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ORIGINAL ARTICLE

Geometry in early years: sowing seeds for a mathematical definition of squares and rectangles

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Abstract In early years schooling it is becoming common to propose activities that involve moving along paths, or programming robots to do so. In order to promote continuity towards the introduction of geometry in primary school, we developed a long-term teaching experiment (with 15 sessions) carried out over 4 months in a first grade classroom in northern Italy. Students were asked to program a robot to move along paths, to pretend to act as robots and to represent the sequence of commands and the resulting paths. In particular, in this teaching experiment, an overarching mathematical aim was to sow the seeds for a mathematical definition of rectangles that includes squares. Within the paradigm of semiotic mediation, we intended to foster the students' transition from a dynamic perception of paths to seeing paths also as static wholes, boundaries of figures with sets of geometric characteristics. The students' situated productions were collected and analysed together with the specific actions of the adults involved, aimed at fostering processes of semiotic mediation. In this paper we analyse the development of the situated texts produced by the students in relation to the pivot signs that were the beginnings of an inclusive definition of rectangles.

1 Introduction and rationale

Rectangles and squares represent a paradigmatic example of the conflict between the perceptual experience and the theoretical needs of a mathematical definition (on this delicate issue also see Kaur 2015; Tsamir et al. 2015), where squares are to be considered as particular rectangles (we will refer to a definition of rectangles that includes squares as being *inclusive*). There is evidence that such conflict persists in older students (Hershkowitz 1990; Lehrer, Jenkins and Osana 1998; Clements, Swaminathan, Zeitler Hannibal and Sarama 1999; Clements 2004; Lin and Yang 2002; Battista 2007; Koleza and Giannisi 2013; Fujita 2012).

In Mariotti and Fischbein (1997) mathematical definition is considered a true didactical problem, because of the conflict between perceptual experiences—and in particular visual Gestalts[1]—and theoretical needs, between the figural and the conceptual aspects. For example, when the child deals with squares and non-square rectangles, a conflict arises: actually, "from the figural point of view squares and non-square rectangles look so different that they impose the need of being distinguished at least as much as triangles and quadrilaterals" (Mariotti and Fischbein 1997, p. 224). So the didactical problem of mathematically defining "rectangles" (inclusively with respect to squares) is related to the possibility of constructing harmonization between the figural and conceptual aspects, between "the need to differentiate imposed by strong figural structures and the requirement to unify, to generalize imposed by the geometrical conceptualization" (Mariotti and Fischbein 1997, p. 245).

Moreover, it is through developing theoretical control—and not simply growing in age—that a student is able to overcome his/her initial attachment to prototypes (Fischbein 1993). In the case of a square or a rectangle, the prototype frequently has sides that are horizontal and vertical,

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[1] Visual Gestalts refer to theories of visual perception, developed in Germany in the 1920s, that attempt to describe how people tend to organize visual elements into groups or unified wholes when certain principles are applied.

and the square is just as wide as tall, while the rectangle is "long", "fat" or "tall". Reaching a level of harmonious interaction between an inclusive definition of "rectangle" and figural aspects of the prototypes allows the student to "control the meaning of the figure by its formal constraints" (Fischbein and Nachlieli 1998, p. 1197), and, for example, to recognize rectangles with different dimensions (including congruent dimensions), different positions or other different visual influences all as examples of "rectangles". This is how a student can overcome his/her blinding attachment to particular prototypes, especially in the cases of concepts to which a variety of figures (as in the case of squares and rectangles) corresponds. The process is much more straightforward when, instead, the student deals with concepts to which *invariant figures* correspond, for instance right angles (Fischbein and Nachlieli 1998, p. 1209).

Such process of harmonization of figural and conceptual components must be fostered through specific interventions, because research shows that it does not occur spontaneously, and attachment to prototypes is not overcome with age alone.

These results are consistent with those of notable studies conducted decades earlier by Luria (1976), who showed that naming and classifying geometrical figures depends heavily on the level of instruction received by the interviewed subject (pp. 31–47). In particular, he showed that the laws of "natural geometric perception" defined in Gestalt psychology are in fact dependent on the culture that subjects are exposed to. This allows us to state that even the perception of what Fischbein refers to as an "invariant figure", such as the right angle, seems to be such only for people who live in a "carpentered world", in a culture of right angles and lines, where these geometric features are *culturally* important.

These findings highlight the fundamental role of instruction in guiding perception along the lines of a cultural theory we want students to become a part of, and they lead us to believe it is important to design early (cultural) interventions to be initiated during the first years of formal instruction, and to be carried out in a continuous manner throughout the course of primary school. Unfortunately there are many widespread bad practices in school which reinforce the separation between squares and rectangles (for instance, activities with attribute blocks, where squares and non-square rectangles are classified in different sets).

Everyday language may act as an additional obstacle towards the development of a mathematical definition: for instance, in both Italian and English (and in other European languages) the names *quadrato* [square] and *rettangolo* [rectangle] hint at a complete separation of the figures into two different classes (square and not-square rectangles). We note, however, that this is not necessarily the case in all languages. In fact, in Chinese, the sequences of ideograms for the words "square" (正方形) and "rectangle" (長方形) contain two out of three same ideograms, those that indicate "sides" and "shape", while the first indicates "exact" (for the square) and "long" (for the rectangle). So, linguistically, a square is seen as a "shape with exact sides" and a rectangle as a "(same) shape with long sides". In this case language makes explicit that square and rectangle are two kinds of a same thing, deeply related to each other and not separated into distinct categories (also see Lin and Yang 2002). In fact this is typical of Taoism in which a central idea is the evolution of events as a process of change and the ideology of "grasping ways beyond categories" or to "categorize in order to unite categories" (以法通類, 以類相從) (Bartolini Bussi et al. 2013).The rationale behind the design of the study we present is that there can be means for constructing the meaning of squares and rectangles, generalizing the perception of square and not-square rectangular shapes, other than everyday language (that reinforces the conditions for an obstacle against an inclusive definition). A viable means can be programming the bee-bot in order to produce particular traced-paths that allow students to focus their attention on certain aspects of the path, such as changes of direction (in the context of the bee-bot, turns left or right of 90°), which can be put in meaningful relationships with (characterizing) parts of figures (such as right angles). It is exactly upon the shared geometrical property of squares and rectangles of having four right angles that we foresaw the possibility of capitalizing: we expected it to allow the sowing of seeds as a variety of acts of spatial experiences that could be developed into a "library" of paths and gestures, accompanied by verbal descriptions and drawings for developing a meaningful inclusive definition of rectangles (see Bartolini Bussi et al. 2007 for a similar analysis in the context of circles). We advanced the hypothesis that the bee-bot might be particularly appropriate for fostering appreciation of the "4 right angles" property because of how the semiotic activity may be promoted. In particular, the activity can naturally assume *body syntonic* features as well as *ego syntonic* ones (Papert 1980). The former, specifically, refers to Papert's vision that learners would be able to relate the behaviour of the microworld objects to their own sense and knowledge about their own bodies. We will see how this is possible with the bee-bot.

