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ORIGINAL ARTICLE

The act and artifact of drawing(s): observing geometric thinking with, in, and through children's drawings

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Abstract In mathematics education, as in other domains, drawing serves as means to access, assess, and attend to children's understanding. While theoretical accounts of drawings are often based on developmental stage theories, we examine insights gained by considering children's geometric thinking and reasoning from embodied cognitive perspectives. We ask, what if the act of drawing serves as a means by which children become aware of geometric concepts and relationships, rather than being viewed as a product of that awareness? In this paper, we examine three vignettes and inquire into the ways that children come to draw in geometric contexts. We suggest that the children's choice to draw as a mode of thinking, the different ways they draw, the manners in which they attend to the mathematics as they draw, and the conceptions that arise with their drawings, contribute in significant ways to their geometric understanding.

1 Introduction

Lucy and her classmates engaged in a variety of geometric drawing tasks over a period of several weeks. Initially the students were given outlines of geometric figures to copy (adapted from Wheatley 2007). Figure 1a shows a sample of four of Lucy's twenty shape drawings. Over the next couple of sessions, the children copied or created shape compositions with different 2D manipulatives including pattern blocks, tangrams, and pentominoes. When composing shapes with polygons, Lucy created a bird-like image and after several attempts was satisfied with her final drawing shown in Fig. 1b.

Based on the set of drawings produced, we might say that Lucy is reasonably successful at recognizing and reproducing simple figures in familiar orientations including lines, circles, squares, and triangles. The second image in Fig. 1a, comprised of an outer rectangle and an inner vertical and horizontal line (or four rectangles), does not appear to pose significant challenges for Lucy as she makes a reasonable replication. The third and fourth drawings increase in difficulty and we begin to see errors in Lucy's efforts to replicate figures that are less familiar and in non-standard orientations.

Lucy's drawings represent geometric drawings typical of children ages 4–7 (see Piaget & Inhelder 1967; Sarama & Clements 2009). That is, we note in her drawings that she successfully distinguishes between straight and curved lines as well as differentiates simple Euclidean shapes. Lucy's drawings do, however, reflect deficiencies in conceptualizing less familiar shapes (e.g., rhombus), the relationship between shapes (e.g., triangles and rhombus), as well as angle and side length measurements.

Although Lucy's drawings may be typical, our multi-layered data sources presented in this paper suggest that these artifacts, taken as is, overlook critical aspects of her geometric awareness. For example, in the fourth drawing in Fig. 1a, Lucy's drawing of two triangles in the upper right and left portion of the figure are decidedly different. The triangle on the left was drawn second and is more accurately oriented suggesting a growth in awareness through the act of drawing. Also, upon completion she recognized that the "cat ears" were not in the position that she desired and commented that her classmate's drawing was closer to what

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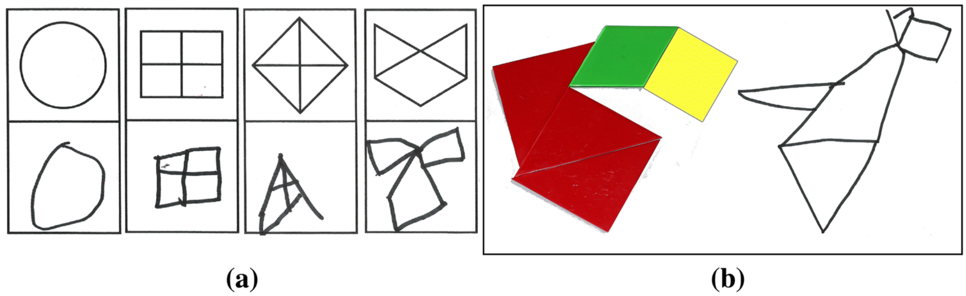

Fig. 1 Lucy's (5 years, 9 months) copies given shapes (**a**), and draws her own design (**b**)

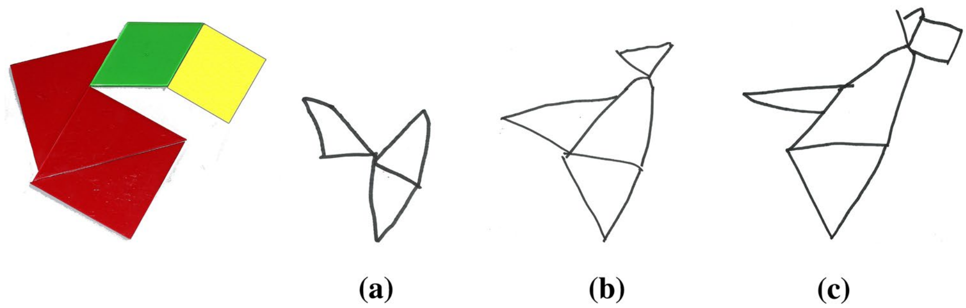

Fig. 2 Lucy's bird and her drawings of it

she intended to draw. Attending to children's mark-making along with their gestures, verbalizations, and transactions offer new insight into their emergent geometric awareness.

For example, Lucy's drawings of her final bird-like image in Figs. 1b and 2c may reveal deficiencies in her awareness of spatial relationships. However, when we look at the discarded drawings (Fig. 2a, b), it is hard to ignore that something is emerging in her act of mark-making.

Lucy slowly draws two "body" triangles in her first drawing (Fig. 2a). She then sits up, declares, "it has a wing" and quickly sketches a shape without referring to her model.

Upon completion of her initial attempt Lucy briefly looks at her drawing, takes another sheet of paper, and says that she wants to try again. This time, she quickly sketches the two body triangles that were her focus in the first drawing, and slows to consider the "wing on the back" (Fig. 2b). She then adds a "head" starting with a horizontal line at the top and then completes a triangle.

In the third and final drawing (Fig. 2c), the first three triangles of the body and wing are sketched quickly, taking on more of a bird-like shape. Lucy slows to examine the "head" and remarks that it is two different shapes. She outlines a rectangle attending to the shape's orientation. A triangular shape is added quickly to complete the drawing.

Here, we observe how Lucy attends to individual shapes, relationships between shapes, composite shapes, and angles—but not necessarily all at once. Each drawing elucidates her constantly shifting geometric attention. What appears to be incorrect or incomplete changes from drawing to drawing all within the span of a few minutes. Rather than attending to an apparent lack of understanding, we are interested in how Lucy's act of drawing brings shape and order to her experience. In this paper we ask, what might we learn about children's geometric thinking if we interpret drawings as a vehicle for thinking and not just an object of reasoning? Also, how do children's mark-making give rise to different ways of thinking geometrically?

We begin by examining past and current literature on children's drawings, visual representations, and geometric thinking. We extend the literature by considering drawing as an act rather than solely an artifact through an embodied cognitive approach. From this theoretical perspective, we analyze three vignettes in primary school settings and reveal the insights gained into children's geometric understandings as they draw.

2 The role of drawings in geometric development and learning

The walls of early years classrooms are often adorned with children's drawings. Drawing contributes to all aspects of child development including fine motor skills, creativity, early writing, storytelling, emotional expression, and autonomy (e.g., see DCSF 2008). Within mathematics, children's mark-making and drawings play a role in meaning making, problem solving, and early symbolism (Carruthers & Worthington 2006). The value of drawing for mathematics understanding is most prominent within the domain of geometric thinking.

Drawing to model, compose, compare, explore, and analyze simple 2D and 3D shapes is a common curriculum expectation (e.g., Clements & Sarama 2009; Inan & Dogan-Temur 2010; NCTM 2006; NRC 2009). Drawing activities support spatial reasoning, 2D and 3D geometry concepts, and problem solving in geometric contexts (Crespo & Kyriakides 2007; Edens & Potter 2008; Nunokawa 2006; Wheatley 2007). Further, drawing tasks, such as replicating shapes and completing spatial structuring images, are a common means to assess geometric understanding and spatial awareness (e.g., Burger & Shaughnessy 1986; Mulligan & Mitchelmore 2009). Throughout this large body of literature, student-generated drawings are often referred to as "visual representations" (David & Tomaz 2012; Diezmann & English 2001), "external visual representations" (Zahner & Corter 2010), or simply, "representations" (Diezmann & McCosker 2011; Woleck 2001). Both visualization and representation are, "at the core of understanding in mathematics" (Duval 1999, p. 3; Presmeg 2006; Rivera 2014; Rivera Steinbring & Arcavi 2014).

Based on Arcavi's (2003) definition,

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (p. 217)

Duval (2014) makes a useful distinction between visualization and visual representations to tease apart Arcavi's definition into two components: *Visualization*, the quick recognition of what is mathematically relevant in an image; and *visual representations,* the tools supporting visualization including diagrams, graphs, figures and drawings.

