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ORIGINAL ARTICLE

Reinventing learning: a design-research odyssey

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Abstract Design research is a broad, practice-based approach to investigating problems of education. This approach can catalyze the development of learning theory by fostering opportunities for transformational change in scholars' interpretation of instructional interactions. Surveying a succession of design-research projects, I explain how challenges in understanding students' behaviors promoted my own recapitulation of a historical evolution in educators' conceptualizations of learning—Romantic, Progressivist, and Synthetic (Schön, Intuitive thinking? A metaphor underlying some ideas of educational reform (working paper 8). Division for Study and Research in Education, MIT, Cambridge, 1981)—and beyond to a proposed Systemic view. In reflection, I consider methodological adaptations to design-research practice that may enhance its contributions in accord with its objectives.

Papert (1980), who was a student of Piaget, champions the pedagogical implications of constructivism: children best learn via engaging in goal-oriented interactions with materials and reflecting on solutions to emergent problems they encounter in so doing. And yet adding a pragmatic twist to constructivism, Papert (1991) coined the term "constructionism" so as to suggest that children best construct knowledge when they construct artifacts in the public domain. I believe that adults, too, best construct knowledge when they construct artifacts in the public domain, and this includes educational researchers constructing experimental learning materials. I thus view design research as a constructionist approach to educational research: researchers best construct theory when they construct artifacts for students and reflect on solutions to emergent problems encountered in so doing. These problems emerge for researchers in making sense of how students engage the artifacts. Researchers' solutions to these emergent problems of analysis are expressed in the form of new or refined theoretical models that attempt to explain educational processes (diSessa and Cobb 2004). Over a succession of design projects, this iterative dialectical process of building artifacts and theory continues, where aspects of effective design practice are identified and articulated into principled frameworks, lessons learned are applied to new contexts, and then new insights are generalized further (Confrey 2005; Streefland 1993).

This reflective article discusses the role of design research in promoting learning theory. The context selected for the reflection is an enduring educational-research problem—the roles that naïve, spontaneous forms of knowing and acting may play in fostering conceptual learning. The article discusses a multi-project dialectical evolution of theoretical models pertaining to this problem. I argue that a succession of design-research studies has brought about iterative transformational change in one researcher's conceptualization of naïve knowing and its didactical affordances. I implicate this transformational change in the micro-analysis of children's behaviors as they engage in cognitive and dialogical problems centered on activities with artifacts designed for these studies. I imply that my subjective experience generalizes broadly to educational scholarship.

To lend structure to this reflection, I will begin by drawing on the literature to trace a historical trajectory in educational scholarship pertaining to the role of naïve knowledge in conceptual learning. I then recount my ontogenetic journey as mapped upon this phylogenetic journey yet

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extending beyond it. This "cartographical" exercise should indicate that my own path has not been a random walk visiting arbitrary points, but rather a reinvention of a historical succession of educational frameworks leading to an emerging new framework that charts new territory. As such, although my design projects per se are idiosyncratic, the congruence of my intellectual journey with the historical evolution of educational theory could lend credence to the design-research practices that fueled and formed the odyssey and, by induction, to the emerging theory.

1 Introduction: reinventing learning theory as a historical and future journey

In his survey of educational reform since Rousseau, Schön (1981) discerns three major pedagogical views on the role of children's naïve conceptions in their instruction. He names these views Romantic, Progressivist, and Synthetic.

The *Romantic* view is passionate and subversive, emanating from a vision of childhood as a safe haven rather than an apprenticeship for adulthood. Schools, by this view, can be harmful to children's development. Whereas schools ought to introduce curricular content by engaging the child directly with natural phenomena, schools in fact use inaccessible formal notations and mechanistic algorithms in reference to these same phenomena. Consequently, the child often encounters difficulty in making sense of instruction and ultimately develops a belief that formal knowledge is divorced from the senses and sensibility (see also Dewey 1938; Kamii and Dominick 1998; Nathan 2012; P. W. Thompson 2013). Rather, the educator should create opportunities for exploration and discovery, always preferring the situated and playful over the symbolic and solemn. A unit of analysis here would be the singular child, with the adult playing an incidental external role as the wise steward. The process of learning is discovery, defined as taking spontaneous actions to apprehend patterns and consistencies in natural phenomena. Through discovery children transition from not knowing to knowing.

Romanticists "invented childhood" and in so doing created legitimacy to treat young humans as deserving age-appropriate instructional regimes, a legacy greatly present in contemporary scholarship, commerce, and recreation. Yet as we now discuss, this *sturm und drang* was met by reactionary forces that shared its passion yet tempered its pedagogical implications.

The *Progressivist* view takes a step forward toward charting an educational program. Endorsing learners' need for situated interaction, Progressivists perceive epistemic continuity from informal to formal perspectives on phenomena. Progressivists believe in the civic mandate and pedagogical potential of educational institutions to foster cognitive continuity from informal to formal knowledge via implementing principled didactical intervention. By the Progressivist view, educators should craft for students structured opportunities to reenact cultural–historical phylogenesis from naturalistic ways of being and knowing to techno-scientific concepts (Brousseau 1997; Freudenthal 1983). Children will thus develop their naïve views into more sophisticated forms of reasoning (von Glasersfeld 1987). In particular, by adopting cultural forms, students will be able to express, organize, elaborate, and appreciate what they already see and know, so that their naturalistic inclinations become better fit for cultural praxis. Children thus enjoy logically planned, scheduled occasions of exercising their natural inclinations to discover and learn in authentic settings, thus expediting their teleological course of cognitive development toward maturity and reason as upstanding participating members of the adult community (see also Froebel 1885/2005). Yet whereas educators craft for children various devices that enhance their development, further intervention should be minimal, because children best learn when left to their own devices. So doing, children creatively reconfigure their situated sensorimotor forms so as better to fit the natural or cultural ecologies they encounter. The unit of analysis here once again is the learning child, the process is that of discovery, defined as recognizing common structures across diverse situations, and knowing consists of being able to apply the emerging models to new contexts.

