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ORIGINAL ARTICLE

Developing a network of and for geometric reasoning

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Abstract In this article, we develop a theoretical model for restructuring mathematical tasks, usually considered advanced, with a network of spatial visual representations designed to support geometric reasoning for learners of disparate ages, stages, strengths, and preparation. Through our geometric reworking of the well-known "open box problem", we sought to enrich learners' conceptual networks for optimisation and rate of change, and to explore these concepts vertically across curricula for a variety of grades. We analyse a network of physical, geometric spatial visual representations that can support new inferences and key understandings, and that scaffold these advanced concepts so that they could be meaningfully addressed by learners of various ages, from elementary to university, and with diverse mathematical backgrounds.

1 Introduction

Spatial visual reasoning is recognised as an essential skill for functioning in the modern world (National Research Council (NRC) 2006; Newcombe 2006) and plays a central role in learning and expertise in mathematics (Natsheh and Karshenty 2014). In fact, developing one's spatial visual reasoning can have a direct positive effect on mathematics achievement (Uttal et al. 2013). Spatial visual and geometric approaches allow for wider accessibility of "advanced" mathematical concepts than typical mathematics investigations usually offer, both vertically across various ages and horizontally across the differentiated needs of learners within a particular grade. Through the specific task we discuss in this article, we sought to enrich learners' conceptual networks for optimisation and rate of change, two topics required for calculus and many other fields of study. We developed geometric representations that could support new inferences and key understandings, and that could scaffold these advanced concepts so that they could be meaningfully addressed by learners of various ages and mathematical backgrounds. This objective was motivated by our broader research on geometry learning and spatial visual reasoning, which has shown how pre-service teachers were able to make sense of optimisation and rate of change in more connected, conceptual ways when they engaged with representations that elicited a geometric, spatial visual approach. This present research stems from our consideration of the following interrelated questions:

i. How can we think about the restructuring of typically computation-based tasks so as to support and foster geometric approaches and spatial visual reasoning?
ii. How can restructuring create interconnected and accessible avenues for engaging with "advanced" mathematical concepts for students of varying ages and mathematical preparation?

In this article, we develop a model for restructuring advanced mathematical tasks with a network of spatial visual representations which supports geometric reasoning for learners of disparate ages, stages, strengths, and preparation. We analyse decisions and considerations to shed light on how attention to cognitive, content, and pedagogical facets of learning experiences can interweave to support geometric ways of knowing. In particular, we focus on the "open box problem", which centres on optimising rates of

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change of volume, and whose reconceptualisation from a number-crunching exploration to a geometric investigation (Whiteley 2007a) afforded new and powerful ways for learners to reason with and about this important content. We introduce and develop the construct of a *task network*, wherein accordance with the 22nd ICMI study on task design, we use the term task to mean a teacher designed purposeful 'thing to do' using tools for students to activate an interactive tool-based environment to produce mathematical experiences. As we illustrate below, a task network can be a powerful tool for teachers and teacher educators in providing engaging mathematical learning experiences that can enrich lessons while respecting the constraints of curricular agendas. Such an approach has implications for teacher preparation, both within faculties of education and departments of mathematics.

2 A spatial visual approach to the open box problem

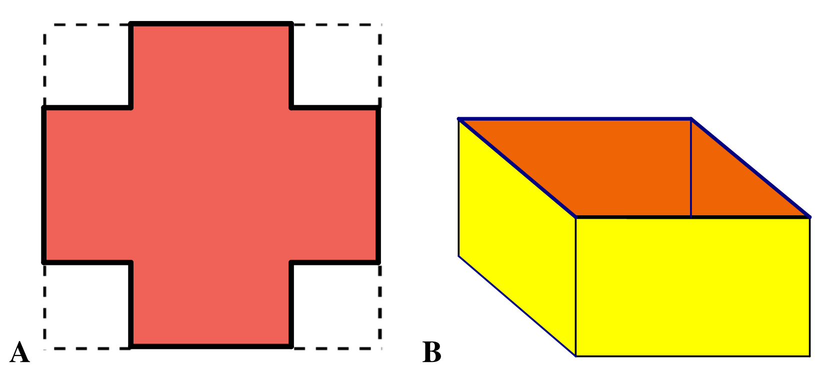
Initiated by Whiteley (2007a) and developed in Whiteley and Mamolo (2011, 2013), the origins of this research began with a perceived need to foster relational understanding (Skemp, 1976) of key content in university and senior high school mathematics students. The open box problem is a long-standing favourite for introducing rates of change and optimisation to secondary school students (Ontario Association of Mathematics Educators (OAME) 2005). The approach promoted by official curriculum documents in Ontario is one of computation and data display, and was found to be disconnected from the key underlying conceptual structure of optimisation problems, even amongst university graduates (Whiteley 2007b). The problem is presented in Fig. 1.

To recast this problem from a numeric activity to a geometric one, the concepts of optimisation and change in volume were represented by tactile models to invite comparisons, through its affordances, of changes in volume between two boxes. Explorations of volume are inherently geometric, and indeed are often included in geometric strands of curricula in various parts of the world. The intention was to invite geometric reasoning in the form of geometric transformations, physical movements, symmetry, and comparison of dimensions that could give direct and convincing evidence as to whether the volume lost is smaller than the volume gained between two similar boxes—leading to locating the potential maximum (where there is no change). In resonance with Sullivan and colleagues' (Sullivan et al. 2013) notion of a purposeful representational task, the recast problem uses a tactile model or representation to demonstrate a mathematical idea, which was then explored, along with a network of associated representations. Through this engagement, students develop a mathematical concept with multiple representations and actions which move among the representations in a relational mode. The key representations developed for this task included pairs of plastic boxes cut out from identical rigid square pieces of plastic sheets. As in the original task, we cut the corners out of each square sheet of plastic (Fig. 1a). Each box is held together with elastic bands (Fig. 2a) so that with their removal, the boxes can be deconstructed into their original nets. Deconstructing the boxes into their original nets both allows for comparisons of area of different surfaces and for a spatial visual representation of the differences in cuts from the corners, as presented in the original problem. In addition, previously made worksheets and complementary activities we made for Geometer's Sketchpad (Jackiw 1991) are available on the wiki site on Optimising with Geometric Reasoning (Whiteley 2011). For a detailed discussion of the materials and their affordances for conceptual development, see Whiteley and Mamolo (2011, 2013).

