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ORIGINAL ARTICLE

Using dialogic talk to teach mathematics: the case of interactive groups

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Abstract This article explores the types of interactions that take place within "Interactive Groups" when individuals come to a meaningful understanding of mathematics. We discuss the possibility for dialogic talk to unveil the process of learning, and we explore the role that tutors may play in making this process happen. We postulate that learning-with-understanding may occur more likely in dialogic spaces where individuals use dialogic talk. We use a methodological tool to analyse interaction, drawing on video-recorded data. We conclude that dialogic talk may generate meaningful learning situations that may improve children's mathematics learning. However, dialogic interaction may also prompt students to use claims that are not mathematically valid. The role of the adult to guide interaction within such situations needs to be further explored.

Keywords Argumentation · Interaction · Dialogic learning · Dialogic talk · Rational numbers · Scaffolding

1 Introduction

Teaching in small groups has been highly appreciated among teachers all over the world (Elbers & Streefland 2000; Elboj & Niemelä 2010; Kieran & Dreyfus 1998). This research is linked with the assumption that individuals working in pairs or small groups are able to solve problems that they might not otherwise be able to do on their own (Mercer & Sams 2006). According to Bruner, the crucial condition for learning to take place is "intersubjectivity." He claims that *intersubjective meaning making* is at the core of any process of learning, and it depends enormously upon contextual interpretation and negotiation. Unveiling this process, which is deeply embedded in human interaction, is one of the present and most pressing challenges for researchers (Bruner 2012).

However, what is not clear is whether all types of interactions will lead individuals to build such *intersubjective meaning making*. Researchers already know that interaction is present in a number of different forms, and we cannot claim that all social interactions (within small groups) may lead to an increase in learning with understanding (Garcia Carrión & Díez-Palomar 2015; Mercer & Howe 2012). What types of interactions take place when individuals come to experience meaningful understanding? Can we rely on small group collaboration as a perfect context for such interactions to appear? Being able to answer such questions would provide enormous insights for teachers to think about and design their lessons in the most effective and productive ways possible.

In this article we use dialogic talk to analyse the interactions that occur within the *Interactive Groups* (IGs) in a lesson on rational numbers. We have chosen IGs because research suggests that this type of classroom arrangement produces large positive impacts in children's learning (Valls & Kyriakides 2013). We aim to discuss these questions in order to elucidate the kind of interactions that encourage meaningful mathematics learning.

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2 Theoretical Framework

2.1 Dialogic talk

Previous studies in mathematics education confirm that language is more than a means to communicate an idea; it also has a higher psychological function. However, as Zack and Graves (2001) claim and quote Ernest's words, "*spoken words* do not equal *thoughts in the mind*" (p. 265). This leads us to focus on talk and interactions. The process of going from the use of language (speech) to effective learning is complex. Based on Vygotsky's work, Zack and Graves suggest that even though the speaker may have the "whole thought" in his/her mind, this "thought" needs to be developed successively in speech. In other words, the notion of rational numbers being built in the learner's mind is a result from successive interactions with the teacher and/or other [more capable] peers. Drawing on these interactions, the learner internalizes the voices of different people. Solving tasks about rational numbers, listening to other people, engaging in dialogue with peers, teachers, etc., are things that allow learners to develop their thoughts about rational numbers. This development is framed by an inter-subjective process (Bruner 2012) characterized by the multiplicity of voices.

Flecha (2000) proposes the "Dialogic Learning" approach to study how interaction works. He draws on Habermas' (1984) concept of communicative action. Habermas argues that we can distinguish between actions that are defended with power claims (i.e., the person who performs these actions holds the position of authority), and actions that are justified with arguments based on validity claims. Thus, Flecha (2000) claims that learning could be dialogic when participants use valid claims, rather than power claims, to justify their utterances.

Drawing on this theoretical approach, we define "dialogic talk" as the type of dialogue established between two or more people who are capable of speech and are oriented to reach agreements on mathematical tasks or activities. Dialogic talk involves the use arguments based on validity claims that are verifiable by individuals who participate in the dialogue. The innovative part of this approach is that it is explicitly based on the interactions that seek understanding, in which the strength of arguments prevails over the speaker's position of power. Dialogic talk only occurs when people use arguments based on validity claims. The speaker should not use his/her position of power over other participants. Thus, participants should not feel coerced during dialogic talk.

2.2 Scaffolding

According to Vygotsky (1978), the development of the higher psychological processes is the result of speech acts mediating the use of artefacts. Not only do speech acts denote the existence of a cognitive development, speech acts also produce cognitive schemes (higher psychological processes). Vygotsky finds that children speak about what they do, and speech act itself is also part of the same complex psychological function. If a child is prevented from speaking when s/he is solving a task, then it is possible that s/he would be unable to solve it. He coins the concept of Zone of Proximal Development (ZPD) to describe that type of situations.

Later, Wood, Bruner, and Ross use the concept of scaffolding to explain the process of educational support between the "expert" adult and the child. According to them, scaffolding is a "process that enables a child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts. This scaffolding consists essentially of the adult "controlling" those elements of the task that are initially beyond the learner's capacity (…)" (Wood et al. 1986, p. 90). In their article, Wood et al. explain how teachers had to explicitly guide 3-year-old students to solve a task, because they were not able to do it on their own. They describe the scaffolding process by highlighting the directive role adopted by the teacher (tutor), who simplifies the task by reducing the number of constituent acts needed to solve it, and encourages children to keep working on the task until the "end."[1] However, in this study we cannot find any incidents of the type of interactions between teachers and students. An analysis that uses dialogic talk as a framework opens the possibility to specify the learning process, since it explicitly focuses on the types of interactions that produce learning with understanding (Garcia Carrión & Díez-Palomar, 2015).

2.3 Challenges of using dialogic talk in Mathematics Education

In Mathematics Education there is a long tradition to use discourse to analyse how students reach understandings of the mathematical concepts and their connections (Edwards 1993; Greeno 1997). Sfard (2002) specifically presents "communication" and uses the metaphor of *thinking-as-communicating*. She asserts that "communication should be viewed not as mere aid to thinking but as almost tantamount to the thinking itself" (Sfard 2002, p. 13). Later, she affirms that "learning mathematics" has to be defined as precisely as an "initiation to [the] mathematical discourse" (Sfard 2002, p. 28). Sfard uses the "interactivity flowchart" as an instrument to

[1] Wood et al. (1986) distinguish between recruitment, reduction in degrees of freedom, direction maintenance, marking critical features, frustration control and demonstration, as steps of the scaffolding process.

analyse how students engage in dialogues among themselves and how they support each other by looking at the utterances the students use to explain or justify their answers. Sfard uses "meta-rules" to determine when and how students develop reactive or proactive utterances that activate or inhibit the interaction between them. Similarly, she also analyses what she calls "the objectified discourse." Sfard uses this concept to distinguish between the use of mathematical words as "things in themselves" or as "pointers to some intangible entities" (Sfard 2002, p. 46).