As a general plan of action, the Progressivist view underlies much of what we call reform-oriented practices in education. And yet from a theoretical point of view, sometimes Progressivists under- or mis-represent the overwhelmingly formative role of cultural intervention in shaping conceptual development. For researchers, this means that experimental designs are far more than the "materials" and "procedure" detailed in methods sections of empirical reports describing pedagogical studies of essentially individualistic learning. Rather, *everything about the researcher–student interaction is relevant to making sense of the student's behavior*.

The *Synthetic* view, like its historical antecedents, embraces the pedagogical utility of both leveraging children's naïve knowing and introducing disciplinary structure. However the Synthetic view problematizes the possibility of cognitive continuity from naive to disciplinary constructions of the world, even through guidance. Whereas the literature presents a variety of positions with respect to the affinity or dichotomy of naïve and scientific knowledge, the Synthetic view generally resonates with sociocultural theory (Newman et al. 1989; Sfard 2002), social constructivism (Yackel and Cobb 1996), and perspectives from psychology (Chi 2013; Kahneman 2003).

The Synthetic view cites naïve and disciplinary constructions as epistemologically incompatible. These informal and formal constructions might overlap only loosely ("transitively") via common discursive reference. That is, a teacher and student may refer to the same object in a perceptual display even as they construct the object differently. Development toward formal constructions implies reconfiguration in the visualization of situations. As such, any instructional intervention seeking to introduce formal constructions necessarily requires students to *re*-see a phenomenon in ways that depart from naturalistic orientation. By way of supporting this re-seeing, teachers should guide students to attend to, parse, visualize, reify, and label the latent properties and aspects of phenomena in ways that hitherto may have never occurred to them as remotely relevant to the task at hand, yet are vital for professional practice (see also Arcavi 2003; Bamberger 1999; Goodwin 1994). Yet whereas the teacher's role is to steer the child's appropriation of these cultural–historical ways of seeing situations, the teacher should also optimize the prospects of the child somehow coordinating and perhaps reconciling these vying naïve and scientific constructions of the world (see also Bartolini Bussi and Mariotti, 2008; Job and Schneider 2014; Radford 2014).

Children enjoy structured opportunities to engage in dialogic internalization and appropriation of formal views on familiar phenomena. The unit of analysis here is the tutor–student dyad or teacher–students manifold, and the process of learning is appropriation, defined as adopting the hegemonic visualization of phenomena under inquiry. Knowing emerges through participating with increasing competency in the social enactment of the disciplinary cultural practice. Yet this new know-how should optimally be grounded, via reflection, in naïve orientations to these situations. So doing, learners render transparent any new artifacts they engaged along the way. It is this latter Synthetic view that Schön evaluates as best depicting the objective of educational intervention.

Any historical survey naturally begs the question, What next? Does this intellectual evolution stop at the Synthetic perspective on learning, or is this yet another milestone on an endless journey from Enlightenment toward ultimate enlightenment? In this essay I will be suggesting that the journey continues, and that the next golden age, which builds on the Synthetic view yet expands on it, might be called "*Systemic*".

The Systemic view does not reject the Synthetic view but qualifies, broadens, explicates, and ultimately supports it. In that sense, the evolution in theory from the Synthetic to the Systemic is not as abrupt as from the Progressivist to the Synthetic, but more of a complexification. The Systemic view draws broadly from emerging perspectives in learning sciences: Enactivist philosophy (Varela, E. Thompson, and Rosch 1991), situated cognition (Greeno 1998), distributed cognition (Kirsh 2013; Martin 2009), dynamic-systems theory (Thelen and L. B. Smith 1994), extended mind (E. Thompson and Stapleton 2009), ecological psychology (Gibson 1977), and ecological dynamics, a theory of learning from sports sciences (Chow et al. 2007; Newell 1986). The Systemic view was first developed in Abrahamson and Trninic (2015) and has been elaborated in Abrahamson and Sánchez-García (2014).

In its unit of analysis, the Systemic view maintains the teacher and learner, but reconfigures them into a dynamical system that includes the learner as an agent, some task that the agent is attempting to accomplish, and a general environment that includes the teacher as a sentient interactive element. Left to its own devices, the system is taken to be functioning in some dynamical stability. The process of learning is perceived as the systemic reconfiguration in transitioning from one dynamical stability to the next. The agent is spurred to adapt when it apprehends new environmental constraints on its task-oriented activity. For example, a teacher might deliberately introduce into a learning environment constructive perturbation that problematizes the student's interactions with the world. Achieving synthesis, such as between naive and scientific knowing, is an emergent process, in which task-oriented interactions among the agent and the designed/monitored environment are tight, rapid, volatile, and recursive. What we call learning is the emergence of new systemic stability borne from the agent's concerted efforts to adjust, develop, and refine coordination patterns for availing of newly encountered features or aspects of the environment toward achieving evolving goals. From the Systemic view, conceptual synthesis manifests in more than just binary either/or states but comes by degrees, just as physical skill varies by degree of dexterity within and between individuals. Learning is tightening one's grip on the world (Bernstein 1996; Merleau-Ponty 2005).

The Systemic view presents a historical opportunity to revisit early cybernetics research on cognitive development (Piaget 1970) and thus position genetic epistemology as squarely relevant to emerging theoretical models of human reasoning as simulated motor action (a.k.a. the "embodiment turn", e.g., Gibbs 2011). All conceptual learning begins from the solution of new motor-action problems, whether or not these problems or solutions are overtly manifest as explicit external activity (Melser, 2004). As Vygotsky stated (1926/1997), "Even the most abstract thoughts of relations that are difficult to convey in the language of movement, like various mathematical formulas,….are related ultimately to particular residues of former movements now reproduced anew" (p. 162).