In this present study, we identify learning experiences and representations that can support and prepare younger students in their spatial visual exploration of the open box problem and associated spatial visual reasoning. In our

Fig. 1 The open box problem

Given a square sheet of material, cut equal squares from the corners and fold up the sides to make an open-top box. How large should the square cut-outs be to make the box contain maximum volume?



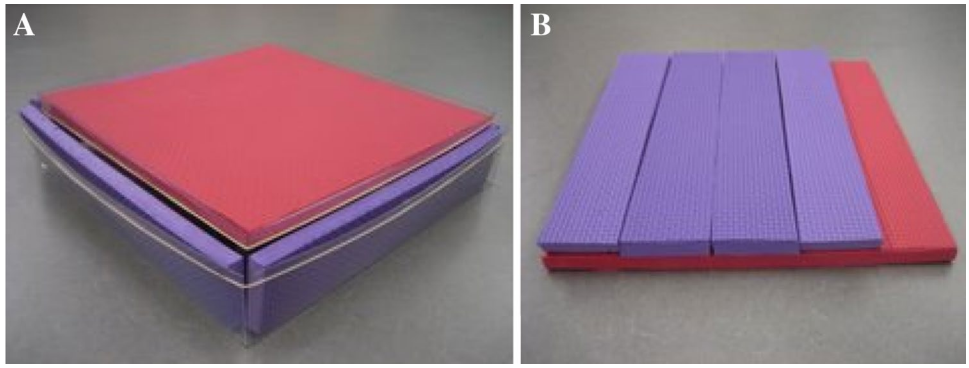


Fig. 2 Models

broader research, we examine variances in learner understandings and difficulties when engaging with this task using a network of purposeful representations. In this, we sought to introduce learning experiences and representations that could scaffold the open box problem for increasingly earlier grade levels. With each occasion, the implementation of our spatial visual approach remains consistent: we give two 'nearby' boxes to each student. Within each pair, boxes differ by 1 cm in the size of the cut of the corners (or equivalently in their height). Volume comparisons via explorations of change of volume are done through the use of purple and red foam inserts. The purple foam inserts surround the four outside sides of the inner box, and at the top of the inner box is a red foam insert (Fig. 2a). The pieces of foam physically represent where volume is being lost (purple) and being gained (red) in moving from the outer box to the inner box (Fig. 2b). The foam affords students ways of physically 'pulling' the changes in volume out of the model and comparing loss and gain directly to identify which box is bigger. For example, in Fig. 2b, the lost volume (purple) is clearly less than the volume gained (red), thus it can be concluded that the inner box has a larger volume.

Rate of change and optimisation are key mathematical concepts that weave through much of the secondary and tertiary curricula around the world and are topics that are utilised in many fields of study including engineering, the sciences and business studies. The understanding of these concepts also supports the learning of other advanced mathematical topics such as derivatives (Weigand, 2014). The learning of rate of change and optimisation is usually addressed computationally first by listing and comparing measurements and then eventually through derivative calculations in calculus. Traditionally, students have found rate of change and optimisation especially challenging because of lack of conceptual understanding of these topics (Herbert and Pierce 2008; Moreno-Armella 2014). Reform efforts in calculus have recommended utilising spatial visual tools to increase student understanding of conceptual ideas (Berry and Nyman 2003). While students of calculus hold many misconceptions of some of the conceptual underpinnings of these topics, geometric contexts can pose additional layers of difficulty for rate problems (Martin 2000). Considering the difficulties underlying conceptual understandings that key concepts in calculus pose for learners in higher education, researchers (e.g. Cuoco and Goldenberg, 1997; Tchoshanov et al. 2002) have recommended building the underlying conceptual understandings of calculus, including rate of change and optimisation, as early as elementary school.

As mentioned, our geometric approach to exploring optimisation was first developed by Whiteley for secondary and tertiary students (and their teachers) as a way to give a meaningful re-examination of the usual computational approaches to optimisation through spatial representations of key ideas and reasoning with their relationships. When implemented with sufficient time, they provide an alternative way to 'make sense' of why the solution found should be the optimum shape. In that context, further activities with imaging smaller cuts lead to an exact solution, still based on the fundamental idea of comparing change in volume of nearby boxes. We have found that younger elementary students are intrigued by the question of 'which box is biggest', and when invited to compare change in volume, knew to pull out the red and purple foams, and overlay them, to discover which box had the greater volume. This representation offers much younger students access to reasoning about some key ideas in optimisation without a symbolic or computational approach.

3 Affordances of spatial visual approaches

Our approach was motivated by research which connects the spatial visual abilities of students to future school success in math and science (e.g. Newcombe 2010), as well as by research which indicates that spatial visual approaches help build conceptual understanding for calculus (e.g. Tall 2007). Spatial reasoning, visual reasoning and spatial visual reasoning have been used interchangeably in the research literature. We rely on Presmeg's (2006) definition of spatial visual reasoning:

> "When a person creates a spatial arrangement (including a mathematical inscription) there is a visual image in the person's mind, guiding this creation. Thus visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics" (p. 206).

