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ORIGINAL ARTICLE

Using enactivism as a methodology to characterise algebraic learning

Maria-Dolores Lozano

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Abstract My purpose in this paper is to illustrate the way in which an enactivist methodological approach guided me as I conducted a two-case longitudinal study where the learning of algebra was explored in different contexts throughout time. Three groups of students in two different schools in the city of Puebla, Mexico, were followed from the last year of primary school (Year 6) to the second year of their secondary education (Year 8). Learning was characterised as the ongoing structural change that allows individuals or groups to act effectively in a changing environment [Maturana (GAIA, a way of knowing: political implications of the new biology. Lindisfarne, New York, pp 65–82, 1987)]. An enactivist methodology, which revolves around the idea of research being a form of learning [Reid (Proceedings of the 20th conference of the international group for the psychology of mathematics education. PME, Valencia, pp 203–209, 1996)], implied that, as I carried out the study, what I was doing was learning about how people learn algebra. My initial questions arose from my experiences with the teaching and learning of mathematics, which gave me a sense of the complexity involved in the learning processes. Later, as I became immersed in the process of investigation of the teaching and learning of algebra, my conceptions continually evolved. The meaning of the phrase 'algebraic learning', which I used as a way of maintaining a wide perspective that allowed me to explore the events in the classroom in a complex way, arose as I engaged with enactivist ideas about learning, with the research literature on the learning of algebra, in conversations with people and in interactions with the participants of my project. I went through a process of continual development and change which I describe in this paper.

Keywords Enactivism · Algebraic learning

1 Introduction

> Science is a human activity and must be understood as such because it is in the domain of human relations that it exists. […] I consider science in its relations to daily life and look at the scientist and his or her desires and aims to understand it. (Maturana 1991, p. 30)

In this paper I describe how enactivism, a theory of knowing which stems from the work of Maturana and Varela (1992), allowed me to investigate and account for different aspects that influence the learning processes in relation to algebra. My emphasis is on the discussion of the methodological approach that guided a research project which consisted of a two-case longitudinal study that had as its purpose to investigate, in a detailed manner, the learning of algebra in different contexts. Discussions and analysis of the results of the study can be found elsewhere (e.g. Lozano 2008).

2 Doing research from an enactivist perspective: the fundamental circularity

In enactivism, researchers are seen as individuals developing their learning in a particular context and the research process is considered a learning process (Reid 1996). This stance has important methodological implications, which

M.-D. Lozano (✉)
Universidad de las Américas Puebla, Sta. Catarina Mártir, C.P. 72810 Cholula, Puebla, Mexico
e-mail: maria.lozano@udlap.mx

are inevitably intertwined with the enactivist view on cognition. I will consider some aspects of this position and later I will relate these ideas more explicitly to the specifics of the design of my study about the learning of algebra.

In enactivism the researcher and the object of research are seen as deeply connected, and constantly changing. From the moment a researcher starts any process of investigation, he or she starts looking at a world that is apparently out there before he or she begins to reflect on it. However, from an enactivist perspective, the world is not separate from the individual. Cognition cannot be characterised by the passive formation of mental representations that correspond to objects in the world (Maheux and Proulx 2015). Rather, in enactivism, cognitive processes are about action (Varela et al. 1991). We do not receive information from the world, we act on it, and our actions constitute our cognitive processes.

Enactivism shows how perceptions are formed by recurrent patterns in sensorimotor activity (Varela 1999; Reid and Mgombelo 2015). The actions that we perform on objects and while interacting with the world give rise to what we perceive and also to structures such as concepts and categories. This means that learning occurs as we actively engage with our environment; it cannot be thought of as absorption of information. Additionally, individuals select particular features from the environment as a result of their structural state in a given moment and the interaction with these particular aspects gives in turn rise to specific cognitive structures.

Moreover, we can say that when we interact with our environment, both our structures and the structures of the environment change. Actions give rise to cognitive structures, and these actions take place in specific settings which are also affected by those actions. The learner and the environment cannot be considered separately; both individual and world are joined through ongoing changes in their structures arising from their interactions. Enactivists use the term *co-emergence* to describe the way in which individual and world specify one another (Davis 1996, p. 10). While acting is a result of an individual's structural state in a given moment, actions occur in a context which is specified by the individual. Learning occurs when an individual acts in particular ways in that given context.

This is the interplay, between an organism and the world, that characterises enactivism. While doing research, the researcher does not encounter a pre-given world, but contributes to its creation in the interaction with others. This is what Varela et al. (1991) call a *fundamental circularity* (p. 3, italics added) and it is at the heart of the methodological approach that enactivism entails. The scientific description of phenomena is a product of our own cognitive system and it is produced in a certain context, one that in turn we modify with our actions.

When, in the beginning of the research process, the researcher selects certain features of his or her experienced world to investigate, this selection is already determined by the researcher's structures, which are in turn shaped by previous histories of interactions. As the research progresses and the researcher interacts with the world, they both go through a dynamic process of evolution; they change continuously in a process of mutual specification. The researcher is influenced by the context in which he or she is immersed, and at the same time, the environment also changes as a result of the researcher's actions (Reid 1996, p. 206).