In the following section the context of the teaching experiment is described, followed by the theoretical frame of semiotic mediation, and a presentation of the research questions and methodology used for the study. The data presented and discussed refer to one class and they represent critical episodes in an ideal trajectory from the semiotic activity around the production of early tasks to a final poster with important shared discoveries (a sort of initial "theory" of the shared discoveries[2]). These aim at sowing seeds for reaching a (inclusive) mathematical definition of rectangles, while focusing on the classroom as a whole.

2 Theoretical frame

The Italian standards (for students aged 3–14) give emphasis to the development of space knowledge and geometry, in parallel with number knowledge (MIUR 2012). An important keyword, crossing all the students' ages, is *mathematical laboratory* that aims at *the construction of mathematical meaning*. In a mathematical laboratory (often realized in the classroom) physical or virtual artefacts are present to foster students' personal involvement in the activity under the teacher's guidance. A typical teaching experiment involving artefacts lasts several sessions, from several weeks to a few months, and it relies on a constant cooperation with teachers (the so-called "teacher-researchers"), made possible by the on-going involvement of the same mathematics teacher with the same group of students for many years. These extended periods with the same group of students make teachers less anxious about the short-term effects of their teaching and encourages them to take care of and to observe long-term processes (Bartolini Bussi and Martignone 2013). From decades of studies on laboratory activities, the theoretical frame of *semiotic mediation* was developed.

We summarize only some elements of the Theory of Semiotic Mediation, focusing on the teacher's role (Bartolini Bussi and Mariotti 2008). The teacher is in charge of two main processes: the design of activities; and the functioning of activities. In the former the teacher makes sound choices about the artefacts to be used, the tasks to be proposed and the pieces of mathematics knowledge at stake, according to the curricular choices. In the latter, the teacher exploits, monitors and manages the students' observable processes (semiotic traces), to decide how to interact with the students and what and how to fix in the individual and group memory.

In this teaching experiment, the chosen artefact is the *bee-bot*, a small programmable robot (see below). Task design is realized drawing on previous experiments made at pre-school level (Bartolini Bussi 2013) and also at primary and secondary school level (Bartolini Bussi et al. 2011).

The design process is represented by the left triangle of Fig. 1 (tasks–artefact–knowledge), where the *semiotic potential* of the artefact is made explicit (that is, the links

[2] For a similar process, see for instance the "theory of gears" as reported in Bartolini Bussi et al. (1999)

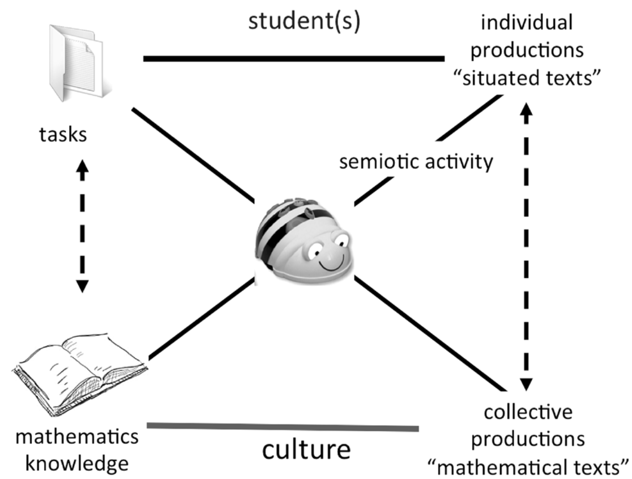


Fig. 1 A diagram of the main processes involved in semiotic mediation

between tasks, the pieces of mathematics knowledge and the chosen artefact).

The other parts of the scheme concern the functioning in the classroom. When students are given a task they start a rich and complex semiotic activity, producing traces (gestures, drawings, oral descriptions, written texts and so on). The teacher's job is first to collect all these traces (observing and listening to students), to analyse them and to organize a path for their evolution towards mathematical texts that can be put in relationship with the pieces of mathematics knowledge. The teacher acts as a cultural mediator, in order to exploit, for all the students, the semiotic potential of the bee-bot, in the left triangle of Fig. 1.

In this last process, Bartolini Bussi and Mariotti (2008) identify three main categories of signs: artefact signs, mathematical signs and pivot signs. *Artefact signs* "refer to the context of the use of the artefact, very often referring to one of its parts and/or to the action accomplished with it […]"; *mathematical signs* "refer to the mathematics context"; and *pivot signs* "refer to specific instrumented actions, but also to natural language, and to the mathematical domain" (Bartolini Bussi and Mariotti (2008), p. 757).

Pivot signs can act as bridges between the artefact signs and the mathematical signs. For example, if we use counting sticks (sticks bundled up in a set of 10), an artefact sign could be "bundles" or "to tie/untie". The corresponding mathematical signs could be "tens" or "grouping/ungrouping". For some time (even weeks) children may use only artefact signs (for example: "I have tied ten counting sticks") or construct hybrid sentences (for example: "I have tied a ten") or directly use mathematical signs (for example: "I have grouped a ten"). In collective mathematical discussions we can observe a variety of utterances related to different uses of these signs.

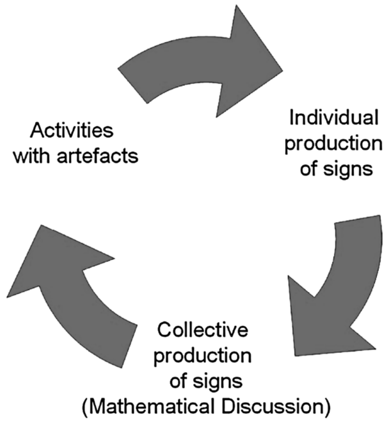


Fig. 2 Diagram of the didactical cycle

Pivot signs, if correctly identified or introduced by the teacher, and intentionally used during discussion orchestration, can be particularly useful for fostering a transition from situated “tests” to mathematical texts. Pivot signs develop and are enriched by their relationships with other pivot signs, hence building a *network* of pivot signs (for example, the “bundle” of ten sticks may be related to single sticks/units or to bundles of bundles/hundreds) and so on. Mathematical signs are not intended to suddenly substitute artefact signs; in fact the latter may survive for some time, especially for lower achievers or in cases in which the formal mathematical definition and the reasoning of the corresponding concepts require long-term processes to be achieved.

The methodology proposed for the classroom process is described by Bartolini Bussi and Mariotti (2008) as the *didactical cycle* (see Fig. 2).