Children come to school with informal spatial visual reasoning skills and with abilities to understand abstract ideas, often before they are introduced in the curriculum (e.g. Bryant 2008; Sinclair et al. 2013). In and out of school, young children actively use spatial visual reasoning in geometric contexts (e.g. navigating, searching, estimating and determining which is the 'bigger piece' of cake). Continuing to develop spatial visual reasoning, and connecting it explicitly to mathematics, may be necessary for achievement and understanding in mathematics (Arcavi 2003; Koehdinger 1992). However, not all students rely on or prefer utilising spatial visual reasoning in performing mathematical tasks (Presmeg 2014). Arcavi (2003) points to three different reasons for why spatial visual tasks may be demanding for children in school: (i) the potentially high cognitive demands of spatial visual processing, (ii) the views different cultures have about the nature of mathematics, and (iii) the roles spatial visual processing plays in different cultures. The lack of attention to strengthening spatial reasoning in schools contributes to the high cognitive demand in essential ways—making the cognitive load higher when the reasoning feels novel to the student. In our context, we developed a network of supporting tasks to reduce the cognitive demands of the open box problem. The network was designed to support spatial visual reasoning and knowledge embedded in school curricula, while attending to the needs and prior experiences of students of diverse backgrounds.

Although the subject of geometry presents a most fitting place to teach spatial visual reasoning, geometry teachers in North American classrooms face distinctive challenges that begin in elementary school and extend throughout pupils' education. Numeracy-focused curricula with rote and shallow approaches to geometric facts, an over-reliance on regularly shaped prototypes, and vague connections between geometric concepts with limited support of geometric and spatial reasoning are some of the concerns often cited by advocates of richer geometric approaches to mathematics. These concerns emerge alongside the usual challenges in mathematics education of meeting the disparate needs of often de-motivated, disenchanted students in an overpopulated, under-supported classroom (Moss et al. 2015). In response, differentiation, the purposeful reshaping of teaching practices to offer multiple approaches in learning environments or materials, has become a key instructional strategy in equitably meeting the needs of all learners in the classroom (Strong et al. 2004; Tomlinson 1999). In what follows, we develop the construct of a task network, which allows for the flexible restructuring of advanced mathematics concepts to support and foster spatial visual reasoning for improved academic achievement and differentiation across grades and learning needs.

4 Theoretical underpinnings and a model for task networks

In developing our notion of a network of learning experiences, we again found ourselves networking—this time with the theoretical underpinnings that inform our understanding of the mathematical content, pedagogical approaches, and cognitive processes of and for mathematics learning. These considerations formed the bases of our thinking and analyses of networked learning experiences, and we apply them with terminology borrowed (and adapted for our purposes) from graph theory to develop a model for conceptualising task design, with particular reference to the design of geometric spatial visual tasks. As with other networks, ours can be modelled by a mixed graph, with nodes that are linked by either directed edges (i.e. there is an orientation flowing from one node to another) or undirected edges (i.e. there is no orientation, and the flow can go back and forth between nodes). We elaborate on this metaphor further below, and first provide an overview of each of the theoretical underpinnings, which we then integrate. In considering mathematical content, we rely on Simon's (2006) construct of *key developmental understandings; conceptual blending* as developed by Fauconnier and Turner (2002) sheds light on the processes used by learners to conceptualise new (for them) mathematics; and we recast Anghileri's (2006) framework of scaffolding interventions for use in task design.

4.1 Key developmental understandings

Simon (2006) introduces the construct of KDUs "to emphasize particular aspects of teaching for understanding and to offer a construct that could be used to frame the identification of conceptual learning goals in mathematics" (p. 360). The construct was developed through the coordination of social and cognitive perspectives on learning mathematics with the intent to shed new light on ways of thinking about understanding. KDUs involve a change, or conceptual advance, in students' mathematical reasoning that often cannot be acquired as the result of an explanation or demonstration. It is an important advance in the development of a concept, and "identifies a qualitative shift in students' ability to think about and perceive particular mathematical relationships" (p. 363–4). Simon (2006) provides as an example the understanding of fraction as a quantity, and notes that this KDU is distinct from the formal definition of a rational number. In particular, KDUs afford new ways to think about and perceive mathematical relationships, which otherwise might not be available.

In their work with adult learners, Sinclair, Mamolo, and Whiteley (2011) identify KDUs for spatial visual reasoning in tasks related to proportional change. We highlight ones that apply more broadly to spatial visual reasoning with geometric tasks, such as making connections between 3-D and 2-D representations, and noticing mathematically significant details and ignoring "distractors" such as physical imperfections of 3-D models. In the context of spatial visual approaches to optimising rates of change, we note that the ability to think about changes in volume as entities which may be compared, and that the result of the comparison reveals information necessary for obtaining the optimum volume, is an important KDU. In the context of task design, we draw on KDUs to inform the provisions and representations included within the task that may be manipulated or acted upon by the learner so that he or she may develop the required understanding. Thus, for our task, we made deliberate choices in the representations of mathematical relationships and ideas to help foster and support thinking about change in volume as an entity. We see a KDU as a new inference—a new piece of mathematical understanding—that is accessible to individuals through their negotiation of new learning experiences. The theory of conceptual blending informs our understanding of how such an inference may develop, and thus also influences choices in task design.

4.2 Conceptual blending

Fauconnier and Turner (2002) offer a theory of conceptual blending to describe how new inferences can arise when two representations and associated ways of reasoning (or 'input spaces') are brought together in a 'blended concept'. The blend can be thought of as a mapping which combines features of the input spaces and projects them onto a third (newly formed) mental space—the output space. In a blending process, some features of the input spaces are mapped, while others are not, thus directing focus of attention and reducing the overall cognitive load for further reasoning. Blends are used to conceptualise actual things such as computer viruses, fictional things such as talking bananas, and impossible things such as time travel. Although sometimes bizarre, "the inferences generated inside them [conceptual blends] are often useful and [can] lead to productive changes in the conceptualizer's knowledge base" (Coulson and Oakley 2005 p. 1513). Blending is not a metaphorical or analogical map, rather it is a specific way to combine and infer from and about information from two or more input spaces (Fauconnier and Turner 2002). The partial representations from an individual's perceptions and concepts that are contained in the prior mental spaces blend by "the establishment and exploitation of mappings, the activation of background knowledge, and frequently involve the use of mental imagery and mental simulation" (Coulson and Oakley 2005 p. 1513).