The work of Sfard is consistent with other research, like the one by Kieran (2002) and Wertsch (1998), who also suggest that we learn through participation. According to Kieran and Dreyfus (1998), when students solve a problem collectively, there are few moments of "universe of thought" in which participants reach the understanding of mathematical concepts. Kieran (2002) asserts that in order for a situation to take place in which two partners form a mutually-productive pair, the existence of a great number of utterances in their dialogue that are object-level oriented (towards mathematics) is necessary. However, it is also possible that none of the participants uses concepts or mathematical connections in his/her dialogue, or that the use is so weak that it cannot be asserted that there is a minimum comprehension of the mathematics being used.

Dialogic talk occurs when participants exchange arguments based on validity claims. Drawing on this approach, participants engage in a discussion in which they try to justify their answers using claims that may be verified by their peers as well. In this sense, participants need to use mathematical objects (and its representations) to support their claims. This type of interaction may have the potential to foster learning among the participants within the group. However, an accurate analysis is needed to clarify how interaction makes learning happen.

3 Research questions

Our hypothesis is that learning (with understanding) may occur when individuals engage in dialogic spaces in which they use dialogic talk.

Our research questions are:

1. What types of interactions takes place when children and/or adults come to a deep, consistent, and meaningful understanding of mathematics ideas?
2. Can dialogic talk clarify how children learn [mathematics] through interaction within IGs?
3. What might we learn from the practice of dialogic talk that will assist us in reinterpreting the role of the tutor?

4 Methodology

In order to discuss our hypothesis, we examine the way a group of children develop their rational reasoning within "Interactive Groups." We have selected a lesson about rational numbers because according to literature, rational numbers appear to be a topic of difficult comprehension for students (Behr et al. 1992; Kieren 1976; Larson-Novillis 1979). Given these difficulties, rational numbers are a great opportunity for researchers to understand how learning occurs in a dialogical way (Flecha 2000), because they offer the possibility of analysing diverse types of arguments students use to justify how they understand the different semantic meanings of rational numbers. For this purpose, we design a lesson unit on rational numbers drawing on Kieren's (1976) work.

4.1 Interactive groups (IGs)

IGs are a way of organizing the classroom characterized by heterogeneous grouping and by the presence of an adult who encourages students to use dialogic talk (Valls & Kyriakides 2013). IGs consist of small groups of students (approximately 6 and 7 students per group, depending on the number of students in class). These students are diverse in their level of cognitive development and their level of mathematical abilities. In order to generate a dialogic space in which dialogic interactions as defined by Vygotsky (1978), Mercer and Howe (2012) and Soler and Flecha (2010) can take place, it is necessary for children to be in different moments of their cognitive development. If all the boys and girls are at the same level of learning, then there is no space for some to help the others, because all of them will face the same problems when they solve a task. Conversely, if they are at different levels, then the children who understand the mathematical task may help their peers. The diversity of levels is a condition *sine qua non* to dialogues in dialogic talk.

4.2 Setting and participants

The study was conducted in a Catalan school in an urban area. The Catalan school is a Learning Community, where community members are encouraged to participate in school activities and agree in applying successful education actions (Elboj & Niemelä 2010; INCLUD-ED 2009). Data was collected in a mathematics classroom where students were organized into IGs. The classroom was composed of a total of 24 children (11–12 year olds). The teacher of the group has wide experiences in primary education. In addition, there were three volunteers who helped to stimulate the interactions of the IGs. One of them was a PhD student from the university. The second was a pre-service primary teacher doing an internship for her undergraduate studies. The third volunteer was a mother. The teacher and the three adult volunteers were in charge of the developing activities for the groups they worked with. There were four IGs in the class. The adults spent about 20–25 min at each IG, and gathered the children's work before moving to another group. After 1.5 h, each adult has worked with four groups, and the children have completed four tasks.

The researchers attended a grade six class during the third semester (from April to June) of the school year 2013–2014. The researchers were present at the IGs every Monday, from 9 to 11 h, during this time period. In this article we present the case of a lesson on rational numbers, which lasted 5 sessions (weeks). The selected episodes belong to a larger study that we have been conducting over the last 3 years (since 2012), in which we have gathered documents and observations from three schools that are Learning Communities, located in diverse areas of Catalonia.

4.3 Lesson design

The lesson was designed by the research team together with the teacher of the group. First, the research team presented several tasks on the semantic field of rational numbers proposed by Kieren (1976). The first proposal was presented in a preparatory meeting with the teacher of the group to ensure that the curricular learning objectives of the group were covered (sixth grade of primary education). As a result of the meeting, several modifications were conducted to introduce a greater emphasis on the idea of "understanding rational numbers as equivalence classes of ordered pairs," and the transformation of mixed fractions into improper fractions and vice versa. Table 1 presents the fit between Kieren's (1976) semantic fields and the two tasks discussed here.

4.4 Data collection

Data include video-recordings of the activity in each of the four IGs organized in each session. Fix cameras were used in each group, audio recorders were placed at the centre of the table to record sound, and a hand-held camera was used to record additional information in each of the IGs. In addition, student notebooks and the researchers' observation diaries were collected. A total of 5 class sessions were recorded. In each data gathering session three members of the research team were present: one was in charge of the fix cameras, another was in charge of the hand-held camera, and the third was free to observe the session's development. An observation script was used for annotations within the classroom session, including (a) setting description, (b) observation of the students interactions from a point of view of dialogic talk, and (c) identification of other relevant contextual elements (additional comments), such as strategies used by students for scaffolding, attitudes or authoritarian reactions during the activity's resolution, etc.

4.5 Analysis of the interaction

Episodes (units of research) are selected from the data collected. All the episodes are transcribed and documented with the data from the video recordings of different cameras, as well as the students' written works and the researchers' field notes.

We have selected two episodes for discussion purposes. The first one focuses on equivalent fractions, and the second one on comparing fraction and arranging fractions in ascending/descending order. We chose these episodes because they belong to the two lessons in the rational numbers unit that best describe the type of interactions that we want to discuss in this article.

We used a communicative methodology approach (Gómez, Elboj, & Capllonch 2013) to analyse the interactions. The communicative methodology postulates that everyone has linguistic communicative competences. Every individual has the capacity to communicate and interact with others, because language and action are innate capacities and, therefore, universal attributes (Habermas 1984). In this sense, the communicative methodology postulates that we (as researchers) have to include the voices of participants in our analysis. For this reason we draw our interpretation on participants' talk.