This interdependence of world and researcher makes the research process a flexible and dynamic one. Research does not occur in a linear fashion; rather, it is seen as a recursive process of asking questions. As the researcher explores the literature, and starts purposefully engaging with the phenomena that are being investigated, the research questions will inevitably evolve. The investigation becomes a continuous process of reflection and modification of the researcher's conceptions.

For my study on the learning of algebra, an enactivist approach meant that I needed to be aware of my initial assumptions. I had to be clear about the way I was looking at the world, and about how my own ideas developed as I went through the research process. This meant considering my history within the learning of mathematics and tracing the development of my thinking as I read the literature, became engaged with the enactivist theoretical and methodological ideas and entered the empirical context.

In the beginning of my investigation about algebraic learning, I had certain questions that made me address the problem in particular ways and that arose from my previous experiences with the teaching and learning of mathematics. I was aware that although students did seem to go through some kind of internal reasoning process, this could be influenced by the learning environment that surrounded them. Teaching techniques influenced, but did not determine, particular outcomes in the learning of mathematics. I considered learning as a complex process where many elements were involved.

I was interested in algebra because, as a teacher, I had encountered many students who found it difficult. In a previous study, I had investigated students' understanding of algebra in an urban school in the UK, where a non-traditional instructional approach was followed. From this work I concluded that individual and collective interpretations of algebraic signs and symbols were determined by the history of each student and each group. As a result of this previous work, I decided to carry out a long-term study in which I could explore these histories. These were the initial ideas that shaped the design for my investigation.

As I immersed myself in the process of research, the choices I made selected particular features of the context I explored. Algebraic learning, as a concept, and the events I investigated, arose as I engaged with the research literature, and in interactions with the participants of my project. Throughout the research study, I went through a process of continual change and, because of my presence and my actions, the people and the contexts I related to were also modified. My questions regarding the learning of algebra were under constant development.

It is this process of change that I want to address throughout this paper. Since the methodological perspective shapes all the different aspects of the research, in the following sections I introduce each and discuss how the enactivist approach guided me.

3 Learning about the learning of algebra

3.1 Exploring the literature

Becoming familiar with the literature is part of the learning process that the researcher goes through when conducting research. The literature shapes the researcher's questions, but also, the researcher interacts with and organises reports according to his or her ideas, highlighting certain aspects and discarding others. Eventually the literature will in turn be modified by the researcher's contributions.

In mathematics education, one of the areas that has been researched the most is the learning of algebra. A large amount of papers have been written on what algebra means, on how students learn it and on different teaching approaches (see for example Otten and Soria 2014; Bell 1996; Dougherty 2001). Different perspectives have been taken, such as analysing students' misconceptions and errors (e.g. Clement 1982), studying the use of language and looking at thought processes (Arzarello et al. 2001). Studies have also been published in relation to the use of IT for the teaching of algebra (e.g. Healy et al. 2001; Hohenwarter and Jones 2007).

I approached the literature from a particular standpoint. Even though I did get a general sense of the different types of studies that have been done, the way in which I organised the research reports was influenced by the ideas I had regarding different epistemological positions I had been contemplating. In particular, I noticed the way in which researchers considered the concepts 'algebra' and 'the learning of algebra'. There are many definitions of algebra and usually researchers take into account a particular characterisation for their projects. Sometimes it is seen as a fixed, independently existing subject, while at others it is considered to be flexible and cultural-relative. Also, reports are influenced, either explicitly or implicitly, by a theory of knowledge or a learning theory. Some consider learning as a constructive process, with an emphasis on individual activity, while others can see it as an activity developed in particular communities or as a historical process. Additionally, in the last decade or so, some researchers have started addressing the embodied character of learning. Most of these studies, however, embrace embodiment from the perspective of 'embodied mathematics', which, as Reid and Mgombelo (2015) point out, is not necessarily enactivist. An example is the work done by Boero et al. (2001), who analyse how students make sense of quadratic inequalities by looking at cultural linguistic embodied categories such as 'going up' and 'going down'. Radford (2014) is also taking an embodied approach by studying algebraic thinking through the relationships between 'the material-ideational components of thinking (e.g. gesture, inner and outer speech) and the manner in which these relationships are organised and reorganised in the course of the students' engagement in activity.' A few studies have used enactivism to investigate the learning of algebra. For example, Brown and Coles (2001) used enactivist ideas such as cognition taken as 'perceptually guided action' (Varela 1999) in their exploration of the development of a school algebra culture in which students found a need for algebra.