An emergent blended space arises in three ways: "through *composition* of projections from the inputs, through *completion* based on independently recruited frames and scenarios, and through *elaboration*" (Fauconnier and Turner 2002 p. 48, emphasis as in original). Specifically, composition creates new relations not previously existent in the separate input spaces, while completion allows the composite structure in the blended space to be thought of as part of a larger structure in the blend, and elaboration, or 'running the blend' consists of cognitive work performed within the blend to exploit and elaborate upon the composite structure (Fauconnier 1997 p. 150–1). The blend continues to offer the individual ways to access each of the original representations, in a flexible manner. In our model for networked tasks developed below, we use directed edges to illustrate composition, completion, and elaboration of new blends as they arise from new or revisited input spaces.

4.3 Scaffolding

As indicated, an objective of this research was to restructure a conceptually advanced task so that it was accessible and applicable to young students. To do this, we relied on scaffolding strategies as they were applied specifically to mathematics learning by Anghileri (2006). Anghileri extended early work by Wood, Bruner, and Ross (1976) who articulated a framework for learning through scaffolding, so as to tailor such provisions to the specific needs of mathematics learners. She developed a three-level hierarchal framework of interventions that includes (i) environmental provisions, (ii) explaining, reviewing, and restructuring, and (iii) developing conceptual thinking. Anghileri's first level, *environmental provisions*, is where the teacher scaffolds the environment around the learner through provisions such as: classroom artefacts (e.g. manipulatives, wall displays, or measuring tools), peer collaboration, sequencing and pacing, and structured or self-correcting tasks. These provisions help realise Wood et al.'s (1976) elements of scaffolding, including reducing degrees of freedom through structured tasks, direction maintenance through deliberately chosen artefacts, and frustration control through sequencing and pacing. The next level, *explaining,* *reviewing and restructuring*, is meant to scaffold the interactions between the teacher and her students. Some examples of scaffolds in this level include: rephrasing students' talk, providing meaningful contexts, simplifying and modelling, and negotiating meanings. Connecting again to Wood et al.'s framing, recruitment can occur through a meaningful context, while rephrasing students' talk can help mark critical features and modelling gives a way to demonstrate. Anghileri's final level, *developing conceptual thinking*, diverges most notably from Wood et al.'s description. It includes developing representational tools (in line with Sullivan et al. 2013), making connections between mathematical ideas, and generating conceptual discourse where the teacher focuses on and extends the mathematical activity through questioning and reflecting.

While Anghileri discusses this framing in terms of classroom interactions and the ways teachers may respond in the moment to promote mathematics learning, we apply her conceptualisation to analyse our process of designing the network of learning experiences. As such, we extend this work to illustrate how attention to scaffolding practices may inform decisions in task design and development, and we highlight the provisions which we view as of particular importance to geometric spatial visual reasoning.

4.4 Integrating perspectives for task design

The means by which one theoretical perspective informed another is quite subtle and varied. We share our approach in combining these perspectives, elaborating on the details in the context of our task network model in the following section. As task designers, we had specific intentions with respect to the mathematical ideas and thinking we hoped to cultivate with our students. We recognised there were several important understandings and representations that could support the open box exploration, and that some of these understandings might require a conceptual shift on the part of the learners. The occurrence of such a shift can be interpreted as a key developmental understanding since it develops the concept in the mind of the learner. Then, in the process of task design, we asked: how might a student be supported in acquiring such a KDU? That is, what different input spaces are, or could be, available for a learner to draw from, such that the experiences, images, and representations of those input spaces afford the composition of a conceptual blend that may yield this KDU? In selecting and designing the provisions of the (external) input spaces of a task, we look toward scaffolding practices for insight. Affordances such as multiple entry points (e.g. Grootenboer 2009), purposeful representations (Sullivan et al. 2013), and negotiating meanings, inform what and how to design a geometric spatial visual learning activity such that diverse learners may develop new ways to think about mathematical relationships and ideas.

5 A networking approach—developing a model for task design

In our task design research, we found a networking approach to be useful on multiple levels:

- In negotiating and articulating relationships amongst the (intended) content understanding to be developed by the learner, the cognitive processes by which such understanding may develop, and the pedagogical considerations and affordances that may support such processes;
- In selecting and integrating curricular content across various strands to support meaningful connections amongst multiple representations (e.g. numeric, visual, physical). Each learning centre for our task focused around one or two ideas and spatial visual representations, which contributed to the individual nodes of the network;
- In applying and integrating scaffolding practices for each individual learning centre.

Curriculum enactment is an important determining factor in the way understanding develops (Cohen and Ball 1999), as such we pay special attention to integrating curricula in our task network. In what follows, we begin with a discussion of the first point and develop a model for a *task network* within the context of restructuring a tertiary-level activity for applicability across various grades. We illustrate ways curricular content may be networked, and then offer an analysis of scaffolding practices that supported our networking of this material for the purposes of meeting diverse learning needs, as it comprised part of a larger mathematical investigation.