The emphasis on drawings as representations tends to entrench the view that a drawing is a "product of creation" or artifact. Student-generated drawings are often assumed to *re-present* information in a problem in a schematic form, and also *re-present* a student's understanding of that problem or the concepts therein. That is, as representations, drawings become a stand in for something else (Goldin 2002; Kaput 1998). This view of drawings as external visual representations of internal cognitive schema has a long history arising predominantly from the psychological literature.

Children's drawings became a focus of study at the beginning of the twentieth century along with the field of child development (Malchiodi 1998). The oddities in children's drawings, such as tadpole drawings of human figures where arms extend from a circular undifferentiated head-body (see Fig. 3), have been used to study individual intelligence, cognitive development, working memory, and fine motor skills (Darling Kelly 2004).

Our modern assumptions about the role of drawing in visual domains in mathematics stem primarily from the work

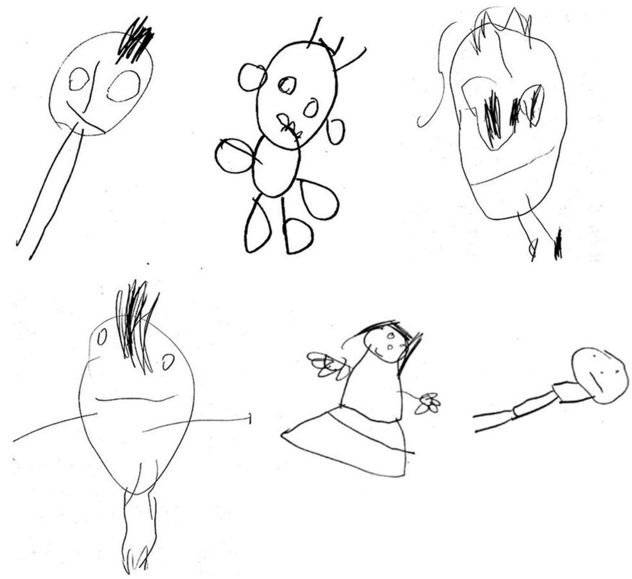

Fig. 3 Draw-a-Child Test. Twins Early Development Study. King's College, London

Piaget and Inhelder (1967). In their studies, drawing tasks such as copying lines, shapes, complex geometric compositions, or scenes played a prominent role in understanding, theorizing about, and assessing cognitive development of spatial concepts. The ability to replicate shapes is assumed to develop in parallel with a child's conceptualization of those shapes. Errors and inaccuracies revealed deficiencies in conceptualizing spatial properties and relationships. "Like a mental image, a drawing is an internal or external imitation of the object and not just a perceptual 'photograph'" (p. 33). That is, children's drawing do not reflect the shapes they 'see,' but represent what they are assumed to 'know'.

Today we continue to see the predominance of drawing tasks used in geometry research as artifacts intended to reveal what children 'know' including their cognitive capabilities, spatial awareness, and geometric understanding (MacDonald 2013; Carlsen 2009; Davis & Hyun 2005). While this literature rarely attends to the role of the body in learning geometrical concepts, the value of drawing and other hands-on activities is prevalent in pedagogical approaches to mathematics learning, albeit from assumptions that learning moves from the concrete to the abstract. However, shifts in theoretical orientation are beginning to question the assumption that drawing serves solely as evidence of cognitive development (Bartolini Bussi 2007).

3 Embodied cognition, mathematics, and drawing

Contemporary theories in cognitive science are transforming long-held assumptions about the nature of cognition and perception. In this paper we view cognition to be a dynamic, contextually contingent, and body-centred phenomenon (e.g., Maturana & Varela 1992; Varela, Thompson & Rosch 1991). Embodied perspectives assume that our cognitive structures and processes emerge with and in our biological bodies, and in the social and cultural contexts in which our bodies are embedded. How our bodies move through space, and our experiences—all of which are historically and culturally specified—have tremendous influence on what it is we come to see and to that which we choose to attend (Thompson 2010).

The implicit role that the body plays in learning both simple and supposedly abstract mathematical concepts, is increasingly being recognized by cognitive scientists. As Lakoff and Núñez (2000) claim, sensorimotor experiences have a significant influence on our creation and understanding of mathematics and "the only mathematics we can know is the mathematics that our bodies and brains allow us to know" (p. 346). For example, the bodily concepts of containment and orientation appear in the definitions and descriptions of early geometric concepts of polygon figures and angles. Even individual mathematicians develop meaning of concepts such as eigenvectors through kinesthetic motion through time (Sinclair & Gol Tabaghi 2010).

Also emerging in the field of mathematics is research that is specific to the role of drawing in mathematical understanding and situated in embodied perspectives of cognition. French philosopher of mathematics, Gilles Châtelet's (2000) notion of mathematical inventiveness that emerges in the interplay of gesture and diagramming provides an alternative to drawing as artifact. He suggests that a diagram is always situated within a narrative and serves as a source of exploration—not a depiction of space (de Freitas & Sinclair 2014). In our work, we attend to the necessary primacy of the body in children's drawings and the potential of mark-making as a site of invention and means of exploration.

Drawing can thus be conceived as a visible trace or tangible by-product of the cognitive processes of our hands, body, and mind in action (Cain 2010); the "kinetic mark" arising from "the physical motion of a drawing body" (Schneckloth 2008, p. 278); and "a method by which to explore the world in its own right instead of being simply a matter of representing an outer world" (Cain 2010, p. 52). This perspective offers an alternative to the dominant view that the brain leads the drawing hand. Rather, an alternative is that we explore the active role that the drawing hand plays in expanding geometric awareness.

We ask what do children (and others) come to know with, in, and through drawings? If drawing is conceived as an act of "becoming aware", a drawing is necessarily never final or complete (Depraz, Varela & Vermersch 2003). Imagined in this way, drawing is the emerging understanding of a body, constituted in the physical, social, and cultural world. Regarding geometric thinking, drawing is not a matter of confirming an external world by fixing it and statically representing it, but a process of "thinking" a world by making information available as part of and through a specific human perceptual experience in the process of becoming. Thus, if a drawing "is not merely a copy nor a perversion, or an expression of a reality; [but that] it *is* a multi-faceted reality itself" (Woodward 2012, p. 14), then drawings as forms of geometric thinking must simultaneously be a matter of making sense in our world as well as making sense of our world.

Studying the whole, drawing, thinking and meaning as nested within physiological, social, cultural and historical perspectives, we conceive of children's drawing as both act and artifact.

4 Methods, data sources, and analysis

Our research focuses on the role of drawing in children's geometric thinking by considering what insights we gain about children's geometric thinking if we interpret drawings as both act and as artifact. The three vignettes in the paper were taken from existing data collected over 2-year periods in second grade classrooms that focused on the embodiment of mathematical knowing and learning. In reviewing this data, three episodes were selected in which children responded with spontaneous drawings to geometric opportunities arising during mathematics lessons that the first author (co)designed and (co)taught. The data collected including video, transcripts, and children's products were reanalyzed to inquire into the potential role of drawings if considered as both act and artifact.

Through the resulting three vignettes, we illustrate our theoretical perspective and examine in depth, the students' drawings as act and artifact. In the first vignette, Nadia and Nathan compose 2D drawings of 3D and 2D objects as they walked around the schoolyard. In the second, Clare and Timothy create drawings based on multilink cube structures in different yet complementary ways. Finally, Mac initiates a series of drawings as he explores shapes composed of or decomposed into triangles. We describe each of the three contexts and specific modes of analysis employed more fully in the vignettes.

In general, our analysis focuses on the children's drawings as act and artifact, their gestures, verbalizations as well as their transactions with objects and others. The same methods of analysis were used for all three data sets. Here we relied heavily on the video recordings, drawings, student-made structures, and field notes collected during the lessons. We (re)viewed and compared the multi-layered data sources of moment-to-moment events (Edwards 2009) several times, verifying our conjectures with each other and against relevant theoretical literature on embodied cognition. In this study, we sought to identify the children's activities in drawing, the geometrical, spatial, and numerical ideas that emerge and the possible relationship(s) amongst these ideas. Moreover, we wanted to gain a deeper understanding for the ways that the children made particular ideas and thinking available to themselves and others, including us as researchers (Nemirovsky & Ferrara 2009) and also how certain understandings impacted the children's geometric thinking. As a result, this research serves to highlight the dynamic, embodied, and contextual ways that the children worked as well as to ground our observations with respect to the emergence of children's geometric ideas and understandings.

