ZDM Mathematics Education (2015) 47:519–529
DOI 10.1007/s11858-015-0700-9

COMMENTARY PAPER

Geometry in the early years: a commentary

Jaguthsing Dindyal[1]

Accepted: 26 May 2015 / Published online: 12 June 2015

Abstract The primary goal of this paper is to provide a commentary on the teaching and learning of geometry in the early years of schooling with the set of papers in this issue as a guiding factor. It is structured around issues about geometry education of young learners, such as: what should we teach in geometry and why; representation of geometrical ideas; the teaching and learning of geometry; and assessment of children's learning in geometry. The author outlines his views based on the literature and the papers in this issue and concludes with an outlook on the future teaching and learning of geometry in schools.

Keywords Geometry · Spatial reasoning · Early childhood

1 Introduction

In what follows, I shall focus on the content of the papers that make up this issue of the journal and try to address some issues pertaining to teaching and learning of geometry in the early years of schooling. There are nine papers based on empirical research from both sides of the Atlantic that involve young children. These nine papers cover various aspects of geometry and/or spatial reasoning (see Table 1): symmetry (Ng and Sinclair 2015), drawings (Thom and McGarvey 2015; Kostopolous, Cordy and Langemeyer 2015), 2D and 3D mental rotation (Bruce and Hawes 2015), definition (Bartolini-Bussi and Baccaglini-Frank 2015), imaginary perspective taking (van den Heuvel-Panhuizen, Elia, and Robitzsch 2015), understanding of and reasoning about properties of figures (Kaur 2015), spatial reasoning (Hallowell, Okamoto, Romo, and La Joy 2015), and spatial and geometrical knowledge in problem solving (Soury-Lavergne and Maschietto 2015). Four of the papers not shown in Table 1 include two papers on Professional Development/teacher education (Tsamir, Tirosh, Levenson, Barkai, and Tabach 2015; Moss, Hawes, Naqvi and Caswell 2015) and two other theoretical papers that includes one on geometry education (Sinclair and Bruce 2015), and another on geometric reasoning (Mamolo, Ruttenberg-Rozen and Whiteley 2015). Some of the papers fall in overlapping categories as they deal with several aspects of geometry and spatial reasoning.

The research reported in the papers shown in Table 1 is mostly qualitative and four of them have used a teaching experiment. The children referred to in these studies are mostly between 4 and 8 years old, except for a few children about 10 years old reported in the study by Kostopolous et al. (2015, this issue). Just a note here that early childhood refers to a child's life between birth and 8 years (Organisation for Economic Cooperation and Development [OECD], 2001). In this paper besides the term early childhood, the term *early years* or simply *children* or *young children* will also be used.

Several issues are raised in the research reported in the papers shown in Table 1. I shall focus more specifically on the following: what should we teach in geometry and why; representation of geometrical ideas; the teaching and learning of geometry; and assessment of children's learning in geometry.

✉ Jaguthsing Dindyal
jaguthsing.dindyal@nie.edu.sg

1 Nanyang Technological University, Singapore, Singapore

Table 1 Empirical research papers involving young children in this issue

References	Context and participants	Study
Soury-Lavergne and Maschietto (2015)	Italy 7-year olds	Teaching experiment with 7-year-olds using *Cabri Elem*. Use of dimensional deconstruction tasks that involve the physical, graphical and geometrical spaces in problem solving
Ng and Sinclair (2015)	Canada Grade 1/2 and grade 2/3 split class	Part of a year-long teaching experiment with children from grades 1 to 3 on symmetry using *Sketchpad* as well as paper-and-pencil environments
Thom and McGarvey (2015)	Canada Grades 1 and 2	Qualitative study, with children from grades 1 and 2, using three vignettes about students' drawings
Kotsopoulos, Cordy, and Langemeyer (2015)	Canada 8–10 year olds (19 in all)	Study about the motion maps created by 8-to-10 year-olds using *iPad* or chart paper. An analysis of their talk, drawings and gesture
Kaur (2015)	Canada 7–8 year-olds, grade 2/3	Teaching experiment involving 7-to-8 year-olds, part of a larger study focusing on students' thinking about different types of triangles. Use of *Sketchpad* and Interactive White Board
Hallowell, Okamoto, Romo, and La Joy (2015)	California, USA 36 grade 1 students (20 F, 16 M)	Qualitative study focusing on a diverse group of grade 1 students, focusing on the composition and decomposition of geometric figures. No gender differences were found
van den Heuvel-Panhuizen, Elia, and Robitzsch (2015)	Netherlands and Cyprus 4–5 year old Kindergartners, N = 334 in the Netherlands and N = 304 in Cyprus	Quantitative study comparing the imaginary perspective taking of 4-to-5 year-old kindergartners from the Netherlands and Cyprus on type 1 and type 2 items
Bartolini Bussi, and Baccaglini-Frank (2015)	Italy Grade 1	Teaching experiment with 6-to-7 year-olds on a mathematical definition of a rectangle that includes squares. Use of *Bee-bot*, a small programmable robot
Catherine D. Bruce, and Zachary Hawes (2015)	Canada 4-to-8 year-olds (42 in all)	Part of a lesson study project, study involving 4-to-8 year-olds of varying abilities focusing on 2D and 3D mental rotation. No gender difference

2 Geometry and spatial reasoning

A wide range of topics are explored in the papers highlighted in Table 1. Amongst others the topics include: symmetry, drawings, dimensional construction and deconstruction, use of definitions, spatial reasoning, and problem solving. Early years' geometry is part of the larger topic of geometry in mathematics taught at upper levels. The issue about geometry in the early years is highlighted by Sinclair and Bruce (2015, this issue) who claim that one of the challenges that mathematics education researchers are facing is to articulate the important educational question of *what* should be taught and *why*.