5.1 A task network: the model

An integral feature of our task network is the consideration of the interplay between the intended teaching of the task and the constructed learning of the student (Stein and Lane 1996). In negotiating and articulating relationships amongst the (intended) content understanding to be developed by the learner, the cognitive processes by which such understanding may develop, and the pedagogical considerations and affordances that may support such processes, we introduce the construct of a *task network*. We borrow terminology from graph theory and illustrate the construct in a generic form in Fig. 3. In Figs. 4, 5 and 6, we model particular task networks and analyse them with respect to our theoretical framing. Specifically, a task network inclu des:

- *Nodes*—these are the fundamental units from which graphs are formed. In graph theory, they may be treated as 'featureless' or they may have an internal structure, representing concepts or classes of objects. The nodes in our model have structure and represent "learning centres" which are then networked. This network of nodes is what we describe as our *task network*, and we illustrate it with examples below. Zooming in, each node may also be represented by a network of KDUs, conceptual blending, and scaffolding practices. That is, each node was designed around a particular representation and context, relevant to the age and stage for which the task was restructured. The representation and context were chosen to illuminate or support an intended KDU, and as such form external or physical input spaces which may be projected by the learner in the creation of a new conceptual blend (Whiteley 2012). Zooming back out again, the intended KDUs for each node were chosen with the task network in mind—every new inference accessible to learners through their negotiation of a particular node was intended to support further inferences that could eventually lead into a conceptual understanding of optimising rates of change in a geometric context.
- *Edges*—these connect nodes and, in a mixed graph, may be either directed or undirected. A directed edge is one with orientation, it can be thought of as an edge that proceeds from one node to another. An undirected edge has no orientation and links nodes without distinguishing one as a predecessor of the other. In our model, we use edges to represent links between KDUs. When the edge is directed, it indicates that to acquire a particular KDU for our task, some previous key understanding must have been developed first. The flow of these edges begins in each task network with the tail (in our case, a paper folding activity) and ends at the head (the open box investigation). This flow is scaffolded in our network by the inclusion of 'intermediate' nodes which are linked to and from the head and tail, as well as amongst themselves. For the most part, these intermediate nodes are linked amongst themselves with undirected edges—that is, the KDU developed in one node need not precede the KDU developed in another. In our ongoing research, we explore the implications for student learning if these edges are treated as undirected versus directed.
- *Arcs*—this is another term for directed edge, yet we distinguish it for our purposes by applying the terminology to the blending process, rather than the progression of KDUs. These arcs may flow in the same direction as the directed (KDU) edges in our model, they may flow in the opposite direction, and they may loop around, revisiting some prior node to thicken understanding at the head. In our models, when the arc aligns with the directed (KDU) edge, we do not distinguish (visually) between the two. We suggest that each node may serve as an input space for conceptual blending as it relates to the KDUs of any other node (in a sense forming 'loops' or 'multi-edges'). Thus, within a task network, there are multiple possible trajectories a learner may follow to acquire the intended KDUs, and correspondingly, multiple possible blends afforded.

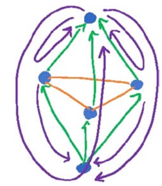


Fig. 3 Generic task network with nodes (*blue*), undirected (*orange*) and directed (*green*) edges, and arcs (*purple*)

6 Task networks for geometric reasoning

In this section, we discuss examples of how the construct of a task network may be used to develop and analyse a network of learning experiences. We begin by identifying our simplest network, which served as the basis from which more complex, and differently structured, task networks were generated. This two-node network was developed for advanced learners, and we illustrate our process of restructuring to meet the diverse needs of novice learners at different stages in their mathematical preparation. This network is linked with directed edges (in green) and arcs (in purple), and is depicted in Fig. 4. The tail node is a paper folding activity (PF) in which the students cut a square piece of paper and create a box they predict will be 'biggest'. Typically, we then incite conversations from the beginning by asking them to create a different box, with a different volume, supporting the key developmental understanding that one shape of paper can create boxes of different volumes. The head node is the open(plastic) box activity (PB) we described above, the models for which open up to shapes of the cut paper—confirming these represent several ways to explore the common boxes. Attending to the head node, we sought to develop the key understanding that changes in volume may be compared directly (as encapsulated entities of loss and gain), and the result of this comparison reveals the direction of change of cut of a given box necessary for moving towards the optimum volume. For this purpose, we considered what available blends might be afforded by the task network and the particular representations and contexts within each node. The input spaces at this node included (i) the word problem of maximum measured volume (*a meaningful context*), (ii) the classical approach to

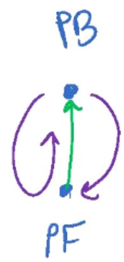


Fig. 4 A two-node network

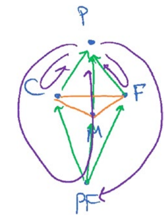


Fig. 5 A task network used with Grade 5 students

optimisation through calculus and rate of change (*making connections* to students' prior knowledge), and (iii) the spatial reasoning (i.e. the variational exploration of change in volume represented physically with pieces of foam—*representational tools*). In italics, we emphasise the scaffolding practices introduced by Anghileri as they may be applied to the task design, and reserve further discussion for the following section. As discussed in Whiteley and Mamolo (2013), these input spaces may be projected onto a blended space which would allow the individual to conceptualise change in volume physically by considering surface area for loss and gain (ΔV in symbols, pieces of foam in the model), and then complete, or run, the blend to focus on the sign of the change, with a simple physical comparison via foam inserts, to determine which changes will make the volume larger.

As mentioned above, one of our objectives was to restructure this task network so that it was appropriate for a variety of learners across a variety of grades. We discuss two such examples of restructuring, one which was used with Grade 5 (Fig. 5) students and the other with Grade 9 students (Fig. 6a). We highlight these two examples as they identify different ways the nodes may be linked in a task network. In these cases, the 'intermediate' nodes were informed by the curricular expectations required for that particular grade, and in some cases were constrained by them (for instance, when directed to cover certain expectations and avoid others).