5 Nadia and Nathan: out in the schoolyard

5.1 Description of the task and Nadia and Nathan's work

The following case features two second grade students, Nadia and Nathan.[1] In the previous day's lesson, their teacher shared a book with the class (Grades 1 and 2) that involved photographs taken from a variety of perspectives of such everyday objects as teapots, tree logs, saxophones, a horse and carriage, and grapes. The students compared the photographs in the book with 3D and 2D figures that they had sorted and classified. The students then went for a walk in pairs carrying a clipboard, paper, and pencils. Deciding whether to walk around the schoolyard, inside the school, or both, Nadia and Nathan go outside and create the following drawings:

By only looking at the students' completed drawings, we notice that except for two words—"Wheel" and "Door", the page is filled with images. The drawings involve for the most part, 2D shapes and possibly, two 3D objects (* and ** in Fig. 4). We identify six of these images as a wheel, a house, a covered area on the playground (*), a door, an ant (***), and a fence (****). The spaces between the images appear to indicate the students' change in focus from one object to another while their drawings of composite figures such as the house, ant, or door, suggest their work distinguishes different parts of the same object.

Below, we feature three consecutive excerpts and discuss how it is only by studying the video data *with* the children's drawings that we are able to identify the actual objects of their focus and gain a deeper understanding of the geometric distinctions that the children and researcher, Mr. Martin, make of, with, and during the drawing the images, *a*–*k*. We also examine how particular concepts and conceptions emerge and shift the thinking of Nathan, Nadia, and Mr. Martin as they walk around the schoolyard. Prior to Mr. Martin joining them, the children had looked at and produced drawings of buildings, a door, leaves, a house, and a road sign (i.e., unlabeled drawings in Fig. 4).

5.2 Exploring Nadia's and Nathan's drawings

5.2.1 Episode A

Mr. Martin and Nathan look at a basketball hoop that Nathan identifies as "[a] square [i.e., painted on the backboard]… and a circle [i.e., hoop]!" Meanwhile, Nadia sits on a bench, drawing. Nathan and Mr. Martin turn to look at what she is drawing:

> Nadia: (pauses briefly) I see electric wi-res with a p-ole (uses pencil to trace the outlines).
> Mr. Martin: And what shape are they, the wires and the pole?
> Nadia: (without making any marks, uses pencil to trace the outlines quickly and not completely). They are little straight lines but this pole is… um… table… pa-per… shaped just like the paper towel roll.

In this short episode, we observe the emergence of certain geometric ideas and understandings that define how Nadia, Nathan, and Mr. Martin think about the schoolyard and the ways that they respond to one another. For example, as Nathan and Mr. Martin shift from thinking about the basketball hoop to what Nadia draws, this occasions visual, verbal, physical and temporal ways of thinking as she explains the image of horizontal lines and oblong ("b" in Fig. 4) to be the "electric wi-res with a po-le". Here, Nadia takes an observer viewpoint as she traces over the 2D images. Mr. Martin then inquires about the geometric shapes of the wires and pole. Nadia responds that the wires are "little straight lines but this pole is… shaped just like the paper towel roll". Here, we notice other aspects of her geometric thinking. Nadia articulates the wires as straight lines (1D) and the pole that appears as an oblong (2D), is *just* like the paper towel roll (3D). It is intriguing that although Nadia states that the pole is just like the toilet paper roll, she draws straight not curved lines. Here, we might say that Nadia decomposes the pole from a 3D object to a 2D figure, "obtained from an intersection plane of the solid" (Duval 2014, p. 166). And just as her use of the word, "but", implies a comparison between the wires and the pole, it also suggests that the lines and the paper

[1] Colleagues, J.-F. Maheux and W.-M. Roth, who collaborated on this research project previously published a part of this excerpt. Their data analysis focused on relationality and mathematical knowing. See, Maheux, J-F. & Roth W-M. (2011). Relationality and mathematical knowing. *For the Learning of Mathematics, 31*(3), 36–41.

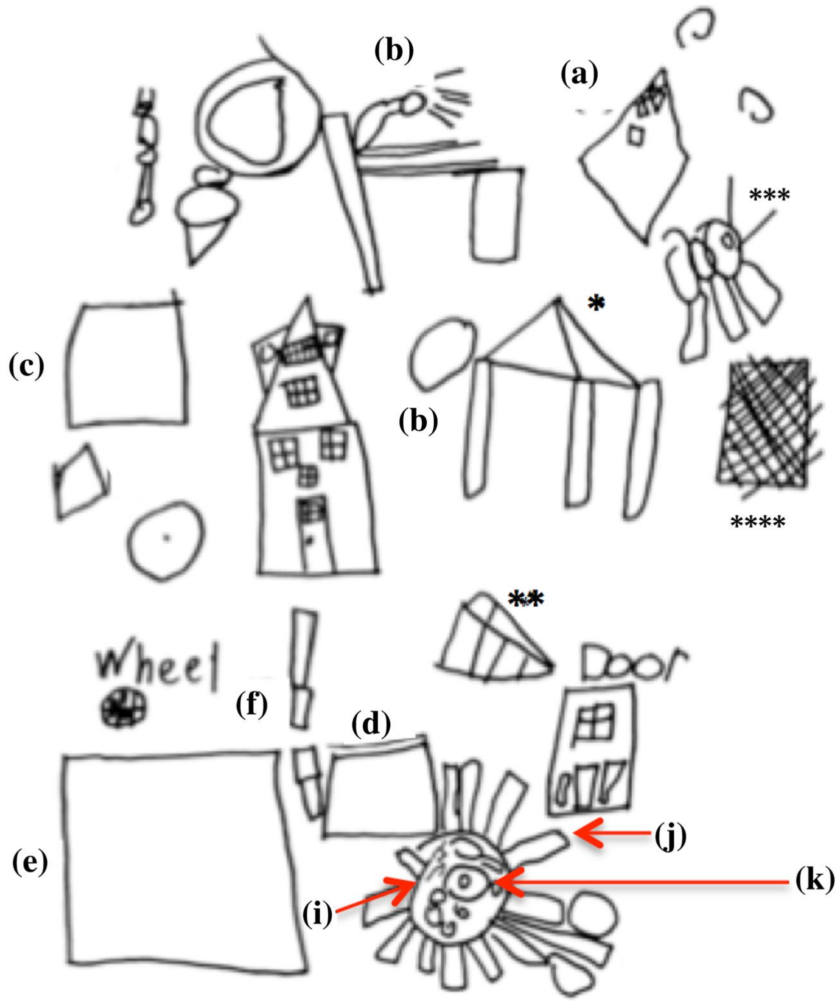

Fig. 4 Nadia and Nathan's drawings and the partial observed sequence, *a–k,* of their drawings

towel roll are different from each other. We find it significant that Nadia traces over the figures with the pencil before she verbally identifies the figures. The drawing of the wires and pole seem to evoke gestures that recursively re-enact the process of drawing itself and it is her work in retracing the drawing that eventuates the verbal identification of the figures in (more) specific geometric terms (Holt & Beilock 2006; Ping & Goldin-Meadow 2010). Although Nadia does not use the term, "cylinder" she relates a previous lesson where the class sorted cylinders that included "long circle[s]", "towel circle[s]", "paper towel roll[s]", "toilet paper roll[s]", and "[r]ound and round t-i-r-e circle[s]"—the latter, a name that originated from Nadia (Thom, Roth & Bautista 2010, p. 81). This connection to shared experiences allows Nadia and the others who participated in the lesson to now conceive the pole as "just like the paper towel roll" and with properties such as curved surfaces, circular tube-like structures, and the ability to roll like a tire. In vital ways, these visual, physical, verbal, and temporal ways of thinking about the wires and pole enable Nadia, Mr. Martin, and Nathan to shift their thinking from everyday conceptions of these objects toward more geometric ones.

5.2.2 *Episode B*

Mr. Martin: What about the field? What shape is it?
Nadia and Nathan turn to face the field.
Nadia: (scans the perimeter of the field) A… (extends her arm, uses the pencil to outline the field) … square!… a big square (draws a square. See "e" in Fig. 4).

In this episode, we are intrigued by the events that precede Nadia's actual drawing of a square onto paper. First, while the students study the field, Nadia visually scans the perimeter. She then holds the pencil in the air and traces its outline. It is only after she does this that she says that the field is a square and draws one. As in the last episode, these visual, physical, and temporal actions of "drawing" seem to be integral to Nadia's verbal identification and physical drawing of the shape of the field as a square

(Grafton & Hamilton 2007; Hostetter & Alibali 2008). In contrast to representational views that assume it is the mathematical concept which leads the hand, we argue for a radically different view; one that allows for the possibility that the conception of squareness as a 2D event, arises out of Nadia's hand as it creates the line drawing (Châtelet 2000).