For my own work, my intention was to investigate the learning of algebra in different contexts also following an enactivist theoretical approach. I started with an open perspective, wanting to see what emerged from what I observed in the classrooms. I used the phrase *algebraic learning* from the start as a way of acknowledging its complex character and in order to address the multiple meanings assigned to algebra and its learning. The initial ideas for my project, which were to look into the learning of algebra through a long-term study in which I could explore histories of interactions in detail, permitted an approach that was open enough for me to account for multiple aspects of learning. This allowed me to deepen my ideas about the learning of algebra and fit well with the thinking of other researchers who are calling for the understanding of how a variety of meanings—sometimes coming from non-mathematical domains—are incorporated into the learning of algebra (Radford 2000). My study was aimed at addressing the need for in-depth studies that is possible to identify in the literature.

3.2 About learning and algebraic learning

Selecting a theoretical perspective which can guide the researcher's ideas is another important methodological aspect which needs to be addressed. As I said before, I used enactivism not only as a methodological approach but also as a theory of learning. From the perspectives I explored, the enactivist one resonated with my experience

the most. I thought it could help me account for the complexity of the learning processes I had already noticed in the classrooms. I knew that the learning of mathematics did not occur in a linear manner, and that it could not be characterised either by looking exclusively at the individual or the context. Enactivism, with its emphasis on the interactions between the individual and the environment, provided me with a middle way in which I could see the learners, their surroundings and the mathematics as not separate.

In what follows I want to discuss more precisely how some of the theoretical ideas shaped my thinking.

3.2.1 The mathematics classroom culture

According to enactivism, when two or more individuals are involved in repetitive interactions within a particular environment, the result will be a history of mutual changes in structure (Maturana and Varela 1992). This means that learning can occur through harmonic changes in structure arising from recurrent interactions. For example, if two students continuously engage in solving mathematics problems, it is likely that they will find a way of working which will change each one's structures in a similar way. The result for each student will always be unique because of individual histories, but there will be a space for coordinated actions that will allow the students to participate in that particular way of problem-solving that they both created. Each student will possess the structures necessary to be able to act in specific ways.

In a given classroom, the participants' structures will change simultaneously as they interact with each other. As a result of these continuous interactions, patterns of behaviour will be created. These patterns, which can be learned by new members of a community through communicative dynamics (verbal and nonverbal communication), will form what, after Maturana and Varela (1992), I called a *classroom culture* (p. 201). A particular culture will be created at every instant in each classroom, as a result of the participants' actions. This culture will in turn shape each one of the participants.

3.2.2 Learning as effective behaviour

The patterns of behaviour created by the participants will constrain the kinds of actions that the community will consider 'acceptable' in a specific moment. Behaviour that allows the participants to continue existing in a location is called *effective behaviour* or adequate conduct. Effective behaviour means to 'operate effectively in the domain of existence' (Maturana and Varela 1992, p. 29), that is, to act in ways that allow the learner to continue existing in an environment, to perform actions that are acceptable.

Different criteria of acceptability will be specified in different contexts. In a mathematics lesson, effective behaviour will be characterised by actions that allow students to participate in their mathematics class. Participating does not mean only to be active and to work with others; being silent can be effective in some classrooms while ineffective in others. Effective behaviour allows students to carry on being students (and teachers being teachers) in the particular classroom in which they are located. Behaviour that is not effective will lead to the interruption of interactions and eventually will prevent the individual from continuing to participate in the particular context in which the actions are not acceptable.

Following these ideas, for my research project I decided to approach the learning of algebra through the exploration of effective behaviours in different contexts, looking at how histories were built through recurrent interactions. My original question, which was to explore the way in which people learn algebra in different classrooms throughout time, evolved into a number of related sub-questions:

- What does effective behaviour mean in different classroom cultures?
- How does effective behaviour change in different classroom cultures throughout time?
- How will the different ways of acting effectively I observe in the classrooms impact on my thinking about algebraic learning?

Research questions in enactivism can be considered more as guidelines than as questions in need of a definite answer which is located in the outside world. They shape the researcher's thinking and guide him or her through the process of data collection and analysis. Methodologically this is important because the theoretical perspective shapes the researcher's views before entering the research empirical context and throughout the process. In this case, enactivism allowed me to consider algebraic learning from a perspective in which its meaning was open for exploration through the observation of behaviours. This allowed for the possibility of the meaning of algebraic learning to emerge from what I observed in the classrooms.

4 Data collection and analysis

4.1 Longitudinal design

With the purpose of exploring effective behaviours in different contexts over time, groups of children were followed as they went from their learning of arithmetic to the learning of algebra. The 3-year study was carried out in two private Mexican schools (School 1 and School 2) which were

located around the same area and which admitted students from middle-class communities, thus providing me with the opportunity to focus on the events in the classrooms without having to explore the students' socio-economic backgrounds. The schools also had the advantage of having groups with a similar number of students, a maximum of twenty in each classroom. Finally, I selected these schools because they had different styles of teaching, School 1 being more traditional and School 2 having a more progressive approach. I wanted to explore effective behaviour in contexts which were different in their teaching styles in order to enrich my perspectives on algebraic learning.

Methodologically, it was important for me to carry out a longitudinal study in which I could explore learning throughout time. This enabled me to address my initial interest in looking at histories of interaction through changes in effective behaviour which might indicate modifications in structures.