In this article we focus on the analysis of participants' utterances. We think that it is necessary to carefully analyse what kind of talk produces [mathematical] understanding, and what does not, in order to really understand how the dialogical component of learning may lead children to learn mathematics. We use the taxonomy inspired by Mercer and Howe's (2012) work, Soler and Flecha's (2010) work and Habermas's (1984) work to characterize these interactions (see Table 2).

Interaction type 1, *exchange of information*, is a type of interaction characteristic of cases in which individuals do not offer any type of argument to validate the utterances provided. For example, saying "$2 + 2 = 4$" without any type of justification carries with it a declarative meaning. No argumentation is attached to this type of interaction.

Interaction type 2, *non-dialogic interaction*, refers to cases in which individuals use asymmetric power positions to argue: someone occupies a power position, while the other(s) occupy a subordinate position. The person in a power position does not justify his/her utterances with validity claims; s/he imposes the ideas on those listening

Table 1 Types of tasks discussed in this article, according to its correspondence Kieren's (1976) taxonomy of semantic meanings of rations numbers

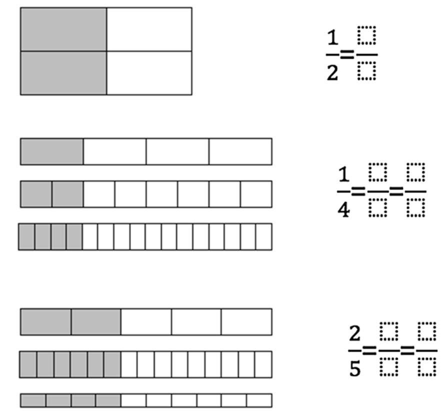
Interpretation	Our focus	Correspondence with Kieren's (1976) taxonomy	Example of task
(1) Rational numbers as equivalence classes of fractions	Ordered pairs Understanding rational numbers as equivalence Use visual representations to illustrate equivalence part-whole relationships (fractions, not ratios)	Ordered pairs Understanding rational numbers as equivalence classes of ordered pairs Symbolic control over part-whole relationships (fractional or ratio)	**Write the correct equivalent fraction** 
(2) Rational numbers as equivalence classes of fractions	Understanding rational numbers as equivalence	Understanding rational numbers as equivalence classes of ordered pairs	**Which fraction is the biggest in each case?** a) $\frac{1}{2}, \frac{3}{6}, \frac{2}{4}$ b) $\frac{4}{5}, \frac{12}{15}, \frac{32}{40}$

Table 2 Types of interaction

Interaction Type 1 Exchange of information	Interaction Type 2 Non-dialogic interaction	Interaction Type 3 Dialogic interaction
No argumentation	Arguments are based on power claims	Arguments are based on validity claims
Example: memorization	Example: authoritarian order	Example: egalitarian dialogue

Source: Garcia Carrion & Díez-Palomar (2015)

to them. The utterance "2 + 2 = 4, because it is like this" and because "I say so" can be taught as a categorical assertion; the justification is the power declaration of the speaker, when s/he states that two plus two equals to four, "because it is like this" and/or "because I say so." In this type of interaction, it is uncertain whether the utterance leads learners to go beyond memorization and understand the concept.

Interaction type 3, *dialogic interaction*, is characteristic of the cases in which individuals who intervene in the interaction establish a dialogue on the basis of validity claims that can be verified in the dialogue. For example, the utterance "2 + 2 = 4" can be taught with supporting manipulative: the teacher shows the student two objects, then adds two, puts them together and then counts them again in front of the student, to show him or her that effectively there are 4 objects in total. In this case, the existence of a verifiable argument opens the possibility to learning with understanding.

We know that human talk usually contains many unclear formulations and sentences. For this reason, the interpretations of the data analysis have been triangulated with data from diverse sources (Denzin 1970) in order to ensure internal validity and reliability. All the data has been analysed and reviewed by at least two members of the team. In all the cases we have codified the data with other researchers of the team. (Mertens & Sordé 2014).

5 Results

5.1 Episode 1: Drawing the graphic representation of equivalent fractions

In this first episode, the adult participants were Javier (researcher) and Ana (volunteer, PhD student), and the student participants were Calvin, Israel, Andrea, Daniel, Nerea, and Laura. The first episode belongs to a session that focused on the semantic concept of equivalence between fractions. The task in this episode asked children to find equivalent fractions using graphical representations as support (see Table 1, case 1). Graphs are rectangles or squares cut into several small square pieces. Ana started by introducing the task and she called the children's attention to the graphical representations. Nerea quickly answered, "It has to be four." Ana promptly asked her to justify her answer. Nerea raised four fingers in her hand, but she did not say anything else.

Ana: Let's do the first one. Look to the graphs of the numbers left… Did you see?
Children: (Many speaking at the same time)
Ana: So, look… What does it mean this graph?
Nerea: That it has to be four
Ana: What number has to be?
Nerea: (raises four finger in her hand)
Calvin: Look… It is multiplying times two… let's see… two times four… is eight. Two times eight, and two… I did a mistake! Yes I did
Nerea: It is four
Ana: Explain it to Laura, not to me
Israel: (Pointing to Laura's sheet of paper)

The excerpt (line 1 to line 6) represents an example of interaction type 1. In the first few seconds of the episode, all of what Nerea said was "four" (and she also raised four fingers with her hand to reinforce her statement). She did not provide any explanation to justify why her answer was four. Her claim was simply a declarative statement, and it was not based on any kind of argument. Nerea was engaged in a communicative situation where she communicated the answer with others, but we do not have enough evidences to assert that the communication was in the sense of dialogic talk, because of the lack of validity claims within the interaction process (Fig. 1).

Then, Calvin, another child in the group, stated that they had to "multiply by two." When he explained his point of view, he realized that he was wrong, and he verbalized it: "Two times eight and two… I did a mistake! Yes I did." By justifying his answer, Calvin realized that he was wrong in his calculation. Using the arguments that he knew, he recognized his mistake: he was wrong when he did the multiplication in the denominator of the second fraction. He quickly rectified his answer as a result of dialogic talk (type 3 interaction).

However, the child who was really having difficulties to solve the task was Laura, a girl in the same IG. Ana, the volunteer, realized that Laura was not following the task, and asked the rest of the children to help her (line 10). In order to help Laura, it was necessary for the children to

Fig. 1 Israel helps Laura to solve the task

understand why $\frac{1}{4} = \frac{2}{8} = \frac{4}{16}$ was the correct answer, and to find a valid claim to explain it. These actions were more cognitively demanding than simply solving the task and providing the answer. Israel helped his peer by being supportive. Calvin also jumped into the dialogue.