The activities of the teaching sequence include the following:

1. *Activities with artefacts* in which classroom discussions are promoted to allow the same artefact to be looked at from different perspectives. In this phase there is a typical sequence of tasks proposed: the warming up task (What is it?) that fosters the emergence of a *narrator voi*ce; the artefact task (How is it made? How could you describe it to a classmate?) that aims at identifying the artefact’s components and naming them in a correct way and at describing the spatial relationships between them, fostering the emergence of a *constructor voice*; the instrument task (How does it work? How can you make it work?) that fosters the emergence of the *user voice*; the justification task (Why does it work in this way?) that fosters the emergence of the *mathematician voice*; new problems (Could we use this artefact to solve a new problem?) designed to foster the emergence of a *problem poser and solver voice* (Bartolini Bussi 2013).
2. *Individual production of signs* (gesturing, speaking, drawing, writing, etc.). Students are individually engaged in the process of the production and elaboration of signs related to the previous activities with artefacts (Bartolini Bussi and Mariotti 2008).
3. *Collective production of signs* (e.g. narratives, mimics, collective production of texts, and drawings). Students are engaged in mathematical discussion under the teacher’s guidance about the previous activities (artefact manipulation and sign production) to compare and to share personal signs (Bartolini Bussi and Boni 2009).

3 The chosen artefact: the bee-bot

The bee-bot is a small programmable robot, especially designed for young students. Its ancestor is the classical LOGO turtle, originally a robotic creature that moved around on the floor (LOGO Foundation 2000). It is not necessary to have an external computer to programme the bee-bot, as the command buttons are on its back (see Fig. 3).

When the programme is executed, the bee-bot moves on the floor: the execution of each command is followed by a blink of the eyes and by a short beep-sound. We have introduced this small robot in dozens of classrooms (pre-schools and first/second grade classrooms) within two on-going projects of teacher development: BAMBINICHECONTANO for pre-school (Bartolini Bussi 2013), under the supervision of the first author; and PERCONTARE (ASPHI 2011) for primary school (Baccaglini-Frank and Bartolini 2012; Baccaglini-Frank and Scorza 2013), under the supervision of the second author.

In spite of its very simple appearance, the bee-bot hints at many sets of meanings and mathematical processes, partly related to mathematics and partly related to computer science, for instance: counting (the commands); measuring (the length of the path, the distance); exploring space, constructing frames of reference and coordinating spatial perspectives (Falcade and Strozzi 2009; Baccaglini-Frank et al. 2014); and programming (e.g. Papert 1993; Noss and Hoyles 1996), planning and debugging. In a long-term teaching experiment, all these sets of meanings are at stake, sometimes in the foreground and sometimes in the background. Which ones to focus on depends on the adult’s teaching intention (see Sect. 3). Our teaching experiment was designed to capitalize on the bee-bot’s potential for fostering awareness of the “four right angles” property of generic rectangles.

The bee-bot walks on the floor and traces paths that can be perceived, observed, described with words, with gestures, with drawings, with sequences of command-icons and so on.

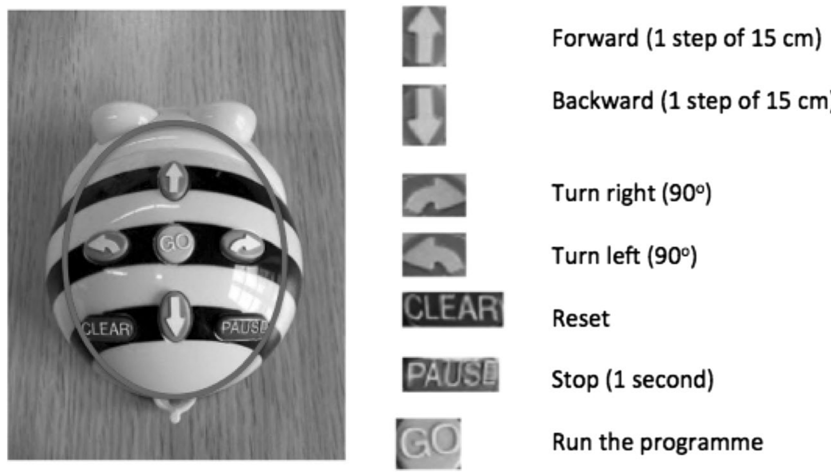


Fig. 3 The bee-bot's back and enlargements of the buttons

Paths constitute a large experiential base to "study" plane figures—not all figures, but only those that can be traced using the available commands. These are polygons with sides measured by a whole number of steps and with right angles only. With the additional restriction of the traced shape being convex, the bee-bot can be programmed to turn only left or only right, and therefore the polygons are always rectangles (including squares). Moreover, in experiences where "pretending to be the bee-bot" is essential, children embrace the robot's perspective: they move with the bee-bot and they see with its eyes. In particular, when walking along a closed convex path and ending up where they started, the children will turn 360° in four equal "chunks" during which their orientation (and ending facing the same direction as when they started) is perceived as essential. These are some features that define the bee-bot's high semiotic potential with respect to the emergence of an inclusive definition of rectangles.

4 The research questions

Within the described framework and with the objectives outlined above we advanced the following specific research questions:

1. How might a long-term process of semiotic mediation that exploits the semiotic potential of the bee-bot with respect to the development of an inclusive definition of rectangles look for first graders?
2. In particular, which kind of pivot signs (if any) can be identified and exploited during such long-term process?

5 Methodology: the teaching experiment

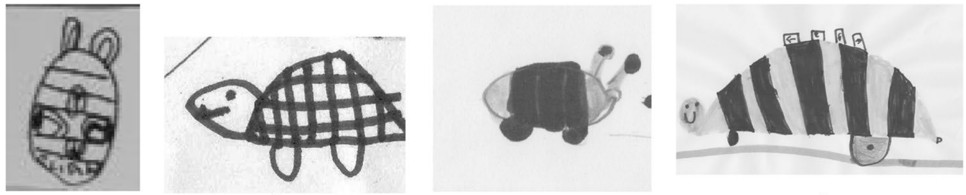
The class was described by the teacher as being average and relatively homogeneous: it has 18 students (aged 6–7). Three adults were involved during the classroom activities: Roberta Munarini, the classroom teacher (T), is a very experienced teacher-researcher, accustomed to taking part in advanced teaching experiments and open to welcoming student–teachers during their internships; her expertise is invaluable when critical choices have to be made, both in the choice of tasks and in the management of tasks. Federica Baroni is a very good student–teacher (S), with a master's degree in education for pre-school, enrolled now in the primary school programme and doing her internship. Anna Baccaglini-Frank, a researcher (R), is active both as a participant in some critical phases of the experiment and as an observer.

Several sessions (15) were carried out at the beginning of the school year, for 4 months (more or less once a week) either in the classroom (C) or in the gym (G), with a careful alternation of whole-class or small-group activity (with adult's guidance) and some individual activity. Each session was carefully observed by one of the adults involved (T, S, R), with the collection of protocols, photos, graphical productions and videos. Transcripts of discussions were prepared by the student–teacher. The tasks were designed by the research team, drawing on the initial intention and on the needed changes to be introduced after classroom experiment.

Due to space constraints it is not possible to report on all the activities, so we will focus on certain sessions in which the production of signs was particularly rich with respect to our objective (further details in Bartolini Bussi and Baccaglini-Frank, 2014).