4.2 Observing algebraic learning

> Everything is said by an observer. (Maturana 1987, p. 65)

Considering research as a process in which the world is investigated through the researcher's structures implies viewing the researcher as an observer of events, an interpreter of phenomena. As an observer, the researcher can only draw distinctions between the observed events; when we examine the world we can only find 'differences or differences between differences' (Bateson 1987, p. 41). The distinctions that observers make are the result of their structures and of their interactions with the world. The research ideas come from a researcher who is interacting with a world, but do not convey direct information about that world. Furthermore, the data that is collected entails already a selection made by the researcher:

> [A]lways and inevitably, there is a selection of data because the total universe past and present is not subject to observation from any given observer's point. (Bateson 2000, p. xxvi)

The researcher, immersed in a context, is the main creator of the theories that originate from the investigation. The role of the observer is therefore crucial, and needs to be acknowledged in any piece of research:

> The observer is a living system and an understanding of cognition as a biological phenomenon must account for the observer and his role in it. (Maturana 1970, p. 9)

Recognising the observer implies making explicit the process through which ideas are generated during the research. We know it is not possible to have access to the structures of an individual in a certain moment, and neither can we examine the complete history of interactions of any observer. However, in order for other researchers to be able to engage with a research study, the context in which the ideas and theories arise ought to be accounted for, and the way in which the observer becomes able to make certain distinctions should be defined. The researcher has to be aware, to some extent, of his or her assumptions, which have to be made explicit together with the procedures of distinction that were used. The researcher's perspective, which he or she takes while 'looking at the world' has to be described, so that others can then 'look' from that perspective. As the research progresses and the observer is able to modify his or her perspectives and 'see' more, other people might be able to also 'see' differently.

When I entered the schools I brought forth certain events. As an observer I tried to specify the criteria that allowed me to distinguish certain events from others. I had to be precise about the way in which I distinguished algebraic learning from other kinds of behaviour. When the criteria for distinction are clearly specified, it is possible for readers to follow discussions with little difficulty. 'You may agree or disagree with someone else […] but if you disagree it means that you are applying different procedures of distinction' (Maturana 1987, p. 69).

In the following section I discuss the way in which I collected the data and specified criteria for the characterisation of algebraic learning. Enactivist approaches, being interpretative in nature, do not presuppose any definitive or fixed set of procedures for the research process. This does not mean that enactivists advocate the abandonment of method; it means that there are no unique ways of doing research. Researchers can be careful and rigorous; this is reflected in the way events are reported rather than in the application of specific kinds of methods.

4.3 Working from multiple perspectives

In enactivism, science does not provide explanations about an objective reality that exists outside the observer (Maturana 1991). Theories are meant to contribute in creating a useful account of the researchers' experience of the phenomena they are investigating. Since, in the process of doing research, we bring forth a complex world, it is important that if we want to propose theories that can be useful in a certain context, we work from multiple perspectives (Reid 1996, p. 207). This is another key feature of the enactivist methodology and it refers not only to the exchange of ideas with other researchers, but to the examination and re-examination of different kinds of data. Different perspectives will open the possibilities for the examination of patterns and commonalities in the data. Events might have

more than one interpretation, but all interpretations must be explicable in a way that other people are able to engage with them (Reid 1996, p. 207). Through the comparison of different perspectives we are able to explain more.

I used multiple perspectives in different ways and on different levels. I studied effective behaviour in different contexts, through time and using different criteria. I collected different kinds of data, the analysis of which involved a constant revisiting. Additionally I worked with other researchers when interpreting the data in order to contrast different findings and enrich my own interpretations.

I investigated effective behaviours from three different perspectives, which are described in more detail below. The first one involved observing behaviour in the classroom, while children interacted with each other. A second way of addressing effective behaviour was the observation of individual actions through interviews. Finally, a third way of looking at effective behaviour was using a test designed by the Mexican Ministry of Education which consisted of mathematical questions and was taken by all the secondary school students at the end of each school year. Throughout the project I observed approximately 100 h-lessons. I was in the schools for 1 month each year for the duration of the study. The first 3 weeks were devoted to classroom observations, and during the last week I conducted interviews.

The use of multiple perspectives in enactivism does not mean that they will be used to reach a unique conclusion or for validation of theories or results. However, it is considered that seeing from different perspectives can allow an observer to see more, to 'widen the domain of possibilities' (Reid and Mgombelo 2015). This is different from other techniques in qualitative research such as triangulation, through which the researcher intends to corroborate his or her findings through the use of different methods of data collection or ways of analysing the data. I wanted to observe effective behaviour from multiple perspectives so that my views could be widened and my characterisation of algebraic learning could be enriched.

4.4 Criteria for observing effective behaviour

In order to observe effective behaviour I specified the criteria that were used in order to distinguish it from other kinds of behaviour. Effective behaviour is relational, it depends on the context in which it occurs and that context is specified by the researcher: 'We human beings assess cognition in any domain by specifying the domain […] and demanding adequate behaviour or adequate action in that domain' (Maturana 1988).