Given this, it also seems important to mention that the (un)conscious ideas and conceptions that Nadia, Nathan, and Mr. Martin articulate and later relate to other objects in the schoolyard cannot be taken to be inherent in the objects themselves but instead are realized transactionally with Nadia, Nathan, and Mr. Martin as they continually relate their thinking in specific ways to the drawings, the process of drawing, and the environment. Their verbal responses in the episode and previous to it such as, "What square? Where do you see a square?", "Ohhhhhh!", "Oh, I see a square on top of the house!", "Oh yeah!" "Oh yeah, there's another one there", and "Square!" as well as Nadia's gestures of tracing the outlines of the drawing of the wires and the pole and the field, for example, evidence their thinking as immanently emergent. It is unlikely that Nadia, Nathan, and Mr. Martin are unfamiliar with the schoolyard and more so, that this is the first time they are deliberately contemplating the schoolyard in a geometric manner.

5.2.3 Episode C

The three walk around and look at the school from different perspectives, identifying a triangle, square, and oblongs. Nadia then draws oblongs ("f" in Fig. 4) while Nathan verbally repeats the names of these.

Nadia: Now what do we see?

They get up and run to the fence and look through it. Nathan points to an object that Nadia identifies as a circle. He draws a circle (see "g" in Fig. 4). Nadia then runs ahead, takes a leap and lands in front of a red plastic lid from a bottle.

Nadia: I see a circle right here! (twirls around)
I'm going to put down a circle! Here… (reaches for the clipboard and draws a circle—see "h" in Fig. 4).
Nadia: (walks then stops and looks down). Flowers… have some sort in it….? (raises, lowers, and raises her voice as she speaks while she points with the pencil and moves it in a circular motion).
Nathan: C-i-r–c-l-e (raises and lowers his voice as he says this).
Mr. Martin: Yeah, a kind of circle.
Nadia: C-i-r–c-l-e (draws a circle, "i" in Fig. 4, raising and lowering her voice as she speaks and draws) and then they have the little things—petals, are the little rolling things (repeatedly uses fingers to outline an oblong).
Mr. Martin: Exactly. Why don't you draw that? Good job!
Nathan: (holds clipboard while Nadia draws oblongs around outside of the circle, "j" in Fig. 4).
Mr. Martin: Those are more like shapes or objects? The flowers, what do you think? They're objects or shapes?[2]
Nadia: They have shapes on top of them but they're objects.
Mr. Martin: Go—od.
Nathan: Almost done (looks at what Nadia has drawn).
Nathan and Nadia draw together.
Nathan: Little circles in it (adds to the drawing, see outer circle of "k" in Fig. 4)
Nadia: Circle… (draws a circle around the smaller circle, see inner circle of "k" Fig. 4).

Here, it is the idea of a circle—first visualized as an object through the fence, then as a container lid and finally, articulated in different ways within the daisy. As in the other episodes, Nadia and Nathan's drawings and their processes of drawing are inseparable from their other ways of knowing. For instance, Nadia's enthusiastic matter of fact way she twirls her body around and declares the container lid a circle resonates with how she effortlessly draws the circle (see "h" in Fig. 4). In contrast, Nathan and Nadia slowly approach the flower, take time to examine it, and eventually draw it with great deliberation ("i–k" in Fig. 4). Connected with this, when Nadia says that, "flowers… have some sort in it…?" while raising, lowering, and raising her voice as she speaks, we take this statement to be both a conjecture and an invitation for Nathan and Mr. Martin to engage. Nadia points to the centre of the daisy and moves the pencil in a circular motion several times. She neither says anything about a circle nor does she draw a circle. It is only after Nathan says, "c-i-r-c-l-e" in a slow and considered way while raising and lowering the pitch of his voice and Mr. Martin agrees that Nadia responds by saying, "c-i-r-c-l-e" just as Nathan had and draws a circle in a synchronous manner as she says its name, starting at one point and creating a line that continuously curves and eventually returns to the point where it began ("i" in Fig. 4).

These moment-to-moment shifts in their thinking beautifully illustrate how the children, their processes of drawing, and the drawings themselves exist "in and of the

[2] In keeping with the provincial curriculum, the class referred to 2D figures as "shapes" and 3D figures as "objects".

flesh"; that is, how they come into being as whole body engagements that require the senses to evoke spoken words, bodily orientations, gestures, movement, kinetic marks, engagement with others, and so on (e.g., as discussed in Thom & Roth 2011). Thus, such bodies are bodies that extend beyond the skin of individual and are embedded and embodied within ever increasing collectives that include the world at large.

When Nadia describes the petals as, "the little things—petals", she adds that they "are the little *rolling* things", repeatedly gestures the outline of an oblong with her fingers, and then draws oblongs, not only do we see articulation of geometric aspects about the petals within her thinking but perhaps, another connection to the electrical pole and toilet paper roll. It is possible that she conceives cylinders and circles in an integrated manner. This is because she previously classified cylinders to be "round and round t-i-r-e circle[s]" (as discussed in Thom et al. 2010, p. 81) and in the above episode, Nadia says, "c-i-r-c-l-e" in the same deliberate and rhythmic way, varying the pitch of her voice as she had done when she said, "t-i-r-e circle". She also explains that shapes are what are *on top* of objects, enabling one to focus on the faces of a cylinder and identify the circles on the ends.

The episode concludes as the children complete the drawing together, drawing smaller circles inside the larger ones ("k" in Fig. 4). Here and throughout the three episodes, Nadia, Nathan, and Mr. Martin exemplify drawing as act and artifact as well as that which is neither an individual nor independent event. The act of drawing and the drawn possesses the potential to provoke the thinking of the person or people doing the mathematics but also others' thinking through engagement. Drawing the circles and the circles as drawn occur with Nadia, Nathan, Mr. Martin, and the environment as they shift and change their thinking, make geometrical-spatial distinctions, and bring into being, the idea and meanings of circles with particular objects.

6 Clare and Timothy: triangles, pyramids, and a number pattern

6.1 Description of the task and Clare and Timothy's work

This excerpt focuses on two other second grade students—Clare and Timothy, and highlights how their geometric thinking about vectors emerges with and in their drawing(s). Together with their second and third grade classmates, Timothy and Clare watch the film, *Notes on a Triangle* (National Film Board of Canada 1969), that features animated compositions, decompositions, and combinations of equilateral, scalene, and isosceles triangles. Several students then decide to work in pairs and small groups to explore, identify, and explain the similarities and differences among the three types of triangles in the film. It is during the class discussion as the students explain their comparisons of the triangles that Clare, Timothy, and three other students leave the carpet and return with the second and third of four 3D multilink cube "pyramids" that they had constructed in a completely different activity earlier that week (Fig. 5). Clare and Timothy turn the pyramids over and show the class that the bottoms of the models "look like triangles too!" (Fig. 6).

The class quickly moves from examining the variants and invariants of triangles to looking at the first four triangular bases of the pyramids and notices that the number of cubes increases from one to three to six to ten. The children conjecture that the number of additional cubes in the base of any next pyramid will be "one more than the last number"; that is, 1 (then add *two* more), 3 (then add *three* more), 6 (then add *four* more), 10, and so on. This geo-numerical connection prompts the children to explore the idea further. In the classroom, there is always a variety of materials on hand, such as geoboards, calculators, plain paper, graph paper, multilink cubes, counters, and so on. Each on their own, Clare and Timothy decide to use dot