Israel: It is four, so you cross out
Calvin: This one, look, this one is…
Israel: Because over here it is: one, two, three, four
Calvin Do you understand? Look. Here you get all this… (pointing with the fingers to the graph, he is counting with his fingers, touching every painted square)

Both of them, Calvin and Israel, tried to explain to Laura why the key element to answer the question was the number four. They used a sheet of paper to show Laura the different steps of their thoughts.

Calvin: Then
Israel: And over here you take all of them
Calvin: (He shows with his fingers the three cases of equivalent fraction represented with graphs.) But all of this is the same! (He insists with his fingers, going over the three figures). All of this is still the same, isn't it?
Israel: Below [i.e. in the denominator] you have to put all the little squares. Eight [they are discussing the case of 2/8]
Laura: Right, it is like what Arantxa says…
Javier: But, why is this equal? Why it is equal?
Calvin: Look, because… eh… Can I have a sheet of paper?
Ana: Here you are, a piece of paper for you
(Calvin is now quiet; he looks embarrassed, not knowing what to do next.)
Israel: Calvin, Calvin! Is like we did with the letters and the Chinese (he refers to a task on equivalent fractions he previously did)
Ana: You have a piece of paper, do you?
Calvin: Can I use this one? (Then Ana gave one to him)
Somebody: Yes!
Calvin: Arantxa told us something… (He takes the sheet of A4 paper and then folds it in half). If we paint this part (one of the halves), all of this…

Calvin and Israel were making an effort to show Laura their thoughts; but it seemed that Laura was not able to follow them. Calvin showed the chart with his fingers on the sheet of paper, noticing that the portion painted is the same every time, although the painted portion was made with a different number of square pieces. Israel persevered, "Below [i.e. in the denominator] you have to put all the little squares. Eight." Laura seemed to link this claim with what Arantxa (the teacher) explained during the lesson. However it was not clear to her why one fraction is equivalent to the other ones. For this reason, Javier (the researcher) asked, "Why it is equal?" Calvin thought for a moment; he was looking for something on the table. He asked for a piece of paper. Ana gave one to him. Then, he started to fold the paper in halves (see Fig. 2).

Calvin kept folding the paper in halves. It was clear to us (the observers) that he was looking for a way to explain why the three fractions were equivalent $\left(\frac{1}{4} = \frac{2}{8} = \frac{4}{16}\right)$.

Calvin: If we paint this part… Then we fold it here… (He keeps folding the sheet of paper), the painted area is still the same…
Israel: And it is the same painted portion, and it is still the same
Javier: Aha
Calvin: That's why one half equals two quarters

Fig. 2 Calvin folds the piece of paper in halves

As soon as he had the piece of paper folded in two halves, Calvin took a colour pencil and painted one of them. Then, he kept folding the paper. At this point Israel made a crucial claim, “And it is the same painted portion, and it is still the same.” Calvin and Israel were justifying why the two equivalent fractions have the same value drawing on Calvin’s folded and painted paper. The rest of the children looked at them (Fig. 3).

Javier: Aha

Calvin: Now, if we fold it again, now… (He folds the piece of paper one more time, and then he unfolds it to show the folding marks on the paper… and starts to count the number of portions he got)… one, two, three, four, five, six, seven, eight. Now we have four eights. Two quarters equals to… it is four eights

Javier: Aha

Calvin’s folded piece of paper illustrates the relationship of two equivalence fractions geometrically. In order to have two equivalent fractions, the following axiom has to be true (1):

$$\text{If } \frac{a}{b} \text{ and } \frac{c}{d} \text{ are common fractions, where } b \text{ and } d \neq 0, \text{ they are equivalent} \quad \text{if } ad = bc. \tag{1}$$

Any valid claim might express this truth, regardless of the representation it may take (oral, written symbols, drawing, manipulative, etc.). We assume that when someone is able to build any type of representation based on such valid claim, this action should indicate that this individual understands axiom (1). In a dialogic interaction (type 3) it is mandatory (by definition) that individuals explicitly use validity claims to justify their statements. Israel and Calvin tried to explain to Laura why the answer was four sixteenths using the multiplicative rule. They looked for valid arguments and transformed the interaction from type 1 to type 3. Calvin pointed to the drawing on the piece of paper with his finger. Then, as we can see in lines 28 onward, Calvin searched for another way to explain why 1/4 = 2/8 = 4/16. He folded a piece of paper to prove the equivalence between the fractions. He painted one side of the folded paper. He kept folding it. Then, he showed the folded paper with the same painted surface even though it now had more folding marks on it. The paper was another representation (a geometrical one) of axiom (1) which Calvin used to sustain his explanation. He claimed that the painted portion remained the same (line 42), and the validity of the claim was the painted surface that remained the same after all the folds.

In the next excerpt (lines 52 to 74), the children were working on a task in which they were asked to determine the missing denominator: $\frac{2}{5}, \frac{6}{\ }, \frac{4}{\ }$. Israel seemed to have some troubles, although he supposedly understood Calvin’s justification. He did not agree with writing “ten” as the denominator. He claimed that the number below the numerator has to be 20. Ana problematized this answer and it opened the floor for additional responses; then Nerea answered “ten!”

Ana: OK, so then… what is the number that we need to write down here? Yes? What number did you wrote? (This is the next task: $\frac{2}{5}, \frac{6}{15}, \frac{4}{10}$)

Israel: Four, and down here… four… Twenty!!!

Ana: Everybody agrees?

Nerea: No, it is, it is… (She counts pointing our each square with the pencil tip)… ten!

Israel: (Insists that it has to be twenty)

Nerea: (Showing her figure to Ana). No, because look… over here there are ten

Ana: Can you explain this to him, Andrea? (She is sitting next to Israel)

Andrea: (She is counting with her pencil how many squares have each figure)

Nerea: But there are ten… he is the one who did it wrong!

Ana: So, what number did you wrote for the second one? (Ana realizes Israel’s mistake, hence she wants to call Nerea’s attention towards Israel’s mistake)

Nerea: Because here we add all the squares in this figure

Ana: Can you explain it to Israel?

