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ORIGINAL ARTICLE

Whole-class scaffolding for learning to solve mathematics problems together in a computer-supported environment

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Abstract We investigate teachers' practices in a whole-class context when they scaffold students' learning in situations where students use technologies that facilitate group learning to solve mathematical problems in small groups. We describe teachers' practices in order to evaluate their contribution to *Whole-Class Scaffolding* in the context of a course that was meant to facilitate learning to solve mathematical problems in small groups. Sixteen and twenty four junior high students took part in two iterations of a design research correspondingly. We identify six teacher practices. These practices include: (1) Presenting strategy-oriented problems in order to familiarize students with mathematical problem solving strategies and heuristics; (2) Decomposing a problem into stages; (3) Modelling the use of the dedicated technologies; (4) Preliminary Learning to Learn Together (L2L2) talk; (5) Routine L2L2 talk; and (6) Summative L2L2 talk. We conclude that scaffolding learning to solve mathematical problem together in a whole-class context is feasible with suitable technologies. This study also provides insights on the Whole-Class Scaffolding theory and on the ways group learning could enhance students' ability to solve mathematical problems individually.

Keywords Mathematical problem solving · Learning to learn together · Whole-class scaffolding · Computer-supported collaborative learning

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1 Introduction

Small group learning is popular in the Learning Sciences. However, research about in-class pedagogies encouraging small group learning in classroom contexts is meager. There are, however, technologies dedicated for learning mathematics together in small groups (e.g., Stahl, 2009). These are called Computer Supported Collaborative Learning (CSCL) tools. These tools foster group learning, mostly in laboratory conditions, and they have hardly been used in full course contexts. In this paper, we investigate the teachers' practices in a whole-class context when they scaffold students' learning in situations where students solve mathematical problems together using CSCL tools. We first portray the theoretical background that lead us to this investigation, followed by descriptions of a novel learning environment and emerging teacher practices in the context of such a learning environment. Last, we discuss these practices and evaluate their contribution to the *Dialogic Whole-Class Scaffolding* theory and practice.

2 Theoretical background

The topic of the present special issue in ZDM is scaffolding and dialogic teaching—two domains that have drawn special attention in education, and especially according to progressive pedagogies. These domains point at the delicate relations that teachers and learners should maintain, in order for productive interactions to occur. In this paper, we focus on practices used by teachers when they scaffold students' learning in order for students to solve mathematical problems in small groups with the help of dedicated technologies. Our research is then at the intersection of Mathematical Problem Solving (MPS) and CSCL. In this section, we provide theoretical underpinnings to our study, and then describe two pieces of software for facilitating collaboration in mathematical problem solving: *Planning tool* and *Geogebra*.

2.1 Math problem solving

Mathematical problem solving (MPS) is at the heart of mathematical activity (Pólya, 1945). Twenty-five years ago, the learning of MPS became a major goal in mathematics education. According to Schoenfeld (1992), MPS is *"…grappl* [ing] *with new and unfamiliar tasks, when the relevant solution methods* (*even if only partly mastered*) *are not known"* (p. 354). These unfamiliar tasks, along with the motivation to reach a solution, trigger a process in which a solver needs to apply various strategies (e.g. solving an equation) and heuristics (e.g. checking a simpler case) in order to solve the problem (Pólya, 1945, Schoenfeld 1985). A problem solver enacts various strategies and heuristics that solve problems, even if this enactment does not always lead to the solution. Schoenfeld (1992) argues that one of the important abilities that distinguishes expert from novice problem solvers, when they solve a difficult problem, is the execution of metacognitive control upon the solution process by carefully monitoring and controlling strategies and heuristics. Veenman and Spaans (2005) examine MPS abilities, metacognitive abilities and intelligence among secondary school students. They show that high-level metacognitive skills predict successful MPS. Moreover, they suggest that metacognitive skills outweigh the contribution of intelligence in the prediction of successful MPS.

Programs for enhancing MPS abilities have led to mixed results (Schoenfeld, 2007; Veenman, Van Hout-Wolters & Afflerbach, 2006). So far, whole-class context has not been found propitious for enhancing MPS abilities. However, three directions seem to provide hopes to foster MPS abilities in the whole-class context. The first is the arrangement of students in small groups. The second is the development of pedagogies based on scaffolding techniques. The third is the development of dedicated technologies.

2.2 Learning together

Many researchers have noted the effectiveness of small group work in learning in general with proper teacher facilitation (e.g., Webb, 2009). When students learn together in small groups, they need to talk to each other and verbalize their thoughts in order to explain them to their peers (Iiskala, Vauras, Lehtinen, and Salonen, 2011). When done properly, this verbalization of thoughts has the potential to facilitate metacognitive behaviors such as monitoring and regulating the strategies and heuristics used in the problem solving process (Whitebread & Pino-Pasternak, 2010). A proper design of learning activities and learning environment may sometimes suffice (Schwarz, Hershkowitz and Prusak, 2010; Schwarz, Neuman and Biezuner, 2000), leading to unguided sense-making and conjecturing towards conceptual change or the acquisition of mathematical strategies. However, unguided talk is generally inadequate and Exploratory talk needs to be learned (Wegerif, 2006). In this paper, we focus on Learn to Learn Together (L2L2) to solve mathematical problems, which is aimed to improve the quality of group talk when students solve mathematical problems together in small groups.

Teaching skills and competences such as L2L2 in groups can be understood as a form of intentional culture change. Cultures have implicit norms that need to be made explicit in order for groups to become aware of them, to reflect on them and to be able to change them (Cobb & Bauersfeld, 1995; Cobb et al., 2001). As explained by Cobb and his colleagues (classroom) norms are appropriated by recurrently enacting practices, and making them explicit through dialogic practices along with the enactment of these practices. Through these iterations, a mini-culture emerges in the classroom.

Three groups of normative practices are essential for the emergence of L2L2 mini-culture (Schwarz, de Groot, Mavrikis & Dragon, 2015 in press). The first group of practices that has been recognized as important for the constitution of a community of learners in general is what Yackel and Cobb (1996) call collective reflection. In contrast to Yackel and Cobb's interests in teacher-led whole-class discussions of past activities, our focus on MPS leads us to also include planning together, as well as regulating or monitoring a plan together, as a way to check or revise an interim plan. The second group of practices is about taking collective responsibility (Zhang, Scardamalia, Reeve, & Messina, 2009). This means that each participant engages as a leader, collaborator, helper, help seeker, etc., in a way that advances the group. The third group of normative practices is peer group assessment.

Schwarz, de Groot, Mavrikis & Dragon, (2015) (in press) explain that peer group assessment could become a part of the L2L2 mini-culture when this practice is enacted iteratively and clear criteria for assessment are made explicit.