By thus embracing the respective theories both of Piaget and Vygotsky, the Systemic view could also stand to serve in promoting a reconciliation of their seminal contributions (Cole and Wertsch 1996). Similar to Vygotsky, Piaget implicated knowledge as both manifest in, and modified by situated interaction. Whether it is couched as discursive or operational, knowledge is inherently immersive, relational, and interactive. Counter to Spackman and Yanchar (2013), I do not view a schema as an internal representation but as a systemically distributed dynamical routine: "Knowing does not really imply making a copy of reality but, rather, reacting to it and transforming it (either apparently or effectively) in such a way as to include it functionally in the transformation systems with which these acts are linked" (Piaget 1971, p. 6). I therefore join Allen and Bickhard (2013) in suggesting the enduring and even increasing relevance of Piaget's epistemological constructs for current research and theorizing of human learning.

The Systemic view thus ascribes a pivotal role to students' reflective motor-action activity in the emergence of conceptual knowledge (Abrahamson and Sánchez-García 2014; Abrahamson and Trninic 2015). This theoretical view bears the practical implication that each agent–child must discover a subjective solution to an embodied interaction task and only then signify it formally. The educator's role is to engage the child in solving an accessible, asymbolic, physical interaction problem. In so doing, the educator attends to the child's sensorimotor exploration via second-person kinesthetic empathy (Depraz et al. 2003). Once the child has demonstrated adequate mastery and explained the solution, the educator interpolates into this new dynamical equilibrium carefully selected or crafted elements, such as symbolic artifacts, that perturb this equilibrium productively. Children enjoy opportunities initially to engage intuitively in free-form problem solving within well-structured learning environments. Next, they tackle new constraints introduced into the environment that shift them into formal models of the same situations. Finally, they describe their actions within what turns out to be mathematical semiotic register. Table 1 summarizes the four views on the role of naïve reasoning in STEM learning.

The Systemic view on the phenomenon of learning is a natural outcome of twentieth century scholarship (Clancey 2008) and accommodates twenty-first century developments in epistemology, pedagogy, and technology (Abrahamson et al. 2012). In particular, the dynamical-systems principle of emergence can account for the constructive role of irrational exploratory behavior in problem solving (Fischer 2001; Lakatos 1976) and vitiate the learning paradox (Bereiter 1985). In turn, emergence resonates strongly with non-prescriptive pedagogy (Turkle and Papert 1991) and the increasing appreciation of error as conducive to learning (Kapur 2014).

The Systemic view is widely encompassing in its attention to structures and processes relevant to making sense of individual learning. The view implies that *researchers themselves are agents of interest in the systemic interpretation of learning* (see also Jaworski 2012; White 2008). We could ultimately achieve a deeper understanding of student learning by stepping back to engage in hermeneutic analysis of our own tacit framing of the research process and our agency therein (Barwell 2009; Guba and Lincoln 1998; Yanchar 2011). This reflective practice is vital to promoting the modeling of educational phenomena under inquiry, because it exposes epistemological and theoretical assumptions implicitly forming our research orientation, rationale, and design (Schön 1983; Tracey et al. 2014). Specifically, researchers, too, may need to synthesize apparently incompatible views of phenomena under inquiry. But how does such synthesis come about? Can design research foster such synthesis? If so, what is the mechanism by which design projects create opportunities for researchers to coordinate new ways of conceptualizing their subject matter of student learning?

In this paper, I describe the role of design-based research in my personal journey toward the Progressivist view, on to the Synthetic view, and beyond to the proposed Systemic view. The objective of the paper is to describe the role of design-based research in the development of theories of learning.

There is a certain professional awkwardness in realizing that one has been reinventing the history of educational reform piecemeal rather than simply accepting current knowledge. But then again, it would be self-defeating for a scholar of constructivist affiliation to expect of his own learning process anything short of ontogeny recapitulating phylogeny. The plan of the paper is to trace this personal development by overviewing several design projects and explaining how they each contributed to forming my current views.

2 Learning theory by three designs

Though they have left a formative mark on the history of Western intellectualism, staunch Romantic views of learning are now often viewed as quaint and untenable. As a student of cognitive sciences, I began my career already disillusioned by the prospects of purist Romantic views to offer viable educational programs. I learned to think of myself as a Progressivist. The following three projects from the past two decades of research are milestones in my theoretical development through design research, from a Progressivist view and onward:

- "Seeing Chance" (probability)—from Progressivist to Synthetic
- "Kinemathics" (proportion)—from Synthetic to Systemic
- "Giant Steps" (algebra)—expanding the Systemic

Table 1 Evolution of views on the role of naïve knowledge in the teaching–learning process

