ZDM Mathematics Education (2017) 49:65–80
DOI 10.1007/s11858-016-0815-7

ORIGINAL ARTICLE

How are questions that students ask in high level mathematics classes linked to general giftedness?

Roza Leikin[1] · Boris Koichu[2] · Avi Berman[3] · Sariga Dinur[4]

Accepted: 4 September 2016 / Published online: 15 September 2016

Abstract This paper presents a part of a larger study, in which we asked "How are learning and teaching of mathematics at high level linked to students' general giftedness?" We consider asking questions, especially student-generated questions, as indicators of quality of instructional interactions. In the part of the study presented in this paper, we explore instructional interactions in two high-school classes for mathematically promising students with specific focus on questions that students ask. The first class included generally gifted students (IQ $\geq$130) who were motivated to study mathematics at a high level (hereafter, a gifted class), and the second class included students characterized by high motivation regardless of their IQs (hereafter, motivation class). We analysed questions asked by the students during algebra and geometry lessons. Two types of questions are considered: elaboration and clarification. We found that students in a gifted class mostly asked elaboration questions, whereas students in a motivation class mostly asked clarification questions. We connect the revealed inclination to ask elaboration questions with intellectual curiosity that characterizes generally gifted students. Accordingly, we suggest that in classes of students who are motivated to study mathematics at high level, students who are generally gifted may create mathematical discourse of higher quality. We also argue that the identified differences in students' questions observed in classes of different types are not only student-dependent (i.e. depend on the students' levels of general giftedness) but can also be teacher-related and content-related.

Keywords Students' questions · Mathematical promise · Motivation · General giftedness · Learning mathematics at high level

✉ Roza Leikin
rozal@edu.haifa.ac.il

Boris Koichu
bkoichu@technion.ac.il

Avi Berman
berman@tx.technion.ac.il

Sariga Dinur
dinurj@bezeqint.net

1 Faculty of Education, University of Haifa, 199 Aba Khoushy Ave., Mount Carmel, Haifa 3498838, Israel

2 Faculty of Education in Science and Technology, Technion-Israel Institute of Technology, Haifa, Israel

3 Faculty of Mathematics, Technion-Israel Institute of Technology, Haifa, Israel

4 Faculty of Education, University of Haifa, Haifa, Israel

1 Introduction

There are many ways to nurture students with high mathematical potential, including but not limited to mathematical classes for mathematically promising students, special mathematical schools, mathematical circles and more (Vogeli 2015). In their review of the ways of treatment of mathematically promising students in Israel, Leikin and Berman (2015) maintain that "Just as a challenging mathematical problem has multiple ways of solutions, so the problems of how to nurture these students have many solutions" (p. 139). The problems include identification criteria, types of ability grouping, approaches to teaching, relevant teacher preparation and more. This paper presents a part of a larger study that analyzes teaching and learning mathematics in two types of classes for mathematically promising students. Classes of the first type include generally gifted (IQ >130) students who choose to study mathematics at a high level (hereafter, GC). Classes of the second type include students who are highly motivated to study mathematics at a high level, but are not necessarily generally gifted (hereafter, MC).

The rationale for our study stems from the following observation. The classroom environment is an important venue for the exploration of teaching and learning in mathematics education research (Kilpatrick 2014). No studies, however, have attempted to document and explore how (if at all) different criteria for forming particular types of classes for mathematically promising students are reflected in classroom interactions.

We focus on the types of questions that students voluntarily ask during the lessons. Specifically, the study is driven by the following research question: What types of questions do the students of GC and MC ask, and what are the differences between their questions, if these exist at all?

2 Theoretical background and literature review

In this section we review the professional literature about: the relationship between mathematical ability, general giftedness, and motivation; types of classes for students with high mathematical potential; classroom interactions in general and types of student questions in particular.

2.1 Mathematical promise, IQ and motivation

The notions of mathematical giftedness, high mathematical ability, and mathematical talent are frequently used synonymously, even though they denote different phenomena (Leder 2012; Leikin 2014). At the school level, these constructs are often connected either to high achievement in mathematics or to general giftedness measured by high IQ. The distinction between the constructs of mathematical giftedness and high mathematical ability is rooted in the debate between static and dynamic perspectives on reaching high achievements in mathematics (Leikin 2014).

To resolve the conflict between static and dynamic perspectives in mathematical giftedness/high mathematical ability, the NCTM (1995) Task Force introduced the notion of *mathematical promise*. Mathematical promise is a complex function of mathematical ability, motivation to excel in mathematics, beliefs about one's capacity to be successful in mathematics, and the learning opportunities. The Task Force notes that students with mathematical promise have the potential to become the leaders and "problem solvers of the future" (Sheffield 2012). Thus, the construct of mathematical promise interweaves mathematical ability with students' motivation and characteristics of students' personality.

Many authors emphasize that motivation constitutes a necessary condition for the development of mathematical ability (Subotnik, Pillmeier and Jarvin 2009). Knuth (2002) claimed that motivation to study mathematics and attitude to mathematics are associated with mathematical curiosity, which can be developed in students through appropriate learning opportunities that are mostly open-ended. Researchers consider motivation to be an important construct that reflects the natural human propensity to learn (Ryan and Deci 1996). In mathematics, this motivation is frequently associated with excitement, courage, and joy in the process of solving a problem or with finding an exciting mathematical discovery.

2.2 Special classes and schools for the mathematically promising

Though there is still no unanimous approach to defining mathematical giftedness, a broad consensus does exist among scholars and educators, proposing that students showing mathematical promise should be given special treatment in order for them to realize their potential (e.g., Colangelo and Davis 2003; Davis and Rimm 2004; NCTM 2000; Sheffield 1999). Thus, the lack of a broadly accepted theoretical definition of mathematical giftedness is not a barrier to opening special schools, classes, and programs for supporting and nurturing 'mathematically promising' students. In different countries, such students are selected for participation in these programs according to diverse operational criteria (e.g., House 1987; Mönks and Pflüger 2005; Vogeli 2015).

For a variety of reasons, special mathematics schools do not exist in Israel. Instead, either special classes for mathematically advanced students are established in schools with heterogeneous populations or university programs are made available to the gifted high-school students (Leikin and Berman 2015). The term "mathematically gifted" is not used routinely in Israel, but others such as general giftedness, high-level mathematical competency, or high motivation to study high-level mathematics. Motivation as a selection criterion opens the programs to a broader population; it also addresses the dilemma of exclusiveness vs. inclusiveness.

Note that mathematics is a compulsory subject in Israeli high schools, and students can be placed in one of three levels of mathematics: high, regular, and low. As a rule, the level of instruction is determined by students' mathematical achievements in earlier grades. Instruction at high level differs from that at regular level in terms of the depth of the material learned and the complexity of the mathematical problem-solving involved.

Our study explores lessons in two types of Israeli classes for mathematically promising students who study high-level mathematics. The classes of the first type are formed using general giftedness (IQ >130) criteria *and* students' motivation to study mathematics at high level, whereas the classes of the second type include students with high motivation to study mathematics at high level regardless of their IQs.

2.3 Classroom interaction and students' questions

Stimulated by Dewey (1933) and Vygotsky (1978), in this we study takes the view that the key feature of the teaching and learning processes is the teacher–student interaction. The quality of interaction determines the quality of learning, and changes in the quality of interactions signify the learning process (Sfard 2001).