6 Findings: the evolution of signs

In this section we give details on selected activity sessions, describing particular signs that the children produced.



a b c d

Fig. 4 Children's initial drawings of the bee-bot

6.1 The early signs

The first session of the teaching experiment involved approaching a new artefact through classroom discussion. Students were encouraged to produce drawings. As expected, students' attention was focused mostly on the narrative aspects (the possible adventures of this small toy) and on the hypotheses as to its "nature" (Is it an animal? Is it a bee? Where are its wings? Is it a "walking bee"? Where could it live? Why has it come to our school?). This activity helped the children to enter a relationship with the bee-bot, at an ego-syntonic level, which probably helped to foster their later identification with the robot at a body-syntonic level (see Sect. 1).

In the second session our aim was to help the children focus on some geometrical and technical features of the artefact: the shape, the presence of some buttons (the command icons) on its back and of others on its belly (on–off). The observation of this session revealed a very rich intertwining of words, gestures (sometimes used for missing words) and, later, upon request, of drawings. The drawings hinted at narratives, highlighting an affective dimension rather than a geometrical-technical one. Figure 4a–d show some examples of drawings.

6.2 Programming the bee-bot: the first signs for representing sequences of commands and possible paths for the bee-bot

After these two early sessions a much more complex semiotic process started with a stronger entrance of symbolic representations. Students introduced sketches hinting at the command-icons and at the traces of the bee-bot's paths that sometimes involved counting (steps). The process was led by the tasks which focused the attention on the paths (What does it do?) rather than on the artefact. The students were able to focus on paths (imaginary trace marks) even when the bee-bot actually did not leave any concrete trace mark (see Fig. 6a–d, later).

In one session, students were given two bee-bots that had been programmed with the same sequence ahead of time. The students watched the twin bee-bots move together, starting facing in the same or in different directions, and then separately. Then the memory of one of the bee-bots

Fig. 5 S is mirroring the children's turning gesture

was erased (CLEAR command-icon) and the students were asked to re-programme it so that it would move in the same way as the other bee-bot and to *describe what they did*. The students' productions concerned both global and local aspects. Global aspects refer to the perception of a path as a whole (as if the bee-bot had drawn it on the floor), whilst local aspects refer to special points of the path. An example of the former is the expression "it did an L"; an example of the latter is "they switched the turn".

Both aspects appeared also in gesturing: the path was represented by a single pointer finger tracing a path in the air (tracing gesture), whilst turning was represented by moving the right hand (for a right turn) or left hand (for a left turn) up and to the right or left in a rotation (turning gesture). The turning gesture was mirrored by S, as she recognized it as relevant, a *pivot sign* that would become very important in the network of pivot signs eventually referring to "rectangle" (Fig. 5).

The local aspect appeared especially when students were asked to programme one bee-bot (after erasing its memory) in order to reproduce the same motion as the other:

Marika: I pressed an arrow to go forwards … then it has to go forwards again.

Martina: We pressed four times forward then turn, then two times forward forward. To know what it had to do we counted its steps.

The students were asked to produce a drawing representing the given commands. They produced drawings in which numbers and arrows were mixed on the page. In

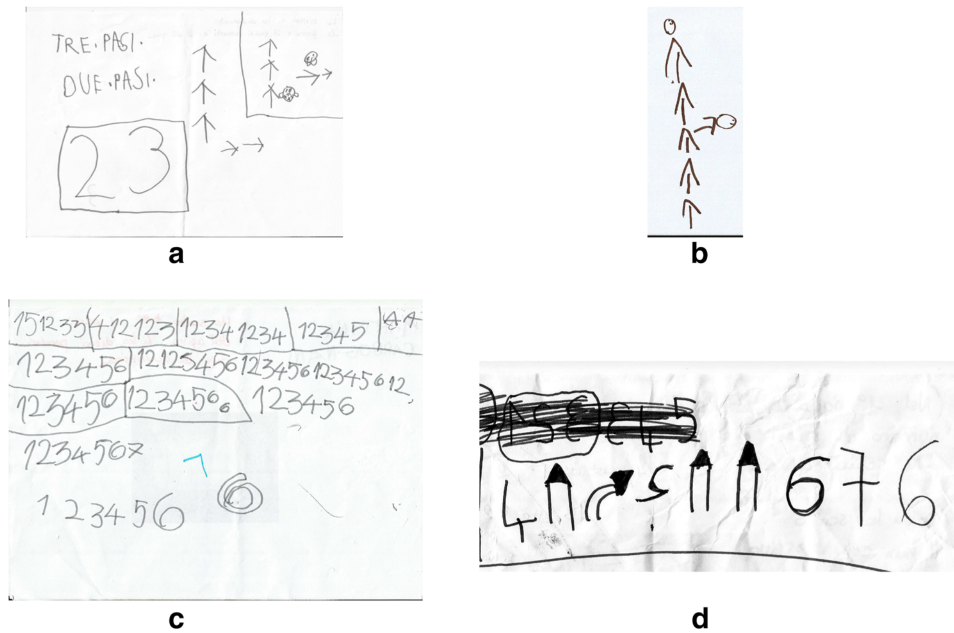


Fig. 6 Students' initial productions of the bee-bot's commands and/or path in the "twin bee-bots" activity

the different drawings it is already possible to perceive different characterizing features of the students' dominant perceptions. The four drawings shown in Fig. 6a–d show the double attempt to draw the shape of the path (global representation) and the list of commands (local representation).

Anita produced a complex drawing with numbers (steps), written descriptions, and arrows arranged to hint at the path (Fig. 6a).

Francesca (as she explained orally) has drawn the arrows: "3 times forward, 1 turn, 2 times forward"; the turn is drawn aside and out of the line as if its status is not yet clear (Fig. 6b).

Martina wrote a small "L" (the path) and only numbers (repeated sequences of numbers: 12345 or 123456, to mean that the little bee restarts) wavering between 6 commands (only the steps) or 7 commands (including the turn) (Fig. 6c).

Simona used a kind of shorthand with either repeated arrows or a number to tell how many arrows (Fig. 6d).

The graphical representations did not depict angles: "turns" are the only aspects (pivot signs relating to angles) that were perceived and represented verbally and physically. However, the perception seemed to be completely dynamic and only functional to the change in direction of the straight segments of the bee-bot's paths. In fact the students were unsure whether or not to even count the turn as a "step" of the command sequence. We remark that this may be the case also because the students have not yet felt the physicality of the "turn". Fostering such physicality was the objective of later sessions. We also note that at this point the paths have a very dynamic connotation for the children: their shape is very seldom recognized as a whole, although their execution seems to be perceived, implicitly, as "a thing" (the children would say "He did *it*" [It: "*L*'ha fatto"]).