2.1 What should be taught in geometry?

There is no argument against the fact that spatial reasoning is important for young children, however, it is important to select what the young children can do and what we ask them to do in geometry. Moss et al. (2015, this issue) have proposed that rather than approaching geometry as a subject concerned mainly with labelling and classifying shapes that geometry be introduced to children in the early years as dynamic, spatial and imaginative.

Kaur (2015, this issue) has cautioned that although reasoning about shapes involves a dimensional deconstruction, for example from 2D to 1D, young children have very little exposure to 1D objects (see also Duval, 2005). As such, young children need more experience working with lines, in particular, with the drawing of straight lines that they usually meet as sides or edges in different 2D or 3D figures. Hallowell et al. (2015, this issue) specifically looked at plane and solid shapes and their representations. From their research they conclude that "We expect that spending time developing children's mereologic, optic, and place operations would increase children's visualization abilities when working with shape diagrams to develop spatial insights." (p. 13)

Visualization was strongly emphasized in the *Principles and Standards for School Mathematics* document published by the National Council of Teachers of Mathematics (NCTM) in 2000. The NCTM document mentioned that spatial visualization—building and manipulating mental representations of two-and three-dimensional objects and perceiving an object from different perspectives—is an important aspect of geometric thinking. This aspect of visualization was an important component of the research by Van den Heuvel-Panhuizen et al. (2015, this issue), who researched an aspect of spatial reasoning through the idea of Imaginary Perspective Taking (IPT) with kindergartners in the Netherlands and in Cyprus. Interestingly, their research showed that in both countries mathematics ability was significantly related to IPT performance; a strong reason for focusing on developing visualization skills of young children.

School geometry is basically Euclidean in nature although at upper secondary levels in addition to the synthetic approach, transformational, coordinate and vector approaches to the study of the subject are quite prominent. In the early years, we do not want children to focus only on trivial aspects such as only identifying plane figures and solids and categorising them. Battista (2007) has reiterated that underlying most geometric thinking is spatial reasoning. As such, spatial reasoning is pervasive in geometry and finding a distinct line between these two might be futile. The National Research Council ([NRC], 2006) report views spatial thinking as a basic and essential skill that can be learned and that can be formally taught to all students using appropriately designed tools, technologies and curricula. Several of the papers in this issue have focused on some aspect of spatial reasoning. For example, Bruce and Hawes (2015, this issue) describe their research with 4–8 year-olds, on performing mental rotations, a type of spatial reasoning. The authors' conclusion is that mental rotation abilities are malleable, and that with practice they can be improved, is quite significant. They even added that it is possible to accelerate the growth of young children's mental rotation skills through a variety of teacher delivered lessons and activities. It is debatable whether acceleration is even necessary or not.

2.2 Why should we teach geometry and spatial reasoning?

One of the reasons we wish to teach geometry in the early years is to develop children's geometric thinking, which can be considered as a form of mathematical thinking within the content domain of geometry. Geometric thinking is inherent in the types of skills we wish to nurture in young children. For example, Hoffer (1981) claimed that when studying geometry we aim to develop five important skills among learners: visual skills (recognition, observation of properties, interpreting maps, imaging), verbal skills (correct use of terminology and accurate communication in describing spatial concepts and relationships), drawing skills (communicating through drawing, ability to represent geometric shapes in 2-D and 3-D, to make scale diagrams, sketch isometric figures), logical skills (classification, recognition of essential properties as criteria, discerning patterns, formulating and testing hypothesis, making inferences, using counter-examples), and applied skills (Real-life applications using geometric results learnt). Although this set of skills seems more appropriate for the secondary level, the development of these skills cannot be delayed and as such it has to start in the early years of schooling.

Another reason for teaching geometry in the early years is that we wish to develop the children's spatial thinking skills. It is important to focus on the development of spatial reasoning of young children as this is a predictor of later mathematics achievement. Although the National Research Council (2006) does not consider spatial thinking as a content-based discipline, the report highlights that thinking spatially entails knowing about the following three things: space, representation and reasoning; all three of which have strong links to geometry. Furthermore, the NRC report strongly points out that spatial thinking can be learned and it can and should be taught at all levels in the education system. A point raised by Soury-Lavergne and Maschietto (2015, this issue) who look at geometry as a model of space and as such consider it as being related to spatial knowledge. Overall, the papers in this issue point to the fact that geometry and spatial reasoning are important aspects that should be included in the education of young children.