Teacher–student interactions in a mathematics class have certain regularities and can be described by their structures (Bauersfeld 1988; Voigt 1995; Mehan 1979; Wood 1998). Mehan (1979) suggested a three-part-exchange structure of classroom discourse composed of three main components: initiation-reply-evaluation (IRE) as the dominant type of teacher-student interaction. In an IRE-structured lesson the teacher is the initiator and evaluator, and the students are respondents. However, in the last decade several studies have demonstrated that the IRE pattern does not necessarily reflect every lesson's structure. For example, in the inquiry classroom the third step may be aimed at expanding students' ideas instead of evaluating them (Forman and Ansell 2001). At the same time, the IRE structure can also indicate initiation by students, response by the teacher and expansion of the ideas by students (Leikin 2005). Leikin suggested a model of instructional interactions according to which teachers' expertise can be reflected in a shift from the teacher to the students (including asking questions). Thus, expert teachers who encourage *pupils to initiate* new learning situations which are meaningful for them develop *pupils*' *motivation* and curiosity when learning mathematics.

Asking questions is one of the most important actions in a mathematics classroom as it promotes interaction, discussion and collaboration (Chin 2004). Tobin and Tippins (1993) argued that it is not easy to find teachers who require learners to generate questions and seek answers. The construction of questions is an important way for learners to build conceptual conflict, and the search for answers may begin the process of resolving that conflict.

Several educators and researchers emphasize the importance of students' questions in the teaching and learning process. For example, Almeida (2012) stresses the importance of moving to student-focused teaching, which suggests that "putting the focus on students' questions rather than on teacher's questions, and valuing students' questions rather than emphasizing their responses is imperative in supporting learning higher levels of thinking" (p 634). Asking questions by students can contribute to the development of their understanding and the construction of their knowledge through a better connection between new and existing knowledge (Almeida 2012; Chin and Osborne 2008; Scardamalia and Beretier 2006; Rosenshine, Meister and Chapman, 1996; Pedrosa de Jesus, Almeida, Teixeira-Dias, and Watts 2003). Moreover, student problem posing is crucial in problem solving and in decision making (Almeida 2012; Zoller 1987), in developing creativity, mathematical thinking and self-confidence (Shodell 1995; Donovan and Bransford 2005; Chin and Osborne 2008), and in promoting the students' motivation and their interest in and commitment to the learning process (Almeida 2012; Chin and Kayalvizhi 2005; Marzano et al. 1988; Pedrosa de Jesus, Teixeira-Dias, and Watts 2003).

Researchers distinguish between different types of questions. Dewey (1944) distinguished between 'genuine' and 'stimulated' questions that could foster 'good habits of thinking'. Watts and Alsop (1996) discussed three categories of pupils' questions as related to different stages of learning: *consolidation questions*; *exploratory questions* and *elaborative questions*. In this framework, consolidation questions are associated with learners' attempts to delineate the rationale for classroom tasks, confirm explanations and consolidate understanding of new ideas in science; that is, to clarify conceptual issues. Exploratory questions are associated with pupils' attempts to expand their knowledge and test constructs they have formed. Elaborative questions are raised when learners search for conviction about their own frameworks of understanding or those being offered to them. They examine claims and counterclaims, elaborating and challenging both their previous knowledge and experience, and that being presented to them. Elaborative questions may have some direct relevance to the classroom topic being taught or go beyond the studied topic.

Another categorization of questions is proposed by Intel Teach Program (2016) that promotes the Socratic Questioning Technique as an effective way to explore ideas in depth. While most of the of questions' types that Intel Teach Program suggests can be considered as elaboration or exploratory questions, the clarification questions category can be considered as an additional one. According to the Intel Teach Program, these questions help participants in a Socratic dialog to better understand other's ideas and questions.

In our view, distinctions between exploratory and elaboration questions are vague and difficult to apply. Thus, in our study we use two categories to analyze students' questions: elaboration and clarification questions as described in the method section.

3 Method

3.1 Participating classes

The participating classes belonged to two different regular schools in the northern part of Israel. Both classes were taught by experienced mathematics teachers who hold a BSc degree in mathematics and an MA in mathematics education. Both teachers had over 15 years of experience in teaching high-level mathematics and were regarded as expert teachers by their colleagues and school principals.

GC consisted of 22 tenth-grade students with IQ $\geq$130 (tested in the 3rd grade)[1] who, in tenth grade, chose to study mathematics at a high level. This choice was motivated either by their enjoyment of mathematics study or by realization that mathematics is important for their future careers. The class studied mathematics 5 h a week, which is the same number of hours as in the rest of the tenth-grade classes, using the instructional approach of deepening the standard curricula.

MC consisted of 28 ninth graders. The class was formed at the end of 6th grade with the criterion of an expressed (in the interview with the child and his or her parents) and confirmed (during a three-day preparatory camp) interest to invest more time and effort in studying mathematics and science than is provided by the regular curriculum.[2] The class studied 8 h of mathematics per week, which is 3 h more than the regular classes. The instructional approach to mathematics in MC included elements of deepening and acceleration. That is, the program enabled 9th-grade students to study mathematical content that is regularly taught in the 10th grade. MC students studied towards taking their matriculation examinations in mathematics 1 year earlier than the regular classes do (as a rule, in 11th rather than 12th grade).

In summary, we treat the GC and MC classes as similar with respect to the students' motivation to study high-level mathematics, but different with respect to the level of students' general giftedness.

3.2 Data collection and analysis

The lessons in MC were videotaped at the end of a school year, and in the GC at the beginning of the following school year. Accordingly, the chronological ages of the students in two classes were close and both classes included 15–16 year-old students. Both classes were videotaped when they studied the same topics.

The data sources consisted of about 15 h of transcribed videotapes; five mathematics lessons of 90 min were videotaped in each class. Most of the time the camera was trained at the teacher and the blackboard; the camera was occasionally rotated in order to capture those students who talked.

Four selected lessons presented below are typical with respect to their structure. All observed lessons started with a short discussion of the homework, after which new material was explained to the whole class. In the explanations, the teachers often used the blackboard to show the students' solutions to a sample problem. This demonstration was usually followed by students' individual work on selected textbook tasks. Sometimes the teacher or the students presented the solutions on the board. The teacher encouraged the students to ask questions and was willing to help by elaborating on the explanations and demonstrations. The four selected lessons were particularly rich in students' questions.

The lesson transcripts were first analyzed by one of the authors and then by all the authors together. We implemented an inductive analysis with partially predefined categories (Dey 1999; Strauss and Corbin 1990). We first scanned the data for episodes in which the students asked questions. Operationally speaking, we looked for the students' assertions/requests that could be interpreted as ending with a question mark expressed by intonation or as invitations to respond. We then isolated those episodes in which the student questions were explicitly addressed by the teacher or the students. At the next stage, we characterized the students' questions in terms of the categories described in the literature. Through a subsequent iterative process of refinement, we eventually abstracted and classified two categories, for which the data seemed to have provided rich and solid evidence and which we found most useful for capturing the differences between GC and MC. As mentioned above, the categories are: *elaboration questions* and *clarification questions*.

3.2.1 Elaboration questions

A student's question was categorized as an elaboration question if it led the classroom discussion towards new mathematical territory, sometimes unforeseen by the teacher. When raising these questions, students seek to examine claims and counterclaims, elaborating and challenging both their previous knowledge and what is being presented to them. Examples: "Well, if that's the case, then…?"; "But what happens if…?" and "How is what we learned yesterday compatible with…?" Questions within this category are attempts to reconcile different understandings, resolve conflicts, test circumstances, force issues, and track in and around the ideas

[1] Gifted classes in regular Israeli secondary schools operate in the framework of a special program of the Ministry of Education (see http://cms.education.gov.il/EducationCMS/Units/Gifted/English).