Fig. 3 Sequence of Calvin’s explanation about equivalent fractions

Nerea: We are in the one down here. And this has ten [portions]
Ana: Come on… in the one down here, and what does "the one down here" mean?
Israel: Four tens
Ana: How much did you got?
Everybody: Four tens
Nerea: (comparing with the previous task: $\frac{1}{4}$, $\frac{2}{8}$, $\frac{4}{16}$) Here it was four and sixteen, and in this one there are four and ten

Now Israel provided a false answer (line 54), because he claimed that the denominator of the third fraction was twenty rather than ten. According to Arantxa (the teacher), Israel was wrong because he misused the algorithm to calculate the denominator of the equivalent fraction (because he used the algorithm mechanically without understanding). At the beginning of the interaction, Israel was just saying "twenty" (interaction type 1), and no additional information to why he was saying that was provided. Then, after the dialogue evolved, we discovered that Israel made a mistake when he applied the "rule of multiplication" to prove that 4/10 = 2/5 (interaction type 3).

For some reason, Israel multiplied the denominator of the original fraction (5), by the numerator of the equivalent fraction (4), so he got 20 (see Fig. 4), which resulted in a wrong answer. Arantxa mentioned that in other occasions Israel had made similar mistakes.

This suggests that even in a dialogic interaction, the "valid" claim may not be true. However, all claims have to be confronted and examined by participants, until everyone agrees on them (according to Habermas' terminology, until everyone reaches an understanding). Nerea's and Andrea's talk was oriented towards demonstrating with evidences that ten is the number in the denominator of the third fraction. They used the graphical representation to support their statement (which is another case of interaction type 3).

5.2 Episode 2: Organizing fractions according to its value

In this episode, the adult participants were Javier (researcher) and Ana (volunteer), and the student participants were Calvin, Israel, Andrea, Daniel, Nerea, and Laura. The task was to identify the fraction with the highest value (see Table 1, case 2). The fractions that the children worked with were: 1/2, 3/6, and 2/4. Ana showed the three fractions one beside the other on a sheet of paper.

Ana: Now look at this (she shows the students a piece of paper with three fractions, see Fig. 5). We are going to tell which one is the biggest…
Nerea: (pointing to 3/6 with her pencil) This one!
Daniel: (points to 2/4)
Andrea: (also points to the same one, 2/4)
Calvin: (pays attention to what everybody is saying)
Nerea: No, this one!
Ana: Which one is the biggest?
Nerea: No… is this one! Because the 3 is bigger than the 2; and the 6 is bigger than the 4
Calvin: All of them are the same
Nerea: No, because look, 3 is bigger than 2 and bigger than 1; and 6 is bigger than 2 and 4
Calvin: All of them are the same
Daniel: All of them are the same
Ana: Why?
Calvin: Because all of them are equivalent
Nerea: It is true! Right, right… because it is twice
Calvin: Yes, because this one multiplied two times is that one, and that one is the same as this one (he points to 1/2 and then 2/4). All of them are halves, halves…
Andrea: Right, it is the same. Yes, it is twice, and it is the same (she folds the paper as Calvin did in the last task). It is the same

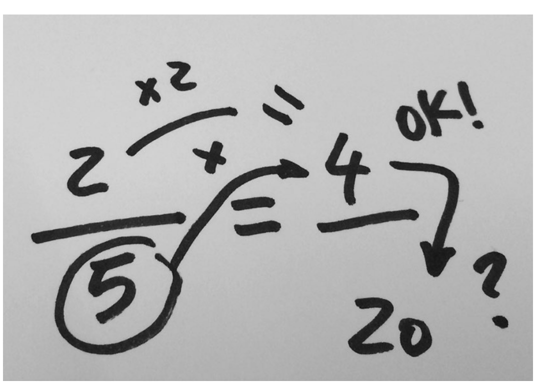


Fig. 4 Alternative interpretation of Israel's mistake

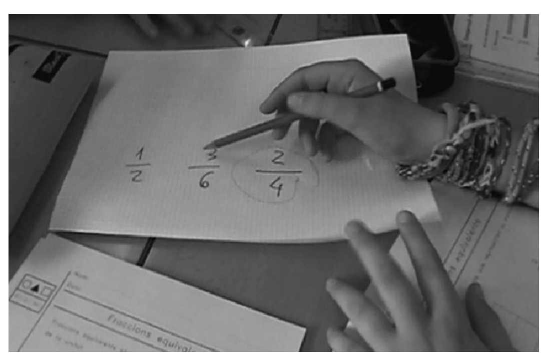


Fig. 5 Nerea points with her pencil to the fraction which she believes is the biggest

Ana: So Israel, what do you think?
Israel: All of them are the same
Calvin: Then, all of them are the same

Nerea answered Ana's question quickly, claiming that the biggest fraction was the one with the biggest numbers in it, i.e., 3/6. But Daniel disagreed; he claimed that the biggest one was 2/4. In the meantime, Calvin looked at his peers. He did not say anything; he was thinking carefully. Nerea justified her answer and claimed that 3 is bigger than 2, and 6 was also bigger than 4; hence she was rebutting Daniel's solution. Then, Calvin jumped into the dialogue and claimed that all the fractions were equivalent. Daniel immediately changed his answer, agreed with Calvin, and said that all are the same. Nerea silently observed the situation. She did not agree with them, but she had doubts. At this point Ana asked, "Why are they the same?" Calvin answered, "Because all of them are equivalent." Now Nerea recognized that those fractions were equivalent: "because it is twice ("*porque se dobla*"). Calvin said that 2/4 resulted from multiplying 1/2 by two. Then, he added another component to the justification: "All of them are halves, halves…" Calvin seemed to know that one over two is half; the same as 2 over 4 and 3 over 6. Andrea agreed with Calvin, and she also used a strategy which Calvin previously used (folding a sheet of paper) to support her claim.

Looking closely to this vignette, we think that this task also provoked the emergence of another episode of interaction type 3. We know that in mathematical terms a set of numbers (in our case, rational numbers or $\mathbb{Q}$) has a partial order, $\leq$, if $\mathbb{Q}$ is reflexive, antisymmetric, and transitive:

$$\forall x, y \in \mathbb{Q}, (x \leq y) \vee (y \leq x) \quad (2)$$

This is the definition of a relation of total order. All valid claims have to fulfil this proposition. Otherwise, it is not valid at all. Similar to episode 1, the discussion in episode 2 began with interactions type 1. Neither Nerea nor Daniel justified their statements. Calvin affirmed that 1/2 = 3/6 = 2/4 because all of them were "equivalent" fractions. Does this mean that he understood "equivalent" as "the same"? Unfortunately, no further explanation was added. Then Nerea added the word "twice" (line 18), introducing another concept to justify her answer. (Now it seemed that the dialogue proceeded to an interaction type 3.) Calvin recalled the rule of multiplication to further explain the task. Andrea recognized that this task was the same (line 23) as the last set of tasks (episode 1). Can we assume here that Andrea, when she said, "It is the same", she recognized a relationship of transitivity between the three fractions (and consequently, using this "knowledge" as a justification for her statement)? We do not know. But it seems that we can maintain that an interaction type 3 happened in this episode.