There are other strong reasons why geometry and spatial reasoning should be included in the school curricula of young children. For example, the *Principles and Standards for School Mathematics* (NTCM, 2000) proposed that instructional programs from prekindergarten through grade 12 should enable all students to

- analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;
- specify locations and describe spatial relationships using coordinate geometry and other representational systems;
- apply transformations and use symmetry to analyze mathematical situations;
- use visualization, spatial reasoning, and geometric modeling to solve problems.

The above list, which is overarching for the whole range of schooling from pre-K to 12, is not specific enough for the children in the early years. It is difficult to encompass the objectives for teaching geometry to various groups of learners from primary to secondary in the same set. Some geometrical concepts related to measurement are not highlighted in the above list as the NTCM (2000) document treats measurement as a separate content strand. On the other hand, Usiskin (1987) claimed that there are four aspects of geometry: (1) visualization, drawing, and construction of figures; (2) study of the spatial aspects of the physical world; (3) use as a vehicle for representing nonvisual mathematical concepts and relationships; and (4) representation as a formal mathematical system. Except for the last aspect, the other three aspects mentioned by Usiskin are certainly important in the early years.

3 Representation of geometrical ideas

The papers highlighted in Table 1 use representations either in paper-and-pencil environments or in computer environments or in both quite extensively. It is not possible to teach geometry without using any figures. We can only study geometrical objects through their semiotic representations (Duval, 1999). Duval also added that in geometry we use language, symbols or figures which he respectively categorised as the register of natural language, the register of symbolic language and figurative register. Why are figures so important in geometry? Perhaps, the single most important reason for using figures in geometry is that a figure demonstrates several connections and relationships at the same time which would otherwise be difficult to bring forth through normal discourse.

A geometrical figure is nothing but a representation of some abstract concept like a triangle. Young children have difficulties making connections between different representations of the same geometrical concept. Worse, children sometimes adhere to some prototypical representations, what Vinner and Hershkowitz (1980) have referred to as concept images. Children use prototypes to categorise shapes (see Hershkowitz, 1989). This may be due, at least partly, to the geometrical experiences that children have in their lessons or what they glean from standard textbooks. Kaur (2015, this issue) who explored the idea of prototypicality in her study, concludes that static media may prompt a particular kind of drawing that limits what children may produce but on the other hand technology can offer more affordances. In addition, Tsamir et al. (2015, this issue) point out that ideal examples (prototypes) are acquired first and it is often the non-critical attributes (e.g. size or orientation) which contribute to the makings of prototypical examples. Moreover, children may hold different concept images for the same concept definitions. For young children, the use of several positive and negative examples may help them to get a firmer grasp of a geometrical concept. The advent of technology now provides very excellent opportunities to teachers to present geometry in a more dynamic manner than previously possible.

Besides the term figure, we commonly use the terms drawing and diagram in geometry. The term drawing which is quite often used synonymously to mean 'figure' appears prominently in the paper by Thom and McGarvey (2015, this issue). The authors, who see drawings as internal or external imitation of objects, highlight that drawings, as forms of geometric thinking, must simultaneously be a matter of making sense in our world as well as making sense of our world. On the other hand, the term diagram is used differently by some authors. For example, Diezmann and English (2001) claimed that diagrams are structural representations and differ from pictures and drawings by their lack of surface details.

Concepts in mathematics are abstract and geometrical concepts are no exception. The figures that we draw in geometry are mere representations of geometrical objects and not the actual objects. This issue is also highlighted by Mesquita (1998) who claimed that we always represent a concrete object, even if we are interested in the abstract one. For example, when a triangle is drawn, the figure can represent either an abstract geometrical object or a particular concrete example. Thus, figures used alone do not enable one child to distinguish between the two cases which is a major problem for beginners learning geometry. Children have a problem differentiating between the particular and the abstract. Other researchers have also commented on the idea of figures, for example, Herskowitz, Parzysz and Van Dormolen (1996, p. 164) have shared that:

> "The word 'figure' itself is ambiguous, because it can refer either to a geometrical object or to a graphical representation of such an object. This ambiguity is a well known source of difficulty for younger students because they do not understand that the objects referred to by their teacher are not the drawings (diagrams) which they can see in their textbooks, or on blackboard, or that they realize themselves."

At a time when young children are struggling with the representation of geometrical concepts, some caution need to be exercised in the content presented to them. Lowrie (2002a) who was working with 6-year-olds in a computer environment concluded that:

> At this point in their physical and cognitive development they [children] cannot appreciate that 3D objects need to be represented in a particular 2D form in order to look 3D in nature. Moreover, these students' understanding of diminishing line and depth perception have not been sufficiently developed to interpret depth cues. (p. 446).

4 The teaching and learning of geometry in the early years

Children are naturally gifted to learn new things but their learning is quite different from that of adults and so children differ from adults not only in the amount of knowledge that they possess but also in the quality of that knowledge. The papers in this issue describe the use of various kinds of objects used in teaching children geometry and spatial reasoning, which can be broadly categorized as coming from computer environments or from non-computer environments. Using De Moor (2005) classification of the three aspects of geometry: orienting, constructing and operating with shapes and figures, several types of resources for teaching geometry and spatial reasoning come to the fore.