6.3 Pretending to be the bee-bot

During this session the students were asked to work in pairs: one would pretend to be the bee-bot and the other would give her commands to move according to some undisclosed (to the bee-bot student) path. The intention was to focus their attention towards the turn command, that seemed to have an uncertain status in the earlier session. Typical words used would be "Straight Ahead", "Left", "Right", "Backwards", usually without quantifying the number of steps, and frequently combining a translation with a change of direction. For example, when a student would say "Left" the bee-bot-student would frequently not only turn left, but also take a step in that direction, or even just take a step to the left without even turning in that direction. S's intervention that asked the students to compare this particular behaviour with that of the bee-bot is fundamental because it led the students to attend to the "turn" command and to start to explicitly consider rotations. One child stated: "… so when I turn I move but I don't walk" [It: "… allora quando giro mi *muovo* ma non *cammino*"] and she gestured the turn with her whole body, while standing on the spot. This denotes rotations as important elements per se, "motions" without having to be translations or "steps", an important seed for the evolution of our sought definition.

6.4 Experience with paths and the "squarized 0"[3]

In these four sessions, realized in the gym, the students physically constructed paths to programme the bee-bot to move along and described the paths explicitly with sequences of command arrows lined up horizontally. The students were learning to trace letters of the alphabet, so we decided to elicit considerations about the possibility of drawing letters with the bee-bot, and, in particular, signs that would eventually refer to *angles* and *sides*.

Various signs were produced in verbal, graphical-written forms and through gestures. Below we show an excerpt from an important exchange between a child and S, from which stemmed a pivot sign that was eventually embraced by the class and that would eventually be related to "angle":

S: How does the bee-bot turn?

Alessandro: It turns like this [he picks up the bee-bot and rotates it through a 90° angle].

S: But if I want it to do a smaller curve, can I?"

Alessandro: No, because he turns completely.

S: So can I make him do the curve of "2"?

Alessandro: No, because he only does a curve like this [he draws a right angle as shown in Fig. 7] … He can't do a little curve like this … or like this.

In this excerpt a combination of gestures (turn), words (the oral description) and drawings appear. Figure 7 shows a combination of a dynamic and static perception of this "piece" of a path imaginarily traced by the bee-bot.

The children generated many signs aiming at distinguishing the shapes (letters) which could be drawn from the others (the letters with "sharp" points, or the rounded ones). Soon the discussion became about the possibility of programming the bee-bot to make it draw particular letters. During this discussion a student (quickly mirrored by other students in the class) named a figure that is the ancestor of all the squares and rectangles that the students will talk about:

S: … Did you do an "O"?

Student: No. Then it could do like this this this and this [he gestures four consecutive right angles] a *square "O"*. Ah, then it can make a square!

Other students then mirrored the statement using the term "*squarized Os*" and they started talking about how it would be possible to make other "squarized letters", meaning letters that include one or more *squarized Os* within them (e.g. P, B). These "squarized 0 s" would become the main pivot sign around which the network of pivot signs would develop. The *squarized O* was the perfect pivot sign to use for the notion of "rectangle" that we were after, because it was characterized exactly by the presence of the "4 right angles" property.

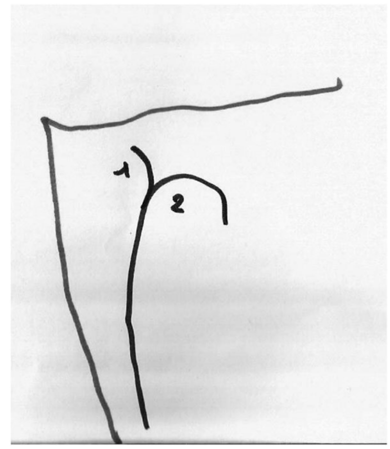


Fig. 7 A student represents both a dynamic component and static "piece" of a path; the turn to trace a right angle

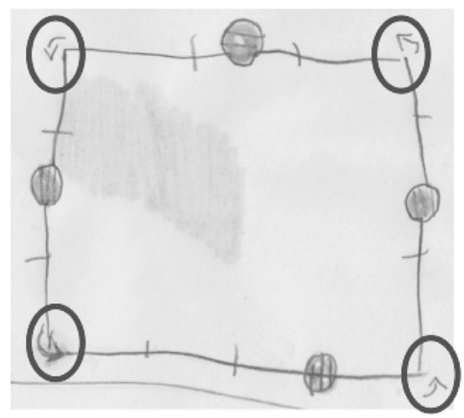
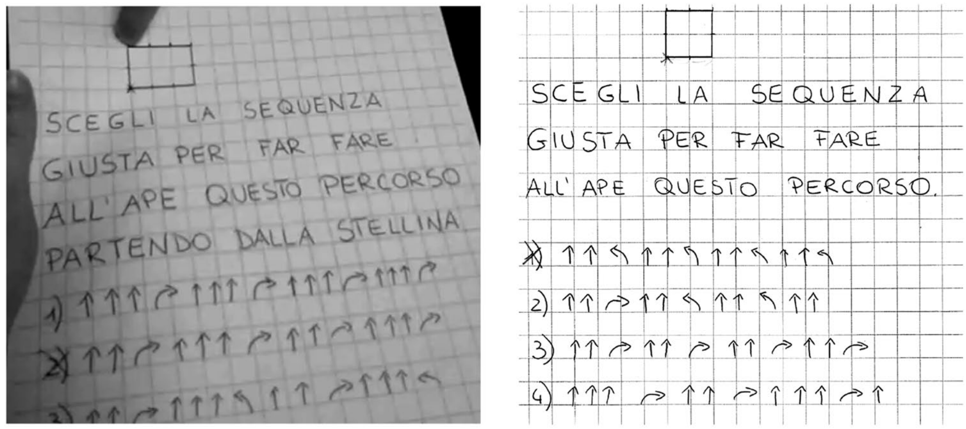


Fig. 8 A student used a pivot sign (highlighted with circles) for "turn/angle" in his drawing of a path for a particular squarized O, which R mirrored and re-proposed to the class successfully. The circles on the sides of the path represent the bee-bot tracing the path

6.5 From "squarized 0" to rectangles and squares

In these sessions, realized in the classroom, the student–teacher, the researcher and the teacher picked up on the students' verbal, gestural and graphical representations of the "squarized O" to foster their evolution into "squares" and "rectangles" with characterizing sets of geometrical properties involving sides and angles. As the students described

[3] We want to translate a non-existing Italian word ("quadratizzato") invented by the students and later used as a pivot sign, so we will use a similar non-existing English word.



a b

Fig. 9 Tasks 1 and 2 on choosing the correct sequence (translated above) to match different representations of paths for two different squarized Os

and represented how sequences of commands could be put in relationship with paths traced for squarized Os, a student graphically introduced "a turn like this" (highlighted arrows in Fig. 8) to indicate the turning points of the path for a particular squarized O. R saw in this sign the potential of becoming a pivot sign for the class with respect to the notion of "angle" (external angle): the sign reminds one of the command-icon on bee-bot's back, but it is slightly decontextualized and used both in drawings that depict the path as a figure and in horizontal sequences of commands that represent how the bee-bot was programmed. The students re-appropriated quite easily the sign re-proposed by R for "turn/angle".