Since effective behaviour is such that 'will enable a living being to continue its existence in a definite environment as it brings forth its world' (Maturana and Varela 1992, pp. 29–30), I distinguished it when I observed conduct which allowed students to continue participating in a certain environment. As stated above, I explored this type of conduct in the classroom environment, in a one-to-one interaction with the students and using an instrument designed outside the classroom environment. In each case I had to be clear about the way in which I was distinguishing effective from non-effective behaviour.

Patterns in effective behaviour seemed a useful way to characterise learning; however, as an observer I was interested in the learning of algebra, which constitutes a particular kind of learning. The enactivist ideas emphasise that, since learning occurs in the process of interactions between the individual and the world, there are no fixed meanings for concepts or disciplines. However, I wanted to specify an initial criterion that allowed me to distinguish behaviour that could be considered to be algebraic from that which cannot. For this purpose I used Kieran's (1996) three-component characterisation of algebraic activity as a starting point through which I can classify behaviours as algebraic or non-algebraic:

- *Generational activities*—formulating algebraic expressions and equations.
- *Transformational activities*—manipulating and simplifying algebraic expressions, solving equations, studying equivalence and form.
- *Global, meta-level activities*—ideas of proof, mathematical structure, problem-solving.

Kieran's (1996) categories allowed me one way of contrasting the different types of conduct that were considered adequate in each of the different contexts I investigated. It is possible for certain activities to be effective in a certain context without them being algebraic. Conversely, behaviour classified as algebraic might not be effective in some environments. It is important to note that algebraic activity is only part of algebraic learning; the latter involves also general behaviours which, in the classroom, support students to act algebraically.

The methods I used for data collection allowed me to have different perspectives through which I could characterise effective behaviour (Fig. 1). I see the different

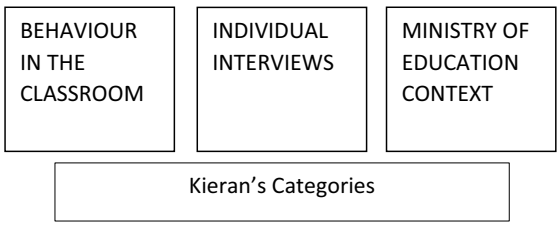

Fig. 1 Algebraic learning from different perspectives

perspectives as windows through which I explored effective behaviour, lenses that enabled me to bring forth a richer sense of what algebraic learning is and through which I could relate behaviours to Kieran's categories. In what follows I describe each perspective in more detail.

4.5 Collection of data

4.5.1 Observations

When investigating effective behaviour, at first I wanted to observe what was happening in the classrooms. The fieldnotes I produced as I observed the lessons were shaped by my particular way of viewing events, which, as I have mentioned already, was the result of my previous history. In order to allow the reader to 'see' what I saw in the classroom I needed to explain the way in which I recorded the data.

I started the observations using an open approach. I first tried to get a general idea of the classroom's ambience. I described the seating arrangements and the mathematical content and types of problems that were posed to the students. I then focused my attention on those kinds of behaviours that I found to be effective in each classroom. In order to do this, I attended to those actions that allowed the participants to continue existing and interacting in that particular environment and distinguished them from those that did not. For example, in a given classroom, I found that asking questions was effective, giving rise to dialogue and further interactions. In another classroom, a question asked by a student was considered to be an interruption and therefore was not encouraged. Behaviour that is effective in a certain environment will be encouraged, and hence occurs frequently.

Effective behaviours are not the final outcome of learning; rather, they show what the participants do in an environment as a consequence of their structural state and at the same time also shape future interactions. In a classroom culture where asking questions was effective, the same process of asking questions triggered further changes in the students, and therefore shaped their learning. When I observed the lessons, I characterised actions as effective whenever they occurred repeatedly and when their presence did not disturb what seemed to be the 'normal' flow of events in a lesson. A great number of particular incidents were described at length in order to illustrate each type of behaviour. It was always important to maintain a level of detail in order to be able to distinguish patterns through which I could 'see' more.

In enactivism, cognition is related to 'the operational effectiveness of living beings in their domain of existence' (Maturana and Varela 1992, p. 29) and this standpoint is different from other approaches which might consider what is effective in relation to external criteria. Effective behaviour did not always relate to Kieran's categories. However, as I said before, as an observer I was interested in a wide characterisation of algebraic learning which included non-mathematical actions.

Becoming familiar with the events in the classrooms helped me to decide on some of the questions I asked in the interviews. Observations also helped me relate their activities to the questions in the test designed by the Ministry of Education.

4.5.2 Interviews

As a way of observing from a different perspective, I wanted to have a closer look into individual accounts of what was happening in the classroom. Interviewing some of the students gave me the opportunity to explore their actions in detail and also allowed me to look for patterns within the same classrooms. Some questions focused on the direct exploration of algebraic activity, which might have been missed by my observations but nevertheless might be part of the students' behaviour. In this case, adequate conduct allowed students to engage with the questions I posed to them.