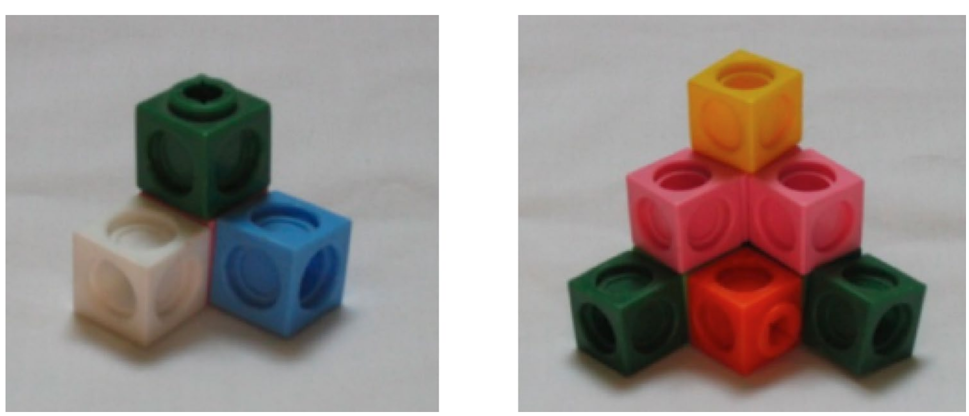

Fig. 5 Photographs of the second and third multi-link *cube pyramids*

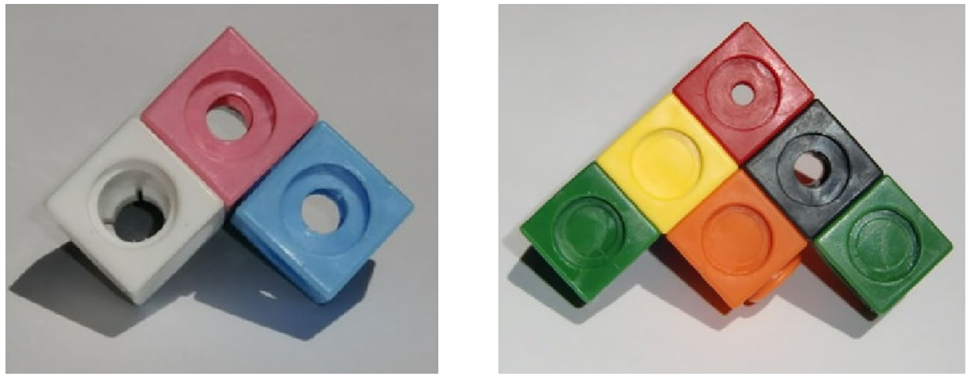

Fig. 6 Photographs of the *bases* of the second and third multi-link *cube pyramids*

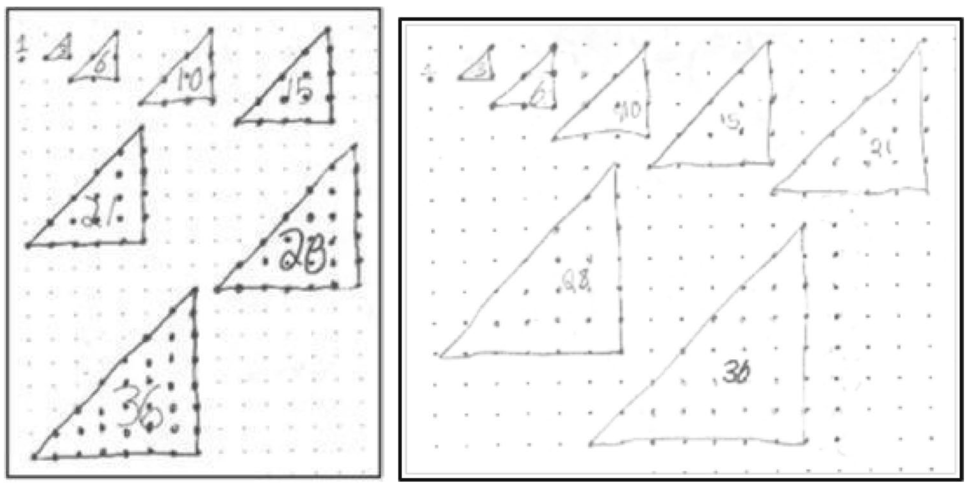

Fig. 7 Timothy's and Clare's drawings of the first through eighth *cube pyramid bases*

paper, telling Jennifer (researcher/teacher) that they will draw in order to continue the number sequence observed in the bottom of the built pyramids. When Jennifer asks how they will use the dot paper, they explain that they could join four dots together to form a square for each cube in the bottom of a given pyramid or count one dot as one cube. Clare and Timothy decide that the latter takes less time and they create the following drawings shown in Fig. 7.

6.2 Exploring Timothy's and Clare's drawings

Clare's and Timothy's drawings, identical in appearance, demonstrate their recognition, conception, and presentation of the cube pyramid from a different perspective; that is, they use a triangular area model to translate the first four pyramid bases from a 3D cube structure to a 2D one and as well, draw the unbuilt fifth through eighth pyramid bases quantitatively as dots.

Timothy uses his drawing as he explains what he thinks the next number in the pattern will be: "I think the next number will be 45 because one, two, three, four, five, six, seven, eight, and… thirty-six [points to "36"] plus nine [sweeps his index finger along the dots that are one dot away from the hypotenuse of the eighth triangle] equals forty-five [points to the middle of the eighth triangle]. So forty-five is the next one". He then places the edge of each of his hands along sides *b* and *c* of the eighth triangle in a stationary 90° position and moves both of his hands in an outward motion, re-enacting how he drew the two sides of the triangles (Fig. 8).

Alternately, Clare notices that the sum of dots of each pair of consecutive triangles—e.g., the first and second triangles, the third and forth triangles, and so on, within the series follows an "odd, odd, even, even" sum pattern. She continues as Jennifer records: "The one dot. The second number has two [more], the third you add three, the fourth you add four, the fifth you add five, the sixth you add six, the seventh you add seven, the eighth you add eight. The next one you add nine to… So thirty-six, thirty-seven, thirty-eight, thirty-nine, forty, forty-one, forty-two, forty-three, forty-four, forty-five". Then, pointing with her two index fingers at the corner of each triangle, Clare slides her fingertips along line segments *a* and *c* in an outward motion to show how she drew the lines while making the triangles. She then extends the two lines on the eighth triangle by one dot each (Fig. 9), explaining that "The one dot. The second number has two [more], the third you add three… the fourth you add four… add nine…" and points to the vertical

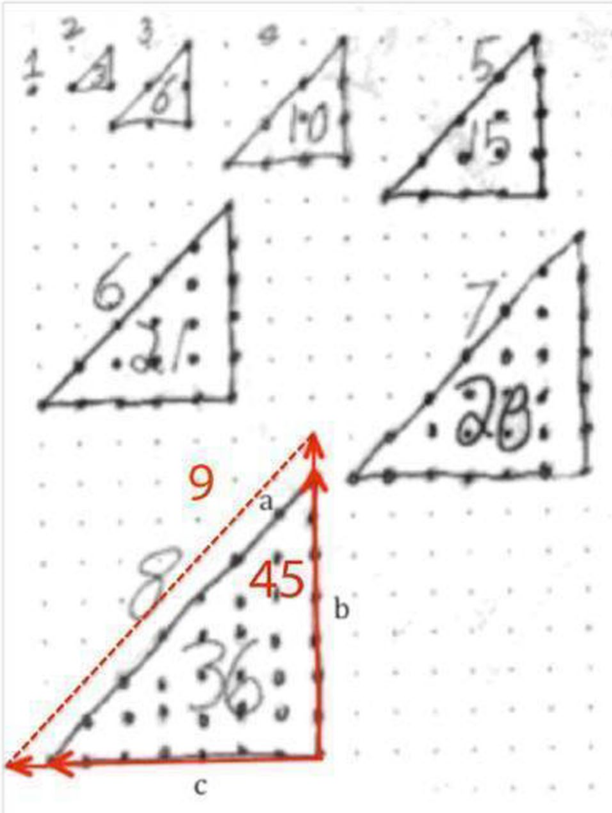

Fig. 8 Timothy's solution for the ninth term/*pyramid base*

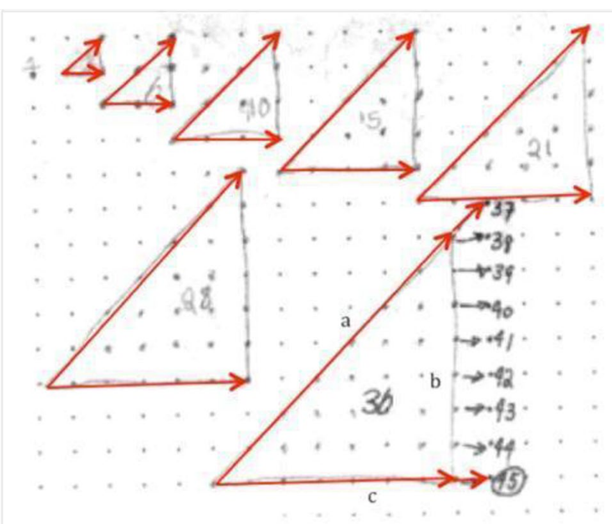

Fig. 9 Clare's hand gestures for each *triangle* and her solution for the ninth *pyramid*

line segments indicated by "b" on the triangle. When she counts 36–45, she points to "36" inside the eighth triangle, identifying the previous term in the numerical sequence and 37–45 as the dots that produce the next line segment *b* which will be 9 dots in length. Together, the sum of the dots is 45.

