the narratives about how the pre-service teachers would have framed their mathematical communications prior to their learning in the course.

PT 1: Show my students a sphere and tell them the formula for volume of sphere and do practice questions.

PT 2: I would use the textbook approach, tell students the formula and get them to use the formulae and practice questions on surface area of spheres. Simply my approach would be teacher talk, student work at their desks and checking of answers for correctness. Maybe show them what is a sphere and that's about it.

PT 3: Show my students examples of quantities that are in direct proportion and do practice questions from the textbook.

PT 4: Tell my students that average speed is total distance divided by total time, and practice lots of textbook questions.

PT 5: Demonstrate on the chalk board a few questions from the textbook on inverse proportion, get students to do similar type of practice questions from the textbook.

From the content analysis of the narratives it is apparent that the pre-service teachers prior to their acquisition of knowledge from the course would have adopted an expository approach to communicate mathematical knowledge, mainly via telling the 'formulae', demonstrating how the formulae is used, and giving their students adequate

Table 1 Framing of mathematical communication after attending the course

Subject topic	What guided me?	Why did I do it this way?
PT 1: volume of sphere	Textbook Learning during the tutorials	To show visually, using manipulatives, how the surface area of a sphere is linked to the volume of a sphere (see Fig. 4) To engage my students in higher order thinking
PT 2: surface area of sphere	Learning during the tutorials	To show visually how the surface area of a sphere can be related to the curved surface area of a cylinder with the same radius as the sphere and height = diameter of sphere (see Fig. 5) Using models to create knowledge is impactful
PT 3: direct proportion	Curriculum document Textbook Past lesson planLearning during the course	Real life context will interest students and engage them (see Fig. 6) Kept to textbook type of examples so that students will be able to relate to the topic in the textbook Teach for understanding
PT 4: average speed	Learning during the tutorials Ideas from the internet	Get students to work collaboratively Engage students with hands-on activities to make mathematical sense of concepts (see Fig. 7)
PT 5: inverse proportion	Textbook Learning during tutorials	Get students to work collaboratively Engage students with hands-on activities to make mathematical sense of inverse proportion (see Fig. 8)

practice to develop procedural fluency as they felt "practice makes perfect". It also appeared that mainly the textbook would have guided them in knowledge about the formulae, selecting and using tasks for demonstration and practice.

5 Discussion and conclusion

The data analyzed and presented in this paper are from five pre-service teachers that participated in the study. The data were collected using reflective journals and narratives which were guided by an interview. The prompts used for the reflective journals as well as the interviews may have had an impact on the responses of the participants and therefore may not represent their uninfluenced perceptions. Hence, based on the limitation posed by the quantity of data and the inability to ensure that the data are comprehensive, generalizations cannot be made. However, it may be said that the data do provide insights into the development of pedagogy of pre-service secondary mathematics teachers attending the "Teaching and Learning of Mathematics (Grades 7 and 8)" course at the National Institute of Education, Singapore. In the following sections I present the findings and discuss them.

5.1 Pre-service teachers' perception of mathematical problem solving before and after the curriculum studies course

From the analysis of the data of the pre-service teachers, about their learning from the tutorials on mathematical problem solving it was apparent that prior to the tutorials they used the word 'problem' for all types of mathematical tasks and also did not give much thought to how they were solving 'problems'. The instruction on mathematical problem solving comprising learning tasks, the circular flower bed and solving two problems, inducted them into clarifying the characteristics of problems and reflecting on the process of problem solving. In solving the problems they also realized that problem solving heuristics were more than merely *using equations* and *guess and check*. Introduction to Polya's framework deepened their understanding of problem solving holistically. After the tutorials on mathematical problem solving it was apparent that there was a significant difference in the pre-service teachers' knowledge about mathematical problem solving. Knowledge about the characteristics of a problem made them sensitive to the use of the word 'problem' and the induction into Polya's framework for problem solving also provided them with a much needed guide to the thinking process when solving problems. They also realized that there were many tools (heuristics), other than algebra and guess and check that may be used to arrive at a solution. They also articulated the need for them as teachers to integrate their knowledge of problem solving into their teaching when the opportunity arises in the future.

It is apparent from the data that pre-service teachers began their transition from being a student to being a teacher during the tutorials (Brown and McNamara 2011). When they were students, mathematics was all about solving tasks that they were given by their teachers, who inevitably labeled all of them as 'problems'. However, now in their role as teachers they have been introduced to mathematical vocabulary that is precise and has characteristics. Furthermore the inquiry into how the process of problem solving is guided by Polya's framework led them to realize that guidance can be provided in a very systematic way when inducting students into problem solving. This finding concurs with that of Wasserman and Ham (2013, p. 70–96), as the experience of the pre-service teachers during their mathematics pedagogy course appeared to have engaged them in examining their knowledge of mathematical problem solving and in correcting their misuse of the word 'problem' and also helped them to realize that there is a framework that may guide problem solvers in their approach when solving problems. Their articulation of their intention to use the term 'problem' to reflect its true meaning and also to introduce students to the wide range of tools for problem solving and the guiding framework appears to hint that they were not seeking to replicate their views based on their past experiences. This finding is contrary to that noted by Zaslavsky and Sullivan (2011, 1–19).