As mentioned above, I observed lessons in classrooms during one month each year and I interviewed, individually, a number of students (approximately one-third of the class) from each of the groups. The fact that I conducted individual interviews does not mean that I considered that learning occurred in isolated individuals. My intention was to explore patterns of actions that were effective in a variety of contexts and situations, as I was hoping for this exploration to give me a wider view of algebraic learning.

During the interviews I wanted students to talk about their experiences in the classroom in order to view events from a different perspective. For this purpose I asked questions such as 'Can you tell me about the maths lessons you've enjoyed the most?' and 'What do you think is needed to be good at maths?' I generated an account of each classroom culture complementing my interpretations of the events I observed with the individual accounts students gave me during the interviews. I also added questions in which students could act algebraically, according to Kieran's (1996) definition. Examples of these questions are the following:

- Think of a number, add 3, multiply by 2, take away 3, add your original number, divide it by 3, take away your original number. Did you get 1? […] Do you think it works for all the numbers? How can you be sure that it works for all the numbers?
- What do each one of these mean to you? $(2 + n)$ $(2n)$ $(2 + n = 8)$

It is from both observations and interviews that specific themes emerged as I looked for patterns in students' activities in the different groups. Algebraic learning occurred in the context of those effective behaviours.

4.5.3 Test

During the last year of my study, I gave all the students in the groups I observed sections of an old version of the test from the Mexican Ministry of Education. For my analysis, I focused on the last five questions of the test because they were more related to algebraic activity than the first ones, which were mainly arithmetic problems. Examples of these questions are:

- The solution to the equation $3x + 8 = x - 4$ is:

  a) 6 b) 2 c) -6 d) 10.

- In algebraic language 'Pedro's age is twice Juan's' is written (if Pedro's age is represented with x and Juan's with y):

  a) $x = 2y$ b) $2x = y$ c) $2x = 2y$ d) $x + 2y$.

I used the test to explore the implications of the different kinds of learning from another perspective. Since students had to complete a version of this test at the end of every school year, in a way it was already part of their learning experiences. It was an important criterion for them, since it was used in all schools in order to assess learning. I wanted to incorporate this perspective as it might give me examples of effective behaviours I had not encountered before. I was interested in relating these actions to Kieran's algebraic activities in order to further explore algebraic learning. Again I looked for patterns in the different classrooms and found differences between groups with different histories.

The use of this individual test as one perspective through which I could look at actions did not mean that I was trying to use it as a final assessment of learning. I wanted to explore effective behaviours in this particular context as one that was already part of the students' environment and which could give me an idea of what students could do when facing those kinds of algebraic questions. This is consistent with the enactivist perspective which considers knowing as doing in specific domains and does not mean that I was taking learning as occurring in individuals who were isolated from their environment. Effective behaviours in the different classrooms could include (or not) acting adequately in the specific domain determined by the questions in the test, which were what conventionally it is considered as algebraic.

> If someone claims to know algebra, that is, to be an algebraist, we demand him or her to perform in the domain of what we consider algebra to be, and if according to us she or he performs adequately in that domain, we accept the claim. (Maturana 1988)

4.6 Quality of research

> The fact that science as a cognitive domain is constituted and validated in the operational coherences of the praxis of living of the standard observers (…) without reference to an independent reality, does not make scientific statements subjective. (Maturana 1991, p. 41)

Traditional measures of the quality of a piece of research such as validity and reliability are often implicitly based on the assumption that research is a process of finding out information about a world that is independent from the researcher. From the enactivist methodological perspective the researcher and the world are deeply intertwined and cannot be viewed separately; therefore different criteria need to be used in order to appreciate different theories and distinguish them from opinions or beliefs.

The enactivist methodology implies that, even if a piece of research is not intended to be valid everywhere, the explanations that we formulate through our research should always be explicable (Reid 1996, p. 207), allowing the possibility for other people to engage with the research. It is important, therefore, that the research includes a clear account of how theories and ideas emerge in the process of doing research. This will then permit and enhance the usefulness of the research project.

Although I have already stressed the importance of being clear about the research process, and I have mentioned the use of multiple perspectives, here I want to go over some methods that I used in more detail. All of them are compatible with the enactivist position and are ways that I have found helpful when thinking about research quality.

4.6.1 Being in the context

A researcher must be able to continue being a researcher at least for the duration of the study. That is, his or her actions as a researcher must be effective in the environment in which he or she is immersed. Because of this, it is important that the researcher is to a certain extent familiar with the context that is being investigated. I was already familiar with the kind of environment I was exploring, since I worked in similar schools for 7 years in Mexico before starting my research project. Additionally, I was in prolonged contact with the communities I accessed, both because of the longitudinal nature of my study and because

of the in-depth type of investigation I developed. This enabled me to be at ease while I did my research, and also it made it easier for students and teachers to feel comfortable with my presence. From an enactivist perspective, since the research process is a learning process (Reid 1996), and learning occurs as structures are modified as a result of histories of interactions, it is desirable that the researcher has a history of being in the context, interacting and reflecting on what is observed not just occasionally but persistently.