4.1 Resources from non-computer environments

First, for *spatial orientation* concepts and relationships children should not be always within the four walls of the classroom. They need to have experiences both inside and outside of the classroom. Children should be given opportunities to progressively read, make and use simple ground plans. For example, Thom and McGarvey (2015, this issue) report how children went out in the schoolyard to create their drawings. They conclude that when drawings are pulled from the very contexts in which they arise—the gestures, verbalisations, actions, transactions, and so on—then multiple threads of meaning are severed. In other situations as well, children need to see objects from different positions, what De Moor (2005) terms "taking a point of view". Van den Heuvel-Panhuizen et al. (2015, this issue) who explored "taking a point of view" using the idea of Imaginary Perspective Taking (IPT) with kindergartners used static pictures of objects from different perspectives. Whether static pictures are used or whether the children need to have personal experience in the environment, points to the fact that experiences for young children within the classroom only are not enough.

Second, for *constructing*, children should be given experience with: free construction materials (clay, plasticine, ropes, boxes), geometric construction materials (lego, pattern blocks, meccano, tangrams), constructing with paper (paper-folding, paper cut-outs), constructing on paper (drawings of shapes, patterns). (See Thom and McGarvey 2015, in this issue)

Third, for *operating with shapes and figures*, children should be given the opportunity to move models of geometrical objects and notice what happens during geometric transformations like sliding, reflecting and rotating. This is highlighted in the research by Bruce and Hawes (2015, this issue) which focuses on the mental rotation of 2D and 3D objects in which they hypothesise that improvement in the spatial thinking of students can have a "two-for-one" effect where improvements in spatial reasoning may also be seen as improvement in overall mathematics.

4.2 Resources from computer environments

Children are nowadays exposed to a large number of technological devices such as iPads, smart phones, computers and many types of electronic games and other software

Table 2 Research papers on teachers' professional development

References	Context and participants	Study
Tsamir, Tirosh, Levenson, Barkai, and Tabach (2015)	Israel Teachers with a first degree in education who were teaching 4- to 6-year olds	In-service early-years teachers' concept images and concept definitions for triangle, circles and cylinders
Moss, Hawes, Naqvi, and Caswell (2015)	Canada Kindergarten and first grade teachers (4 + 1), a principal and two other officers. Caucasian female teachers with a B.Ed. degree	Professional learning team using an adaptation of the Japanese Lesson Study that involved teachers engaging in mathematics, conducting task-based clinical interviews, designing and carrying out exploratory lessons with researchers and creating resources for other educators Working with ELL children from Syria and Iraq

which are meant for both entertainment and information. These digital natives have developed subtle ways of dealing with geometrical concepts and spatial reasoning at large. It seems normal to use technology in teaching young children. Several of the papers in this issue have reported the use of technology. For example, Ng and Sinclair (2015, this issue) used the *Sketchpad* in their research on symmetry. Kostopolous, Cordy and Langemeyer (2015, this issue) report the use of iPads in their research whereas Kaur reports the use of interactive whiteboards. Bartolini-Bussi and Baccaglini-Frank (2015, this issue) used the Bee-bot a small programmable robot. On the other hand Soury-Lavergne and Maschietto (2015, this issue) used the *Cabri-Elem* which is a dynamic geometry software.

Clements, Nastasi and Swaminathan (1993) very aptly pointed out that technology used thoughtfully and creatively rather than as a teaching machine, can engender and support educational environments that will empower children to flourish and they also added that there are high levels of spoken communication and cooperation as young children interact at the computer. While technology is extremely important in teaching young children, Lowrie (2002b) cautioned about the use of computers:

> …it may be more worthwhile to encourage young children to develop important foundation understandings away from computer-based environments or provide learning experiences on the computer that challenge children to consider links between 3D, simulated 3D and 2D worlds. (p. 445)

The significant use of technology in teaching certainly brings to the fore issues about equity for various groups of young learners. However, technology now offers affordances hitherto not possible by traditional means. Unless activities involving technology for children are planned with extreme care, chances are that irrelevant cues can become the focus of the children's attention rather than the intended mathematics to be learned. Teachers need some kind of confidence in geometry, in the pedagogy and certainly working with the technology, what has been termed Technological Pedagogical Content Knowledge (TPACK) by Koehler and Mishra (2009).

4.3 Teaching of geometry and PD of teachers

The papers by Moss et al. and Tsamir et al. in this issue (see Table 2) looked at the professional development of teachers. Teaching geometry at the early childhood level requires that teachers are confident about the content that they have to teach at this level. This point was raised by Perry and Dockett (2002) who claimed that many early childhood teachers do not have a strong mathematics background and they added that: "At this time when children's mathematical potential is great, it is imperative that early childhood teachers have the competence and confidence to engage meaningfully with both the children and their mathematics… (p. 107)". The child should be working in collaboration with other children and the teacher and as such, instruction should take a wide variety of approaches that, amongst others, includes play. Moss et al. (2015, this issue) have strongly pointed out that a major barrier to the implementation of a strong early years mathematics programme lies in the inadequate preparation of early years educators.