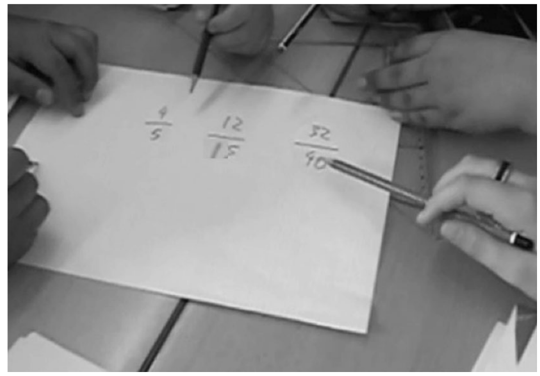


Fig. 6 The second group of fractions

Next, Javier showed one more group of fractions (see Fig. 6), and he asked the group to determine which number was the biggest. The numbers used to make up the fractions in this question were bigger ($\frac{4}{5}$, $\frac{12}{15}$, $\frac{32}{40}$) than those in the previous task.

Javier: So, what about those ones… Which one is the biggest?
Daniel: Those are more difficult
Javier: Aha
Nerea: This one! (She highlights the 32/40 with her pencil)
Calvin: No, all of them are…
Daniel: No… all… all of them are the same, like before
Calvin: Yes, but it is like… if we would multiply by… eh? Wait…
Daniel: Multiplying by this one? (He points to the 4/5)…
Calvin: No, wait a moment (he is thinking)
Ana: (Waits for about 5 s) So, which one is the biggest?
Calvin: But-but-but… Wait, wait a moment (he looks like doing mental calculation)
Ana: Daniel, which one is the biggest?
Daniel: The last one (32/40)
Ana: Why?
Daniel: Because it's the biggest: 32 and 40
Andrea: (pointing to the number 32). Yes, but he eats a bigger piece
Daniel: Yes, but it is the same than here… there are more pieces, but smaller. However, over here there are 4 pieces, but you have just 5 parts. And… mmmm… I mean, you eat less pieces, but you eat… you eat more quantity
Nerea: Right… over here there are more pieces, you eat more pieces, because there are 40 (she starts to draw a circle and she tries to cut it in slices, and she counts them)…
Ana: Laura, which one is the biggest?

Daniel: Then, there are 5 parts, and she eats one, and… (He draws another circle and he cuts bigger pieces than Nerea in her circle… and he paints them according the proportion that it is supposed to eat)

Nerea: Aha… but… (She is failing drawing the circle because she cannot cut it up to 40, so she starts to draw a rectangle and she cut it in small squares)

Daniel complained that the fractions were more difficult. Nerea made the same mistake again: she chose the fraction with the biggest numerals (32/40). Calvin was the most cautious; he hesitated before answering the question (because he wanted to find the number that he needed to change 4/5 to 32/40 first, and he did a mental calculation to figure it out). In the meantime, Daniel went ahead and claimed that similar to the previous task, all fractions were the same. However, while Calvin looked for the number to multiply the numerator and denominator with, Daniel allowed Nerea to persuade him with her claim and decided to change his previous answer. He stated that 32/40 was the biggest fraction, because the numbers were the biggest ones. When Nerea and Daniel tried to justify their claim, they started to mix things up. They talked about the size of the pieces and the number of the pieces there were, and tried to connect these ideas with the idea that one fraction has bigger numerals than the other one. Then, Nerea started drawing and looked for a different representation to better support her argument. But she failed because dividing a circle into 40 slices was not easy work. She moved to rectangles, but it was also difficult to cut 40 pieces from the original rectangle. Daniel looked at her and tried to draw the fraction; he knew that the sizes of the pieces were not the same; the slices in 4/5 were bigger than the ones in 32/40. His drawing was unfinished and imprecise.

Andrea: Here you have eight left

Javier: What do you mean, Andrea, when you say that here there are eight left? I do not understand

Andrea: That here (she points to the 4/5; and then to the 32/40)… here from 32 up to 40 there are 8 left. You have eight left. And over here (in the case of the 4/5) you get only one left

Javier: Aha

Ana: But, how is possible that there is one piece left? (No answer)

Ana: Nerea, what do you want to do with your drawing? Can you explain it to us? (Nerea kept drawing for a while)

Nerea: There are 40. Over here there are 40

Calvin: Four times… which one in the number down there?

Ana: Could you explain it to us?

Nerea: You eat 32, right? You eat 32. You eat all of those… (She points to the squares/rectangles she drew). So you eat **mooore** (emphasis in her words)

Calvin: This is the result of eight times four

Nerea: Would you let me talk? Look, here there are 40. You eat 32, and you have left… you have a lot left… You still can eat… Yes, here, from 5… aha! Over here, from 5, you eat 4, so you can eat 1… but here, this, this… Then, this one is better, because you have more left, for next day

Javier: But the pieces that you have left are bigger ones, or smaller ones?

Nerea: The same

Javier: The same pieces [are the pieces the same size], or the same their quantity [you have the same quantity of pieces]?

Calvin: No, look! The pieces are smaller… but you have the same…

Javier: What do you mean?

Nerea: No, but you have more pieces… [More quantity]

Calvin: But look, there are smaller… If you put all of them together, they are like the one big that you have left here (he points to the 4/5)

Nerea: Oh, well…

Calvin: It is because they are equivalent… it is five times eight; and four times eight… the same

Andrea: It is the same piece, but cut more times

Daniel: Times eight. You have eight pieces

Nerea: Yes, smaller ones, but you have more left for another day

Calvin: But same quantity, look (he points to the drawings)

Nerea: (she looks at him; silent)

Ana: Did you understand?

Andrea's comment ("here you have eight left") determined the group's next interactions, and Nerea found support in those words to justify that 32/40 is bigger than 4/5. Nerea's answer suggested that maybe she did not understand the multiplicative relation between the numerator and the denominator in a fraction. Traditionally teachers explain the difference between numerators and denominators by telling students that the denominator contains the number of pieces "in the cake", whereas numerator includes the number of pieces that "you ate." Hence, the difference is the number of pieces (slices) that you leave for later. Based on what Andrea said, this was Nerea's thinking. For Nerea, the most important thing in the case of 32/40 was the eight pieces for tomorrow, whereas in the case of 4/5 she only got one piece left. It is clear that Nerea did not consider the size of the pieces. Nerea was making an effort to justify her utterance (interaction type 3) which we know was not correct because even though the difference between the numerator and denominator gives more "pieces" in the fraction 32/40, those given pieces are smaller than that in the fraction 4/5. It is not a matter of subtracting, but dividing in a way that the value of the three fractions remains identical.