4.6.2 Obtaining and recording data

In enactivism, obtaining the data is already an act of interpretation which is done through the researcher's system and through a history of interactions. 'The world everyone sees is not *the* world [or any world] but *a* world which we bring forth with others' (Maturana and Varela 1992, p. 245). What can be described as data is a product of the researcher's interactions with the context in which the research takes place, and not pieces of information, as could be considered in other qualitative approaches.

What is important is that the researcher describes how he or she produced and selected the data that was used in the explanation of the phenomenon that is being investigated, in this case algebraic learning. In my research study I tried to be clear about the methods I used to collect information. For that purpose, all the interviews were tape-recorded. In order to explore effective behaviour I needed to observe events for relatively long periods of time. It is through persistent observation that a researcher can identify those behaviours which occur repeatedly and which form patterns. Relevant features can only be appreciated if the researcher has time to distinguish irrelevant anomalies from important issues.

4.6.3 Transcribing and translating

In order to make the data available for prospective readers of my research, and also to be able to work with whole texts during the analyses, all interviews were transcribed and then translated from the original Spanish. Both transcribing and translating are interpretative processes which need to be addressed explicitly when engaging in those activities, researchers select certain features while ignoring others, as a result of their structural state. Since in enactivism research is not about describing an objective reality, what is important is that the procedures of selection are described so that other people can follow the researcher's decisions.

When I transcribed, I first typed the students' words. I paid attention to detail, recording pauses and moments of silence. I also incorporated the written text that students produced during the interviews. It was important, both for my own and for the reader's interpretation of the students' work, to provide the timings in which the students wrote symbols and procedures when they answered the interview questions.

During the process of translation, I worked with an English native speaker, which provided me with a different perspective with which I could contrast my interpretations. The process of transcription and translation, because of its interpretative nature, is necessarily incomplete. I tried however, to be consistent throughout the years of the study, using the same coding system and criteria.

4.6.4 Analysing

I have already mentioned how, for enactivist research, working from different perspectives is desirable and important not for reaching consensus but in order for the researcher to see more (Reid 1996). When analysing data, the use of multiple perspectives can be done in different ways. For example, different researchers can examine the same pieces of data, or data collected in different ways can be used when exploring the same phenomenon.

In the process of analysis, which started when I transcribed and translated, working with different researchers was useful and also gave credibility to the research. While doing the analysis I talked to other researchers about my data and, when possible, I worked with them on the development of categories.

When looking at the interview transcripts, I started by identifying the themes that were recurrent. My purpose was to get a sense of the richness and variety in the students' responses and of the patterns that were possible to find within and across the different groups. I discussed some of the interview transcripts with colleagues in order to talk about what we considered to be relevant, pointing out differences and similarities in our findings. After this initial approach, I was able to group different sections of transcripts together under specific headings and subheadings.

The themes that I found when I analysed the interview transcripts were contrasted against the ones that were developed from the lesson observations. In most cases, students' responses supported what I observed in the classrooms; and on some occasions, they provided me with additional issues that I then explored in subsequent classroom observations and interviews.

The analysis of data, through which the different themes that characterised algebraic learning were identified, is also a particular interpretation which does not reflect an independent reality and is not intended to be an absolute view. Since my perspective from the beginning was open to a wide characterisation of learning, during the analysis I considered themes which might have been discarded by other researchers. Talking to colleagues and teachers, even after the project finished, highlighted different possibilities for

analysis and also coincidences in ways of thinking about events in the classrooms.

4.7 Algebraic learning

Although the focus in this paper is the discussion of the methodological approach I took during this investigation, I want to briefly talk about what I found, as this will help me explain how the enactivist perspective shaped my results and conclusions. I start with the aspects of effective behaviour that differentiated the classroom environments. I then discuss how I characterised algebraic learning in order to finally reflect on my own learning process.

4.8 Aspects of effective behaviour

As I analysed the data I collected through observations, interviews and the test, some themes emerged. Table 1 contains those aspects of behaviour which I found to be the most relevant in the classrooms I observed. The patterns I found during the observations and interviews revolved around these themes. Certain kinds of behaviour were directly observable, while other categories emerged from the participants' discourse, both during the lessons and in the interviews.

More categories can be added to this list—these are not the only aspects that can be considered when thinking of a learning environment. It is impossible, however, to describe the complexity of any environment in its entirety. These themes reflect some of the principal differences between the different learning environments, both from my perspective and that of the students.