It is only by inquiring into how Clare and Timothy create their drawings that important insights into their geometric conceptions and reasoning are possible. We now discuss how the students make geometric sense of the pyramids, their drawing(s), and the number pattern.

It is clear from the embodied ways that the students conceptualize the terms in the sequence in relation to the built pyramid bases and how they draw and extend these involve not only an area model but also the notion of the triangles as arising out of their drawing of vectors across time. What we initially regarded as static, distinct, and sequential triangles are now 2D figures emerging from vectors that express in a dynamic way, a geo-numerical sequence in which each triangle exists as a specific moment within the continuous increase in magnitude and direction of the vectors.

We also notice that although Clare and Timothy's drawings appear identical and share overlapping conceptions, the students conceive the triangle in subtly different ways. Clare visualizes the triangle arising from vectors *a* and *c* while Timothy sees it emerging from vectors *b* and *c*. Given this, we realize how the students' focus on a particular corner of the triangle offers both a point from which the pencil can begin to draw lines as well as how the lines once drawn, enable hand gestures as vectors. Further still, even though Clare and Timothy's ways of thinking are geometrically and numerically compatible, it is their focus on different corners of the same triangle that occasions critical differences in what is possible for them to see, feel, and think as they make sense of what they are drawing or have drawn (as discussed in Thom 2011). For instance, how Clare draws, gestures, explains or even perceives what she is drawing|has drawn|might draw unfolds as she creates vectors that produce a 45° angle. Likewise, Timothy's actions and knowings are defined during the embodied moments in which he creates triangles that originate from 90° corners. Akin to Châtelet, the students' drawing(s) and their spoken words, written text, and gestures, "are embodied acts that constitute new relationships between the person doing the mathematics and the material world" (de Freitas & Sinclair 2012, p. 134).

Clare and Timothy's work that involves conceptualizing the pyramid bases as a number pattern is neither trivial nor simple but significant and complex. The ways they engage in drawing enable them to work connectively with, in, and through several dimensions—0D with single points or dots, 1D with lines segments and vectors, 2D with triangles, and 3D with the horizontal base of the built pyramids. Also remarkable is the dynamic and fluid quality of vectors that afford Timothy and Clare to work simultaneously in arithmetic and geometric ways to make sense of the identified patterns. Thus, their drawings do not serve as just representations of their knowledge but rather, each drawing created and situated within a narrative, exists as such and enables critical insight into the profound ways in which the children body-forth their mathematics.

7 Mac: "What other shapes can be made from triangles?"

7.1 Description of the task and Mac's work

The following excerpt features Mac—Clare and Timothy's grade 2 classmate, as we observe his geometric reasoning in relation to composite 2D figures. Unlike his classmates who choose to compare the three triangles after watching the film, *Notes on a Triangle* (National Film Board of Canada 1969), Mac returns to his desk, picks up his pencil, reaches for a large piece of plain paper, and asks, "What other shapes can be made from triangles?"

Mac produces the following drawings shown in Fig. 10.

Mac identifies the four 2D shapes as: hexagon, oblong, square, and trapezoid. Interestingly, while all of the shapes are composed of smaller triangles, none of these appear in the film. There is mirror symmetry among the triangles that form the oblong and square as well as other lines of symmetry among the smaller triangles within the trapezoid and the hexagon. Additionally, the trapezoids, squares, and smaller triangles that constitute the hexagon exhibit rotational symmetry.

Now, if we look at how Mac solves for the tasks, we observe the following Figs. 11, 12, 13 and 14.

The order that Mac solves for the figures is: (1) square, (2) oblong, (3) trapezoid, and (4) hexagon. He first draws