5.2 Pre-service teachers' perception of how mathematical knowledge is communicated before and after the curriculum studies course

From the data analyzed and presented it is apparent that the pre-service teachers, prior to their acquisition of knowledge from the course, would have adopted an expository approach to communicate mathematical knowledge, mainly via telling the 'formulae', demonstrating how the formulae are used, and using students' adequate practice to develop procedural fluency as they felt "practice makes perfect". It also appeared that mainly the textbook would have guided them in knowledge about the formulae, and in selecting and using tasks for demonstration and practice. The pedagogy course "Teaching and Learning Mathematics (Grades 7 and 8)" appears to have engaged the pre-service teachers in meaningful activities and experience of how mathematical knowledge may be communicated in ways other than just telling. It is apparent from the synopsis of the lessons that were conducted for the topics Arithmetic and Mensuration, presented in this paper, that the content of the topics for grades 7 and 8 was the focus of all the deliberations during the tutorials and an array of activities were used

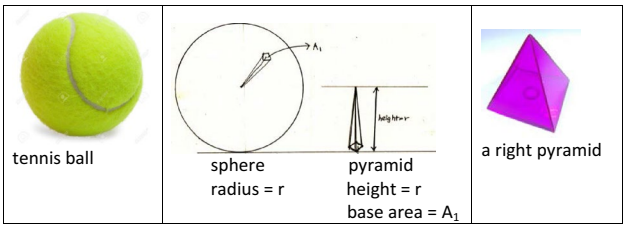


Fig. 4 Volume of a sphere

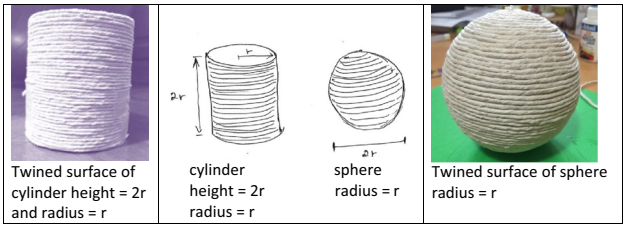


Fig. 5 Surface area of a sphere

by the instructor of the course to demonstrate the span of the pedagogical tools that may be used by the pre-service teachers when doing the same with their students. The tasks, as shown in Figs. 1 and 3, shed light on the nature of pedagogical content knowledge the teachers were engaged in co-constructing amongst themselves and with their instructor. They did this in groups of four and three often role playing, and searching for as many alternative possible solutions as they could. The whole class discussions challenged and deepened their understandings.

It is apparent from their reflections on their learning during the tutorials, as shown for Arithmetic and Mensuration, that their limited depth of content knowledge and ways of communicating knowledge were definitely deepened and widened respectively. Furthermore from the tasks, as shown in Figs. 4, 5, 6, 7 and 8, framed by the pre-service teachers for their micro-teaching, it is apparent that they drew on their learning during the course to do so. PT 1, PT 2 and PT 4 went beyond the examples

Fig. 6 Direct proportion

Example 1 – Total wages earned
(Working hours from 8 am to 8 pm)

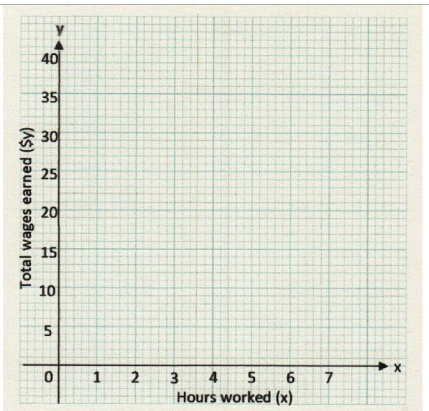
Hours worked (x)	1	2	3	4	5
Total wages earned (y)	5	10	15	20	25
$\frac{y}{x}$					

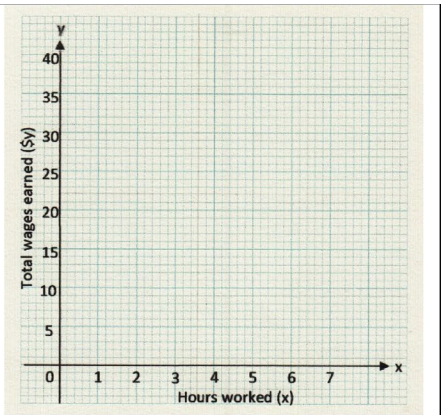
- Compute the value of $\frac{y}{x}$ for the table. What did you get?
- Can you write an equation expressing y in term of x?
- Represent the data from the table on the graph below. Did you graph a straight line? Does the line pass through the origin?