Pedagogical perspective	Principle of learning	Process		Unit of analysis
		Child's experience	Teacher's role	
Romantic	Discovery via semi-spontaneous situated inquiry of mechanisms underlying phenomena	Unschooled exploration of engaging phenomena	Selects interesting phenomena; provides inspiration, security, and general support and guidance	"Solo": student operates essentially alone within a "bubble" created by teacher
Progressivist	Guided reinvention of cultural–historical knowledge; naïve knowledge vital for instigating process	Relatively straightforward adoption of formal methods as means for (re-)articulating informal knowledge	Sets up didactical situations; intervenes to foster student's engaged epistemic continuity from informal to formal knowledge	"Solo with accompaniment": child operates essentially alone, but design and peripheral steering are acknowledged as vital, artful didactical activity
Synthetic	Dialogic negotiation of naïve and scientific views on phenomena; knowledge is co-constructed via contested then negotiated discursive forms of reference to shared situations	In the course of attempting to perform a task, encounters unfamiliar problem; recognizes, in scientific views and discourse, potential utilities for enhancing productive participation; engages in concerted appropriation efforts	Essential dialogic participant in presenting and mediating scientific views as complementary alternatives to naïve views	"Chamber music": student learning is a dyadic or ensemble process and achievement ("obuchenie", Cole 2009)
Systemic	Proactive response to constraints (didactic perturbation) so as to achieve new dynamical stability in the task-oriented agent–environment ecology	In the course of engaging a situated sensorimotor task, elicits available schemes yet encounters unexpected feedback; via explorative interaction establishes new attentional anchors that are hence signified professionally	Elicits student's operatory schemes, challenges their manifest adequacy for problem solving, introduces new constraints, and—via attentive kinesthetic empathy—guides toward functional reintegration, then adoption of symbolic artifacts as new attentional anchors	"Concerto": student as agent in complex situated activity of multiple goal-oriented participants constantly tuning to each other. Participant–researchers as designers/facilitators whose tacit epistemic orientations pervasively and reciprocally shape the environment, ongoing intervention, *post facto* analyses, conclusions, and scholarly dialog

The narratives for each project explain our design rationale and hypotheses, our surprise when things went counter to our expectations, and our inferences from these surprises (see also Lobato et al. 2015 (this issue) on constructive failures in design research). If I use the plural "we" rather that just the singular "I", it is because much of the work described herein was done in collaboration with graduate-student researchers.

2.1 From progressivist to synthetic view of learning: Seeing Chance (probability)

The Seeing Chance project (Abrahamson 2012a) was founded on the Progressivist rationale of creating opportunities for learners to articulate informal judgments of likelihood in terms of formal sample spaces. Given appropriate materials, activities, and facilitation, I initially assumed, children would experience cognitive continuity along this learning path—they would see the formal framing of likelihood as elaborating on their informal view.

2.1.1 Design problem and rationale

The mathematical content of probability has long challenged students of all ages (Jones et al. 2007). Specifically, there have been no empirical demonstrations, to date, of children spontaneously reinventing correct sample spaces of compound events by noticing latent formal structures in the learning materials themselves. Perhaps those studies failed to elicit and engage early conceptions (Smith et al. 1993).

We know that infants are able to judge correctly the relative likelihood of binomial random samples vis-à-vis the population from whence they are drawn (Denison and Xu 2014). For example, infants register a sample of four green balls emerging from a mixed green/blue collection of balls as more surprising than a sample of two-green-and-two-blue balls. The infants use what Tversky and Kahneman (1974) have called the representativeness heuristic, that is, they register the sample's internal binomial ratio (e.g., the ratio of green balls to blue balls) and compare this ratio to figural, structural, compositional, or procedural properties of the randomness phenomenon from which it was generated. So doing, however, the infants do not attend to the set of independent outcomes composing the sample, as in the formal classicist-probability procedure of combinatorial analysis (i.e., the specific *order* of green and blue balls in the sample). Thus when they make their judgment calls, infants do not bear in mind the entire sample space of all such possible sets (all permutations, or all variations on all possible combinations). Indeed, even adults tend to believe that a coin tossed four times is more likely to land on heads–tails–heads–tails than heads–heads–heads–heads—that is, HTHT appears to them more representative of a two-sided coin than HHHH does—whereas according to mathematical theory, in fact, these two outcomes are precisely equiprobable. In what follows I refer to "H" or "T' independent outcomes that compose the binomial outcome as singleton outcomes (e.g., HTHT is composed of four singleton outcomes).

Our design rationale began from noting that an outcome with 2 heads and 2 tails *in any order* is indeed more likely than an outcome with 4 heads—it is six times as likely. (The binomial event of 2 heads and 2 tails has six different favorable outcomes that are each a discernable variation on the 2H2T combination of four singleton outcomes, whereas the event of 4 heads has only one favorable outcome). As such, people who ignore the specific order of singleton outcomes appear to be answering the question correctly, only that they are answering in accord with their subjective, non-normative understanding of the question (Borovcnik and Bentz 1991).

2.1.2 Design solution: the marbles scooper and combinations tower

We decided to create a binomial experiment that would enable students both to leverage their informal perceptual judgment and appreciate how mathematical formal analysis elaborates on their informal judgment. We therefore designed a random generator that highlights the variable order of singleton outcomes *during the experiment itself* (see Fig. 1a), and we provided media (see Fig. 1b) for creating and assembling its event space (see Fig. 1c). We believed that making the order of singleton outcomes an integral structural property of the random experiment itself, not just a logical property of its formal analysis, would make the formal analysis more accessible. Presumably, by embedding the analytic forms into the phenomenon proper, we were anticipating and thus preempting challenges of accepting formal analysis.

2.1.3 Findings and conclusions

But we were wrong. At least, as anticipated, our participants—elementary school, middle school, undergraduate, or graduate students—did all offer correct informal judgments of the relative likelihood of experimental outcomes, for example stating that a sample with two green and two blue marbles is more likely than a sample with four green marbles. Also as anticipated, they were able, at our request, to distinguish visually among variations in the spatial configuration of green/blue marbles, and they were able to "go through the motions" of building the experiment's sample space as based on these combinations and variations. And yet initially *they did not appreciate the relevance of*