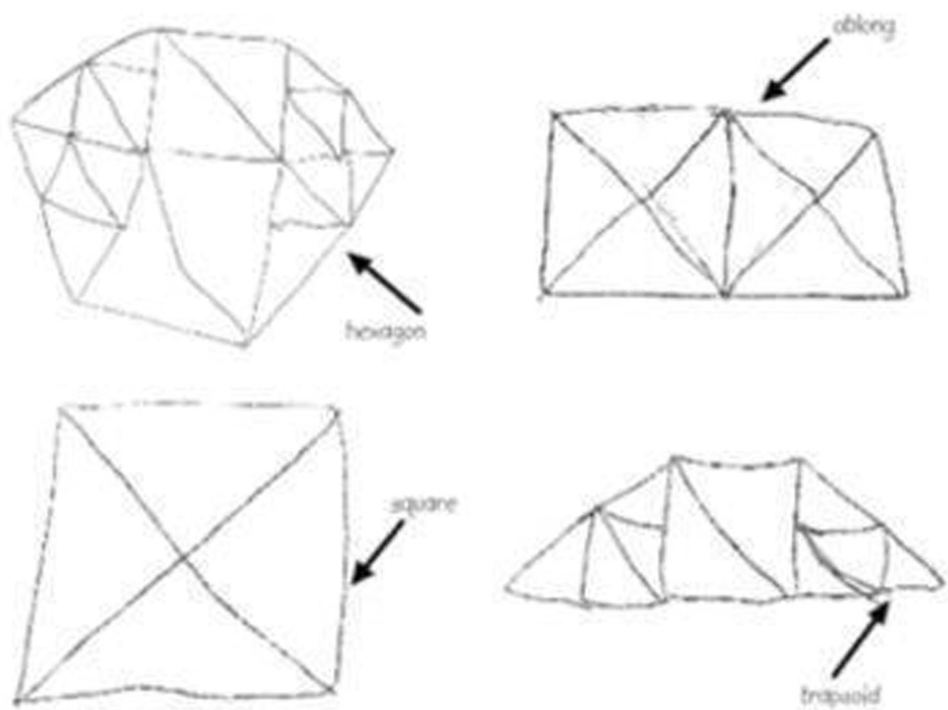

Fig. 10 Mac's drawings of other shapes that can be made from *triangles*

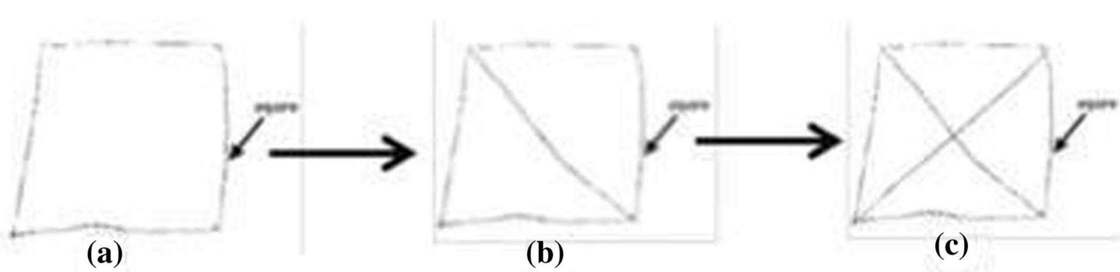

Fig. 11 Mac's sequence in drawing beginning with a *square*

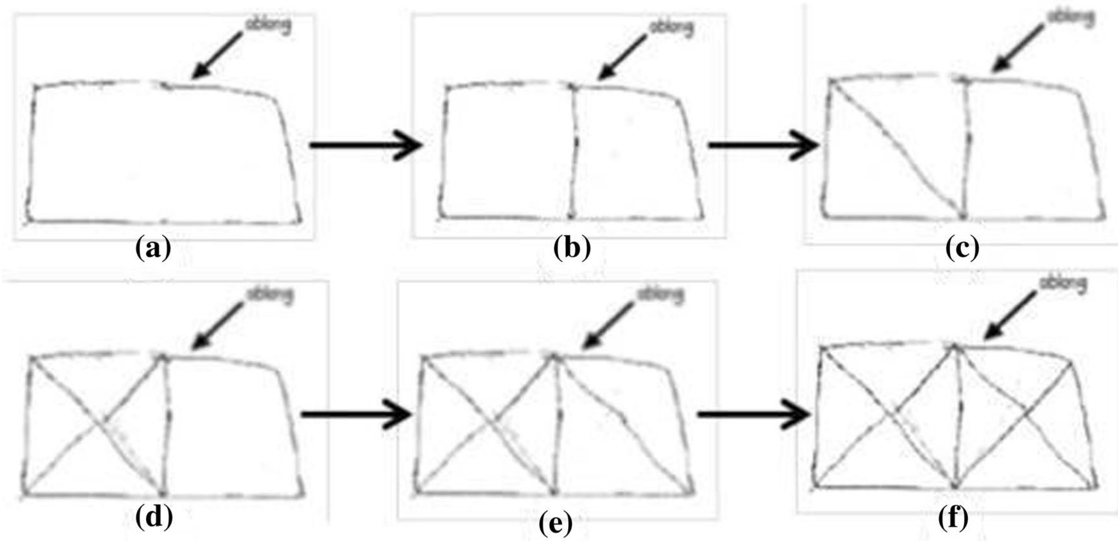

Fig. 12 Mac's sequence in drawing beginning with an oblong

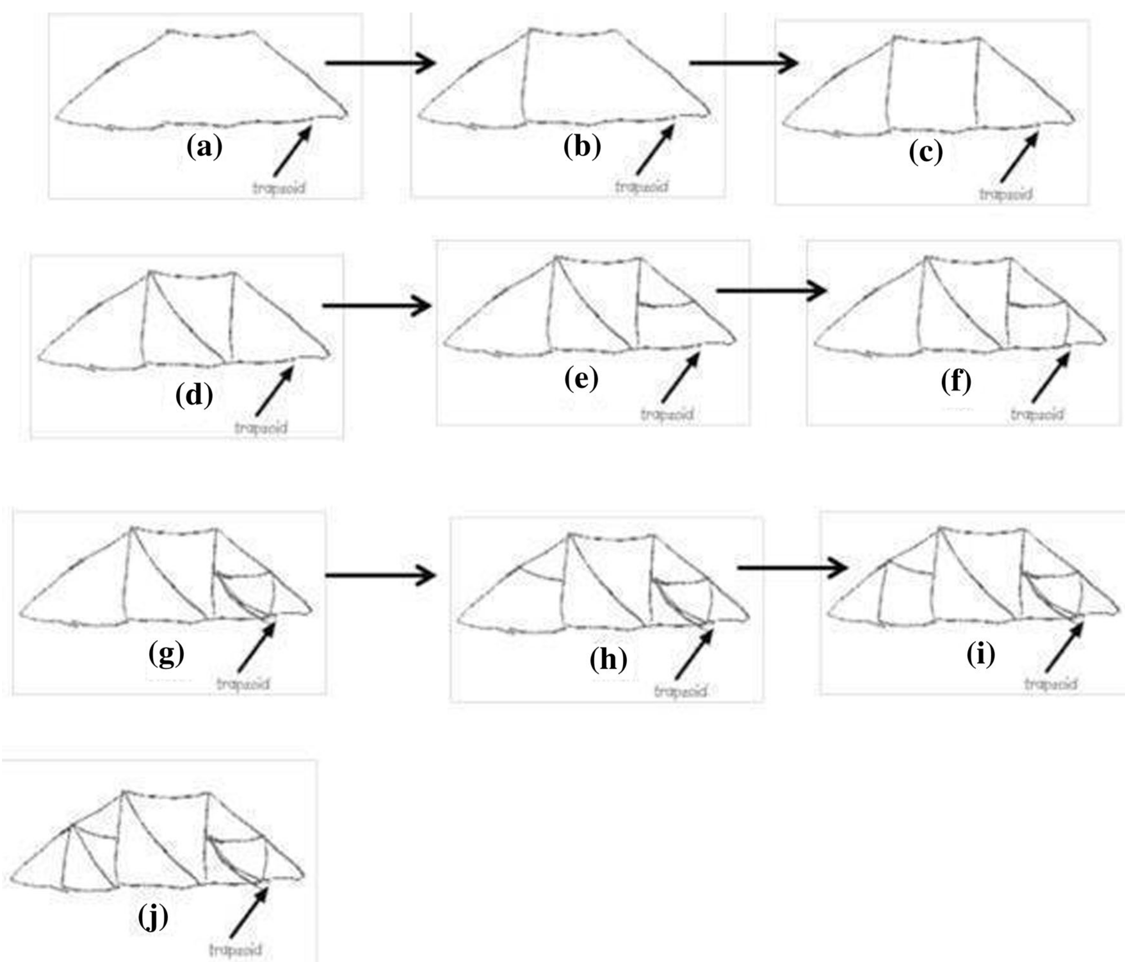

Fig. 13 Mac's sequence in drawing beginning with a *trapezoid*

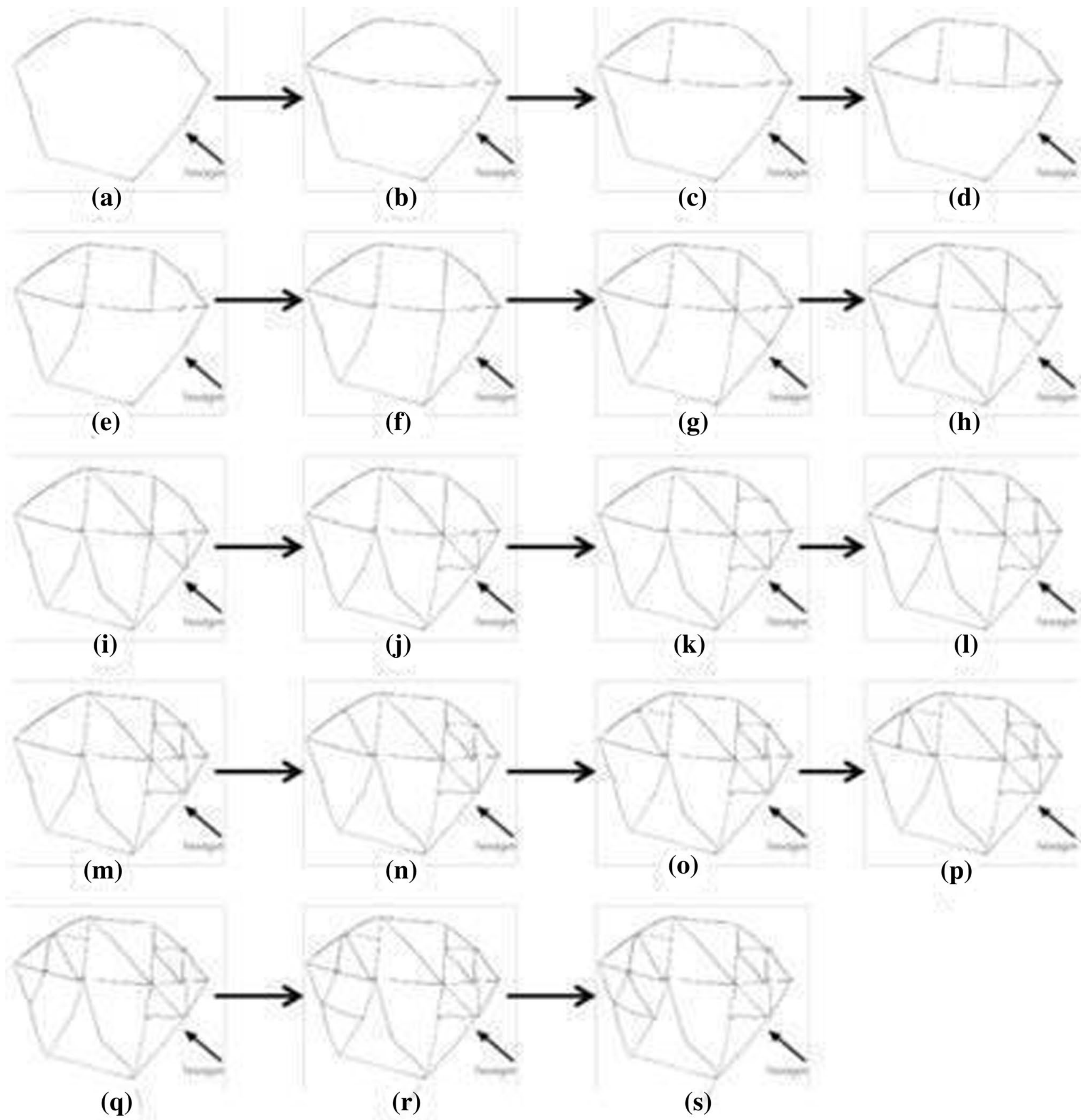

Fig. 14 Mac's sequence in drawing beginning with a *hexagon*

2D figures other than triangles and in so doing, appears to use a process of decomposition to solve the problems. We wonder if Mac simply demonstrates what he already knows about these shapes. However, as we examine Mac's drawing(s) in detail (Figs. 11, 12, 13, 14), we see the acts and artifacts to be his geometric thinking as it evolves with the triangle and in relation to other 2D shapes. In brief, Mac makes new and continuous sense of other shapes that can be made from triangles as he draws. An emergent event then, we cannot assume Mac's geometric understanding of the triangle in relation to other 2D shapes as that which can be predetermined by him or us. We consider several observations to support this view. First, Mac does not approach the work with any definitive statement such as, "Shapes can be made of triangles" or "Other shapes that can be made of triangles" or even asking, “What shapes *are* made of triangles?” All of these would prompt Mac to draw what he already knows to be true and likely what he sees in the film, for example, a hexagon composed of six equilateral triangles. In contrast, we take Mac’s question as genuine and the geometry he achieves originates within his inquiry. Third, as Mac draws, he purses his lips and makes comments like “let’s see…”, “hmmm…”, and “I wonder…” Several times, Mac pauses and either looks at the figure he is working on, tilts his head in a different direction, or looks at what he has drawn. These verbalizations and physical actions suggest his unfamiliarity with the task, possibly his efforts to make sense of something that is not yet drawn, as well as a playful curiosity and intentionality for what he draws next. Fourth, we surmise that Mac’s consistent approach to the square, oblong, trapezoid, and hexagon by beginning first with the ‘answer’ and then working backwards is not so much an act to re-present what he already knows but rather, the contrary; that is, the four figures are first conjectures and what he draws inside these shapes is the development of his geometric thinking as he explores whether and how these figures might be composed of triangles.