More specifically about geometry and spatial reasoning, Ginsberg et al. (2006) have reiterated the lack of teacher preparation, lack of content knowledge and lack of interest. Perhaps a good approach to enhance early childhood teachers' professional knowledge is to adapt and use the Japanese Lesson Study which is described by Moss et al. (2015, this issue). The good thing about this model of professional development is that teachers work in teams rather than individually to plan, research and implement a lesson thereby learning through the process. It is obvious that teachers should have a good understanding of the geometry they teach. The paper by Tsamir et al. (2015, this issue) focuses on the concept images and concept definitions of early childhood teachers from Israel. Their research demonstrates that some teachers had difficulties with the definitions of some figures and could not even identify some examples and non-examples. This deficit in teachers' knowledge may not be only true only in Israel but elsewhere as well. When teachers are not confident about their own geometrical knowledge then this can have a long-lasting negative impact on children's learning.

As such, teaching geometry is more complex and often less successful than teaching numerical operations or elementary algebra (Duval, 1998). Regarding early childhood perspectives on teaching geometry, Fuys and Liebov (1993) claimed that one guiding principle should be that instruction should involve the "whole child" and his or her cognitive, affective, social, and physical needs and characteristics. To this effect the authors highlighted three principles:

- The goals for teaching geometry should enhance children's knowledge, skills, dispositions and feelings.
- Instruction should be based on how children learn, namely: by doing, reflecting on their actions, sharing their ideas with classmates and the teacher.
- Instruction should feature a variety of methods (play, learning center, projects, direct teaching) (p. 215)

Teaching of geometry cannot be *only* about enhancing a child's knowledge and skills but should also focus on the child's dispositions and feelings.

4.4 Theories about children's learning of geometry

The paper by Mamolo et al. (2015, this issue) proposes a theory about restructuring mathematical tasks with a network of spatial visual representations that support geometric reasoning for different groups of learners. Theories in any field have an important role in explaining phenomena whether they have been developed within or outside that field. These theories ultimately become the foundations for further research until some new theory, that provides new insights, is developed. For geometric thinking, two theories are often cited: the Topological Primary Thesis (TPT) by Piaget and the van Hiele levels.

4.4.1 Topological Primary Thesis

Piaget had worked with young children on tasks such as touching real objects and paper cut-outs, naming and drawing the figures as well drawing shapes of shadows (see Fuys and Liebov, 1993). Piaget and Inhelder (1967) claimed that: (1) a child does not represent space by directly perceiving the environment but builds it up by prior manipulation of that environment; and (2) the progressive organization of geometric ideas follows a definite order, called the Topological Primacy Thesis (TPT). The TPT postulates that first, topological relations (inside, outside, connectedness, and continuity) are constructed then projective relations (rectilinearity) are constructed, and finally Euclidean relations (angularity, parallelism, and distance) are constructed. One of the main criticisms of the TPT is that every figure possesses both Euclidean and topological characteristics whereas Piaget and Inhelder's experiments depended on a mutually exclusive classification of figures into these two categories (see also Clements and Battista, 1992).

4.4.2 Van Hiele theory

The van Hiele model of geometrical thinking can be used to guide instruction as well as assess students' abilities (Crowley, 1987). This model with five levels was developed by the van Hieles (husband and wife) in the Netherlands in the 1950s. Initially, the levels were numbered from 0 to 4 but later the numbering was changed to from 1 to 5, with level 5 as the highest level. The levels from 1 to 5 are sequentially: recognition, analysis, informal deduction, formal deduction, and rigour. Van Hiele (1986) acknowledged that the roots of the theory are found in Piaget's work and he was also concerned about the difficulty in tracing the levels of thinking in geometry. He claimed that the levels are not situated in the subject matter but in the thinking of man and as such the levels are not age-bound as in the Piagetian cognitive developmental theory.

Most of the children in grades K-4 will demonstrate geometric thinking at levels 1 or 2 (Fuys and Liebov, 1993). For the teacher, this implies that the content has to be planned to suit the needs of the children for a given level and also how to help the children to move to the next level. The van Hiele levels although used widely are not free of controversy. However, it should be noted that the van Hiele levels were developed at a time when school geometry was primarily Euclidean.

Based on their research, Clements, Swaminathan, Hannibal and Sarama (1999) suggested the existence of a precognitive level before van Hiele level 1 and claimed that the van Hiele level 1 should be reconceptualised as syncretic (a synthesis of verbal declarative and imagistic knowledge, each interacting with the other). Some researchers have found that a student can possibly develop two consecutive levels of reasoning at the same time (see Guitiérrez, Jaime, and Fortuny, 1991) and others like Pandiscio and Orton (1998) claimed that the van Hiele theory lacked generality. There have been attempts to link the van Hiele theory with the Structure of the Observed Learning Outocmes (SOLO) taxonomy and Skemp's model of mathematical understanding (see Olive, 1991). On the other hand, Pandiscio and Orton (1998) argued for a synthesis of van Hiele and Piaget's perspectives. While there is agreement about some sort of hierarchical levels in geometric thinking, the van Hiele theory may not on its own provide a complete picture. Other theories that can also encompass work with technology should be considered. Mamolo et al. (2015, this issue) have proposed a network of and for geometric reasoning. However, these authors do not make links to other models of geometric reasoning.