[2] Motivation classes, such as the one that took part in our study, operate in Israel within different projects. The class that took part in our study operated under the auspices of the MOFET project. MOFET is an Israeli public association that provides methodological and pedagogical support to regular schools interested in attracting and nurturing STEM-oriented students (see http://www.reshetmofet.org/en/ for project details).

Fig. 1 Task 1 in a GC calculus lesson

Task 1: Explore the function $f(x) = \frac{2-x}{x+3}$ and draw its graph.

and their consequences. Such questions may have an indirect connection to the teacher's explanation or be triggered by tangential issues. Note that the *Elaboration Questions* category, as defined above, unites exploratory question and elaboration questions as defined by Watts and Alsop (1996).

3.2.2 Clarification questions

A student's question was categorized as a clarification question when it asked for support to gain a better understanding of the learning material (e.g., concept definitions, theorems or mathematical problems) presented by the teacher or other students. Additionally, students' utterances were classified as clarification questions when students needed approval for their ideas (e.g., a solution strategy they planned to use) related to the teacher's assignment. For example, the following questions belong to this category: "What do you mean by…?" "I don't understand this step, please explain it again…" and "Could you please remind me…" Questions within this category do not challenge what is being presented; rather they attempt to narrow the gap between the learner's knowledge and the knowledge required by the presented material or explanation. Such questions have direct relevance for what has been presented and are stimulated by the students' wish to meet their teacher's requirements. Note that the defined above *Clarification Questions* category unites Watts and Alsop's (1996) categories of clarification questions and consolidation questions.

4 Findings

In this section we present and analyze four classroom episodes. The first two episodes (one occurred in GC and another in MC) contain classroom discussions of tasks in the context of function explorations. The third and the fourth episodes (again, one episode from each group) revolve around geometry proofs.

4.1 Pre-calculus lesson in GC

4.1.1 Episode 1: classroom discussion of the homework task in GC

Prior to the lesson under discussion, the class was taught how to qualitatively (i.e., without using limits or derivatives) investigate rational functions and draw sketches of graphs. Specifically, the students were taught to sketch the graphs using mainly the following information: functions' intersections with the coordinate axes, the functions' signs and their behavior at infinity. The students did not have any knowledge about asymptotes at this stage, so their decisions about the functions' behaviors at infinity were based on informal reasoning. In addition, the students were familiar with the idea of linear translation of the graphs of functions.

At the beginning of the lesson, Tom, a student, asked TG (henceforward the mathematics teacher in GC) to discuss the following homework task (Fig. 1):

In response, TG invited Tom to solve the task on the board.

Tom confidently found the point of intersection of the function $f(x) = \frac{2-x}{x+3}$ with the axes and domains in which the function was positive or negative. Then he asserted that the function approaches -1 when x approaches infinity.

1. Tom: The function approaches -1 when x approaches infinity.
2. TG: Okay, Tom, now let's look at what happens to the behavior at infinity. How did you see that this is close to -1?
3. Tom: Because 2 minus infinity is negative infinity and three plus infinity is also pretty much infinity so it's kind of… it approaches -1.

Following Tom's explanation, the TG asked the class:

4. TG: Class, does negative infinity divided by infinity always tend to -1?

This TG's question that used a student's mistake to initiate a vivid discussion demonstrates TG's ability to stimulate students' reasoning and interactions among students. TG and the students brought several examples to show that this statement is not always true (e.g., $\frac{x^2}{x}$ and $\frac{3x}{x}$).

Then TG asked:

5. TG: If negative infinity divided by infinity is not always -1, how can it be shown that this function $\left(f(x) = \frac{2-x}{x+3}\right)$ approaches -1 when x approaches infinity?

Another student, Orion, approached the board and wrote:

6. Orion: $-\frac{x-2}{x+3} = -\left(\frac{x+3}{x+3} - \frac{5}{x+3}\right) = -1 + \frac{5}{x+3.}$ And when this is… when x approaches infinity this $\left(\frac{5}{x+3}\right)$ is equal to 0… [Writes and talks]

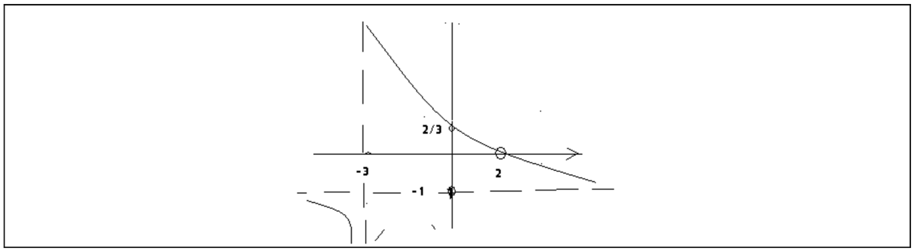


Fig. 2 A sketch on the board in a GC calculus lesson

Orion's solution was unplanned by TG, but her openness to students' ideas allowed other students to be exposed to his solution. TG accepted Orion's explanation and repeated it to the class. Then she offered another explanation, in line with previous lessons.

7. TG: $\frac{2-x}{x+3}=\frac{x\left(\frac{2}{x}-1\right)}{x\left(1+\frac{3}{x}\right)}=\frac{\frac{2}{x}-1}{1+\frac{3}{x}}\rightarrow\frac{0-1}{1+0}=-1$ [writes on the blackboard and talks]

Maya expressed her concern about the TG's solution:

8. Maya:I have a problem; here $\frac{\left(\frac{2}{x}-1\right)}{\left(1+\frac{3}{x}\right)}$ the function is not defined as 0 and there $\left[\frac{2-x}{x+3}\right]$ it is not defined as −3.
9. TG: Okay, you're right, but now we're doing an algebraic manipulation to check what happens at infinity and at negative infinity…, thus the equal sign should be replaced by the "tends" sign

$$\frac{2-x}{x+3}\rightarrow\frac{x\left(\frac{2}{x}-1\right)}{x\left(1+\frac{3}{x}\right)}\rightarrow\frac{0-1}{1+0}=-1$$

This concern of Maya demonstrated her attentiveness and understanding that algebraic manipulations of functions should result in an equivalent algebraic expression. Thus, request of preciseness in mathematical language during the lesson can be considered as an elaboration question rose by the student.

Then TG drew a sketch of the function on the board (Fig. 2).

At this point Uri entered the discussion:

10. Uri: It has to be semi-linear, right?
11. TG: What do you mean?
12. Uri: It has to become more and more linear. As the x gets bigger, −2 and −3 become smaller in this particular place, it needs to be linear.
13. Tamar: I don't understand what "semi-linear" is.

It seems that the idea of semi-linear function was new also to the teacher, that is, the discussion moved to the territory unforeseen by TG. In response, TG reiterated what Uri said. Then he returned to the Orion's solution and showed how it was related to the idea of linear translation of the graph $y={}^{1}/_{x}$, which was discussed at the previous lesson. Some students found this method to be simpler than the method of reducing by x (see row 7), but TG noted that the former method is less general because it cannot be applied to all functions. For example, it would not be useful in drawing a graph of the function $g(x)=\frac{x}{x^2-1}$.