2.3 Scaffolding

Vygotsky (1935/1978) explores the relationships between the organization of a learner's environment and learning. In order to do so, Vygotsky develops a theoretical framework for the assessment of a child's performance—the Zone of Proximal Development (ZPD): "*…the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers*" (p. 86). Wood, Bruner and Ross (1976) suggest the term Scaffolding as the situational and adaptive support that is given by the caregiver. Scaffolding is later related to the idea of ZPD by Cazden (1979) as a support system that creates opportunities for the learner to move within the ZPD—from the actual developmental level to the level of potential development.

Over the years, the scaffolding metaphor became highly appealing as a means to describe teaching practices intended to facilitate learning by a student, and the term Scaffold became useful in order to describe learning materials and technological aids that serve as objects that support learning (Pea, 2004). Puntambekar and Hubscher (2005) identify four functions in scaffolding: (1) Modelling—showing the student how to do an assignment; (2) Ongoing diagnosis—monitoring the student's actions; (3) Calibrated support—recruiting the student's interest, reducing the degrees of freedom of the task by simplifying it, maintaining direction, highlighting the critical task features, controlling frustration, and demonstrating ideal solution paths; and (4) Fading out—gradual transfer of the responsibility for learning, from the teacher to the student. As Puntambekar and Hubscher (2005) notice, while dedicated technologies can help in the enactment of these functions of scaffolding, supporting learning is not straightforward, even in the presence of these technologies.

Smit, van Eerde and Bakker (2013) conceptualize Whole Class Scaffolding (WCS) theoretically and empirically. This conceptualization does not deny the existence of an individual learner's ZPD, but it attempts to address the class as a unit of analysis. Three main WCS practices are identified: Diagnosis, Responsiveness and Handover-to-independence. Diagnosis is needed in order to assess the current state in learning. It is not directly connected to the response in a timely manner. Responsiveness is the teacher's adaptive and calibrated response to the actions of her students. *Handover* is an overall process in which the teacher gradually fades out and takes less responsibility for the learning; she makes considerably less diagnostic and responsiveness moves in a way that leaves the students with more responsibility (and desirably, more competence) than before. Effective WCS should manifest itself as a distribution of diagnosing and responsiveness acts over time, which ultimately cumulates in order to serve the handover to independence. In addition, Smit, van Eerde and Bakker (2013) distinguish between the online (thus, in-class, with students) and offline (off-class, mostly asynchronous) emergence of these three practices.

In the spirit of Smit, van Eerde and Bakker's (2013) WCS theoretical framework, in the current paper we ask which practices are performed by teachers when they scaffold the learning of metacognitive monitoring and regulation upon MPS, and L2L2 behaviors in service of MPS in a whole-class setting. Educational technologies, we suggest, could help in this endeavor.

2.4 Dedicated educational technologies for orchestrating small group work

The integration of educational technologies into classrooms raises opportunities and challenges. CSCL software are a family of technologies that afford group learning through the use of various learning resources, interaction management tools, and multiple representations of both problem solving domains and interaction tasks (Andriessen, Baker & Suthers, 2003). There are examples of success when one small group works in a laboratory context. For example, Stahl (2009) successfully used Virtual Mathematics Teams (VMT), a virtual space for unguided geometrical problem solving, in a laboratory context (with one group of students). Recently, the classroom context has been verified. Cuendet, Dehler-Zufferey, Ortoleva and Dillenbourg (2015) have given an example of what they call the *orchestration* of collaborative behaviors between students using CSCL software and have shown the complexity of such an endeavor.

2.4.1 The Planning tool

Between the years 2010–2013, we participated in the EC-funded R&D project *Metafora* (257872). Metafora provides a CSCL platform that includes tools intended to encourage L2L2 in science and mathematics. One of the tools is the Planning tool. This virtual shared-space is designed to provide technological affordance for scaffolding small-group construction of plans and reflections upon students' learning process. This is accomplished by a creation of a constantly revised map with the use of a set of icons—Visual Language Cards—that employ a

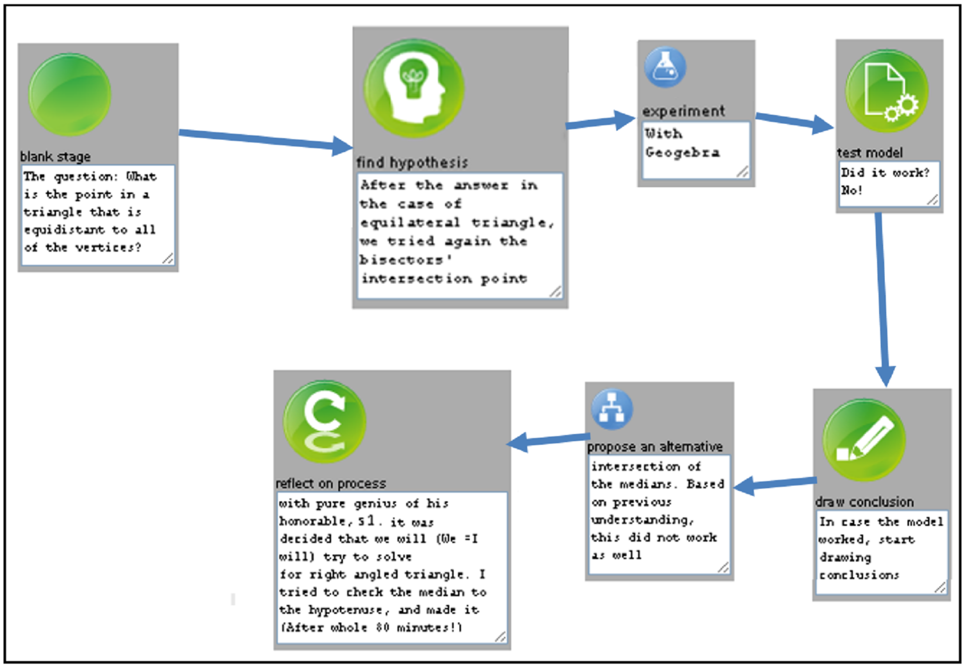


Fig. 1 A map, made by two students, in which they plan and reflect upon their collaborative MPS process

specific set of vocabulary (Abdu and Schwarz 2012; Schwarz, de Groot, Mavrikis & Dragon, 2015 in press). The cards are based on models of MPS (e.g. Pólya 1945). The Planning tool allows the plan to be written in the form of a map. However, Abdu & Schwarz (2012) show that the written plan does not necessarily represent the solvers' actual plan. Rather, it demonstrates the fact that the Planning tool affords metacognitive talk. The Planning tool affords (group) reflections that is constrained by the visual language to concepts that are relevant to the MPS process (Abdu and Schwarz 2012). It, thus, affords shared-metacognitive discourse (Iiskala, Vauras, Lehtinen, and Salonen, 2011) upon learning processes. Figure 1 displays a reflection upon and the planning of a solution process constructed by two 8th grade students. The text in the "test model" card, in Fig. 1, is noteworthy. The students in this example chose a card that symbolizes the refutation of a hypothesis. Thus, the Planning tool also affords reflections on non-productive paths to the solution in order to explicitly refute them (Abdu, 2015). This use yields a product that is intended to describe the MPS process, to allow the teacher to diagnose group learning in order to provide responsiveness to each group. Thus, the Planning tool has the potential to afford delegation of responsibility for learning to students (handover to independence), when they solve mathematical problems together.