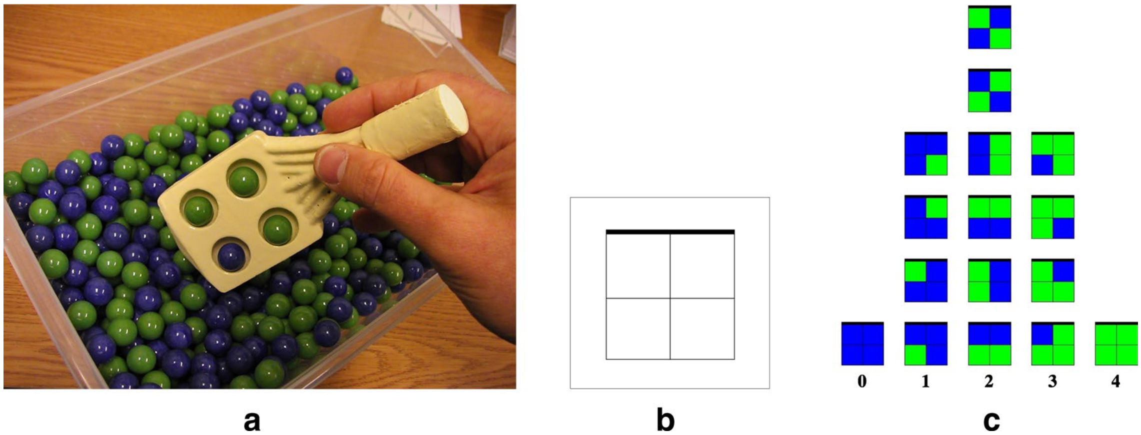

Fig. 1 Materials used in a design-based research project investigating relations between informal intuitions for likelihood and formal principles of the event space: **a** a "marbles scooper", a utensil for drawing out ordered samples from a box full of marbles of two colors; **b** a card for constructing the sample space of the marbles-scooping experiment (a stack of such cards is provided, as well as a *green* crayon and a *blue* crayon, and students *color* in all possible outcomes); and **c** a "combinations tower", a distributed event space of the marbles-scooping experiment, structured so as to render quantitative relations among the events conducive for heuristic perceptual inference

these variations for articulating their sensation of likelihood. These study participants were inclined to consider the formal analysis of the phenomenon only due to the mild pressure of the social interaction. Nevertheless, they ultimately did accept the probability space (the "combinations tower" in Fig. 1c), because they saw that the events they had judged as more likely were modeled as bearing more variations. Elsewhere, we have characterized this cognitive process as abductive reasoning (Abrahamson 2012b) leading to heuristic–semiotic leaps (Abrahamson 2009). Only once they had appropriated these mathematical forms as acceptable methods of reasoning, the students retroactively accepted the combinatorial analysis *process* that had led to creating these products (Abrahamson 2012c).

We concluded that the property of variation (the internal order of singleton events within an order-less combination), which is critical for scientific analysis of binomial experiments, is deeply at odds with a naïve view on randomness phenomena. This conclusion was supported by a follow-up controlled experiment, which demonstrated that invoking people's probabilistic schemas blinds them to event variation (Mauks-Koepke et al. 2009). We thus rejected the general Progressivist view, because we were witnessing blatant and apparently incompatible discontinuity from naïve to scientific understanding of probability. And yet at the same time, all the students did ultimately accept the classicist view on the randomness experiment. What was going on? Did we confound our own study by using an interview protocol that created a learning path for our participants? Perhaps the protocol, ostensibly a "cold" research tool for eliciting and measuring knowledge, in fact instantiated an essential pedagogical practice. Perhaps the interviewer inadvertently played a critical role in enacting a productive, commendable, and culturally authentic social practice known as "teaching".

If the interviewer's dialogical participation was essential for the ultimate success of the learning process, this finding could thus be understood as vindicating the design rationale. Yet for this, we reasoned, the very notion of facilitation would warrant fundamental rethinking. In this emerging understanding, the interviewer is "reappointed" from the outer methodological periphery of the learning event into the inner pedagogical circle, as an integral component of the activity design proper. Hence, we would not need to excuse ourselves for teaching the student, as though engaging in dialog with a student marked a shortcoming of the instructional materials. Rather, we would need to interpret the interviewer's function as critical in facilitating students' negotiation of naïve and mathematical visualizations of the objects in question. That we actively led the students to understand the activity's target content was cause for celebration, not shame. And yet recognizing this meant that something was missing in our fundamental conceptualization of learning. Our experiments were studies not of learning per se, but of education.

2.1.4 Shifting to synthetic view

In hindsight, the transition from Progressivist to Synthetic views on learning was the most challenging passage in my recapitulation of modern educational scholarship. I faced the resounding failure of my prized pedagogical artifact, the marbles scooper, to elicit from any person an articulation of their unmediated perceptual judgment in the form of its mathematical counterpart. Despite having embedded the mathematical form as an inherent structural property of the interaction device, my study participants ignored this form—they did not differentiate objects in the shared perceptual display on the basis of this property. There, right before our four eyes, were the variations—a set of alternative spatial layouts of a certain compound event, each creating a distinct figural pattern—and yet whereas I attended to this phenomenal property as bearing information that was critically relevant to the task at hand, my interlocutors disregarded this property as entirely inconsequential to the task.

I thus experienced a harsh breakdown in the unreflective flow of implementing my design rationale, and as a direct consequence of this breakdown my implicit pedagogical worldview "announced itself afresh" (which is how Heidegger describes breakdowns). I became aware of my implicit Progressivist belief in an individual's would-be capacity for unguided reinvention of culture and, then and there, let go of that belief. I also gave up the convenient Progressivist notions of a teacher's essentially marginal role in the process of guided reinvention, as a facilitator who sets a course of progress but then does not intervene significantly along this course. I now recognized the complexity of the interviewer's multimodal discourse and came to think of education as a joint achievement of teacher and student (Cole 2009). Disillusioned with the Progressivist model, I scrabbled for a new one that would align with my cumulative empirical findings. But first I had to unravel an apparent dilemma.