This is a common mistake in elementary schools (Behr et al. 1992). However, the mistake comes from an earlier dialogue, when Daniel (who started the episode claiming that all fractions were the same) changed his mind and claimed that 32/40 was the biggest fraction. Andrea seemed to disagree with Nerea and Daniel (line 43), when she noticed that in the fraction 4/5, a bigger piece was eaten She knew that the size of the slices matters. However, she allowed herself to be persuaded by Daniel and Nerea, hence later on (lines 62–64) she also highlighted the difference between 32 and 40 as a determination factor in deciding which fraction is the biggest one. Although this dialogue was dialogic (in our terms), the claims used by Nerea, Daniel and Andrea were not valid.

Javier intervened by asking whether the pieces were of equal size. That was a strong intervention. Nerea continued to avoid this variable in her thinking. Then, Calvin drew on the sheet of paper and used his drawings to show that the 40 pieces of the fraction 32/40 occupy the same surface as the ones of the fraction 4/5. Thus, the amount of cake that one eats in both cases is the same, although one may get "more" pieces left in one of them. Nerea, after that "evidence", admitted that Calvin was right, and remained silent while looking at the picture.

6 Discussion

Vygotsky believes that guidance from adults and more capable peers are crucial for children to develop their potential skills. Later on, Wood et al. (1986) design a detailed protocol of tutoring that involved family members as facilitators to guide their children in the construction of a pyramid with wooden blocks. The protocol provided a clear set of procedures to inform family members how to proceed to the children's reactions to the task.

However, later research move away from intervention. Instead of intervene, the adult acts as a facilitator and orchestrates learning, leaving children the freedom to build their own mathematical meanings when solving the task (Elbers & Streefland 2000). But if we leave children to their free will, how can we ensure that they reach the right arguments? While the presence of a tutor may help children to scaffold their thinking (as Wood, Bruner and Ross claim), the tutor can also limit children's discovery learning, which Bruner (1961) believes to be especially meaningful. Thus the key here seems to be a balance between intervention and discovery learning: On the one hand, the teacher needs to take care of the children by supporting them, and warning them when they go off topic. On the other hand, the teacher needs to provide children with enough freedom to discover the answers by themselves.

Since dialogic talk draws on the exchange of arguments based on validity claims, it may have the potential to move beyond the traditional concept of scaffolding. But how does this interaction work? This is difficult to identify. Rogoff, Turkanis and Bartlett (2001) claim that leaving children alone without any kind of tutoring may lead them to a uncertain situation in which they can be lost in the middle of a discussion, and adopt wrong answers as valid ones. Moreover, Mercer and Howe (2012) affirm that not all interactions may result in effective learning. In fact, there are interactions that produce exactly the opposite. For example, if a child uses an invalid argument in front of his peers and everybody accepts that argument without questioning it, we cannot affirm that it is effective (in terms of acquiring a correct knowledge) even though a learning process occurred and the child may be sincere in his/her effort to look for a valid argument. Conversely, it could be the case that children engage in a dispute when solving a particular task and exchanging contradictory claims, accepting the argument of the leading child not because the validity of his/her claim, but because his/her power position in front of his/her peers.

Our data is consistent with Mercer and Howe's (2012) claim. Among the interactions produced within the IGs, we see some that were non-productive. We think that this is clear in the case of type 1 interactions. Similarly, some type 3 interactions illustrate highly productive discussions in which students provided meaningful explanations denoting their understanding of specific properties of rational numbers. However, the classification of the sequences of talk is not always clear, as we can see in the last fragment of the second episode (Nerea and Calvin's "dispute"). Nerea's dialogue in lines 73 onward may be seen either as type 2 or 3 interactions. It could be type 2 interactions, because we can interpret Nerea's actions as fairly egocentric and she was resistant to negotiate her claims with her peers. However, it could also be type 3 interactions, because we can interpret her use of language as a way to explore the task and to look for a valid explanation. In any case, Nerea's reasoning about using subtraction to find the highest fraction was not mathematically correct.

Then Javier asked about the size of the pieces. Perhaps Javier's intervention truncated the possibility for Nerea to realize her apparent mistake. This brings us another question: Should a teacher (or a volunteer) intervene within the dialogue to redirect the discussion? If so, in what moment is the intervention of the adult most appropriate?

Literature suggests that adults' guidance (tutoring) is crucial for children to justify their claims. For this reason we decide to recover the work by Wood, Bruner and Ross (Wood et al. 1986) regarding scaffolding and place it in a new context. We share Wood et al.'s idea that there must be an adult guiding the children, leading them during their process of reasoning, but not to limit them to freely find their own claims. The dialogic talk postulates that the adult should encourage students to argue without giving them the answers, because if the teacher (or the adult who is facilitating the task) gives them the answer, then we cannot say that children struggled in their process of learning. In the same vein, it is uncertain to state that "learning-with-understanding" emerges from a situation in which the teacher provides the answer explicitly to the children. In lines 10, 52, 55, 60, 63 and 67 (first episode), and lines 24, 36, 39, 41, 51, 68, 72 and 98 (second episode), Ana used dialogic talk. She asked the children to further explain their claims to her or to other children in order to justify their claims. The children engaged in interesting dialogue in order to do what Ana requested. And we can conclude that the children learned together.

This is consistent with Elbers and Streefland's (2000) claim that students learn collaboratively. According to them, understanding results from "cycles of argumentation", to which many children contribute their thoughts in a recursive process. In this process, children discuss a particular topic repeatedly until they generate such "understanding" of the idea. Elbers and Streefland's conclusions agree with Forman's et al. (1998) proposal about collective argumentation. Zack and Graves (2001) draw on the sociocultural tradition and arrive at the same idea: learners reach understanding after a series of interactions in which they dialogically build meaning on the "shoulders" of their peers. These researchers find that when a teacher acts in a nondirective style, the students are able to find valid justifications to support their thinking by themselves. The teacher orchestrates the discussion among students by recruiting attention to certain aspects of the discussion and encouraging students to use argumentative claims. It seems that the dialogic talk approach is consistent with this contribution in some ways.

7 Concluding remarks

Regarding the first research question, our data suggest that we can find examples of the three types of interactions in all the sequences. Interactions within IGs are complex. Although IGs are egalitarian spaces (in Habermas and Flecha's terms), where participants intervene looking forward to reach understanding, certainly we come across situations in which participants seek either to hold the truth or just to dominate the whole discussion. Therefore real classroom situations where students work in IGs will produce the three types of interactions described in this article.