2.4.2 Geogebra

Dynamic Geometry (DG) tools afford the construction of geometrical objects in a dynamic way. Inquiry in DG software can support the learning of concepts in Euclidean Geometry as well as mathematical proofs (Hadas, Hershkowitz & Schwarz, 2006). In this study, we use Geogebra because of its ubiquity. In addition to its DG qualifications, Geogebra also affords the creation of dynamic mathematical models. This function helps students to test their hypotheses in the problem-solving process (Abdu, 2015; Stahl 2009).

3 Research question

Combining advanced technologies (the Planning tool and Geogebra) and pedagogical objectives (fostering *MPS* and *L2L2*) into one learning environment is an ambitious enterprise. In this paper, we describe the difficulties and opportunities by combining these technologies and pedagogical objectives in a classroom context. Our research question is:

RQ Which practices are performed by teachers when they scaffold "Learning to solve mathematical problems together" in a whole-class, computer intensive, setting?

4 Context of the study: an R&D project based on a research design paradigm

This study is a part of a 3 years project, Metafora. Metafora is a joint effort of seven European institutes. Metafora, aimed to develop tools for Learning to Learn

Together through MPS and Scientific inquiry, and to determine to what extent small groups of students realize the expectations of the designers. One of these tools is the Planning tool described above. The methodological approach follows a design research paradigm (Collins, Joseph & Bielaczyk, 2004), where cycles of development punctuated by implementations lead to refinements in tools and in theory. Among the salient characteristics of this approach is that the methodology is not based on controlled experiments, but on descriptive studies that attempt to illustrate how particular learning goals can be achieved. Also, the researchers are often involved as participants. In the present study, one of the teachers is a researcher (the first author of the present paper). We will later elaborate on possible weaknesses of this decision, and the ways we addressed them.

The theory and practices of WCS for group learning of MPS are developed during the Metafora project through the following cycle of experiments (Metafora team, 2013): First, we conducted two small-scale seminars with students and one with teachers. These seminars were led by experienced teachers (Abdu and Schwarz 2012). A course for training pre-service teachers in the practices of teaching was conducted and moderated by the second author (Schwarz, de Groot, Mavrikis & Dragon, 2015 in press). Finally, two iterations of a full year course for junior-high school students were developed. The present study is based on these two iterations.

The First iteration of the course was held in the 2011–2012 school year, in an orthodox-Jewish school for boys in Israel. Sixteen 13–14 years-old male students took part in this course. The students were chosen based on their high grades in math, and on their willingness to participate in the course. The class met in weekly meetings of 90 min long during school hours. The focus of this iteration was on MPS, with minor attention to L2L2. The teacher who taught this course has taught mathematics for 10 years. The Second iteration of the course was held in the 2012–2013 school year. Twenty-four 14–15 years-old students took part in this course. Similarly, the students were chosen based on their high grades in mathematics and on their willingness to participate in this course. Students met in weekly meetings of 90 min long, after their regular school day. In addition to the focus on promoting MPS, teaching was directed to explicitly encourage L2L2 behaviors in order to solve mathematical problems. The teacher (TR), the first author of this paper, has taught mathematics for 3 years at the time of the study. This was the first time he taught MPS.

In both courses, all meetings took place in computer rooms equipped with both the Planning tool and Geogebra. The students had constant access to these technologies. The teacher prompted students to use these tools based on their needs. In order to promote discourse on MPS strategies that students employed (for the sake of simplicity, we consider MPS heuristics as MPS strategies), the teacher utilized four kinds of dialogical-learning interactions (Alexander 2008; Palinscar and Brown 1984; Webb 2009; Wegerif 2006). First, after the teacher presented the problem, he offered MPS strategies that students should apply or notice. Second, the teacher led a whole class discussion at the end of every lesson, during which the students reflected upon the emergence of MPS strategies suggested in the beginning of the lesson in their own solution process. Third, the teacher initiated small group discussions that focused on MPS strategies. Finally, the teacher instigated unguided small-group discussions while students worked on the problems. An indirect way to instigate such discussions was to ask students to use the Planning tool.

4.1 Research methodology

Our methodology is a mix of design-based research (Collins, Joseph and Bielaczyc 2004; Design-Based Research Collective 2003) and action research (Cohen and Manion 1980, Feldman and Minstrell 2000), as we attempt to be both situational and transformational, as well as to be informed by theory and to inform theory. Teachers contribute in the development of the learning environment iteratively by reflecting upon the design after each of these iterations (Feldman and Minstrell, 2000). The aim is for teachers to progress to the "best practice" through reflection on their own teachings (e.g. Rowland, Turner and Anne 2014). Accordingly, TR (the first author) first worked as a design researcher in the dedicated-seminars and the first iteration of the course, and reflected upon these iterations (Abdu, De-Groot and Drachman 2012; Abdu and Schwarz 2012; Abdu 2013, 2015). Then, based on what he learned from these reflections, he conducted a course to elaborate on practices he found effective for facilitating MPS in small groups. As mentioned earlier, TR decided to incorporate explicit talk about L2L2 into his teachings.

4.2 Data collection and validation

Both courses were video-recorded. In both courses, we focused on the teacher practices during whole-class discussions and on one to two groups of students when they learned together. Geogebra files and Planning tool maps were collected from every group after each class. In both courses, a researcher was present in the room and took notes. The teacher in the second iteration (TR) also watched the recorded videos and reflected upon the lessons on the days the lessons were conducted.

In congruence with Derry, Pea, Barron, Engel, Erickson, et al., (2010), we analyzed recorded episodes in which teachers performed WCS practices with particular interest in their contribution to the theory and practice of learning in order for students to solve mathematical problems in small groups. We exemplify these practices in actions taken by TR (the teacher in the second course, and the first author) and some actions taken by TZ (the teacher in the first course) in this paper.

Following Feldman and Minstrell's (2000) work on tools for overcoming possible shortcomings in the validity of action research, we validated our interpretations of the selected episodes through triangulation of information from various sources—recordings of the lesson, researcher's notes, and teachers' reflections. These episodes were analyzed accordingly by the third author and an additional researcher who took part in the design research in accordance with the WCS theory.