5 Assessment of children's learning

Young children are at a very critical age struggling with the learning of language and at the same time with the learning of mathematical concepts. It is important for teachers to find out what exactly the children have learned and how these children can demonstrate that learning. The focus cannot be only on the products of learning but also on the processes. Furthermore, Fleer and Quiñones (2013) argue that children's thinking should not be the only aspect that is assessed but that it is necessary to as well capture how the children are emotionally experiencing the assessment situation. Accordingly, the authors suggest that the teacher as the assessor has to work together with the child who is assessed in a dynamic assessment situation and thus these researchers position assessment as being in motion and mediated through interaction in social situations rather than being static in time.

Assessment of children's learning is quite challenging. Some suggestions for improving assessment from Realistic mathematics Education (see van den Heuvel-Panhuizen, 1996, p. 27) seem fitting for assessing children's work in geometry:

1. Help teachers observe learning processes so that they are aware when learning is taking place and when not.
2. In order to develop tests with a diagnostic purpose, use observation as a point of departure for test development.
3. Conduct discussions with the children to find out more about what they know and understand.
4. Emphasise formative assessment that will help the teacher become more knowledgeable about the learning process.

None of the papers in this issue specifically addressed the idea of assessment although assessment of children's learning was somehow present in the different studies. For example, the study by van den Heuvel-Panhuizen, Elia and Robitzsch (2015, this issue) assessed the Imaginary Perspective Taking (IPT) by using multiple choice items. Although, the authors argue that the response was above chance levels, it is not possible, in this case, to gauge individual children's learning which is most valuable to teachers to improve their teaching.

Often, young children have uneven backgrounds as far as their learning prior to formal schooling is concerned. Also, young children come with unequal linguistic skills and may as well be coming from families with disadvantaged backgrounds which limit the children's access to certain types of resources. It will be difficult to smooth out all entry level differences among the young learners. However, for teachers to help young children individually, assessment should not only inform the teacher about the current learning of the children but also about their future learning; an idea termed *potentive assessment* by Fleer (2010).

6 Concluding remarks

In this paper, I set out to explore some of the issues surrounding the teaching and learning of geometry in the early years of schooling. My discussion is structured around ideas gleaned from the literature and the papers in this issue. Some points raised include:

- Geometry and spatial reasoning remain important areas that should form part of any curriculum for early childhood.
- There should be a strong focus in school curricula on geometrical figures and their representations.
- Learning as well as teaching of geometry and spatial reasoning are quite complex. These complexities are compounded by the inadequate level of teaching training and professional development. It is important to pay close attention to existing theories about geometric thinking but also to be alert to find new ways of conceptualizing geometrical thought.
- Assessment of children's learning must take place in an environment of trust with the young children and should not focus only on the products of learning.

Regarding the papers in this issue, we can note a wide coverage of topics on geometry and spatial reasoning. It should be noted that the researchers and the research contexts are mostly from North America or Europe and as such highlight content and pedagogies prevalent in these regions.

6.1 Moving forward

Beyond the content of the individual papers, the different authors in this issue show a strong interest in developing the geometric thinking of young children. We need to carefully think and rethink the issues pertaining to the curriculum, learning, teaching, assessment, teacher preparation and research in general in the early years of schooling.

- The curriculum for the early years of schooling should cover the essential concepts children will need to explore their surroundings as well as prepare them for the more advanced concepts they will meet in the later years of their schooling. We have to be careful in trying to include too much content that may have a negative effect on the young children's motivation to learn more mathematics. What would constitute a good geometry curriculum for the early years of schooling has to be developmentally appropriate for the children. Perhaps the approach suggested by De Moor (2005) is worth considering.
- We now have a large amount of research about how young children learn in general. However, we still need to find out more about how children learn in specific content domains of mathematics such as geometry. The types of basic objects in spatial and geometrical thought (physical object, sensory object, perceptual object, conceptual object, concept definition) proposed by Battista (2007) may be worth exploring further. Regarding a framework for geometric thinking, the Van Hiele levels may not adequately capture children's learning of geometry at this very young age and perhaps a new framework that is more comprehensive is needed.
- Teachers need to continually assess children's learning; however assessments cannot be limited to paper and pencil tests only. The emphasis should be on formative assessment that will help teachers to gauge the children's learning and plan learning activities that will further enhance the children's learning. (see van den Heuvel-Panhuizen, 1996) The idea of potentive assessment by Fleer (2010) is worth exploring for children's learning of geometry.
- Attitudes towards learning in general and more specifically towards mathematics are formed very early in the life of a child; a reason why we should be very careful about what we teach to young children and how we teach it. There is no doubt, early childhood teachers should be not only be well-qualified but more specifically have a strong background in geometry and a strong interest generally in mathematics education. The challenge will be to select, prepare and retain good teachers who can confidently teach geometry.