4.1.2 Analysis of Episode 1

The structure of the episode is as follows: Tom asked a question about a homework task—TG invites Tom to present his solution—Tom presented and explained—TG disagreed with Tom's explanation and asked the class for counterexamples—the class offered counterexamples so that the problem with Tom's explanation became clear—TG repeated Tom's initial question—Orion presented his solution—TG presented an additional solution—Maya asked a question about TG's solution—TG modified and completed her solution—Uri asked a (seemingly) unrelated question—TG asked Uri to clarify—Uri explained his question—Tamar asked TG to explain Uri's question—TG just reiterated Uri's explanation—TG returned to the solution of Orion, links it to the material studied in the previous lessons and discussed the affordances of her and Orion's solution.

Accordingly, the episode includes discussions of three solutions to the same problem (Tom's solution [2–4], Orion's solution [6] and TG's solution [7–9]) and a discussion of Uri's question [10–13]. Overall, the discussion is essentially driven by the students' questions. We now turn to characterizing them.

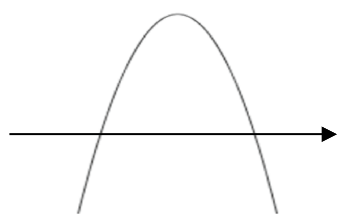
It is difficult to say whether Tom's initial question was aimed at clarification or elaboration of his solution, but TG used this question as an opportunity to elaborate on a vague term "function behavior at infinity." Maya was clearly seeking an elaboration when expressing her concern about TG's solution [8]. By the above definition of what an elaboration question is, Maya challenged an element of the presented solution that had not been suggested by TG. Uri's question [10] was also an elaboration question, because it concerned a term which had not been articulated before in GC lessons, but apparently invented by Tom as his way to make sense of the

In order to solve the inequality $ax^2 + bx + c > 0$, find domains of the function $y = ax^2 + bx + c$ for all types of inequalities: $y \geq 0$, $y \leq 0, y > 0$, $y < 0$

Fig. 3 Reminder on the board in MC

Task 2: Solve the following inequalities:
(1) $-x^2 + 4x + 5 \geq 0$; (2) $-3x^2 - 6x - 5 > 0$; (3) $x^2 - 10x + 25 > 0$;
(4) Given the parabola: $y = (-k^2 - 2k + 3)x^2 + (k - 1)x + 3$
Find: Values of k for which the parabola has
(a) the point of minimum; (b) the point of maximum.

Fig. 4 Task 2—in an MC algebra lesson



$$-x^2 + 4x + 5 \geq 0$$
$$-x^2 + 4x + 5 = 0 \mid \cdot (-1)$$
$$x^2 - 4x - 5 = 0$$
$$(x - 5)(x + 1) = 0$$
$$x_1 = 5, -x_2 - 1$$
$$-1 \leq x \leq 5$$

Fig. 5 Solution to Task 2.1 in MC

previous discussion. Maya's and Uri's questions both exemplified their prior knowledge and inventiveness. Only Tamar's question [13] could be classified as a clarification question, as she asked for an explanation of what had been explicitly said. Some of the students' elaboration questions and original ideas were accompanied by TG's clarification questions [11].

4.2 Algebra lesson in MC

4.2.1 Episode 2: solving quadratic inequalities

This episode took place in an MC algebra lesson. The lesson began by reviewing one of the homework tasks. The discussion centered around the concept of positive/non-negative and negative/non-positive functions, in connection to the graphs of quadratic functions. After checking the homework, TM (henceforth teacher in MC) moved on to the subject of the lesson, quadratic inequalities. TM connected this topic to the previous lesson by noting that the students actually already knew how to solve quadratic inequalities, and wrote on the board as a reminder (Fig. 3):

After reviewing the solution stages, TM handed out a worksheet with the following exercises (Task 2) (Fig. 4):

TM and the students solved the first exercise together: TM asked questions about what should be done at each step of the solution and emphasized what was important to write, draw and make note of (Fig. 5).

TM went back over the solution and explained all the solution stages.

After this repeated explanation, the students were encouraged to ask questions.

14. Moshe: I didn't understand.
15. TM: What did you not understand?
16. Moshe: How we solved this?
17. TM: Go back; what did we do here?
18. Moshe: We solved a function [meaning, the equation $ax^2 + bx + c = 0$], but which parabola do we have?
19. TM: Here?
20. Moshe: Does it intersect with the x-axis?
21. TM: Yes.

22. Moshe: Ah…so why didn't we write that?
23. TM: [Repeats the whole solution.]
24. Moshe: And what do we do now?
25. TM: Which parabola do we draw?
26. Moshe: A negative one.
27. TM: Inverse. We drew it. Where are the positives? Above the x-axis. Where are the negatives? Below the x-axis. Yes, so now…? Which segment do we choose?
28. Moshe: What do you mean by "which segment"?
29. TM: Here, what do we write for the answer? Which inequality do we solve? Greater than 0. So we solve, choose a segment with a plus sign…

This transcript (lines 14–29) demonstrates that students asked clarification questions. Similarly to TG, TM tried to answer the student questions by questions addressed to the students. However, these questions did not lead to the discussion elaboration. Afterwards, TM offered another way (so she said) to solve the problem.

TM explained that in the case of a negative coefficient of x^2 in a given inequality, it is efficient to multiply the inequality by -1, change the inequality sign and solve the new inequality. (Note that in the first solution, multiplication by -1 was applied to the equation, not to the inequality).

Like TG during Episode 1, TM presented the students with the solution by using "the second way" and emphasizes that there is no need to look at the original inequality but rather at the inequality obtained after multiplying it by -1. At the end of the process TM notes that with both of these methods you reach the same solution.

30. TM: Now let's compare the answer in the end with this answer. *It is the same answer, right*?
31. Peter: Yes, yes.
32. TM: This [second] way is simpler because we are always looking at the minus sign. It bothers us. We multiply by -1 and solve it as usual. The number of possibilities [of the parabola forms] can be reduced, so I prefer this way. … Now, who has questions?
33. Moshe: I don't…I don't understand. I have a problem with this parabola.
34. TM: You don't have to draw it precisely. You draw it like this and afterwards you mark the points -1 on the left side and 5 on the right side. Understand?

Once again, TM encouraged students to ask their question, and the questions raised by the students were clarification ones. Later, when the students had solved the inequality $-3x^2 - 6x - 5 > 0$, the teacher asked them to solve it using the second way and justified this request by saying that with the second way there is less chance of making mistakes.

35. Peter: The teacher, which way did you say we should use?
36. TM: The second. Here, multiply by -1 right away and solve the inequality and *we draw the straight parabola*. We won't forget to draw. Sometimes students solve and forget the inverse parabola and draw the straight parabola and make a mistake.

4.2.2 *Analysis of Episode 2*

The episode can be summarized as follows: TM presented a way of solution and exemplifies it—TM repeats the explanation—Moshe asks questions about the presented solution and TM answers—TM offers another solution—Moshe asks questions about the second solution and TM answers—TM compares the solutions and states that the second one is preferable—to be sure, Peter asks which way they should use—TM reiterates that the second way is preferable.

Overall, the episode includes discussions of two solutions one of which was produced by students with the TM help and the second was presented by the teacher. It is worth noting that while TM encouraged her students to ask questions and listened to students ideas, the students of MC felt free to expose their lack of understanding and ask even very basic questions. As a rule, the questions are directed to the teacher, and the teacher answers the questions without attempting to open the discussion to the whole class.