5 Results

In this section, we provide representative examples of teacher practices that are based on WCS. Among them, we notice three practices pertaining to MPS: *Choosing problems to exemplify strategies, Decomposing a problem into stages,* and *Modelling the use of the educational technologies*. In addition, we describe a set of practices meant to facilitate Learning to Learn Together (L2L2): *Preliminary L2L2 talk, Routine L2L2 talk,* and *Summative L2L2 talk*.

5.1 Scaffolding mathematical problem solving

5.1.1 Choosing problems to exemplify strategies

In the first iteration of the course, the design-research team decided that in order for the students to be able to solve complex problems in late stages of the courses, *degrees of freedom* should be increased gradually. For this reason, early stages of both courses were dedicated to focus on one MPS strategy/heuristics per meeting. We observed three main types of scaffolding practices in the course.

The first practice, and also the most common one, was to present 1–2 MPS strategies at the beginning of the lesson, right after the teacher presented the problem to solve (and as we will see later in this paper, the same approach was adopted by TR in the case of L2L2 behaviors). A detailed example for such a scaffolding practice is given in Sects. 5.1.2 and 5.1.3.

The second practice is one that was enacted mainly towards the end of the year, when students were familiar with several MPS strategies. Groups of students received a set of short problems, but this time in addition to solving problems, the students were also asked to state a particular strategy they used in order to solve these problems. TZ applied this practice and asked students to choose a known strategy suitable for the problem in the two meetings that preceded the last problem of the course: the Gardener problem. TZ explained that the objective of this practice was to review the MPS strategies learned throughout the year and to choose a suitable strategy among others to solve the Gardnener problem.

In the third practice, the teacher showed students a set of strategy-oriented problems that could be effectively solved with one specific strategy in order to bring forth the strategy. For example, in the 13th lesson of the second course, TR chose five problems that could be effectively solved with the heuristic "Working backwards" (Pólya 1945). TR did not tell students the intended MPS heuristic to be learned. Instead, students tried to solve the problems, first with limited success but when the strategy was revealed by one group of students, TR allowed the information about the success of the heuristic to circulate in the class. As a result, students immediately tried to apply this new and successful strategy to other problems given to them, entailing by such immediate attempts to transfer the application of one successful strategy to other problems. This method proved to be effective, as groups generally were able to solve 3–4 of such problems in a row in that lesson.

In addition to their functions in scaffolding a solution for a given problem, these practices could also be seen as scaffolding students' learning in the course level. The ultimate goal is to hand-over the ability to solve mathematical problems together in small groups over the course. The iteration of these practices allowed students to solve problems with increasing levels of difficulty.

5.1.2 Decomposing a problem into stages as a scaffolding practice

We now present a problem proposed at an early stage of the course (lessons 6–7)—*The drowning man*. We then show how TR, the teacher in the second iteration, prepared himself to support students to reach the solution of this problem. Four MPS strategies were planned to be used in the lesson in the following order: Understand the question, Define variables, Build a mathematical model, and Build a model with Geogebra. The first and last strategies have already been enacted in former problems. The second and third strategies have not been enacted yet. The formulation of *The drowning man* problem was:

> *Chaim is drowning. Elifelet is on the beach. He is a lifeguard. What is the fastest way for Elifelet to reach Chaim if he needs to run to the water and then swim to Chaim?* […] *Build a Geogebra model with which*

you will be able to assess an appropriate entrance point [A] for the general case in which (1) the placement of Chaim and Elifelet could change, and (2) Elifelet's Swim and Run speeds could also change.

In order to scaffold the solution to this problem, TR devised an epistemological analysis of the problem (Dreyfus, Hershkowitz and Schwarz, 2015) which included five steps, as was recorded (and translated) in his notes: (1) *Understand the problem*: Elifelet needs to run on the beach and then swim in the sea, and therefore the velocities are probably different in each zone. The total time needs to be minimal since Haim is drowning. (2) *Define variables:* define the velocities [Vr for running] and [Vs for swimming]. Define the distances [a for swimming] and [b for running]. (3) *Build a mathematical model* for the total time it takes for Elifelet to reach Haim. Running time in the land is $Tr = b/Vr$, and swimming time is $Ts = a/Vs$. Thus, the total time is $T = b/Vr + a/Vs$. (4) *Build a model in Geogebra:* according to this mathematical model, and the drawing (Fig. 2), "Elifelet" and "Haim" are fixed points, "b" is the distance between Elifelet to the edge of the water (A), and "a" is the distance from that point to Haim. Vr and Vs are defined as independent variables, and could be any value from 1 to 30 (Km/H). (5) *Solve for a given state:* once a Geogebra model is constructed, the students are able to investigate different cases of the problem and give a qualitative estimation—where should Elifelet enter the sea.

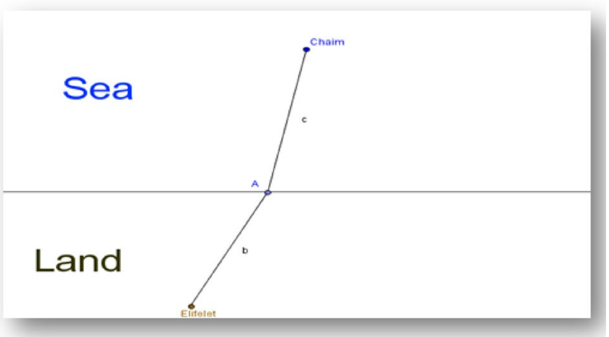


Fig. 2 A Geogebra model of the drowning man problem

This epistemological analysis allowed TR to scaffold his students' solution process by decomposing the solution path into segments, offline. Geogebra was expected to serve as a scaffold for the students' creation of a model that would help them in solving the problem.

5.1.3 *Modelling the use of the planning tool and Geogebra*

In early stages of the project, we realized that students encountered difficulties when they wanted to use educational technologies. In order to palliate these problems in various occasions, teachers chose to model the use of these educational technologies by utilizing: (1) the vocabulary of the visual language in the Planning tool, and (2) basic functionalities of Geogebra. The Drowning-man problem was solved in lessons 6–7. Lesson 6 was the first encounter with the Planning tool and the second encounter with Geogebra. After that lesson, TR watched the recordings and studied the Planning tool maps and Geogebra models created by the students. TR noticed a gap between what he observed in the recordings and the quality of students' Planning tool maps and Geogebra models. In other words, the maps did not reflect entirely students' problem solving process. Therefore, TR started lesson 7 with a whole-class discussion in which he *modeled* the use of the Planning Tool and Geogebra, and reflected upon students' solution processes. He used a projector in order to present a Planning tool map based on the maps of several groups and the epistemological analysis we presented above. The following episode clarifies this blend:

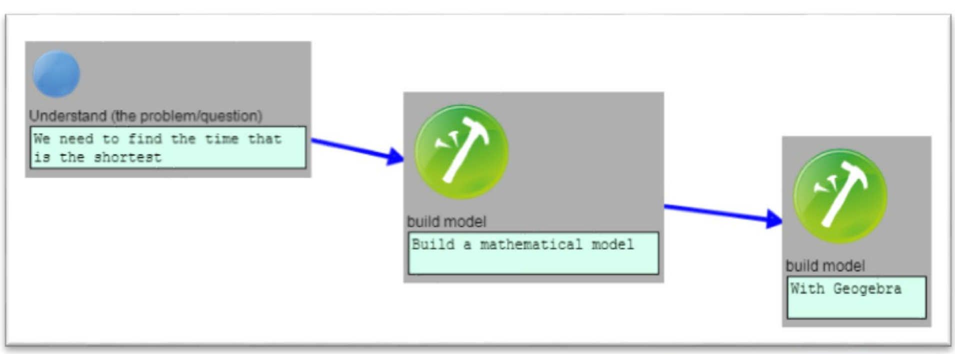


Fig. 3 A planning tool map, created by TR in order to model reflection on the problem solving process

Episode 1

TR puts the *Understand (the problem/question)* card on the Planning tool (Figure 3). After a short discussion about the meaning of the problem, he explicitly states the problem: *We needed to find the time that is shortest*. He then builds a mathematical model for the solution, on the whiteboard, based on the mathematical models that were created by the students (*T=b/Vr+a/Vs*) in the last lesson. He then adds the first *build a model* card to the Planning tool and writes *Build a mathematical model* in it. Then, TR takes this mathematical model and shows how he creates a Geogebra model that fits with the mathematical model [see Step 4 in the epistemological analysis]. Finally, TR adds a third card to the Planning tool – another *Build a model* card, and writes into it *With Geogebra*. Figure 3 displays the final map. After TR finishes the map, he concludes: *…This is all that we did…We are now aware of three strategies we used on our way to the solution. One, we understood the problem, two, we created a mathematical model, three, we created a model in Geogebra.*

There is an obvious difference between the ways Geogebra and the Planning tool are presented by TR. Geogebra is a piece of software that creates dynamic models of geometric and algebraic phenomena; the students were familiar with mathematical concepts that were represented by Geogebra, such as points and variables. As for the Planning tool, while the mere use of this tool is simple, we learned in earlier phases of the research that it requires higher-order levels of thinking in terms of the awareness of strategies used and their temporal deployment in the process (Hamilton, Lesh, Lester & Yoon, 2007). Consequently, TR undertakes metacognitive reflections on the learning processes through *modelling*, emphasizing three principles: First, cards appear in a chronological manner. Second, placing the cards is not enough, there is a need to include a contextual meaning (e.g., understand the problem: *We need to find the shortest time*). Third, the card "Build a model" appears twice, and they are interpreted differently (Build a mathematical model/With Geogebra), implying that the cards could appear more than once in the Planning tool maps.

In addition to the fact that this example was closely related to the students' solution, this modelling process also served as a reminder of "what happened last week". Thus, TR's act of *offline diagnosis* of students' solutions led to *online responsiveness*, which was manifested as *online modelling* at the problem level.

5.2 Learning to learn together

In early phases of our study, we came to realize that merely putting students together in small groups to solve mathematical problems does not necessarily guarantee any productive interaction. Conversely, we observed several group-learning behaviors that enhanced MPS, such as *Peer Scaffolding*—a form of leadership move in which a student progressing in the solution process scaffolds his peer's solution (Abdu, 2013), and peer scaffolding is decisive for further group MPS. For these reasons, TR decided to integrate explicit prompting of L2L2 ground rules in the second course. These rules were applied and discussed in three forms of whole-class talk: Preliminary L2L2 talk, Routine L2L2 talk, and summative L2L2 talk. TR adopted a dialogic approach in which students had the opportunity to talk about their behaviors within the context of group learning, and to target their group MPS behaviors by following these L2L2 rules.

5.2.1 *Preliminary L2L2 talk*

The preliminary L2L2 talk was held during the sixth lesson, just before solving the Drowning-man problem. TR wrote on the whiteboard a set of nine rules of talk, including some reasonable ones and some that sounded foolish (e.g., *Let the oldest start*). The class talked about them and decided which rules to accept and which to reject.

TR adopted a dialogic approach and elicited explicit discourse upon group learning behaviors. By doing so, the students became aware of L2L2 practices, and discussed them in order to decide whether these rules should be incorporated into their learning. From a WCS perspective, such a practice also serves as online monitoring of students' naïve opinions about L2L2.

5.2.2 Routine L2L2 talks

TR chose to integrate talks about desired L2L2 behaviors into the lessons as routine practices in order to instill L2L2 norms. Routine L2L2 talks were enacted in each lesson. At the beginning of the lesson, TR introduced a L2L2 rule and asked students to pay attention to it. At the final plenary discussion, TR asked students to reflect upon their experience with this L2L2 rule during their group work.

The following excerpt is taken from lesson 7. TR presented the rule, *Encourage other group members to participate*, in the beginning of the lesson:

> TR: *"Collaboration. In the previous lesson we talked* [...] *about all kind of rules that are good for collaboration, for efficient learning in groups* [...] *throughout this year we will focus on this issue of collaboration. One of the things that you mentioned, here, it is written [reads from his papers]: To encourage other group members to participate. So, in terms of collaboration, this is another focus that I would like you to have... [Writes on the board]: To encourage other group members to participate"*

TR asked the students to pay attention to this *peer-diagnostic* behavior while they solved the problem together, and by doing so, he promoted the *handover to independence* process. The rule remained written on the whiteboard, along with the relevant MPS strategies. Before the end of this lesson, TR gathered the class again, and asked them to talk about their experience with that particular L2L2 rule. In the following excerpt, TR asked his students to reflect upon a situation in which someone encouraged his/her friend to participate in the solution to the problem. Shir and Tamar were two girls from the same group who decided to talk about their L2L2 experience.

Episode 2

TR:	"*I would like to know if there was a situation ... In which someone had to encourage his team members* [to participate]...?"
Tamar:	"*Shir wasn't paying attention ... She was all on another computer ... And I told her to join in* "
[Applause in the class]	
TR:	"*What happened ...what interests me here with Shir is, what happened after that? What happened ...?*"
Shir:	[points to Tamar] "*She forced me to learn...*"
TR:	"*And what happened then? Did this intervention help or... you were still not paying attention? ... It's okay...*"
Shir:	"*kind of, uh, floating ...*"
TR:	"*floated*"
Tamar:	"*But you did help us*"
TR:	"*Well, Let me ask, was there an improvement?*"
Shir:	"*Yes*"

In this episode, a dialogue facilitated by the teacher helped Tamar to talk about how Shir did not contribute to the group solution. She encouraged Shir to participate. Shir stated that nothing really changed after that elicitation, but Tamar used a more positive approach and stated that Shir did help the group later on. At this point, TR stopped encouraging the dialogue and ended the discussion with a closed question: Was there an improvement? Tamar answered "yes". Thus, she felt that the fact that she encouraged Shir to participate resulted in a positive outcome.