The research reported in this issue touches a wide range of topics in geometry and spatial reasoning at the early childhood level. Sinclair and Bruce (2015, this issue) have provided a good summary of the research themes in this issue. Qualitative approaches are predominant in this sample of research from both sides of the Atlantic, with several of the studies carried out using some kind of technological tool. Battista (2007), referring generally about the development of geometric and spatial thin king at various levels, points to some interesting areas to be considered for further research that is relevant for early childhood researchers. There is a need to:

- Understand children's difficulties examine the effects and effectiveness of alternate instructional approaches.
- Focus on not only the cognitive factors but also socio-cultural and affective factors in geometry learning.
- Investigate how technological enhancements affect children's learning.
- Carry out more comparison studies, both quantitative to investigate generality) and qualitative (to investigate difference in cognitive processes (pp. 903–904).

Geometry is an important branch of mathematics that is taught to children from the very early years and is prominently present in school curricula all over the world. Children have experiences with geometrical objects well before they enter school. As such, in schools we need to make the necessary connections to the real world of the content that the children learn in geometry. In addition, we should make the learning enjoyable for the children.

References

Bartolini-Bussi, M. & Baccaglini-Frank, A. (2015). Geometry in early years: sowing seeds for a mathematical definition of squares and rectangles. *ZDM Mathematics Education*, *47*(3) (this issue). doi:10.1007/s11858-014-0636-5.

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). Reston: National Council of Teachers of Mathematics.

Bruce, C. & Hawes, Z. (2015). The role of 2D and 3D mental rotations in mathematics for young children: What is it? Why does it matter? And what can we do about it? *ZDM Mathematics Education*, *47*(3) (this issue). doi:10.1007/s11858-014-0637-4.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook for research in teaching and learning of mathematics* (pp. 420–464). New York: Macmillan.

Clements, D.H., Natasi, B.K., & Swaminathan, S. (1993). Young children and computers: crossroads and directions from research. *Young Children*, 56–64.

Clements, D. H., Swaminathan, S., & Hannibal, M. A. Z. (1999). Young children's concepts of shapes. *Journal for Research in Mathematics Education, 30*(2), 192–212.

Crowley, M. L. (1987). The van Hiele model of the development of geometric thought. In M. M. Lindquist, & A. P. Shulte (Eds.), *Learning and teaching geometry, K-12. 1987 Yearbook of the National Council of Teachers of Mathematics* (pp. 1–16). Reston: NCTM.

De Moor, E. (2005). Domain description geometry. In M. van den Heuvel-Panhuizen & K. Buys (Eds.), *Young children learn measurement and geometry* (pp. 115–144). The Netherlands: Freudenthal Institute, Utrecht University.

Diezmann, C. M., & English, L. D. (2001). Promoting the use of diagrams as tools for thinking. In A. A. Cuoco & F. R. Curcio

(Eds.), *The roles of representation in school mathematics: 2001 NCTM Yearbook* (pp. 77–89). Reston: NCTM.

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century: an ICMI study* (pp. 37–51). Dordrecht: Kluwer Academic Publishers.

Duval, R. (1999). Representation, vision and visualization: cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt, & M. Santos (Eds.), *Proceedings of the 21st annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 3–26). Columbus: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Duval, R. (2005). Les conditions cognitives de l'apprentissage de la géométrie : développement de la visualisation, différenciation des raisonnements et coordination de leurs fonctionnements. *Annales de Didactique et de Sciences Cognitives, 10*, 5–53.

Fleer, M. (2010). *Early learning and development: cultural-historical concepts in play*. Cambridge: Cambridge University Press.

Fleer, M., & Quiñones, G. (2013). An assessment perezhivanie: building an assessment pedagogy for, with and of early childhood science learning. In D. Corrigan, et al. (Eds.), *Valuing assessment in science education: pedagogy, curriculum, policy* (pp. 231–247). Dordrecht: Springer.

Fuys, D. J., & Liebov, A. K. (1993). Geometry and spatial sense. In R. J. Jensen (Ed.), *Research ideas for the classroom: early childhood mathematics* (pp. 195–222). London: Simon & Schuster/Prentice Hall International.

Ginsburg, H. P., Cannon, J., Eisenband, J., & Pappas, S. (2006). Mathematical thinking and learning. In K. McCartney & D. Phillips (Eds.), *Blackwell handbook of early childhood development* (pp. 208–229). Massachusetts: Blackwell.

Guitiérrez, A., Jaime, A., & Fortuny, J. M. (1991). An alternative paradigm to assess the acquisition of van Hiele levels. *Journal for Research in Mathematics Education, 22*(3), 237–251.

Hallowell, D., Okamoto, Y, Romo, L., & LaJoy, J. (2015). First-grader's spatial-mathematical reasoning about plane and solid shapes and their representations. *ZDM Mathematics Education*, *47*(3) (this issue). doi:10.1007/s11858-015-0664-9.

Hershkowitz, R. (1989). Visualization in geometry: two sides of the coin. *Focus on Learning Problems in Mathematics, 11*(1), 61–76.

Hershkowitz, R., Parzysz, B., & Van Dormorlen, J. (1996). Space and shape. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education: Part 1* (pp. 161–204). Dordrecht: Kluwer Academic Publishers.