Example 2 – Total wages earned
(Working hours from 8 pm – 8 am)

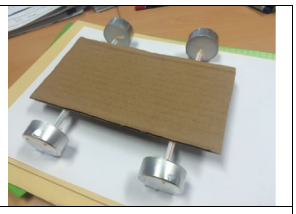
Hours worked (x)	1	2	3	4	5
Total wages earned (y)	10	15	20	25	30
$\frac{y}{x}$					

- Compute the value of $\frac{y}{x}$ for the table. What did you get?
- Can you write an equation expressing y in term of x?
- Represent the data from the table on the graph below. Did you graph a straight line? Does the line pass through the origin?





Every group of 4 students was given the following materials to make a toy car.
- Cardboard - 1 piece
- Tea candles – 4 pieces
- Straws – 2 pieces
- Satay sticks – 2 pieces
- Some scotch tape



Make your toy car.
Use your toy car, metre rule and stop watch to do the following experiment.
Give your toy car a push and record the measurements in the table below.

	Round 1	Round 2	Round 3
Distance (cm)	d1 = 30	d2 = 40	d3 = 50
Time taken (sec)	t1 =	t2 =	t3 =
Speed (cm/s)	s1 =	s2 =	s3 =

Calculate the average speed using the following:
Average speed = [s1 + s2 + s3] ÷ 3

Next use the average speed (you found) and the distance 120 cm to find the time taken. Is your answer = t1 + t2 + t3 ?
Can you explain why the answer is the same or not?

Fig. 7 Average speed

In your group of four students you are given 100 beads.
Please share the beads in the following ways:
First when there is only 1 student, next 2 students, three students and lastly 4 students.
Record your observations in the table given.

Number of student/s (x)	1	2	3	4
Number of beads per student (y)				
Total number of beads (xy)				

Does the total number of beads change?
How does the number of students and beads change?

What does x multiply by y represent?

Write down an equation involving x and y?

Draw a graph showing the relationship between x and y?
What is the shape of the graph?

Draw a graph showing the relationship between 1/x and y?
What is the shape of the graph?

Fig. 8 Inverse proportion

and manipulatives they worked with during the tutorials, to examine sources for ideas to communicate mathematical knowledge during their micro-teachings, while PT 3 and PT5 used the knowledge they had acquired during the tutorial and crafted real life contextual examples to firmly establish the exact understandings of direct and inverse proportion. All five pre-service teachers attributed their wanting to communicate mathematical knowledge the way they did so for micro-teaching to their belief, linked to their experiences during their tutorials, that visual models created using manipulatives, hands-on activities, real life contextual tasks, and tasks to engage students in higher order thinking were promoters of meaningful learning.

It is apparent from the data analyzed and presented that pre-service teachers brought to the course their beliefs that were implicitly constructed from their personal experiences as learners of mathematics (Green 1971). As the course tutorials engaged them in learning mathematics and mathematics pedagogy in a constructivist environment, their beliefs were challenged and they appeared to modify their beliefs. This finding concurs with that of Ball (1988) and Feiman-Nemser and Featherstone (1992). Furthermore, engaging the pre-service teachers as active learners during the tutorials appeared to develop their pedagogy and this is coherent with the work of Wasserman and Ham (2013). As noted by Kinach (2002), to transit pre-service teachers from "telling-math" explanations to "teaching-for-understanding" explanations, mathematics educators must constantly and persistently negotiate with them two ideas, namely the *hows* and *whys* of knowledge being explained. From the brief description of episodes of the tutorials conducted by the instructor, reported in this paper, it is apparent that the focus of the deliberations during the tutorials was the why and how of mathematical knowledge. This focus was also coherent with the work of Charalambous et al. (2011) as recurrent opportunities to engage in providing explanations were provided and reflective journaling was used to focus students on how best to communicate mathematical knowledge.

6 Conclusion

The data presented in this paper affirms that the course "Teaching and Learning of Mathematics (Grades 7 and 8)" was situated squarely between the experiences of pre-service teachers as students in classrooms and their future experiences as teachers in the same venues (Kennedy 1999). The course engaged pre-service teachers in making sense of the mathematics they will be teaching in school, thereby facilitating their transit from being a student to being a teacher in two main aspects. The aspects are (1) developing an understanding of mathematical problem solving which is the primary goal of the school mathematics curriculum in Singapore, and (2) shaping their ideas of communicating mathematical knowledge in ways that address the why and how of it.

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