Regarding the second research question which asks whether dialogic talk is useful to unpack how children learn with understanding, our answer is yes, but we should proceed with caution. While it seems that dialogic talk assists learning-with-understanding, it could also be the opposite case, as we have discussed in the previous section. However, learning-with-understanding is not possible without dialogue (in the sense of providing valid claims). Hence, in a sense, dialogic talk clarifies how children learn, because it is a *necessary, though not sufficient* condition for learning to happen.

Finally, regarding the third research question, our data suggest that the role of the adult is not neutral. This is a crucial point on how to conduct a lesson. The decision to provide students with instruction is a dilemma in our profession (Lobato, Clarke & Ellis Lobato et al. 2005). Previous research suggests that tutors should encourage children to justify their answers to engage them in a dialogic process. Our data is consistent with this claim.

Many questions arise from this discussion. First: dialogic talk is useful to understand how children learn; but is it enough? What happens with children who monopolize the discussion based on their own misunderstandings of mathematics? This question forces us to reconsider our initial assumption that valid claims are sufficient to understand children's learning. While "valid claim" (and the theory of argumentation) play an important role in dialogic talk, interactions are also mediated by feelings, status, confidence, etc. There is an affective dimension embedded in the interaction. The boundaries between "valid claims" and "power claims" are sometimes blurred. Second: when this happens, what is the role of the adult (tutor)? Should s/he intervene to get the students back on the track (as Javier did)? Does this type of interaction (dialogic interaction) facilitate the establishment of the "cycles of argumentation" set by Elbers et al. (2000)? This suggests that our taxonomy has to be more refined to cover these types of aspects, because human talk usually contains many unclear formulations and phrases. Answering these questions demands new research to further explain the link between interaction and learning.

References

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296–333). New York: Macmillan Publishing.

Bruner, J. S. (1961). The act of discovery. *Harvard Educational Review, 31*, 21–32.

Bruner, J. (2012). What psychology should study. *International Journal of Educational Psychology, 1*(1), 5–13.

Denzin, N. K. (1970). *The research act: A theoretical introduction to sociological methods*. Chicago: Aldine.

Edwards, D. (1993). But what do children really think? Discourse analysis and conceptual content in children's talk. *Cognition and Instruction, 11*(3–4), 207–225.

Elbers, E., & Streefland, L. (2000). Collaborative learning and the construction of common knowledge. *European Journal of Psychology of Education, 15*(4), 479–490.

Elboj, C., & Niemelä, R. (2010). Sub-communities of mutual learners in the classroom: The case of interactive groups. *Revista de Psicodidáctica, 15*(2), 177–189.

Flecha, R. (2000). *Sharing words: Theory and practice of dialogic learning*. Lanham, MD: Rowman & Littlefield.

Forman, E., & Larreamendy-Joerns, J. (1998). Making the implicit explicit: Classroom explanations and conversational implicatures. *Mind, Culture, and Activity, 5*(2), 105–113.

Garcia Carrión, R., & Díez-Palomar, J. (2015). Learning communities: Pathways for educational success and social transformation through Interactive Groups in Mathematics. *European Educational Research Journal, 14*(2), 151–166.

Gómez, A., Elboj, C., & Capllonch, M. (2013). Beyond action research. The communicative methodology of research. *International Review of Qualitative Research, 6*(2), 183–197.

Greeno, J. G. (1997). On claims that answer the wrong questions. *Educational Researcher, 26*(1), 5–17.

Habermas, J. (1984). *The theory of communicative action: Vol. 1. Reason and the rationalization of society*. Boston: Beacon.

INCLUD-ED Consortium. (2009). *Actions for success in schools in Europe*. Brussels: European Commission.

Kieran, C. (2002). The mathematical discourse of 13-year-old partnered problem solving and its relation to the mathematics that emerges. In C. Kieran, E. Forman, & A. Sfard (Eds.), *Learning discourse* (pp. 187–228). Dordrecht: Kluwer.

Kieran, C., & Dreyfus, T. (1998). Collaborative versus individual problem solving: Entering another's universe of thought. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd PME* (Vol. 3, pp. 112–119). Stellenbosch, South Africa.

Kieren, T. E. (1976). On the Mathematical, cognitive and instructional foundations of rational numbers. In R. A. Lesh & D. A. Bradbard (Eds.), *Number and measurement. Papers from a research workshop* (pp. 101–144). ERIC (ED 120 027).

Larson-Novillis, C. (1979). Locating proper fractions on number lines: Effects of length and equivalence. *School Science and Mathematics, 53*, 423–428.

Lobato, J., Clarke, D., & Ellis, A. B. (2005). Initiating and eliciting in teaching: A reformulation of telling. *Journal for Research in Mathematics Education, 36*(2), 101–136.

Mercer, N., & Howe, C. (2012). Explaining the dialogic processes of teaching and learning: The value and potential of sociocultural theory. *Learning, Culture and Social Interaction, 1*(1), 12–21.

Mercer, N., & Sams, C. (2006). Teaching children how to use language to solve maths problems. *Language and Education, 20*(6), 507–528.

Mertens, D., & Sordé, T. (2014). Mixed methods research with groups at risk: New developments and key debates. *Journal of Mixed Methods Research, 8*(3), 207–211.

Rogoff, B., Turkanis, C. G., & Bartlett, L. (2001). *Learning together: Children and adults in a school community*. New York: Oxford University Press.

Sfard, A. (2002). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. In C. Kieran, E. Forman, & A. Sfard (Eds.), *Learning discourse* (pp. 13–57). Dordrecht: Kluwer.

Soler, M., & Flecha, R. (2010). Desde los actos de habla de Austin a los actos comunicativos. Perspectivas desde Searle, Habermas y CREA. *Signos, 43*(2), 363–375.

Valls, R., & Kyriakides, L. (2013). The power of Interactive Groups: How diversity of adults volunteering in classroom groups can promote inclusion and success for children of vulnerable minority ethnic populations. *Cambridge Journal of Education, 43*(1), 17–33.

Vygotsky, L. S. (1978). *Mind and society: The development of higher mental processes*. Cambridge, MA: Cambridge University Press.

Wertsch, J. V. (1998). *Mind as action*. New York: Oxford University Press.

Wood, D., Bruner, J. S., & Ross, G. (1986). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry, 17*(2), 89–100.

Zack, V., & Graves, B. (2001). Making mathematical meaning through dialogue: "Once you think of it, the Z minus three seems pretty weird". *Educational Studies in Mathematics, 46*(1–3), 229–271.