Hoffer, A. (1981). Geometry is more than proof. *Mathematics Teacher, 74*, 11–18.

Kaur, H. (2015). Two aspects of young children's thinking about different types of dynamic triangles: prototypicality and inclusion. *ZDM Mathematics Education*, *47*(3) (this issue). doi:10.1007/s11858-014-0658-z.

Koehler, M., & Mishra, P. (2009). What is technological pedagogical content knowledge (TPACK)? *Contemporary Issues in Technology and Teacher Education, 9*(1), 60–70.

Kotsopoulos, D., Cordy, M., & Langemeyer, M. (2015). Children's understanding of large-scale mapping tasks: an analysis of talk, drawings, and gesture. *ZDM Mathematics Education*, 47(3) (this issue). doi:10.1007/s11858-014-0661-4.

Lowrie, T. (2002a). Making connections between simulated and "real" worlds: young children interpreting computer representations. In B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.), *Mathematics education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 441–448). Sydney: MERGA.

Lowrie, T. (2002b). Visual and spatial reasoning: young children playing computers. In D. Edge, & B. H. Yeap (Eds.), *Mathematics education for a knowledge-based era* (Proceedings of the 2nd East Asia regional conference on mathematics education, Singapore, vol. 2, pp. 440–446). Singapore: National Institute of Education.

Mamolo, A., Ruttenberg-Rozen, R., & Whitelely, W. (2015). Developing a network of and for geometric reasoning. *ZDM Mathematics Education*, *47*(3) (this issue). doi:10.1007/s11858-014-0654-3.

Mesquita, A. L. (1998). On conceptual obstacles linked with external representations in geometry. *Journal of Mathematical Behavior, 17*(2), 183–195.

Moss, J., Hawes, Z., Naqvi, S., & Caswell, B. (2015). Adapting Japanese Lesson Study to enhance the teaching and learning of geometry and spatial reasoning in early years classrooms: a case study. *ZDM Mathematics Education*, *47*(3) (this issue). doi:10.1007/s11858-015-0679-2.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston: NCTM.

National Research Council (NRC) (2006). *Learning to think spatially: GIS as a support system in the K-12 curriculum*. Washington, DC: National Academy Press. http://www.nap.edu/download.php?record_id=11019.

Ng, O., & Sinclair, N. (2015). Young children reasoning about symmetry in a dynamic geometry environment. *ZDM Mathematics Education*, *47*(3) (this issue). doi:10.1007/s11858-014-0660-5.

Olive, J. (1991). Logo programming and geometric understanding: an in-depth study. *Journal for Research in Mathematics Education, 22*(2), 90–111.

Organisation for Economic Cooperation and Development. (2001). *Starting strong: early childhood education and care*. Paris: OECD.

Pandiscio, E., & Orton, R. E. (1998). Geometry and metacognition: an analysis of Piaget's and van Hiele's perspectives. *Focus on Learning Problems in Mathematics, 20*(2&3), 78–87.

Perry, B., & Dockett, S. (2002). Young children's access to powerful mathematical ideas. In L. D. Enhlish (Ed.), *Handbook of international research in mathematics education: direction for the 21st century* (pp. 81–111). Mahwah: Lawrence Erlbaum Associates.

Piaget, J., & Inhelder, B. (1967). *The child's conception of space*. New York: W.W. Norton.

Sinclair, N., & Bruce, C. (2015). New opportunities in geometry at the primary school. *ZDM Mathematics Education*, *47*(3) (this issue). doi:10.1007/s11858-015-0693-4.

Soury-Lavergne, S., & Maschietto, M. (2015). Articulation of spatial and geometrical knowledge in problem solving with technology at primary school. *ZDM Mathematics Education*, *47*(3) (this issue). doi:10.1007/s11858-015-0694-3.

Thom, J., & McGarvey, L. (2015). The act and artifact of drawing(s): observing geometric thinking with, in, and through children's drawings. *ZDM Mathematics Education*, *47*(3) (this issue).

Tsamir, P., Tirosh, D., Levenson, E., Barkai, R., & Tabach, M. (2015). Early-years teachers' concept images and concept definitions: triangles, circles, and cylinders. *ZDM Mathematics Education*, *47*(3) (this issue). doi:10.1007/s11858-014-0641-8.

Usiskin, Z. (1987). Resolving the continuing dilemmas in school geometry. In M. M. Lindquist & A. P. Shulte (Eds.), *Learning and teaching geometry, K-12:1987 Yearbook* (pp. 17–31). Reston: National Council of Teachers of Mathematics.

Van den Heuvel-Panhuizen, M. (1996). *Assessment and realistic mathematics education*. Utrecht: Centre for Science and Mathematics Education, Freudenthal Institute.

Van den Heuvel-Panhuizen, M., Iliade, E., & Robitzsch (2015). Kindergartner's performance in two types of imaginary perspective taking. *ZDM Mathematics Education*, *47*(3) (this issue). doi:10.1007/s11858-015-0677-4.

Van Hiele, P. M. (1986). *Structure and insight*. Orlando: Academic Press.

Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *Proceedings of the 4th international conference for the psychology of mathematics education* (pp. 177–184). Berkeley: University of California.