By the above definition, the students' questions are clarification questions. For instance, Moshe [14–29, 33] clearly seeks assistance in obtaining an explanation of what had been presented by the teacher. Peter [35] asks TM to verify the way in which she expected the students to solve the task. In response, the teacher goes back and explains the solutions to the exercises several times.

4.3 Geometry lesson in GC

The lesson presented below was the first geometry lesson after a long holiday break. Prior to the holiday, an extension of Thales Theorem was proved, and the homework consisted of the problems that could be solved by the use of this extension. The lesson below is divided into three episodes.

4.3.1 *Episode 3.1: discussion of Thales Theorem*

At the beginning of the lesson TG went over the Thales Theorem again, drew (Fig. 6) on the board and wrote the proportion:

$$\frac{AC}{AB} = \frac{AC'}{AB'} = \frac{CC'}{BB'}$$

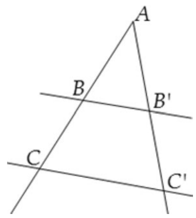


Fig. 6 Drawing to Thales Theorem

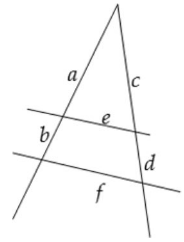


Fig. 7 Modified rawing to Thales Theorem

Neta noted that AB' and $C'B'$ as well as $CB$and AB look equal on the drawing. She asked whether they indeed are equal [37].

TG modified the drawing in order to make it visible that these segments are not necessarily equal (Fig. 7). Neta reformulated the proportion as $\frac{a+b}{b} \stackrel{?}{=} \frac{c+d}{d} \stackrel{?}{=} \frac{f}{e}$ and asked TG whether it was correct [38].

TG confirmed that the proportion $\frac{a+b}{b} = \frac{c+d}{d}$ was correct and asked Neta how she knew that both ratios are equal to $\frac{f}{e}$.

Neta did not respond. TG explained that $\frac{a+b}{b} = \frac{c+d}{d} \neq \frac{f}{e}$ and corrected Neta's proportion to $\frac{a+b}{a} = \frac{c+d}{c}$

She suggested that Neta's mistake was rooted in the first drawing, in which a seemed to be equal to b and c to d. However, Neta did not accept the correction and asked why $\frac{a+b}{b} \neq \frac{f}{e}$ [39].

In response, the teacher explained that Neta's proportion could not be derived from Thales Theorem.

40. TG: Look, we spoke about 4 proportional segments and we defined what the proportion was. Right? The proportion $\frac{a}{b} = \frac{c}{d}$ leads to the other proportions. And from here we cannot deduce that $\frac{a+b}{b} = \frac{f}{e}$. Okay?

Neta was not convinced:

41. Neta: But I asked about …an extension of Thales [Theorem; not about the theorem itself]. How is this related to the segments on parallel lines [f and e]?
42. TG: How did we prove the extension of Thales Theorem? We drew a parallelogram [adds $B'E$—Fig. 8]
43. Neta: Yes.
44. TG: Right? And we look at this angle as an angle [C'] from which two sides come out. This means, we related to the Thales Theorem extension only so that it would be easier to implement more complicated problems. Okay? Now, you see yourself that this is not true, that the drawing does not…

Neta was still not convinced and continued asking why in the denominator of the equation $\frac{a+b}{a} = \frac{f}{e}$ there is a and

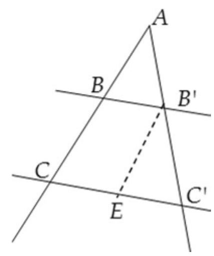


Fig. 8 Drawing to A, the proof of the Thales Theorem

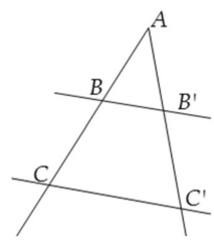


Fig. 9 New drawing to illustrate proportion

not b. TG stopped the dialogue by proposing that Neta go over the theorem's proof once more at home.

4.3.2 Episode 3.2: and what if they are parallel?

TG continued the lesson and formulated the Inverse Theorem for Thales Theorem ("If two straight lines separate the sides of an angle at proportional intervals, then they are parallel"), draws a new drawing and writes the proportion $\frac{AC}{AB} = \frac{AC'}{AB'} = \frac{CC'}{BB'}$ (Fig. 9).

At this point the following dialogue between TG and another student, Ofek, took place:

45. Ofek: …Why does it matter if these are sides of an angle and not just two lines? Because they can also be parallel?
46. TG: Ah…if they are parallel lines… [AC parallel to $A'C'$—Fig. 10]…let's think. And these lines are also parallel (AA', BB', CC').

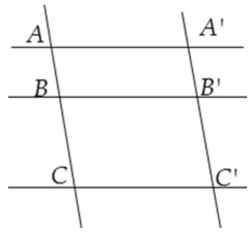


Fig. 10 What if they are parallel lines?

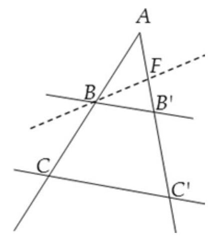


Fig. 11 Drawing to the indirect proof of the inverse Thales Theorem

47. Ofek: This [AA', BB', CC' are parallel] is the regular part of Thales Theorem.
48. TG: If the lines were not sides of an angle, they would be parallel. Now give an example of a proportion… What can you say about these segments? (AA', BB', CC').
49. Ofek: They are parallel.

The discussion continued, and TG, together with the students, reached the conclusion that the quadrilaterals formed by the five lines are parallelograms and so segments AA', BB', CC' are equal. Thus, the ratio between them is 1. In contrast, it is clear that the ratio between AB and BC is not necessarily equal to 1, and this is a contradiction.

50. TG: Therefore, it cannot be that these straight lines are parallel and we must relate to AC and AC′ as sides of an angle.

4.3.3 *Episode 3.3: indirect proof (reductio ad absurdum)*

In continuation of the lesson, TG proved the Inverse Thales Theorem. She used an indirect proof, and Tal asked:

51. Tal: Why do we need to assume this [that BB' is not parallel to CC′]? You can simply say that it is appropriate to the proportions and you don't know if… to say that you don't know if it's parallel. It could be parallel but it could also not be (Fig. 11).

As a result of Tal's question, a dialogue about the meaning of indirect proof ensues. TG explained that the inverse of a theorem is not always correct.

4.3.4 *Analysis of Episodes 3.1, 3.2, 3.3*

Episode 3.1 has the following structure: TG presented an extension of Thales Theorem—Neta inquired whether the drawing for the theorem was correct—TG modified the drawing—Neta suggested an alternative formulation of Thales Theorem and asked whether it was correct—TG answered that Neta's formulation was wrong—Neta asked why—TG answered that it did not follow from the proof of the theorem—Neta was not convinced and repeated her question—TG suggested she go over the proof again at home.

Neta's persistence is worthy of note. As we have seen, Neta did not take the teacher's explanations for granted and repeatedly put forward her ideas. Neta's question [37] can be classified as a clarification question, though it caused the correction of the drawing by the teacher. Her main questions ([39], [41]) were elaboration questions since they were directed at understanding mathematical statements beyond TG's plan and led TG to conduct the class discussion of the meaning of inverse statements.

The structure of Episode 3.2 is as follows: TG formulated the Inverse Theorem—Ofek asked if the theorem would be correct in an additional case—TG opened the class discussion and eventually concluded that it would not. Ofek's question [45] fits the definition of an elaboration question because it is a question of a "what happens if…" structure which directed the discussion towards "new mathematical territory."