This dialogic approach allowed TR to provide online diagnosis for student's reflection upon real situations in which L2L2 rules were applied. This led to further online responsiveness, in which TR talked to his students about the dynamics between his role, and their role in the learning environment. TR then concluded the discussion above:

TR: "*I want to summarize this point…in the following way. I'm your teacher ... one of my most important roles are, when I see ... a student that does not participates, my job is to encourage her, okay? Now if we have the whole class, like this* [Plenary] *it is very easy for me. I see someone which does not participate, and I respond. But when you all work in small groups ... I cannot see what is happening in each group. Therefore, this responsibility, falls on the group members…It is also my responsibility, in general, and I can help, but this responsibility goes to the group. Because the group members should push the group forward.*"

TR explicitly talked about a behavior that was applied at the problem level—Encourage group members to participate—and expanded the handover-of-independence as the responsibility for learning to the course level: When learning together, students have responsibility for their own learning and the learning of the whole group, which should be manifested by encouraging other group members to participate when they don't.

5.2.3 Summative L2L2 talk

In order to diagnose the change in students' approach to L2L2 throughout the course, TR conducted an additional summative L2L2 talk in lesson 17. In this discussion, he brought the same nine rules from the first talk, and asked the students again: Which terms do you accept, which do you reject?

Episode 3

We discuss an episode in which the class talked about the "foolish" L2L2 rule—*the oldest one should start*. At first, the students concluded that they reject this rule, as they did in the preliminary L2L2 talk. But TR adopted a dialogic stance and asked them one more open question: How do you decide who is first to talk? His assumption was that the students would agree that there should be is no such rule. As we can see in the following excerpt, this was the case at first:

TR:	…So, how you decide who is the first to talk in a group? Is it a decision? Is it...
Ofer:	There is no such thing.
Dovev:	It is spontaneous.
Shani:	Someone has an idea and suddenly he starts talking.
Edna:	…to get things started, so obviously he is the one who starts the talking.
TR:	The first person that gets things started.
Edna:	Yes.

However, as TR attempted to sum up with his pre-assumed conclusion, Gad challenged this conclusion:

TR:	[Gets up and goes to the white board] Ok, so we accept it.
Gad:	I do not accept this. I say that the one that is the most unsure should be the one talking.
TR:	Why?
Gad:	I will tell you.
TR:	Explain yourself.
Gad:	Most of the times, the one that is sure of himself is right. The one that is not sure of himself, will probably not be right.
TR:	OK…
Tamir:	Then he will have the opportunity…
Gad:	And then he will have the opportunity to learn from his mistake. Because … If we will let the right one to speak immediately, first, he will be right. And then no one will learn from nothing.

Gad challenged the class' (and teacher's) conclusion by stating what constructivists already know: In order to learn, a student needs to build meaning on his own; and this is impossible when someone else simply provides the student with the correct answer. Teachers deal with this phenomenon (e.g. Dieszmann and Watters, 2000) by first letting nonproductive paths to the solution to be refuted, and then allowing a productive path to be presented. The class, however, did not conform to Gad's idea.

Shlomi:	Not necessarily.
Shani:	Who said he [the student that starts talking first] is right?
Shlomi:	Exactly. He can be wrong.
Ofer:	All philosophical.
Shani:	The fact that he is first, does not mean that…
Gad:	But he is sure of himself.
TR:	So, the one that is most unsure of himself should start talking? Because he will probably make a mistake and it is good that he will learn from his mistake.
Dovev:	How is this related?
Ofer:	No.
TR:	This is what Gad says.
Dovev:	So, he is wrong.
Gad:	My idea is wrong, not me!
Edna:	Why should the one that is the most insecure, like to start talking?

Gad's idea was assessed by his peers. As he defended his ideas, Gad implied that his friends should apply a L2L2 rule of talk—*Criticize ideas and not people*. As the discussion went on, Ofer initiated a Shared-metacognitive discussion about Gad's ideas—Gad was the first to talk, and his idea was wrong. Thus, the first who talked (Gad) was wrong, but we learned something from it. This is not clear for everyone, and TR asked Ofer to state his idea again. This episode ended with a rather humoristic discussion about this paradox:

Ofer:	I would like to mention something that even he [Gad] did not notice.
TR:	Which is?
Ofer:	He said that the first to talk is the one that is not sure [and we agreed that] he was wrong.
Gad:	I did not understand what was my mistake?
TR:	Did you hear what Ofir said?
Gad:	[Smiles] It was all planned…
TR:	It was really beautiful…
Daniel:	What did he say?
Edna:	[Smiles] He was the first…
TR:	Ofir says that [Laughs] can you repeat what you have just said Ofir?
Ofer:	Gad said that we should not let the first student to be one that is right [i.e. the surest]. There should be a mistake so everyone will learn. This is what he did. He was the first and he was wrong in what he said.
TR:	But wait a minute, if he was wrong…
Ofer:	This means that his idea was wrong and there is no conclusion!
TR:	But wait a minute, if he was wrong in what he said, why did we accept now what he just said? You accept what he says [Smiles]
Dovev:	This is a paradox
Ofer:	Here, I was wrong
TR:	[To Ofer] So, you are right…

Two additional L2L2 behaviors could be observed here. First, we could see a student's (Gad) online diagnosis of a L2L2 rule (Criticize ideas, not people), which led to a response ("*My idea is wrong, not me!*"). The second L2L2 rule that could be observed here is students' emerging *reflection on current talk*. This summative L2L2 talk allows us to see where students stand now, in terms of their responsibility for the learning of their peers, and in their ability to online monitor, and reflect upon, L2L2 behaviors.

5.3 Preliminary indications for the effectiveness of the course

Learning interactions observed during the full *Metafora* project are reported elsewhere (e.g., Abdu and Schwarz 2012; Abdu 2013, 2015; Kynigos and Moustaki, 2016). Since the topic of the current paper is teacher practices, we do not focus on the effectiveness of the learning experience from the students' perspective. In this paper, we describe some L2L2 affordances, and have not provided evidence on how the aforementioned practices affect MPS. But we feel that we cannot provide readers with a full picture without some indications of the effectiveness of this course. We bring here two examples of evidence that suggest the congruency between iterated enactment of L2L2 behaviors and the development MPS skills.