7.2 Exploring the acts and artifacts of Mac’s drawings

It is by studying Mac’s drawings as they evolve that we see him develop, execute, and justify actions of subdividing, combining, and transforming 2D shapes. We mention earlier that as Mac consistently draws the 2D figures first and then works to identify smaller triangles within the figures, the order suggests he uses decomposition as the strategy for solving the problems. However, in analyzing the artifacts and his mark-making, we also see evidence of figure composition. While Mac draws 2D figures and eventually expresses these in terms of smaller triangles, he also uses drawn figures to comprise the next shape. For example, after solving for the square (Fig. 11), Mac draws the oblong as two squares (Fig. 12). While it is unclear if Mac’s reasoning arises as a form of decomposition or as composition, we cannot help but notice the mutual way in which the squares and the oblong appear with and enfold each other.

We also observe how Mac makes use of spatial memory and geo-spatial visualization in the figures he draws. For instance, as he produces 2D figures, there is coherence across the figures he creates and investigates further. Mac begins his inquiry of other shapes that can be made from triangles with a square and then proceeds to solve for an oblong. Following this, he considers the trapezoid and finally, the hexagon. Thus, we can say that Mac’s work progresses from one figure to the next in a simple to more complicated manner. Differently, we can say that it is the square that is nested within and from which all triangles emerge across the four figures. Both cases require Mac’s recognition of 2D figures and his ability to relate these shapes to one another by applying them to new situations, using this knowledge to inform what he draws next. Mac also demonstrates specific geometric relationships with and in his drawing(s). For example, he draws the hexagon as a six-sided 2D figure that includes two identical trapezoids that share a horizontal line of symmetry.

When we study the actual pencil marks that Mac makes to form lines within a figure, we notice that while certain marks resemble earlier ones, these also bring forth new geometrical actions and objects. For instance, as Mac solves for the trapezoid and later the hexagon, he draws a diagonal line, dividing the square in half, and produces two isosceles triangles. Upon making the next mark, he does not draw a second diagonal line to make four triangles as he did with the oblong and before that with the square. Instead, he creates squares in the triangles on either side of the first square and diagonally divides these into smaller triangles (“e–k” in Fig. 13). And, when Mac solves for the hexagon and draws a line across it so as to deal with the figure as two trapezoids, he makes a line that again divides the square diagonally in half in the trapezoid but this time, continues the line beyond the square and into the triangle (“g” in Fig. 14). The result surprises Mac (i.e., he exclaims, “Oh?!”) and prompts him to draw and reason anew. Here, in order to conserve the idea of a square composed of two isosceles triangles inside a larger triangle of the same kind, he draws two new lines (“i–j” in Fig. 14) that are of a rotation or reflection of the lines in the previous trapezoid (“j” in Fig. 13).

In these manners, each time a figure is drawn and each time Mac draws, new possibilities arise. Thus, what and how he draws next is always in some way changed. It is this aspect of same-yet-different that continually engages Mac to evolve his thinking in response to the shapes and marks as they emerge on the paper. Our further reflection on Mac’s work reveals how the drawing itself and Mac’s geometric thinking co-evolve in unpredictable, mutually responsive, and recursive ways. Just as the drawing is brought into being through Mac’s drawing of it, it is the drawing as act and artifact that provokes Mac to respond to it. Together and in radical ways, both (in)form the other as co-emergent selves, very much engaged in conversation with each other where what has been drawn invites that which is yet to be drawn. If we look closely at Mac’s drawing of the hexagon, we might say that it is the drawing’s empty spaces (“c” in Fig. 14) that occasion Mac to make other lines within the smaller and larger trapezoids. As well, new symmetries arise in the hexagon and thus, in Mac’s geometric thinking, as he creates three new lines ("d-f " in Fig. 14). And, when Mac is caught by surprise as he draws and watches the diagonal line (“g” in Fig. 14) not stop but continue and pass through the corner of the square, he responds to the new and unexpected line by drawing a similar line in the space below and as a result, creates parallel lines.

8 Concluding remarks

Drawing activities are frequently invoked in primary classrooms to support geometric learning and spatial awareness, and to assess what has been learned. In many instances, children's drawings are assumed to serve as stand-ins for or representations of children's internal cognitive schema. In this paper we use the theoretical orientation of embodied cognition to dissolve the internal-schema/external-representation divide by examining children's drawings as both act and artifact. From this theoretical perspective, we offered and analyzed three vignettes involving the spontaneous drawings of second grade students in response to spatial-visual and geometry lessons. We attended to the drawings themselves as artifacts and also to the act of drawing including the temporal marks on the page, the children's gestures, and their spoken words.

Across the three vignettes, we observed how the act of drawing objects allowed the children to bring forth geometric-spatial distinctions in 2D and 3D shapes. That is, the children did not appear to be representing what they already knew, but the geometric properties became evident to themselves and others through the drawing event and upon reflection of that event. We noted that multiple drawings by one or more of the children expanded their understanding of pattern, shape composition, and geometric relationships allowing new insights through the drawing and in the accompanying narrative as they made comparisons within and between drawings. The multiple drawings also offered opportunities to attend to a series of different geometric aspects and spatial relationships revealing emergent understandings and an evolution of thinking. The marks on the page provided fodder and provocation for new acts of drawing that extended the geometric experience. The whole body engagement of hand, gesture, movement, mark-making, and interaction provided a means of exploration and invention as the children made their geometric insights available to themselves and others.

From a pedagogical perspective, our research suggests that drawing products do not necessarily provide evidence of children's geometric understanding and may, in fact, overlook critical aspects of their understanding; however, providing opportunities for spontaneous and prompted, iterative, or multiple drawings may serve an even more important purpose as a space for geometric exploration and conceptual invention. Educators may not always be able to witness the drawing event, but by inquiring into the emergence of children's drawings through individually and collectively sharing their descriptions and re-enactments keeps the drawing narrative intact and enables the possibility for more complex understandings of children's geometric thinking.

Our research points to the significant role that the drawing body plays in children's mathematical inquiries. Not only do the students spontaneously choose to draw as a mode of thinking, but also the different ways they draw and the conceptions that arise with, in, and through their drawing(s), contribute substantially to their individual and collective geometries. The children as well as peers, teachers, and researchers use the acts and artifacts of drawing as visual and kinetic geometric tools with which to present, conceptualize, and solve for the problems posed. Just as Châtelet (2000) theorized that mathematician's drawings emerged out of and through gesture, we observed the children's drawings as whole body acts of meaning, sense-making, and intentionality. Their drawings and their activity in drawing, thus, do not necessarily re-present particular concepts or give rise to geometric ideas retrospectively. Nor do the mathematics or the children's mathematical conceptions inherently exist in the materials or contexts. As Cain (2010) proposes and the children in this study demonstrate, it is their geometric reasoning which takes place within the flow of drawing and creating diagrams, making it "a matter of learning as much as it [is] a matter of thinking" (p. 32). If we pull the drawings from the very contexts and narratives in which they arise—the gestures, verbalizations, actions, transactions, and so on—multiple threads of meaning are severed. Geometric ideas and conceptions emerge through and connect with precisely what is seen, heard, touched, felt, and moved. This is the locus where students, the subject-object of inquiry, and their understandings of geometry come into being. It is where the articulation of meaning is ever-constant, always partial, and inevitably unfinished.

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