6.6 Matching trajectories and sequences of commands

At this point two tasks were given by T concerning the relationship between different representations of paths for squarized Os (Fig. 9):

Task 1: Choose the right sequence to make the bee-bot walk on the drawn path starting from the small star.

Task 2: Choose the right sequence to make the bee-bot walk on the drawn path.

As this was a crucial session, T herself orchestrated the discussion following some of the students' individual solutions.

We report on the moments of the discussion in which important considerations leading to the characterizing "4 right angles" property of the squarized Os emerged. Here, the discussion was on the first task:

1. T: You said that to do this kind of path … is it an open or closed path?
2. Cecilia: Closed
3. T: Closed. Our bee had to always do what?
4. Cecilia: Four angles
5. T: Four angles. What does "angles" mean?
6. Cecilia: Angles means the points here.
7. T: The points. And what do these points correspond to in the commands for our bee?
8. Student: To the turns!
9. T: To some turns. So you say it needed to have four[4] turns.
10. Alessandro: This way the figure did not stay open.
11. T: This way the figure did not stay open. And those four turns, Cecilia, how were they supposed to be, you said it before?

[Cecilia has trouble answering immediately, and mumbles something incomprehensible.]

12. T: The turns so that the bee-bot does this kind of path, how do they have to be?

[T refocuses Cecilia's attention on the "points" she had mentioned earlier.]

13. Cecilia: All turned in the same direction.
14. T: All turned in the same direction.
15. Cecilia: So to the right.
16. T: In this case to the right. Could it have been only to the right?
17. Student: Yes, otherwise it would have gone …
18. T: Was there a way to make the bee turn to the left?

[4] Very early on in the activities the students argued that there needed to be four turns because "When she is finished the bee-bot needs to look the same way she started." This argument was supported through mirroring by S and T, since it was important for the desired evolution of the network of pivot signs around the squarized O.

[T lets the students think about her question and different students start shouting out answers. T picks up the comment of one student, mirroring and re-launching a question.]

19. T: It would start from the angle and then what shape did it have?
20. Student: A rectangle!
21. T: If you already know the name, ok, it's called a rectangle.

[T refocuses the children's attention on the direction of the turns. One student is called to the whiteboard and he shows how the bee-bot could have always turned to the left.]

22. T: So it's not true that it has to do all the turns to the right. It could also do them all to the left.

[Murmuring of many children, and T addresses directly one student who was talking to a classmate.]

23. T: How do the turns have to be, Laura? Always to the left?
24. Laura: No. Sometimes also to the right.
25. Student: In any case the turns have to always be the same.
26. T: In any case, for it to be a squarized O, they have to always be …
27. Students together: The same.
28. T: Is that clear?
29. Students: Yes.

T systematically mirrored (Bartolini Bussi and Boni 2009) students' utterances and oriented the discussion towards what is considered important: a first generalization for the squarized Os as figures that always have four turns-angles (3–14); and the turns have to be in the same direction no matter whether it is to the left or to the right (18–29). The generalization was reached through the generation and reflection upon examples of sequences and ways of having the bee-bot trace a same squarized O, and then it was stated in general terms, with an appropriate (mathematically speaking) use of the words "in any case" and "always".

In the discussion after the second task as well, T stressed the important property, that is, having four turns/angles. Then T led the children to agree upon the terminology "sides" and "angles", mirroring words that had been introduced by certain students, and explicitly putting them in relationship with parts of the different representations. T also guided the discussion towards the recognition of similarities between the different types of squarized Os (for this part of the discussion she has on the board a 2 × 2 and a 3 × 2 rectangle with the respective sequences of command-icons).

6.7 Discussing a poster on "our" discoveries

The children engaged in other activities in which they consolidated their discoveries on squarized Os. Then, during the last session, T, in charge of the construction of the classroom memory, introduced a summary poster with the class's discoveries, that she had agreed upon with R and S. The poster is reproduced in Fig. 10 and it aims at fixing "the important things to be remembered for the future". The poster was given to the students both as a large banner to put up on the wall, and as a small A4 copy to be attached to their notebooks. T's intention was to pave the way towards a more precise mathematical definition to be constructed in the second or third grade. She consciously chose to offer four examples, none of which were in the canonical orientation. This choice was intentional, to try to limit the children's rigid development of the typical prototypes (on this issue, with respect to triangles, also see Kaur 2015).

As she walked into the classroom with the rolled-up poster, T declared she had written down the students' thoughts and she asked the children to guess at what they would see on the poster. Many immediately raised their hand and said: "The little bee!", "The squarized Os!", "Squares!", "Rectangles!" As she hung up the poster, the student who had proposed the sign for the turn (external angle or bee-bot's rotation) said: "Those are my turns!", and a student who had mentioned the squarized Os bragged: "See, *I* was right: those are all squarized Os!"

6.8 A post-test

During the last week of school (about 4 months after the teaching experiment was finished) T assigned three post-test tasks to investigate what had "stuck" in her students,[5] as she declared in the interview in which she explained what she observed.

The post-test consisted of three sheets of paper containing the same shapes with the same orientation (see Fig. 11) but arranged differently. The tasks on each sheet were:

[5] Not all students were involved in the post-test activity because it was carried out during the final weeks of school, while various end-of-year activities were taking place. In total 7 of the 18 children were selected (2 high achievers, 3 average achievers, 2 low achievers) and assigned the post-test.