5.3.1 Emergence of MPS strategies and L2L2 behaviors

In lesson 9, Noga and Shelly, two of TR's students, solved the *five lines* problem: what is the maximum number of regions created by five straight lines in a rectangular surface? In order to answer this problem, the students were encouraged to apply the MPS strategy *Check simpler case* and the L2L2 behavior *Listen to your friend's ideas.*

Since early stages of the course, Shelly was usually the one who led the discussion and made leadership moves. Noga was quite introverted; she usually followed Shelly's lead. In the first stages of the problem, Noga and Shelly used Geogebra, which allowed them to constantly check their hypotheses. After they came up with a solution—five lines create sixteen surfaces—they reflected upon their solution process with the Planning tool:

Shelly:	[Browsing between cards] *First…We understood the problem.*
Noga	[Looks at what Shelly does]
Shelly:	[Drags "Understand the problem" card to the board] *We understood the problem* [Starts typing and then refers to Noga and passes the keyboard to her] *You should write.*
Noga takes the keyboard and types. Shelly still holds the computer mouse	
Shelly:	[describes what they did thus far] *We understood what should be done, in the problem, dash. We should build as many areas as possible out of five lines …. After that, we did trial and error* [Looks for a trial and error card] *…Without trying to think of a strategy.*
Noga:	*How do you write "Strategy"?*
Shelly Takes the keyboard and write the word "strategy" while Noga looks at the class board, on which the instructions for the lesson are given.	
Noga:	*Let's try "simpler case"*

The *Trial and error* card allowed Noga and Shelly to acknowledge the fact that although they came up with the number sixteen, they were not systematic and had no strategy. We see here the first signs of a change in mathematical norms: *doing math* is about systematic, intelligible, inquiry, rather than merely 'giving a correct answer' (Schoenfeld, 2007). From a L2L2 point of view we see here two behaviors that helped in shifting the responsibility for the learning from Shelly, the dominant group member, to her friend Noga. First, Shelly *encouraged her peer to participate* when she gave her the keyboard, and then she *listened to Noga's idea* to *check a simpler case*.

As a result of this elicitation, Shelly opened up Geogebra. From this point on, they inspected simpler cases of the problem, starting with one line. They found that one line creates two regions, two lines create four regions, and so on. Several minutes later they came up with the conclusion that in order to obtain the maximum number of regions with five lines, all five lines should cross each other. In such a case the correct answer is again, sixteen. This systematic inquiry was also demonstrated when the two students insisted to understand a rather fundamental axiom: if two straight lines met once, they will never meet again. This was important for them, because if these lines ever meet again, they will create an additional region. Only then the students drew conclusions and reported them in the Planning tool. As we could see in Fig. 4, their final map is closely related to their MPS process. The way they used the Planning tool was closely related to TR's modelling—they organized the cards chronologically and elaborated on their specific solution process in every card.

5.3.2 Collaboration, Cooperation and mathematical problem solving

Another aspect of the relationship between L2L2 and MPS could be discerned through the changes in five groups of students underwent during TR's course in solving a well-known mathematical problem—The Billiard problem (e.g. Cifarelli & Cai, 2005). These five groups solved this problem before (pretest) and after (posttest) the course. Figure 5 displays the problem presented to the students.

During the course, we distinguished between two types of group-learning behaviors: *Collaborative* learning is learning that is done together, with joint-attention towards the same artefact or cognition; *Cooperative* learning is an act in which labor is distributed, and individuals work on sub-tasks (Dillenbourg, 1999). In addition, we measured MPS in terms of claims [e.g. the number of hits at the sides of the table is zero if the Billiard table is a square], arguments [e.g. the previous claim is backed with data—examples of square billiard tables], and shared-valid arguments—mathematically valid arguments that were explicated or referred to by all group members. We asked the following questions: What are the changes in MPS and L2L2, from the pretest to the post test? What are the relationships between alternating cooperation, collaboration, and the emergence of MPS competence?

Results show a statistically significant increase in the number of claims and arguments produced by group members after the course. In addition, students significantly more cooperation in the pretest. When we analyze all pretests and posttests (n = 10), we find correlations between the percentage of cooperation episodes out of the total episodes, and the number of claims (r = .5), arguments (r = .5) and shared-valid arguments (r = .04) produced. Thus, students were more inclined to produce claims when they applied some cooperation—individual MPS within group—into their solution process.

An increase in the amount of shared-valid arguments, however, was observed for all groups except for one group.

Fig. 4 Reflection upon the solution to the five lines problem, created by Shelly and Noga

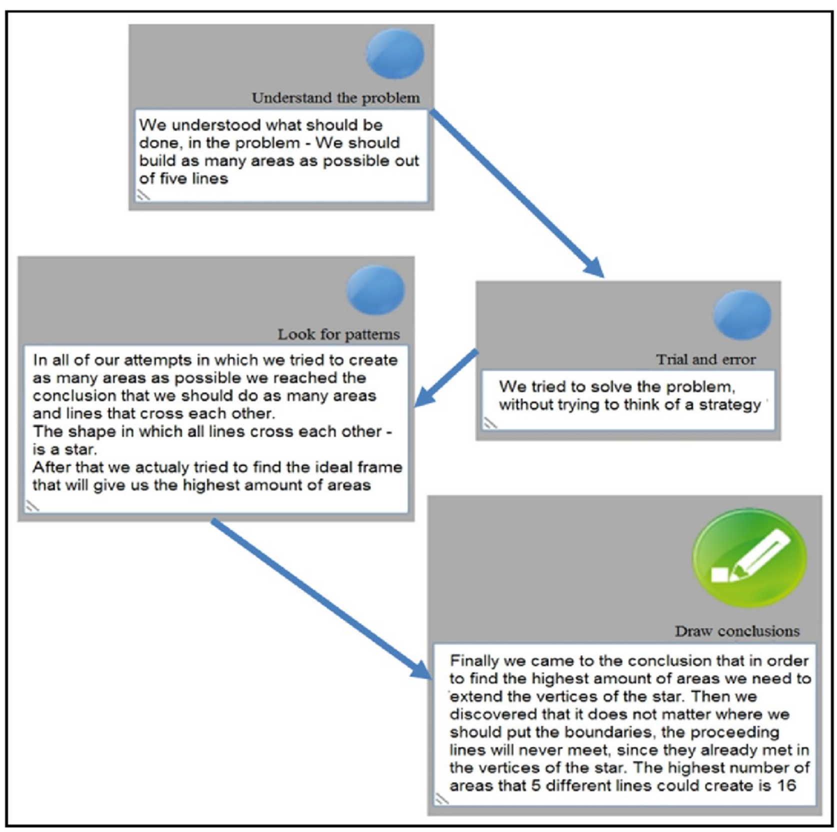
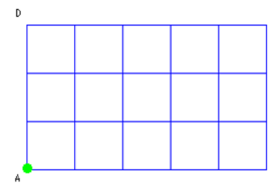


Fig. 5 The Billiard problem

The Billiard game

The billiard game in this problem is played in rectangle tables, as illustrated in the following drawing.