Tal's question [51] in Episode 3.3 is another example of an elaboration question. This is because Tal did not merely accept what TG had explained, but challenged her explanation and suggested an alternative course of reasoning.

Overall, Neta's, Ofek's and Tal's questions manifested their intellectual independence, analytical skills, highly-developed hypothetical reasoning and the wish not merely to follow TG's explanation but to assimilate the presented proofs into their existing knowledge.

4.4 Geometry lesson in MC

4.4.1 *Episode 4.1: multiple solutions to a geometry problem*

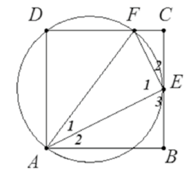
After checking the homework, TM handed out a worksheet containing Task 3 (Fig. 12).

TM remarked that there was only one way to prove Task 3(a) and at least two ways for Task 3(b). The students were given 4–5 min to think of the problem. TM circulated among the desks and the students asked her questions.

Fig. 12 Task 3—geometry problem in MC

Task 3

$ABCD$ is a square. A circle that is tangent to CB at point E, passes through vertices A and D and intersects CD at point F.
Prove: (a) $\angle AEF = 90°$; (b). $\angle A_2 = \angle A_1$; (c) $BE = CE$



Proof b.1, by Elad, was as follows:

$\angle FEC = \angle FAE = x$ (an angle between the tangent and the chord equals a corresponding inscribed angle.)

$\Rightarrow \angle AEB = 180° - 90° - x = 90° - x$

$\Rightarrow \angle EAB = 180° - 90° - (90° - x) = x$
sum of angles in a triangle AEB

$\Rightarrow \angle FAE = \angle EAB = x$

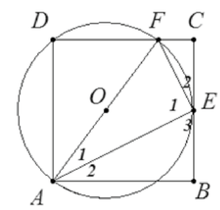


Fig. 13 Proof b1 Task 3 in MC

52. Yossi: TG, can I mark the center of the circle? Can I mark O and the radius?
53. TG: Yes you can.
54. Katya: Do I have to mark the center?
55. TG: It's not necessary…
56. TG to Peter: Use the square.
57. Peter: What does that give me?
58. Katya: How do you solve that?
59. TG: Using the theorems… Given a square—you have to prove that angle AEF is 90°. Think—what do you need to prove in order to arrive at 90°?
In about 5 min, Moran presented the solution to Task 3(a) of the problem, as follows:
59. Moran: Because angle *ADF* is an inscribed right angle, it relies on the diameter of the circle; thus angle *FEA* relies on the diameter and hence it equals 90°

For Task 3(b), two proofs were found and presented: Proof b.1 (Fig. 13) and Proof b.2 (Fig. 14).

Elad explained this proof from his desk, and TM wrote the proof on the board. Katya asked to explain the proof once more, and the teacher did so.

TM was enthusiastic about this proof and praised the student. As in the previous case, she worked near the board while the student explained his proof. TM required the student to provide justification. The other students required additional explanations, and TM answered their questions.

58. Yossi: Angle *F*?
59. TM: Here, here?
60. Yossi: Angle *F* is also x.
61. TM: Ah, great, nice. This is also a method.
62. Katya: Why?
63. TM: Explain why. Why? Same rule.
64. Yossi: The circumferential angle that leans against the hypotenuse is equal.
65. TM: The angle between the tangent and the hypotenuse, this is the hypotenuse [*AE*]. Where is the circumferential angle that leans on this hypotenuse?

F_1, right? If x is here, so here x continues.

66. Katya: Why is x there?
67. Yossi: And then I made it so that *B* is 90°.
68. TM: Yes. So here?
69. Yossi: $90° - x$
70. TM: And here? $90° - x$.
71. Katya: Why? Why is this x?
72. TM: Just a second, we're…it's right, very nice. Great job. And this method is also very good. And correct. Look, if we start by marking x here. This is the tangent, this is the hypotenuse. E is the angle between the tangent and the hypotenuse? Hypotenuse, tangent, and hypotenuse. Where is the circumferential angle that leans on the hypotenuse?

Proof b.2 was presented by Moran.

$\angle AEB = \angle AFE = x$ (because an angle between a tangent and a chord equals a circumferential angle relying on the same chord).

$\angle FAE = 90 - x$ total of the angles in triangle AFE

$\angle EAB = 90 - x$ total of the angles triangle AEB

Therefore $\angle FAE = \angle EAB = 90 - x$

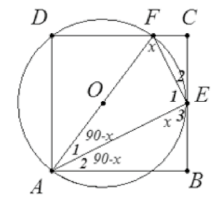


Fig. 14 Proof b2 Task 3 in MC

Fig. 15 Proof c.1 Task 3 in MC

Proof c.1 was presented by several students who enthusiastically spoke out all at once.

The proof was as follows:

$FCBA$ is a trapezoid

$\angle OEC = 90°$, because a radius is perpendicular to tangent at the tangent point.

Next, $\angle ABC = \angle OEC = 90°$, and thus OE is parallel to AB.

$OF = AO$, and thus OE is a midline of the trapezoid $FCBA$ and $EB = CE$.

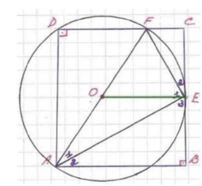


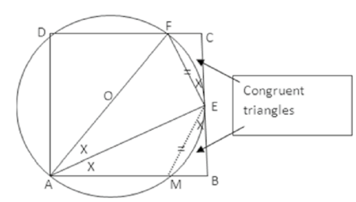
Fig. 16 Proof c.2 Task 3 in MC

Proof c.2 by Peter, was as follows:

Equal chords correspond to equal inscribed angles (see Part (b)),

Therefore $FE = ME$.

Thus, triangles FCE and MBE are congruent,
which concludes the proof: $EC = BE$.



For Task 3(c), two proofs were presented as well: Proof c.1 (Fig. 15) and Proof c.2 (Fig. 16).

TM tried to organize the discussion so that each student could be heard. She was pleasantly surprised, enthusiastic and praised the students.

73. TM: The midline in the trapezoid? This is something new!—So, okay. …
74. TM: Nice, great. I didn't think of this—good for you! Great, good job, good job, nice, I hadn't even thought about this method. …

The other students ask to explain the proof again, but this time TM did not encourage them because the lesson was nearly over and she was interested in letting Peter present the second proof for Task 3(c) (Fig. 16).

Once again, TM was enthusiastic about the proof and praised Peter who presented it. The other students wanted to show other proofs, but the bell rang. The discussion and presentations continued, however, during the break.

4.4.2 Analysis of Episode 4

The above lesson had the following structure: TM gave the students a problem for independent solution—the students worked on the problem and consulted TM—Moran presented her solution to Tasks 3(a)—Elad presented the first proof to Task 3(b)—Katya asked to explain Elad's proof—TM explained—Moran presented her proof—TM encouraged the rest of the students to ask questions—Yossi and Katya asked questions about the proofs and TM answered—the first proof to Task 3(c) was presented by several students—the other students asked to explain the proofs again—TM did not do so because of the time constraints—Peter presented the second proof—other students wanted to present their proofs and ask questions about the presented proofs but the lesson was over.