In their informal reasoning, my study participants engaged not a cultural form but a primitive cognitive capacity that has been documented even in neonates. On the one hand, the radical-constructivist view could not suffice, because I repeatedly observed non-continuity between this early capacity and the complementary analytic construction of the same phenomenon—students could not reinvent the analytic construction without heavy-handed steering (see also Bereiter 1985). On the other hand, neither could the sociocultural view suffice, because the participants had first to engage their unmediated perceptual skills—if they had not done so, then they could not know what the activity was *about*. Somewhat reluctantly, I realized that my dilemma could be dissolved, but at the price of my Progressivist identity. My educational worldview thus became Synthetic, a synthesis of constructivist and sociocultural views of learning.

Having arrived at the Synthetic view, my research focused on the general design problem of creating situations in which teachers can foster synthesis, that is, enable and encourage children to see a phenomenon from both naïve and scientific perspectives as well as to ground the new concepts, that is, to coordinate these alternative constructions via heuristic–semiotic reconciliation. This Synthetic view framed my design for the Kinemathics project, as follows.

2.2 From synthetic to systemic views of learning: the Kinemathics project (proportion)

The *Kinemathics* project (Abrahamson et al. 2014b) took on the design problem of students' enduring challenges with proportional relations (Davis 2003). The objective was to create for students' opportunities to discover informal solutions to interaction problems centered on coordinated proportional motions in space and then synthesize these solutions with a formal mathematical re-articulation of these solutions. It was expected that the teacher would play an essential role in the critical phases of this process.

2.2.1 Design problem and rationale

When students look at $6{:}10 = 9{:}x$, they are liable to make sense of these symbols through an "additive lens" instead of a "multiplicative lens". That is, they might attend only to the *differences* among the numbers: seeing a difference of 4 between 6 and 10 (or seeing a difference of 3 between the 6 and 9), they infer that the other pair has the same difference, so that the unknown number is 13 (whereas it should be 15).

We assumed that students have little if any presymbolic action imagery as personal meaning for proportional equivalence (Pirie and Kieren 1994; Thompson 2013). Thus, students would need first to construct informal, presymbolic action imagery pertaining to proportional equivalence, and only then, per Synthetic rationale, they would coordinate additive and multiplicative views on this phenomenon. Our design solution was the Mathematical Imagery Trainer for Proportion (MIT-P).

2.2.2 Design solution: the mathematical imagery trainer for proportion

We seat a student at a desk in front of a large screen and ask the student to "make the screen green". The MIT-P remote-senses the heights of a user's hands above the datum line (see Fig. 2a). When these heights (e.g., 2" and 4"; Fig. 2b) relate in accord with an unknown ratio set on the interviewer's console (e.g., 1:2), the screen is green. If the user then raises her hands in front of the display maintaining a fixed distance between them (e.g., keeping the 2" interval, such as raising both hands farther by 6" each, resulting in 8" and 10"), the screen will turn red (Fig. 2c), because the pre-set ratio of 1:2 has been violated. But if she raises her hands appropriate distances (e.g., raising her hands farther by 3"

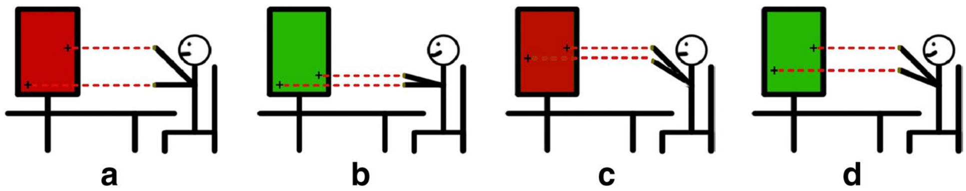

Fig. 2 The Mathematical Imagery Trainer for Proportion (MIT-P) set at a 1:2 ratio, so that the target sensory event (a *green background*) occurs only when the right hand is twice as high along the monitor as the left hand. This figure encapsulates the study participants' paradigmatic interaction sequence toward discovering the proportional operatory scheme: **a** while exploring, the student first positions the hands incorrectly (*red* feedback); **b** stumbles on a correct position (*green*); **c** raises hands maintaining a fixed interval between them (*red*); and **d** corrects position (*green*). Compare **b** and **d** to note the different vertical intervals between the virtual objects

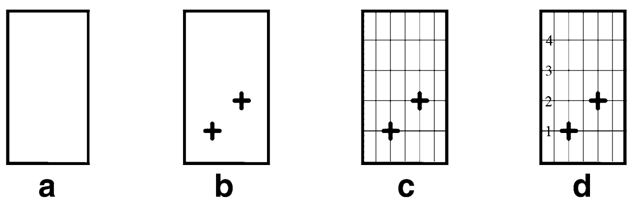

Fig. 3 MIT-P display configuration schematics, beginning with **a** a blank screen, and then featuring a set of symbolical objects incrementally overlaid by the facilitator onto the display: **b** cursors; **c** a grid; and **d** numerals along the *y* axis of the grid. These schematics are not drawn to scale, and the actual device enables flexible calibrations of the grid, numerals, and target ratio

and 6", respectively, resulting in 5" and 10"), the screen will remain green (Fig. 2d). Participants are tasked first to make the screen green and, once they have done so, to maintain a green screen while they move their hands. This new way of moving the hands is the action imagery that later becomes signified as proportion.

The activity advances along a sequence of stages, each launched by the introduction of a new display overlay (see Fig. 3) immediately after the student has satisfied each of successive protocol criteria. For example, consider a student who is working with the cursors against a blank background (Fig. 3b). Once he articulates a strategy for moving his hands while keeping the screen green, the activity facilitator introduces the grid (see Fig. 3c).

The instructional intervention was initially conceptualized from a Synthetic perspective. Our participants were first to engage in naïve exploration of the problem space by which they would discover a sensorimotor coordination pattern for keeping the screen green while moving the hands. Next, we would introduce for them the alternative scientific technique that utilizes the grid (for an additive visualization) and then also the numerals (for a multiplicative visualization). These symbolic artifacts could serve as resources for parsing the blank space into discrete units so as better to control the manual actions and predict their outcomes. That was our activity plan.