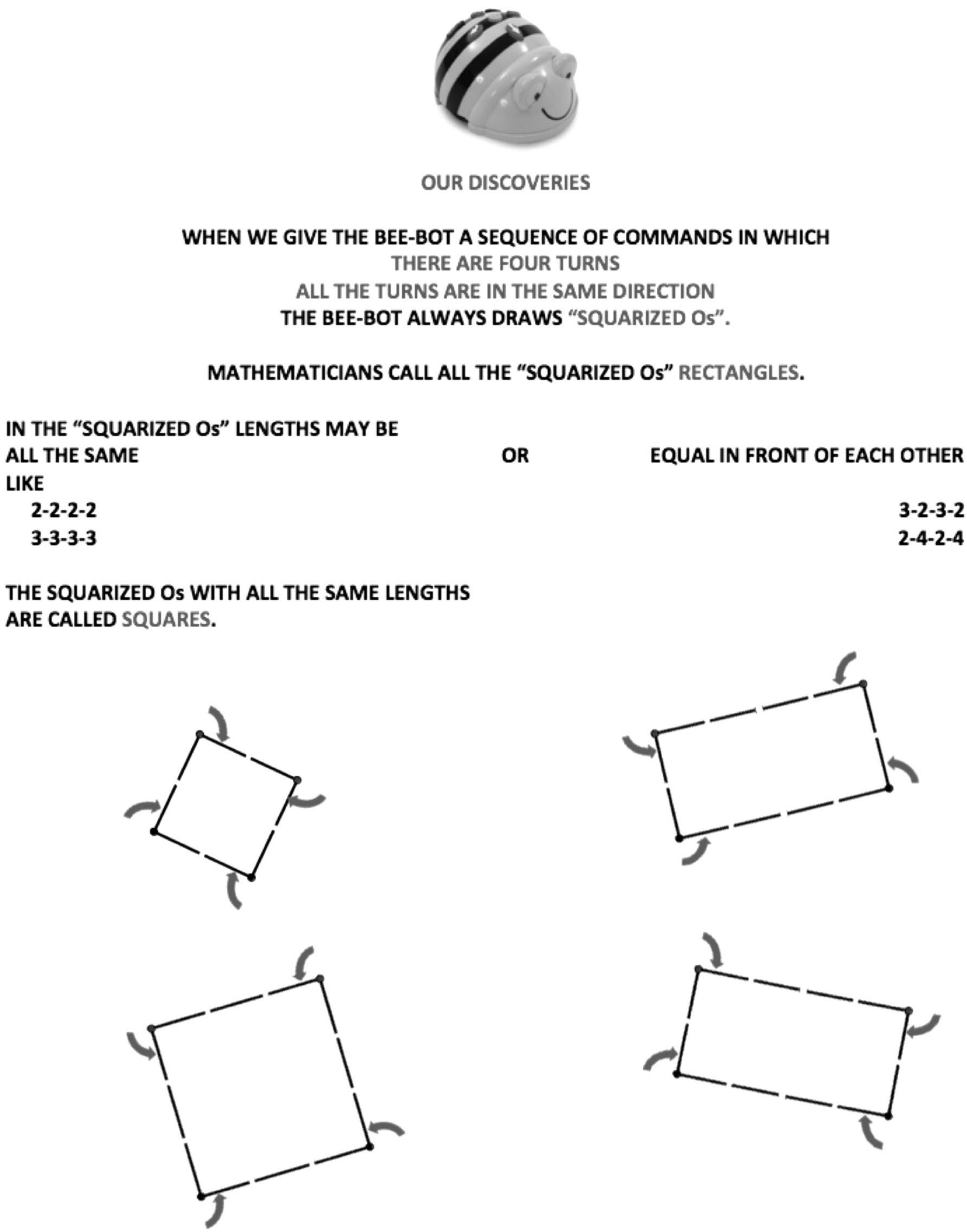


Fig. 10 The final poster (translated)

1. Circle all the squarized Os that you see below.
2. Circle all the rectangles that you see below.
3. Circle all the squares that you see below.

When T shared her observations with R in an interview, she described three tests that in her opinion were representative of "interesting types" of answers that her students gave:

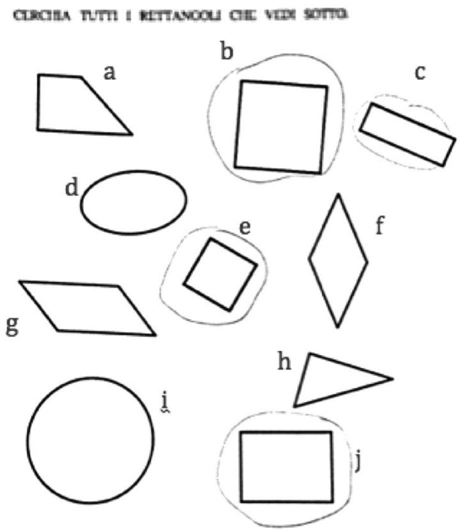


Fig. 11 A student's answer to task 2 from the post-test given by T at the end of the year. The letters were added in this picture in order to identify the shapes in the comments

Type 1[6]: In the first task the student circles the four shapes that represent rectangles according to an inclusive definition (b, c, e, j); the student recognizes the same shapes in task 2 even though they are in different places on the sheet, and sometimes even states: "They are the same as before!" In task 3 the student only circles the two shapes he (correctly) recognizes as squares (b and e).

Type 2[7]: In task 1 the student circles all four correct shapes and states: "They are the ones with four equal angles." In task 2 she selects three of the four rectangles (b, c, j); and in task 3 she correctly identifies the two squares (e and b).

We will briefly comment on these answers in the discussion.

Type 3[8]: In task 1 the student circles only b and j, the "ones that look more square"; in task 2 the student identifies three of the four rectangles (b, c, j), but not e; in task 3 the student circles only b.

7 Discussion

In this teaching experiment our aim was to exploit the semiotic potential of the artefact bee-bot with respect to sowing seeds for an inclusive definition of rectangles. In particular, we wanted to observe what a long-term process of semiotic mediation would look like for first graders, especially what pivot signs (if any) might be identified and exploited during such process.

Within the classroom activities proposed it was in fact possible to observe the evolution of a network of pivot signs associated with the "squarized O". We will describe how this network of pivot signs emerged from the signs developed to describe the bee-bot's paths and to describe commands given to the bee-bot, that became more and more focused on rectangles, thanks to the design of the activities and the guidance of S, T and R.

7.1 The bee-bot's paths as wholes

At the beginning students produced drawings containing different signs (numbers, words, lists of commands). The rationales for the choices were frequently not clear. We hypothesize that the students had a global image of the experience and that they produced signs based on what caught their attention the most and also on their individual skills. Very seldom (see the little L in Fig. 6c) did they draw the "whole" path, as a figure, even though they described the "way the bee-bot walked" as an "L". Frequently, especially in this initial phase when specific terminology had not been developed and shared by the students, they used gestures in the air. One interpretation of this might be that the children were compensating for missing words. On the other hand, it could be that in this situation the gestures were perceived as being more effective at communicating, perhaps especially in self-communication (this is consistent with findings in Kaur 2015). In this sense the turn gesture mirrored by S became part of the shared signs, and it was a first instantiation of a sign to eventually refer to the idea of "angle". Still the dynamic aspect of the signs seemed to be prevalent. In some cases, although the dynamic component was dominant, students perceived the path as whole static elements even when there was no permanent sign (e.g. no trace left by a marker). We hypothesize that this depends very much on the confidence the students had with the particular shape (e.g. L) they perceived in the path (or that they made the path in the shape of). Also, the presence of rhythmic eye blinks and sounds produced by the bee-bot and ended by a double number of beeps and blinks may have fostered the perception of a path, in terms of a sequence of steps with a starting point and end point, as a body syntonic whole.

The definite transition to seeing paths as "wholes" and being able to represent them was fostered by the activities in which students programmed the bee-bot to "trace" (even though it did not actually leave a mark) particular letters, conceived as stand-alone figures. In particular, the children came to attend to specific characteristics of the letters when

6 This type of response was provided by the two high-achieving students and by one of the average-achieving students.

7 This type of response was provided by the other two average-achieving students.

8 This type of response was provided by the two low-achieving students.

discussing which ones the bee-bot was able to trace or not. The conjectures advanced by the children in this context were relatively advanced and showed an ability to distinguish "curves" from "segments" and "angles", and even to distinguish between angles greater than or smaller than right angles, with the right angles being a strong perceptive landmark. We believe, however, that this is not evidence for the fact that right angles are universally recognized as invariant figures, but rather of the fact that the children we worked with are immersed in a culture that values right angles and lines, a "carpentered world", as Luria would say (1976, p. 31).