An imaginary ball is hit from the bottom left (pocket A) at the angle of 45°. The ball also bounces from the sides of the table, at the angle of 45°. The ball continues to roll until it drops into a pocket. There are pockets in every corner (A, B, C and D).



1. *Try the game on the illustrated 5X3 table. In which pocket will the ball stop? How many times (not including the start and end points) will it hit the sides of the table until it drops into one of the pockets?*
2. *Select tables in other sizes, as many as you like, and answer section 1 for them.*
3. *What should be the size of the billiard table, if the ball enters one of the holes after eight hits (not including start and end point)? Is there more than one solution? What are they?*
4. *Explain, then, what are the relationships between the dimensions of the table and the amount of hits required for the ball to drop into one of the holes? How did you come to this conclusion?*
5. *If you came up with such a relationship, why does it work this way?*

Table 1 Ground rules brought into the preliminary L2L2 talk

Accepted	Rejected
Encourage other group members to participate	There should be no jokes
Listen to others' ideas	Always try to reach an agreement
Criticize ideas, {in a constructive way} not people	If a wrong decision is made someone should take the blame
Change your mind if you hear a good reason	Let the oldest start
Give reasons for one's ideas	

Accepted rules are listed on the left hand column and rejected ones on the right hand column

In the posttest, 55 % of the episodes for this group were cooperative and they were the only team that showed a *decrease* in shared-valid arguments. For the sake of argument, if we remove the results of this particular posttest from our data set (n = 9), we find very high correlations between the percentage of cooperation episodes out of the total episodes, and the number of claims (r = .89), arguments (r = .86), and shared-valid arguments (r = .92) produced. Thus, in addition to our conclusion about the effectiveness of the course in terms of MPS, we learn about the relationship between group learning behaviors and MPS: More cooperation in group learning is advisable as long as it is not "too much" in a way that impedes coordination among group members.

6 Discussion

To answer our research question, *what practices do teachers perform when they scaffold learning to solve mathematical problems together in a whole class, computer intensive, setting?,* we observed and analyzed teacher practices that aim to scaffold group learning of mathematical problem solving, with the aid of educational technologies that support shared-metacognitive talk (the Planning tool) and building mathematical models (Geogebra). We analyzed these practices in terms of the WCS theoretical framework, in particular with relation to the attempts to *hand-over the responsibility* for solving mathematical problems together to the students by conducting whole-class explicit talks about MPS strategies and L2L2 behaviors. The results of complementary analysis show clear evidence of the effectiveness of this course. These results will be fully described in later publications.

In congruence with Smit, van Eerde and Bakker (2013), WCS was *distributed over time* in both MPS strategies and L2L2 behaviors. L2L2 behaviors *cumulated*, as manifested by the students' responses to TR's dialogic prompts, and encouraged delegation of the responsibility to solve mathematical problems (handover to independence), as manifested in the summative L2L2 talk. In that sense, the mini-culture for learning to solve mathematical problems together does not resemble what Cobb, Stephan, McClain and Gravemeijer, (2001) see as an inexorable appropriation of norms and skills orchestrated by the teacher. We report here a rather dialectical process in which the norms and skills to be appropriated were negotiated by all participants who behaved as responsible actors.

L2L2 rules are meant to advance solutions to mathematical problems, in which the roles of the group members are balanced. For example, the rule *the one that is the most unsure should start talking* suggested by Gad in the summative L2L2 talk, is closely related to the need for cooperation in mathematical problem solving situations. Gad's underlying assumptions is that group members sometimes have different ideas with regard to the solution of a problem, and that all of the ideas should be heard—even if one of the students is sure that he has a productive idea. Studies that investigate group-interaction show that when individuals are given time to generate ideas before the joint solution (cooperate) and are given the chance to present these ideas in order to receive feedback (collaborate), better ideas emerge (Wit, 2006). Wit explains that when working collaboratively, people tend to conform to ideas of the group. He concludes that when time is given to get acquainted with a problem, generate new ideas and reflect on them, it is more likely for an individual to explicate his ideas in front of the group since he is more capable of defending them. Thus, the ability to incorporate individual meaning making while solving mathematical problems in small groups is a critical means to develop an individual's ability to solve mathematical problems. This is particularly true in the case of groups that are unbalanced, as was stated by Gad.

Reflecting back to Smit, van Eerde and Bakker (2013)'s WCS model, we find it useful to add an additional classification in WCS: (1) at the *course* level, learning to solve mathematical problems together includes actions such as modelling the use of educational technologies, applying a gradual responsiveness in terms of the number of strategies and heuristics used, and eliciting L2L2 talks. (2) At the *problem* level, the aim to solve a particular problem leads to practices such as decomposing a problem into segments, offline diagnosis of the students' progress in the problem with an appropriate responsiveness, and a dialogic approach that affords online and offline diagnosis and responsiveness for L2L2 behaviors.

Finally, we would like to discuss the particular role of *Modelling* in a WCS setting. Our findings show that this is not a straightforward quest. Data in this study show that there is a difference between modelling MPS strategies and L2L2 behaviors. Whereas MPS strategies could be demonstrated by one person and could be exemplified in a specific context of a particular problem, L2L2 behaviors could usually be demonstrated in the context of a dialogue with peers. From the perspective of a single student, social interactions such as L2L2 behaviors could not really be planned in advance, since they emerge as a result of interactions between peers. Social interactions, then, could only be monitored as they occur and/or reflected upon. For these reasons, modelling L2L2 behaviors in WCS might be a challenge for teachers decided which rules accepted and which to reject (see Table 1).

7 Conclusions

Fostering inquiry-driven practices in mathematics is not an easy endeavor, and many teachers fail in understanding and in conveying to their students the explorative nature of *doing mathematics* (Schoenfeld 2007). This paper presents an attempt to palliate this shortcoming. Three design decisions were taken. First, we arranged students in small groups. Second, we developed a novel learning environment to facilitate Learning to Learn Together in order to Solve Mathematical Problems. Third, we integrated technologies that were aimed to facilitate such learning. Attaining the goal of Learning to Solve Mathematical Problems was not separated from the goal of Learning to Learn Together.

We observed teachers' practices as they enacted scaffolding practices in the context of the whole class, according to the combination of (1) the mathematical dimension that fosters metacognitive skills to use strategies and (2) the social dimension that encourages group learning. We show here that this combination was possible, and that L2L2 positively influenced student's ability to solve mathematical problems together.

In spite of the specific learning environment under which this experiment was conducted, it provides evidence that designers should be encouraged to create meaningful learning environments and educators should be encouraged to use them in order to scaffold L2L2 behaviors in service of learning to solve mathematical problems. This paper suggests that these two goals are attainable when they are combined in a suitable environment.

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