6.1 Restructuring the open box problem: an example for Grade 5

When working with Grade 5 students, we introduced three 'intermediate' nodes for comparing volumes, linked by undirected edges (depicted in orange)—a computation activity (C), a measuring activity (M), and a filling activity (F). Directed edges (green arrows) stemmed from the paper folding activity (PF) to each of the intermediate nodes, and other directed edges stemmed from each of the intermediate nodes to the open box activity (PB). To reduce the complexity of presenting the model here, we do not identify all of the arcs representing possible blending (depicted in purple), but one may imagine them there. Again, each of the nodes was designed to support the volume comparison KDU, and each afforded its own KDU around which scaffolding provisions and blending opportunities were designed. For instance, the filling activity (F) included a story context of a vacation with the family in which boxes of snacks were to be prepared in advance. The activity involved comparing volumes of boxes by filling them with rice crispies and comparing the 'overflow' as the rice crispies were poured between boxes of different cuts. For a Grade 5 student, this can be seen as a KDU, as it allows a new way to think about volume comparison that does not require calculating one volume first and then the other. By simply attending to the overflow of rice crispies, one can deduce which box is the bigger between the two, within a margin of error in the models (such as bulging sides), as well as infer the direction of change required to move towards an optimum. With respect to the affordance for conceptual blending, input spaces included the physical representations, the context, and prior experiences baking at home (where strategies for measuring ingredients lent themselves to this context). Comparing volumes of two boxes is the common theme in all of these nodes, and the open box models support more exact comparisons when the prior pouring comparisons were inconclusive. A goal is that all these representations of changing volume blend into a richer concept of volume change.

6.2 Restructuring the open box problem: an example for Grade 9

Another important representation that lends itself to a richer concept of volume is a graphical one, which is appropriate for students who have seen graphic displays, even of data (senior elementary, secondary, and tertiary) and which can follow physical explorations of volume change. In Fig. 6a, we illustrate how this representation was networked with the other representations for Grade 9 students. This representation connects to the KDU observed by Sinclair et al. (2011) of negotiating physical constraints of 3-D models. Specifically, the open box activity (PB) as presented is constrained by the thickness

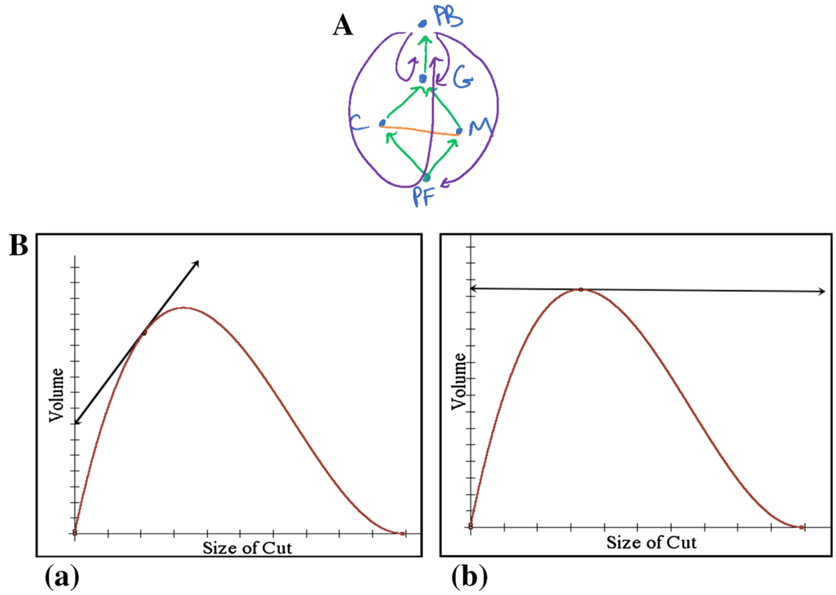


Fig. 6 **A** A network used with Grade 9 students. **B** Graphical representations and instantaneous rates of change: **a** not a possible maximum (non-zero rate), **b** a possible maximum (zero rate)

of the foam and because of this affords blends that support understanding of average rates of change, but not instantaneous rates of change. One approach is to make the differences of the boxes very small—the thickness of Bristol board, for instance. This provides a physical form of taking limits of changes in cut size by focusing attention on comparing surface areas. The G node in Fig. 6a can both precede and follow the PB node (creating a loop or multi-edge), and allows for an integration of input spaces that can support one another. Preceding the PB exploration is an activity that begins with a table of volumes, structured by the size of the cut, and graphed as a scatter plot. This representation supported the understanding that the extremes (cut nothing, cut half-way to the centre of the edge) were of zero volume, and that there would be a maximum that was not the (expected) cube. Following the PB node, the graphical representation may be extended and refined using GSP to depict what happens with smaller and smaller steps (cut sizes) until eventually focusing on the sign of the local change (qualitative instantaneous rate of change) by comparing surface areas (as illustrated in Fig. 6b). These graphical representations highlight important connections between average rates of change and slopes of secant lines, and instantaneous rates of change and slope of tangent lines—notions typically reserved for university calculus, but which are made visible and made a source of reasoning for younger students through the networking of geometric spatial visual and graphical representations. They allow an elaboration of the conceptual blend relating to which boxes cannot be the optimum (there is a change which increases the volume) and why there is a single 'max box shape' (as opposed to the existence of two differently cut, but equally optimal boxes).

6.3 Restructuring the open box problem: further examples and a generalised approach

In this section, we shift focus to view the general within the particular (Mason and Pimm 1984)—that is, through our consideration of particular examples, we illustrate how our construct may be operationalized in general for the purpose of restructuring other tasks. Specifically, we explicate a process by which an "advanced", task may be restructured in a network to foster and elicit more accessible spatial visual approaches. Key to this is that the task designers (as a group) bring multiple representations, typically spatial visual, to the topic—so that an initial network of representations can be explored. These representations are chosen with curricular/learning objectives in mind. As an example, the networks depicted in Fig. 7 highlight how the restructuring of a simple network can yield different possible opportunities via different possible intermediate nodes. Further, in Table 1, we exemplify specific spatial visual representations that were networked in tasks to support particular KDUs in optimisation, proportional reasoning, and groups of symmetries, respectively. In