7.2 Paths as sequences of commands

In parallel with the evolution of the signs representing paths as figures, the children elaborated signs to represent paths in terms of sequences of commands, a separate but related semiotic register (Duval 2000), within which important properties were identified. Initially these signs were copied from the command icons on the bee-bot's back and arranged in various manners (orientation and position on the page; see Fig. 6a–d): they appeared to be mixed with other signs, including words, expressing the commands or actions realized by the bee-bot. Eventually the class agreed to write the sequence of commands as arrows lined up in a horizontal line.[9]

When relating the arrow signs to the paths drawn as figures, the students would transcribe the arrows into a sequence "around" the figure. In these activities a great synergy between the different representational registers emerged, and spontaneously some children gestured "turns" with the pivot sign discussed above, while simultaneously drawing a sign like the "turn" command-icon on the bee-bot's back at the vertex of the path represented as a figure. This way the same sign, simply rotated in different places on the whiteboard or on paper, would represent *both* a command in the programmed sequence and a turn in a certain point, *and* the vertex of an angle, in the drawn path (see Fig. 8). This pivot sign reminded the children, on one hand, of the command-icon on the bee-bot's back, the perceived rotation of the bee-bot, and the action of turning in the path on the floor; and, on the other hand, the mathematical sign for angle (an external angle, in this case).

When specifically analysing the process of evolution of this pivot sign, we noticed that it originated with the turn gesture (Fig. 5) and evolved during the experience of turning while pretending to be a bee-bot, when the children would feel with their own body the action of turning, thanks to the design of the activities that exploited the body syntonicity of the bee-bot's microworld. The process was reinforced by the children noticing that when they turned they "move" but they did not "walk": it can be interpreted as the children realizing that turning does not involve any movement of the centre of rotation. The gesture (with a hand and/or with the whole body) was then related to the pivot sign developed from the "turn" arrow on the bee-bot's back. This sign was then used systematically in the following sessions proposed by R, T and S, and it was reinvested in the characterization of *squarized O*s.

The representation of sequences of commands as horizontal arrays of command icons (little arrows, straight or curved) had the effect of fostering, for some students, the shift towards a kind of pre-algebraic notation: "Three steps then three then three then three we make a square," or "two forward, turn right, three forward, turn right, two forward, turn right, three forward, turn right."

7.3 The network of pivot signs around *squarized O*s

The emergence of the sign "squarized O", a verbal sign, accompanied by a number of figural representations, was a very happy episode, upon which R, S and T capitalized heavily, seeing in it the potential of sowing the seeds for an inclusive definition of rectangles, as desired. This sign quickly became a pivot sign and was put in relationship with other signs that had emerged, in particular those related to the right angles present in squarized Os. We hypothesize that such a sign was proposed and embraced so quickly by the whole class because of the children's immersion in a "digital world", in which rectangular shaped figures can be experienced frequently (for example on digital displays or on supermarket labels to represent "zeros"). Possibly, this is an example of how technology affects processes of meaning-making. In a different culture it is likely that children would not have suggested such a sign, nor might they have understood it even if it were proposed by an external adult. As Luria described in his studies that involved non-educated adults in their thirties (1976), possible pivot signs (according to our framework) might have been "windows with different frames" (Luria 1976, p. 38), that is, square or non-square rectangles.

An important characteristic of the squarized Os, that was noticed by the students, and re-proposed in whole-class discussions by T, is their having "turns" all in "the same direction" (see the excerpt in Sect. 6.6). Moreover, it was possible to exploit again the body syntonicity of the bee-bot's microworld when reflecting upon the number of necessary "turns". While one child initially argued that the squarized O was "closed" even after only three turns, many children objected to the fact that the bee-bot would not have "finished" if it was not "seeing the same things"

[9] This convention is also a design feature in the app Mak-Trace (see Baccaglini-Frank et al. 2014).

as when it started, going "the whole way around". This was the reasoning we embraced to reach the characterization of squarized Os summarized in the final poster.

The final poster introduced by T (Sect. 6.7) uses some of the students' expressions and the network of pivot signs developed around the main one—the squarized O—aiming at fostering a gradual transition towards mathematical definitions. Squares are not yet seen as special rectangles, as it is too early to invite students to see the possible inclusion of the set of squares into the set of rectangles; however, some common features are emphasized as special discoveries (Fig. 10), including the denomination of all the bee-bot's paths as "squarized Os" (the ancestors of rectangles according to the mathematical definition).

Finally, in the post-tests, we gladly noticed how two of the three main "interesting types" of answers (types 1 and 2) included (with one small exception) an equivalence between "squarized Os" and "rectangles". The exception was in type 2 where the students selected as a squarized O and as a square but not as a rectangle. Unfortunately, although we could make inferences as to why this might be the case, we do not have enough data to advance meaningful hypotheses on these results at the present time.

8 Concluding remarks

Although the Italian (and English) language does not foster the conception of a square as a particular kind of rectangle, the expression invented by the students, "squarized O", actually seems to unify under a same name all convex polygons that the bee-bot can trace (in mathematics these correspond exclusively to rectangles, including squares). In other words, the children seem to have spontaneously produced a conception of the common ancestor (in the context of the bee-bot's paths) of the figures we were interested in defining. Although this particular pivot sign was not expected from the a priori analyses of the activities, we believe that similar signs can be either produced by other students or easily accepted if proposed by an adult (as we did in the other classroom). Moreover, because of the digital culture in which the children of many countries are raised today, we believe that a sign such as the "squarized O" could easily be introduced by the teacher and it would be picked up swiftly by the students. In this sense this pivot sign was a very important finding that we believe a significant part of our community can capitalize on for other teaching sequences.

Finally, we observe that the didactical sequence outlined is only one of many possibilities for a teaching experiment aimed at sowing seeds for an inclusive definition of rectangles. A very different—but equally rich—venue is offered by dynamic geometry software that can and should be used to design similar teaching experiments. In fact, these students will very likely also work with dynamic geometry. The means by which the semiotic potential of these different microworlds can be exploited are, of course, different. Efforts should be made by the teacher and by the designers of the activity sequences to construct meaningful relationships between the situated texts that can emerge from the students' experiences within these different environments around the same mathematical meanings.

Our choices were based on the particularly rich context the bee-bot offered for very young students, knowing that it would open doors to many other mathematical meanings: we do not believe that otherwise it would have made sense to invest this long a period of time on trying to define rectangles. In the following years the teacher will pick up different seeds planted (for example, those related to introducing measure or coordination of spatial perspectives) to develop other mathematical meanings.

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