2.2.3 Findings and ontological innovation

But we were wrong. At least, as anticipated, the participants initially discovered how to move their hands in the "green zone" by gradually increasing the vertical distance between their hands while raising the hands. Yet once we had introduced the grid onto the screen, the students did not need us to demonstrate, mime, or explain how to use these symbolic artifacts to enhance their actions—they figured out the process on their own, without dialogic mediation, by tinkering with the new elements introduced into the working space. Instead of moving their hands continuously and simultaneously while adjusting their distance, the participants switched spontaneously to moving their hands discretely and sequentially, such as alternately raising the left hand 1 unit and the right hand 2 units, for the 1:2 setting. Later in the interview we helped the students reconcile the various visualizations of proportional motion (Abrahamson et al. 2014b).

Thus a new sensorimotor coordination pattern, the *ratio*-oriented action-based solution, emerged in the course of engaging the grid as a mediating auxiliary stimulus for accomplishing the *difference*-oriented solution. We named this phenomenon *hook and shift*. At first, during the hook stage, the problem solver detects within new features of the environment affordances for performing the task, where these affordances might be pragmatic, epistemic, or discursive. Yet then, during the shift stage, in the micro-process of engaging and adjusting these features to serve performance subgoals, a new action sequence emerges that is coupled with the environmental resources. The problem solver's *post facto* awareness of this shift is what we typically call "learning" (Abrahamson et al. 2011). The construct of hook-and-shift bears ties to the theories of distributed cognition (Martin 2009) as well as instrumental genesis (Vérillon and Rabardel 1995). And yet, whereas the learning opportunity was deliberately designed and implemented, the micro-event of reinventing ratio was unguided.

We were fascinated to witness students bootstrapping mathematical solutions to interaction problems without direct dialogic intervention. At the same time, this bootstrapping occurred not in a cultural void, but in an extremely structured environment. Yet what exactly was the role of the researcher–interviewer in this process? Apparently, the researcher took specific measures to bring the participant to the point that the bootstrapping could occur. And yet what were those measures? We re-analyzed our video data to develop a coding scheme of *tutorial tactics* for fostering hooks and shifts. Comparing this scheme to psychological (Bruner 1966) and cognitive-anthropology models of professional perception (Goodwin 1994), we concluded that theories of teaching and learning need to accommodate recent pedagogical, technological, and epistemological advances in the practice of mathematics education (Abrahamson et al. 2012), and that designers and teachers should respond to students' emerging cultural practices around technological devices (Negrete et al. 2013).

2.2.4 Shifting toward a systemic view

Coming into this study, we had espoused a Synthetic view on learning. And yet we then documented cases of sensorimotor exploration leading to spontaneous emergence of culturally appropriate operatory schemes. Our attention was drawn to these micro-moments of serendipity that, we believe, are powerful grounding experiences that could be fostered widely via learning activities using embodied-interaction technology (see also Lindgren and Johnson-Glenberg 2013). Our interest in the emergence of knowledge via embodied interaction led us to dynamical-systems approaches to individual development (Abrahamson and Trninic 2015). In turn, our increasing interest in the systemic co-construction of knowledge among agents operating in embodied-interaction learning spaces led us to consider perspectives from Phenomenology and Enactivism on how people tune toward each other (Depraz et al. 2003).

The final design project case study, which I describe more briefly than the earlier cases, led us to perceive the researchers themselves as bona fide components of the empirical data and, as such, legitimate subjects of an expansive systemic investigation.

2.3 Expanding the systemic view: the Giant Steps project (algebra)

The Giant Steps for Algebra project (Chase and Abrahamson 2013) addressed the pedagogical problem of introducing algebra to first-time learners. This study took place in parallel to Kinemathics (Sect. 2.2) and was thus conceived as a Synthetic project.

2.3.1 Design problem

The subject matter of algebra is challenging for many students who struggle with the ontological nature of a variable quantity as well as its symbolical notation and related solution algorithms (Kieran 2007). Our Synthetic rationale was to create opportunities for students to reinvent normative scientific forms for organizing algebraic activity via initially building naïve models of situated problems and then reflecting on the systematicity of their models. We began by inquiring into common instructional methodology for algebra and in particular canonical situations, forms, and metaphors.

2.3.2 Design rationale

The logic of algebraic propositions, such as $3x + 14 = 5x + 6$, is often grounded in action schemes pertaining to the twin-pan balance scale (see Fig. 4a). Therein, equivalence is maintained via commensurate changes to the two expressions on either side of the equal sign. Whereas this approach discloses the logic of algebraic algorithms, it is not presented as emerging directly from a familiar situation. Further, material masses are one step removed from the early developmental constructions of number as quantity (de Hevia et al. 2014). Finally, the balance-scale model is perhaps unnecessarily complex, in that it constructs equivalence between measures of two different sets, the collective masses of two groups of objects. An alternative model (Dickinson and Eade 2004, see Fig. 4b) constructs algebraic propositions as the equivalence of two different expressions for the measure of one and the same object—the length of a single line segment. The reader is invited to appreciate how the number-line model supports mental solution.

2.3.3 Design solution

In the Giant Steps for Algebra project ("GS4A") we are evaluating through design experiments the following conjecture: Given appropriate technological mediation, students will be able to reinvent the number-line system for solving situated algebra problems and then signify this system in symbolic notation. Working in GS4A, students first build a diagram in an attempt to solve a word problem. For example, Fig. 4b, above the line, could depict the journey of a giant who departed from the left and walked three

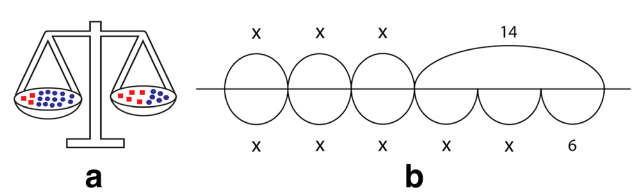

Fig. 4 **a** Balance scale and **b** number-line instantiations of "$3x + 14 = 5x + 6$"

steps followed by 14 additional meters to a destination on the right, where he buried treasure. Figure 4b, below the line, shows that this giant traveled again the following day, beginning from the same point of departure on the left and arriving at the same destination on the right, only this time walking five steps and an additional 6 meters, where he located his treasure. How large is his step?