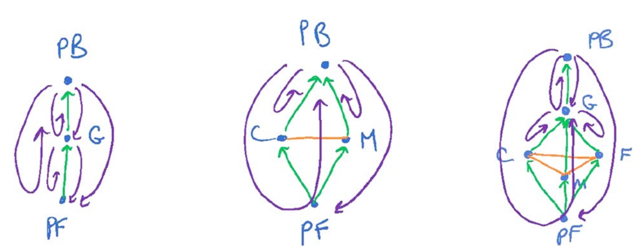
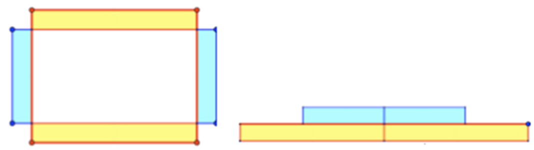
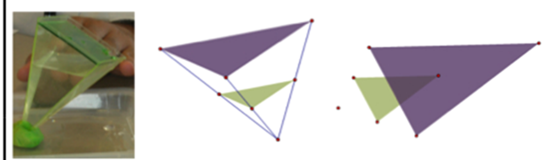
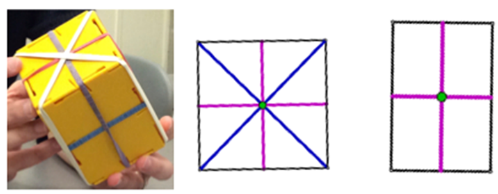


Fig. 7 Other task networks

Table 1 Examples of spatial visual representations for networking tasks

Concepts	Inputs for Blending	KDU
Optimum Area rectangle with fixed perimeter		Optimum occurs when loss and gain are the same
Proportional Reasoning via Dilation (e.g., see Sinclair et al., 2011, for details)		Scaling across dimensions: length k, area k^2, volume k^3 Seeing 2D shapes as encoding 3D information
Group of Symmetries – Composition of symmetries		Two reflections guarantees a rotation

general, each node is designed around a particular representation relevant to the age and stage for which the task is being restructured. Representations are chosen based on their ability to support or illuminate an intended KDU. For instance, the KDU of making connections between 3-D and 2-D representations could be supported by two nodes—one focused on a 3-D exploration with filling boxes, and one focused on a 2-D analogue which encodes the same information via a flat representation. As we describe below, considerations then turn to available scaffolds that could form parts of external or physical input spaces in a projected conceptual blend. Such scaffolds could include physical artefacts (e.g. boxes to measure), contexts (e.g. a family picnic), or opportunities for peer–peer teaching. The intent is to make available possible connections or experiences which can then be consolidated or 'networked' in the mind of the learner. Each node is then considered in connection to other possible nodes, with particular attention paid towards how one node might support another and whether the relationship is reflexive or hierarchal. In the former case, a KDU associated with one particular node need not precede that of another node—that is, the conceptual blends that may be composed or completed at one node need not constitute input spaces for blending at another node. In the latter case, the intended KDU might require the running of a blend composed in one or more prior nodes (e.g. if representations build off of one another).The arcs which illustrate the 'flow' of the network (e.g. Fig. 7) correspond to how the task network can be implemented—they highlight the different, but complementary, trajectories which comprise the network.

7 Supporting the network: a look at scaffolding within nodes

We now zoom into consider some of the structural features of the nodes. These scaffolding practices (and others) supported our task networks and allowed us to provide entry points to multiple learners of diverse mathematical ability and background. As indicated, we used scaffolding on multiple levels—both to restructure a tertiary-level task network, and to inform the provisions of particular nodes within that network. In each case, the external 'scaffolds' may be removed—in the former case, nodes for Grade 5 students may be omitted when using the task with Grade 9 students, and so on, in the latter case provisions such as a particular context may be removed when it is no longer needed. We note that while the external supports of the scaffolds may be removed, the representations made available to the learner via conceptual blending remain accessible and useful. In this section, we turn our attention toward the process of networking scaffolding practices within particular nodes of our task network for the purpose of supporting geometric spatial visual reasoning of volume change. Our approach diverges from Anghileri's

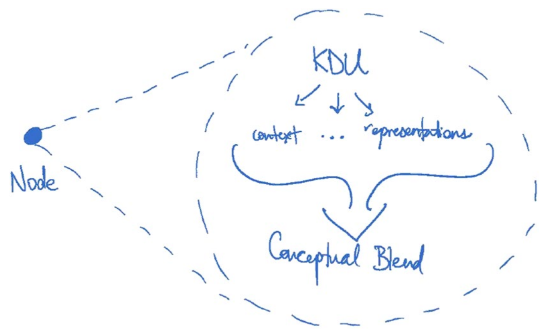


Fig. 8 Scaffolding practices provide structure to nodes, are informed KDUs, and afford opportunities for conceptual blending

(2006) positioning of scaffolding practices as having a hierarchal structure, and extends the applicability of scaffolding from supporting classroom interactions to orchestrating them. In Fig. 8, we illustrate how our theoretical underpinnings informed the design and structure of each node. Essential to the development of our task network was the integration of scaffolding practices, and our process began with what Anghileri (2006) describes as *level 3 scaffolding—developing conceptual thinking*. This was a fundamental goal of ours, and it informed all of our subsequent decisions regarding *environmental provisions* (*level 1*) and *task restructuring* (*level 2*). Indeed, the hierarchical positioning attributed by Anghileri to these three categories of scaffolding was not something that for us translated into distinctive categories or privileged strategies—rather, it was through networking strategies that we were able to leverage the original task to foster the aforementioned KDUs and opportunities for conceptual blending. We illustrate our process by focusing on some of the scaffolding techniques used for Grade 5 students in the task network depicted previously in Fig. 5.

7.1 Developing conceptual thinking through task restructuring

Restructuring is described as a scaffolding strategy in which teachers "introduce modifications that will make ideas more accessible, not only establishing contact with students' existing understanding but taking meanings forward" (Anghileri 2006, p. 44). Our initial approach was to restructure the original open box exploration so that students with minimal formal exposure to volume, 3-D shapes, and nets could eventually reason meaningfully about volume optimisation. We also saw a need to not only take meanings forward, but also to extend them laterally—making connections with other strands such as number sense, measurement, and data management. Fostering spatial visual reasoning as students (re)addressed geometric concepts or arithmetic algorithms was an important predecessor to the volume comparison strategies (Battista 2003) afforded by the open box materials, and an essential element in promoting conceptual thinking. As Anghileri notes, "*making connections* is crucial as a strategy to support mathematics learning" (Anghileri 2006, p. 48 emphasis as in the original text).