Even though I cannot describe the themes or the analysis of each one of them in detail in this paper, I include them in order to illustrate how I was able to account for my experience with teaching and learning in the different classrooms. The themes do not represent dichotomies; there were elements of all the different aspects in each classroom. For example, students in a given environment were free to do certain activities and constrained in other ways. All these aspects supported algebraic learning in different ways, which is what I showed in the analysis when I related the different themes to Kieran's categories of algebraic activity. The enactivist framework and the analysis I carried out led me to characterise algebraic learning as the structural change that occurred in individuals or groups as they acted effectively in a culture in which algebraic activity, according to Kieran's (1996) definition, was *needed*. This took place in classroom cultures where a history of interactions, in which algebraic activity became part of the students' behaviour, was created. The way in which the different aspects were found to be related to algebraic activity indicated that algebraic learning emerged in classrooms in which the history of interactions included an acknowledgement of the rational, emotional, embodied and social character of learning. It was promoted in cultures where individuals were offered opportunities in which they could engage, in the company of others, in ways of working algebraically. In these contexts, students were encouraged to explain and justify their ideas, because it was through the process of explanation that algebraic meanings were produced, shared and modified.

Table 1 Effective behaviours in the classrooms

Directly observable effective behaviours
Active/passive
Attentive/inattentive
Working with others/working individually
Freedom/constraint
Correct answers/explanations
Non-observable effective behaviours
Understanding, thinking reasoning/remembering

It was found that in these environments, the different kinds of algebraic activities occurred in the process of collective exploration of mathematical problems which addressed students' preferences and already existing ideas. Transformational and generational activities did not occur in isolation, but always in the process of exploration of structure. Therefore, they occurred through global, meta-level activities. In the cultures where algebraic learning was fostered, the exploration of mathematical structure was also linked to generalisation, which could occur without the use of algebraic symbols. Generational and transformational activities were still important, however, because algebraic symbolism was taken as a powerful tool for expressing and exploring mathematical structure. Algebraic learning was supported in classrooms where students were encouraged to use algebra to explain and justify mathematical situations. Stress was put on global, meta-level activities without abandoning the more mechanical aspects.

In classroom cultures where algebra was used in the process of exploration of structure together with the elaboration of explanations and justifications, a need for the use of algebraic symbols and procedures was automatically created, therefore promoting algebraic learning. When students naturally engaged in algebraic activity, as a result of their previous history of interactions, they were able to shape their already existing meanings and to integrate different concepts and procedures into their behaviour, that is, they were involved in *algebraic learning*.

5 Learning about algebraic learning

The circular character of the research project in which world and researcher constantly specify each other implies,

as I said before, that I did not attempt to find definite truths about the world I was investigating. In any kind of empirical research 'we find ourselves learning new things within a context which is partially of our own creation' (Reid 1996, p. 205) and therefore it does not make sense to try to think about objects and ideas that are completely separate from us:

> One of the most important characteristics of enactivism, is that it does not try to produce explanations of reality: '[T]heories and models of enactivist research … do not purport to be representations of an existing reality. Rather they are theories for; they have a purpose […] and it is their usefulness in terms of that purpose which determines their value.' (Reid 1996, p. 208)

I did not try to write a theory of the learning of algebra. Rather I tried to explain what I observed in a coherent manner, one that a community of researchers might consider useful. As part of my research, I talked about my project to other researchers and, ultimately, this work had to meet certain criteria for it to be acceptable. What is significant here is that I did not attempt to provide an explanation for the 'truth', but I acknowledged my role in the process of research in order to produce consistent statements.

Throughout this investigation I developed my ideas about algebraic learning in a way that I could continue existing as a researcher, which, according to enactivism, is the definition of learning. As a result I was able to formulate a richer characterisation of algebraic learning, which included elements that emerged as I engaged with the events I observed and as I interpreted them through theoretical ideas.

Even though from the start I had an idea of the complexity of the learning processes, this study allowed me to deepen my own understanding about the learning of algebra. Characterising algebraic learning as an emerging behaviour which was promoted in specific contexts modified my initial ideas. Aspects which I had not considered before became relevant, such as the emotional and embodied aspects which I observed were important. The characterisation of algebraic learning that I ended up with felt deeper and more encompassing.

The ideas I developed were the ones which were adequate when I finished this project. They are the ones that allowed me to explain what I observed in a coherent manner, that is, they were 'good enough' (Zack and Reid 2003, p. 43). My understanding might complement results from other studies in the literature. For example, Britt and Irwin (2007) concluded, in a longitudinal quantitative study, that students who developed awareness of algebraic structure of operational strategies that they used to solve problems in arithmetic were engaging in algebraic thinking. They therefore advocate a teaching approach that includes 'seeing algebra within arithmetic'. My in-depth study, which attempted to maintain a very open approach to algebraic learning, and which incorporated aspects that are not typically considered, might enrich such conclusions and suggestions.

As I reflect on my ideas they inevitably change and they will change more in the future. The account I have presented here, however, allowed me to change my thinking in substantial ways and to 'see' more, which is what doing research is about from an enactivist perspective.

> [A]s explanations are experiences of the observer … all explanatory domains constitute expanding experiential domains in which the observer lives new experiences, asks new questions, and unavoidably generates new explanations in an unending, recursive manner, if he or she has the passion for explaining. (Maturana 1991, p. 31).

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