2.3.4 *Reverse scaffolding and situated intermediary learning objectives*

In the technological incarnation of GS4A currently under development, the dialogic teacher is recast in software procedures. Students work in a computer microworld that includes a virtual toolbox and a modeling space. The students read a brief story about a giant who journeys twice along the same path, and then they attempt to represent that story in a form that is similar to the image in Fig. 4b. The toolbox includes different settings for the giant to advance either in steps or meters.

The activity has students reinvent the model through a succession of activity levels of increasing difficulty, for example story items that include subtraction (walking backward). At each level students struggle with what turns out to be a cumbersome interaction function, such as adjusting all the steps to be of equal length, yet once they satisfy a performance criterion for this level they are rewarded with a new function that facilitates those actions. We call this pedagogical design approach *reverse scaffolding*. Scaffolding is enacting for a novice aspects of a complex practice they are not able to perform. Reverse scaffolding is enacting for a novice aspects of a complex practice they *are* able to perform.

We are thus speaking of a type of situated pragmatic knowledge a child develops through tinkering with available resources in an attempt to model a hypothetical situation. An artificially intelligent pedagogical agent monitors and evaluates the student's activity, and it delegates to itself actions and constructions that satisfy our implemented criteria for conceptual adequacy. We call each element of this emerging construction or modeling know-how *situated intermediary learning objective* (SILO). In Abrahamson et al. (2014a) we inquire into how our design team developed the construct of SILO. We conclude that this construct emerged inadvertently in our discourse as a collective pragmatic solution to challenges of coordinating our collaborative activity in a team with diverse backgrounds and complementary objectives.

2.3.5 *Shifting to an expanded Systemic view*

By thus reflecting on and modeling our own design process, we are broadening the scope of what is legitimate and, we argue, vital for a Systemic conceptualization of mathematics education. We know that students' learning opportunities and the sense teachers make of them are enabled yet constrained by the teachers' content knowledge and pedagogical content knowledge (Sztajn et al. 2012). The same holds for design researchers investigating teaching and learning. And yet designers can play a key role in developing comprehensive Systemic models of instruction, because the practice of creating educational artifacts sits at the nexus of teaching, learning, and theorizing. Our task is to conceptualize and build resources for fostering alignment between what we know and what we want our students to know. A Systemic approach could lend structure and process to this task.

3 Reflection

Design research can bring about transformational change to investigators' conceptualization of learning, instruction, and design. This change comes about via the investigators' attempts to solve emerging problems they encounter in the analysis of empirical data gathered in the implementation of their designs.

I began this article by summarizing Schön's historiography of reform-oriented educators' views on learning since Rousseau. Therein, the epochs have been Romantic, Progressivist, and Synthetic. I then foreshadowed a fourth epoch, the Systemic, a dawning era that I associate with the cybernetic, situated, and embodiment turns in the cognitive sciences. Next, I surveyed my personal recapitulation of the phylogenetic process via a sequence of several design projects over a couple of decades of research that ultimately oriented me to expand the journey. Classically speaking, travelers on an odyssey arrive home. I did not embark from the Systemic perspective, but having arrived here, it does feel like a denizen worthy of inhabiting.

Throughout this qualitative meta-analysis, I have attempted to communicate coherence between emerging theory and emerging practice. From one project to the next, as my educational worldview ascended along Schön's *Gradus*, both my epistemological conceptualization and methodological practice became more complex by expanding the unit of analysis. Both epistemology and methodology moved out of the students' heads to embrace their embodied actions and then include also the researchers' actions, knowledge, and beliefs. The latter factors—determining the researchers' tacit assumptions underlying design rationales and solutions—warranted collective dialogic investigation within the design-research team, and this process itself was analyzed. Thus, the business of design studies unfolded iteratively into a larger activity structure that has often remained undisclosed and unconsidered in the practice and documentation of educational research.

In closing, I submit that two adaptations—epistemological and methodological—are required in the practice of design research so as to realize its potential, and I offer that a Systemic view could contribute to both adaptations.

Researchers need the license of their community of practice to indulge and report on their ongoing introspective search for the embodied roots of meanings, which they conduct prior to substantiating these in the form of prototypes (Abrahamson 2014; Schiphorst 2011; van Rompay, Hekkert, and Muller 2005). For this to occur, I contend, the community will need to engage in open discourse on how to align design-research methodology with phenomenological philosophy and embodiment theory (Depraz et al. 2003).

Systemic models would further require researchers to interrogate and foreground their own agendas and conceptual structures, which once again is challenging yet vital (Barwell 2009; Vagle 2010). Laboratories would need to cultivate an egalitarian discursive culture wherein participants reveal their own introspective reasoning processes as bona fide and critical objects of collaborative reflection promoting the collective research effort. Doing so may be particularly difficult to achieve in university laboratories, which often include undergraduate, graduate, and post-graduate students as well as senior researchers.

Collins (1990) authored a historical technical report in which he laid the foundations for an educational science modeled after engineering practice. He ended the report with the following words: "There are many issues that have important consequences for how we should deploy the technologies we develop, and it is important that we start addressing them in a systematic way" (p. 7). I would humbly add that we might address these issues in a systematic way by addressing them in a Systemic way.

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