7.2 Developing conceptual thinking through structures provided in the environment

Anghileri (2006) discusses the purposeful arrangement of the environment outside of teacher/student communication that scaffolds student learning. For example, *peer collaborations* were seen as important (Cohen 1994) scaffolding tools, especially for visual activities with disparate learning needs of students, and as such were key components of all the tasks and activities. Peer collaboration became a significant tool for us in supporting the learning needs of all the students, especially for those significantly behind grade level.

Part of conceptual thinking in geometry involves connections with other subjects, thus we sought to make our task network multi-representational, involving different strands from the curriculum. We included multiple spatial visual representations of the ranking of volume of the boxes, which were then coordinated via data management principles. Students organised the measures of volume data into charts and directions of change were provided as supports, which paralleled the type of charts often used in calculus for investigating first and second derivatives for graphing and optimisation. Our representations also overlapped with ideas from algebra and patterning when we introduced conventions, symbols and tools for recording the data on volume.

Our provisions for *artefacts* included both *structured tasks* and *self-correcting tasks*. Structured tasks can include worksheets, but Anghileri (2006) includes all tasks, even those initiated by the student, that have a structure or challenge imposed on the task. The manipulatives and representations that were chosen to support the foam exploration had both structured and self-correcting aspects to them. We observed the students using self-correcting gestures as they used the rice crispies to compare volume. The boxes the students made were not as rigid as the model with foam. As the students filled their boxes with rice crispies, there were visible sources of errors in the bulging sides of the boxes and in the rounded mounds of rice crispies at the top of the boxes. The visible sources of errors left uncertainty amongst the students. This uncertainty supported the desire for more accuracy in using the rice crispies. We observed the students making a sweeping gesture over the top mound of rice crispies to ensure that there was no overflow. This spontaneous gesture made it likely that the same pattern of ranking would be found with the rice crispies as with the foam comparisons (where that representation has a ‘flat top’ with no overflow).

Students recognised sources of error and used the foam comparisons, which were sharper even with different visible sources of error, to resolve the errors of the rice crispies. The purposefully designed representational materials—the plastic boxes with foam inserts—supported self-correction as well. These materials supported reasoning that was convincing to the students and not reliant on teacher feedback. The foam exploration helped verify student conjectures about the increase of volume to an expected maximum, and then a decrease back to zero as the cut size (height) was increased from zero.

7.3 Restructuring by providing opportunities for negotiating meanings

Beginning with an open-ended task was intended to elicit interest, promote creativity in problem solving, and encourage flexible and communicative approaches (Silver 1997). The open-ended nature of finding a box that ‘would be right for your families’ invites multiple solutions to the problem, and as such allows opportunities for students to negotiate meanings and connect mathematical ideas. We intended to evoke problem solving through spatial reasoning as the students considered the size of their families and connected this information back to the question. Negotiating meaning is again strongly connected to the environmental provisions afforded in the task design—namely the element of peer collaboration. Students began the exploration in pairs, then transitioned to increasingly larger groups, where they had to explain their approaches, negotiate goals for solving the problem (e.g. finding a largest box), justify their spatial visual estimations of size rankings, and then negotiate and implement strategies for verifying their estimations. These negotiations supported connection-making as well as the use of artefacts as representational tools.

8 Concluding remarks

Geometric spatial visual reasoning plays important roles in learning mathematics: it acts as a tool to represent understanding of concepts, and as a process through which to understand concepts (Bruce and Sinclair 2015). In developing the construct of a task network for and of geometric reasoning, we were able to leverage the benefits of spatial visual approaches to mathematics to allow for wider accessibility of advanced concepts, specifically, optimisation and rates of change. These concepts are rarely introduced formally to young students, yet, they are concepts for which many children have an informal understanding and an interest. Optimisation and rate of change are also concepts that allow for multiple approaches and connections amongst more familiar concepts in the elementary school curricula (Cuoco and Goldenberg 1997). In particular, by addressing optimisation via rate of change in volume, our task networks support important geometric reasoning through geometric transformations, including physical movements, symmetry, proportional reasoning, and visual spatial estimation. While relatively little research attention has been paid to children’s practices in using and estimating volume (Vasilyeva et al. 2013), Dorko and Speer (2014) found that the difficulties the students experience with volume can inhibit achievement in mathematics, and may extend as far as university calculus. As such, there is a need for research that attends not only to student difficulties, but also to strategies and approaches that can help provide alternative representations that overcome or bypass these difficulties. Our research makes an important contribution in this direction by developing tasks that network ideas of volume and volume change across various stages of learning—from elementary to tertiary education, as well as by offering a construct for task development that can foster connected approaches to other curricular content both vertically across grades, as well as laterally across strands and ways of reasoning.

We suggest that our task network affords possibilities for teachers and teacher educators to support connections amongst post-secondary, secondary, and elementary school mathematics. Further, our research sheds light on how the complexities of teaching toward conceptual understanding in geometry and through geometric ways of reasoning may be captured and used to advantage in task design. By networking theoretical frameworks, we illustrate the interdependent relationships of content (KDUs), cognition (conceptual blending), and pedagogical sensitivities (scaffolding) in teachers’ disciplinary knowledge of mathematics. We suggest that such relationships are important to foster in teacher preparation, both within departments of mathematics and faculties of education. Looking forward, our research invites further questions specifically connected to these complexities. The question of how learning trajectories may differ depending on the sequence through which learners experience the undirected connections between nodes in the network requires further empirical investigation. At each node in the task network, we may expect different possible and available blends depending on where the student began; however, there is no expectation of transitivity and the question of to what degree does extending a blend depend on the initial blend composition remains open.

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