Overall, different students produced different proofs so that the class was exposed to several proofs. TM and some of the students became very enthusiastic. However, there were also students who expressed their concern that none of the proofs was written in an organized way so they anticipated that they would have difficulty at home when trying to review the proofs. It should also be noted that TM's decision to make room for oral presentation of different proofs only by the most successful students left little opportunity for the rest of the students to ask questions. Those questions that were asked anyway were clarification questions, either when the students worked on the problem ([52], [54], [58]) or when discussing the presented proofs ([60], [64], [68], [73]). Despite the lack of time, TM repeated each proof to the class several times, but it appeared that some of the students were accustomed to receiving a more detailed and slower response to their clarification questions.

5 Discussion

5.1 Classroom discourse in GC and MC and the differences in questions the students asked

This paper presents a part of a larger study that was aimed at characterizing teaching and learning mathematics for students with high mathematical abilities. We examined classroom interactions in two classes of students who were motivated to learn mathematics at high level, but differed in the level of general giftedness. Specifically, we attended to questions that students in two types of classes asked during mathematics lessons. Our study demonstrates that this research methodology (i.e., focusing on students' questions) is effective in the comparative analysis of learning process in classes of different types.

Overall, asking elaboration questions was a distinct characteristic of the GC students in both algebra and geometry lessons. The students' elaboration questions expressed their need to make sense of the ideas presented by others, and to present and discuss their own ideas. These questions reflected the students' curiosity and persistence and contributed to a high level of mathematical discourse.

Sometimes the discourse was truly exploratory (Sfard and Prusak 2005). The GC students were unwilling to accept a mathematical argument just because "it's what the teacher said." Some of the students' questions were unexpected by the teacher and required TG to exhibit flexibility when managing the lesson (Leikin and Dinur 2007; Leikin and Lev 2013). It is also of note that during the discussion the students made an effort to formulate their questions clearly or reformulate them, so that the teacher and their fellow students would understand what they intended to ask.

In contrast, the MC students asked mostly clarification questions, either in the algebra or in the geometry lesson. One of the norms in MC was that the students could ask very basic questions and the teacher patiently responded to these questions, even in cases when the questions led her to repeat what had already been explained. TM treated all the students' questions with respect, but it was evident that she was the exclusive mathematical authority. She decided what was true or not true, which method was the most appropriate for solving a problem, and when it was best to solve a problem using different methods. Furthermore, the students worded their clarification questions in a rather vague and general way (e.g., "Why?", "I didn't understand", "How did we solve this?", "Again", "What?"). On the one hand, this observation implies that the students trusted TM to be able to understand them, but on the other hand, it implies that the culture of asking questions was not well-developed in MC. Probably as a result of vague questions, TM's responses to the students were also quite general. As a rule, she repeated the complete explanations of the solution methods. Overall, the students' questions in MC were not especially helpful for raising the level of discourse and, as a rule, did not trigger the emergence of new ideas or arguments. The discussions based on the students' questions revolved mainly around problem-solving techniques.

An additional difference between the classes was related to who eventually answered the questions. In both classes, most questions were directed to the teacher. However, in GC some of the questions to the teacher were redirected by her to the whole class. In this way, the class became actively involved (e.g., by providing examples and counterexamples) in answering. In MC, the teacher answered the students' questions even in the cases when the questions concerned the solutions suggested by fellow students. In this class the teacher was usually in a dialogue with a particular student, and the rest of the students could not always follow the dialogue.

Finally, there are differences between GC and MC related to the use of multiple proofs and explanations. In GC the approach of presenting different methods and explanations was especially prominent in the algebra lessons and slightly less so in the geometry lessons. In MC the opposite was true. In the geometry lessons the approach of providing different proofs for problems was very prominent, whereas in the algebra lessons the teacher preferred the students to solve problems in one specific way.

5.2 Where do the between-class differences come from?

We cannot predict what would happen if the teachers were to switch classes. Even so, the data enable us to suggest that some of the described differences are student-dependent, content-dependent and teacher-dependent.

An expression of *student-dependent* differences between mathematical discourse in GC and MC is associated with asking elaboration vs. clarification questions presented above. Students in GC clearly exhibited mathematical curiosity, persistence and originality of mathematical thought when asking questions that were related to, but not necessarily based on, the material that was presented to them. Note here that a combination of creativity, persistence and high achievements fits Renzulli's (1986) definition of giftedness and thus the revealed difference in asking elaboration questions vs. clarification questions can be considered a student-related characteristics of the discourse in GC. Moreover since curiosity is considered a special characteristics of individuals with high intellectual potential (Davis and Rimm 2004), we argue that students' innate curiosity enhances the exploratory nature of mathematical discourse, while the open type of instructional interaction promotes their curiosity (Knuth 2002). Leikin and Lev (2013) and Leikin, Waisman, and Leikin (2016) demonstrated that a special characteristic of generally gifted students who study mathematics at high level is mathematical insight. Curiosity and insight can lead to elaboration questions and ideas that a teacher may have overlooked in the lesson planning.

An expression of *content-dependent* differences related to observed classroom interactions can be seen in the differences between the MC geometry and algebra lessons. Each lesson was conducted in the same class of students by the same teacher. While in geometry, the teacher opened the stage for presentation of different solutions to the problems, in the algebra lesson she led the students towards one, safe [in her view] solution method. We find these differences intriguing and plan to conduct a systematic study aimed at documenting and explaining differences in instructional interactions in algebra and geometry lessons conducted by the same teachers.

An expression of *teacher-dependent* differences can be seen, for instance, in the differences between instructional interactions in both algebra and geometry lessons conducted by each of the two teachers. It is worthwhile recalling that TG and TM were appreciated by their colleagues and principles as expert teachers and that both had more than 10 years of experience teaching mathematics at high level. Both teachers were attentive to students' ideas and encouraged them to ask questions. However, the differences in teachers' styles are apparent.

TG was ready to change her lesson-plan spontaneously, and to enter into mathematical territory that was new for her (e.g., TG clarifying question [11], her readiness to discuss the concept of "semi linear" function [episode Y], and discussion on the justification of why the two lines in Thales' theorem are not parallel [Episode 3.3]). We assume that while TG's openness to students' ideas allowed students to raise these kinds of ideas, students' ability to move to new mathematical territory allowed the teacher to exhibit her flexibility in teaching, and her sensitivity to students.

TM encouraged her students to produce multiple solutions to the problems and to ask "any question". At the same time, once different solutions were presented, and even when TM was excited about students' solutions, she felt the need to act as an evaluator (e.g., [61]), and she directed the students to a simpler solution (e.g., [32], [35]). We hypothesize that this characteristic of TM's work was determined by her feeling of responsibility for the students' success. An additional teacher-dependent difference between TG and TM was that TG encouraged student-to-student communication, and TM did not. Rather than promoting students' answers to students' questions, TM responded herself and frequently repeated what had already been explained.

We see the differences between teachers' styles as the main study limitation. As mentioned, we cannot predict what would happen if the teachers were to switch classes. However, since both teachers' were sensitive to the students' needs, we suggest that if TG and TM would indeed be switched, they would adapt their teaching styles to the students' characteristics.

In summary, we argue that while classroom interactions reflect complex relationships between teaching, learning and the content taught and learned, our study enables us to suggest that general giftedness adds meaningfully to classroom discourse.

We believe that our study opens directions for further studies that will be aimed at the analysis of teaching and learning for mathematically promising students who are motivated to study high-level mathematics, but differ in their level of general giftedness, and who study in different types of classes. We hope that these studies would eventually reach unequivocal recommendations as to the ways that best suit the needs and learning styles of mathematically promising students with different additional characteristics.

References

Almeida, A. P. (2012). Can I ask a question? The importance of classroom questioning. *Procedia—Social and Behavioral Sciences*, *31*, 634–638.

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In D. A. Grouws & T. J. Cooney (Eds.), *Perspectives on research on effective mathematicsteaching* (Vol. 1, pp. 27–46). Reston: NCTM.

Chin, C. (2004). Students' questions: Fostering a culture of inquisitiveness in science classrooms. *School Science Review, 86*(314), 107–112.

Chin, C., & Kayalvizhi, G. (2005). What do pupils think of open science investigations? A study of Singaporean primary 6 pupils. *Educational Research, 47*(1), 107–126.

Chin, C., & Osborne, J. (2008). Students' questions: A potential resource for teaching and learning science. *Studies in Science Education, 44*(1), 1–39.

Colangelo, N., & Davis, G. A. (Eds.). (2003). *Handbook of gifted education* (3rd ed.). Boston: Allyn and BacOn.

Davis, G. A., & Rimm, S. (2004). *Education of the gifted and talented* (5th ed.). Boston: Allyn and BacOn.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Boston: D.C. Heath and Co.

Dewey, J. (1944). *Democracy and education: An introduction to the philosophy of education*. New York: The Free Press.

Dey, I. (1999). *Grounding grounded theory: Guidelines for qualitative inquiry*. San Diego: Academic Press.

Donovan, M. S., & Bransford, J. D. (Eds.). (2005). *How students learn: Science in the classroom*. Washington D.C.: The National Academies Press.

Forman, E. A., & Ansell, E. (2001). The multiple voices of a mathematics classroom community. *Educational Studies in Mathematics, 46*(1), 115–142.

House, P. A. (Ed.). (1987). *Providing opportunities for the mathematically gifted K—12*. Reston: NCTM.

Intel Teach Program (2016). Designing effective projects: questioning—The Socratic questioning technique. Retrieved on May 14, 2016 from http://www.intel.com/

Kilpatrick, J. (2014). History of research in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 267–272). Berlin: Springer **(Electronic Version)**.

Knuth, E. (2002). Fostering mathematical curiosity. *Mathematics Teacher, 95*, 126–130.

Leder, G. (2012). Looking for gold: Catering for mathematically gifted students within and beyond ZDM. In H. Forgasz & S. F. Rivera (Eds.), *Toward equity: Gender, culture, and diversity* (pp. 389–406). Advances in Mathematics Education Series, Part 3. Dordrecht: Springer.

Leikin, R. (2005). Qualities of professional dialog: Connecting graduate research on teaching and the undergraduate teachers' program. *International Journal of Mathematical Education in Science and Technology, 36*(1–2), 237–256.

Leikin, R. (2014). Giftedness and high ability in mathematics. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Electronic Version: Springer.

Leikin, R. & Berman, A. (2015). Mathematics for students with high mathematical potential in Israel. In B. R. Vogeli (Ed.), *Special secondary schools for the mathematically talented: An international panorama* (pp. 117–143). Series on Mathematics Education (Vol. 12). Denver: World Scientific.

Leikin, R., & Dinur, S. (2007). Teacher flexibility in mathematical discussion. *Journal of Mathematical Behavior, 26*(4), 328–347.

Leikin, R., & Lev, M. (2013). Mathematical creativity in generally gifted and mathematically excelling adolescents: What makes the difference? *ZDM—The International Journal on Mathematics Education*, *45*(2), 183–197.

Leikin, R., Waisman, I., & Leikin, M. (2016). Does solving insight-based problems differ from solving learning-based problems? Some evidence from an ERP study. *ZDM, 48*(3), 305–319.

Marzano, R. J., Brandt, R. S., Hughes, C. S., Jones, B. F., Presseisen, B. Z., Rankine, S. C., et al. (1988). *Dimensions of thinking: A framework for curriculum and instruction*. Alexandria Va.: Association for Supervision and Curriculum Development.

Mehan, H. (1979). *Learning lessons: Social organization in the classroom*. Cambridge: Harvard University Press.

Mönks, F. J., Pflüger, R., & Nijmegen, Radboud Universiteit. (2005). *Gifted education in 21 European countries: Inventory and perspective*. Nijmegen: Radboud University.

National Council of Teachers of Mathematics (NCTM). (1995). Report of the NCTM task force on the mathematically promising. *NCTM News Bulletin*, *32*.

National Council of Teachers of Mathematics (NCTM) (2000). Principles and standards for school mathematics. Reston: NCTM.

Pedrosa de Jesus, M. H., Teixeira-Dias, J. J. C., & Watts, M. (2003). Questions of chemistry. *International Journal of Science Education, 25*, 1015–1034.

Renzulli, J. S. (1986). The three-ring conception of giftedness: A developmental model for creative productivity. In R. J. Sternberg & J. E. Davidson (Eds.), *Conceptions of giftedness* (pp. 332–357). New York, NY: Cambridge University Press.

Rosenshine, B., Meister, C., & Chapman, S. (1996). Teaching students to generate questions: A review of the intervention studies. *Review of Educational Research, 66*(2), 181–221.

Ryan, R. M., & Deci, E. L. (1996). When paradigms clash: Comments on Cameron and Pierce's claim that rewards do not undermine intrinsic motivation. *Review of Educational Research, 66*(1), 33–38.

Scardamalia, M., & Bereiter, C. (2006). Knowledge building: Theory, pedagogy, and technology. In K. Sawyer (Ed.), *Cambridge handbook of the learning sciences* (pp. 97–118). New York: Cambridge University Press.

Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*(1), 13–57.

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher, 34*(4), 14–22.

Sheffield, L. J. (Ed.). (1999). *Developing mathematically promising students*. Reston: NCTM.

Sheffield, L. J. (2012). *Mathematically gifted, talented, or promising: What difference does it make?* The paper presented at the 12th International Congress on Mathematical Education.

Shodell, M. (1995). The question-driven classroom. *American Biology Teachers, 57*(5), 278–281.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park: Sage.

Subotnik, R. F., Pillmeier, E., & Jarvin, L. (2009). The psychosocial dimensions of creativity in mathematics: Implications for gifted education policy. In R. Leikin, A. Berman & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 165–179). The Netherland: Sense.

Tobin, K., & Tippins, D. (1993). Constructivism as a referent for teaching and learning. In K. Tobin (Ed.), *The practice of constructivism in science education* (pp. 3–21). Washington, DC: AAAS Press.

Vogeli, B. R. (Ed.). (2015). *Special secondary schools for the mathematically talented: An international panorama. Series on Mathematics Education* (Vol. 12). Denver: World Scientific.

Voigt, J. (1995). Thematic patterns of interactions and sociomathematical norms. In P. Cobb & H. Bauersfeld (Eds.), *Emergence of mathematical meaning: Interactions in classroom culture* (pp. 163–201). Hillsdale: Erlbaum.

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Cambridge: Harvard University Press.

Watts, D. M. & Alsop, S. (1996). *The QUESTCUP Project: a study of pupils' questions for conceptual understanding*. Paper presented at the American Educational Research Association, New York, April

Wood, T. (1998). Alternative patterns of communication in mathematics classes: Funneling or focusing. In H. Steinbring, A. Sierpinska & M. G. Bartolini-Bussi (Eds.), *Language and communication in the mathematics classroom* (pp. 167–178). Reston: NCTM.

Zoller, U. (1987). The fostering of question-asking capability: A meaningful aspect of problem-solving in chemistry. *Journal of Chemical Education*, *64*, 510–512.