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ORIGINAL ARTICLE

Mathematical interaction shaped by communication, epistemological constraints and enactivism

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Abstract On the surface, mathematical interaction often appears as an immediately transparent event that could be directly understood by careful observation. Theoretical considerations, however, clearly show that mathematical speaking and conversation in teaching–learning situations are highly complex social structures comprising many preconditions. Communication does not generate direct understanding and the object of communication—mathematics—is, as knowledge of abstract relations, not directly accessible. The learning agents—the teacher and students in the mathematics classroom—have to cope with these difficulties in a way of reciprocal actions between social communication and individual consciousness.

Keywords Enactivism · Autopoietic system · Epistemology of mathematical knowledge · Communication · Understanding

1 Introduction: setting the scene

This paper will contribute to the problem of understanding in mathematical interaction in a twofold perspective:

- Mutual understanding among participants in mathematical interaction is not directly achievable. It requires the embedding of occurring messages into a rational background of common conceptual views and actions (see Sect. 2).
- Interpretative mathematics education research cannot directly elucidate processes of emerging understanding in real mathematical interaction simply by mere observation. Research-based understanding of processes of understanding in mathematical interaction requires research methods and theories as a rational, scientific nexus for integrating empirical observations from mathematical interaction into a coherent theoretical framework (see Sect. 2).

This "twofold problem of understanding" is subjected to the impossibility of the direct mediation of meaning in communication in a dual way. The interacting participants cannot directly transmit their meaning and understanding to other participants; and the researchers cannot gain insight and understanding of real interaction processes by means of direct access: "Everything said is said by an observer" (Maturana 1988a).

Enactivism offers special explanations of how processes of knowing and learning (mathematics) can be understood: "cognition and knowing are explained within enactivist theory as active processes that occur directly through the interaction between the cognizing subject and the environment, rather than as a construction of representations of the environment by the cognizing subject" (Goodchild 2014). What are specific conditions and constraints of the '*mathematical* environment' in teaching–learning situations that the cognizing subject (students and teacher) directly interacts with? Further analysis will show that there is not one unique '*mathematical* environment' that students directly interact with. Two main notions of the '*mathematical* environment' could be: (1) the '*mathematical* environment' has a more or less concrete, thing-like quality; (2) the '*mathematical* environment' has to be engaged with as an abstract and structure-based entity. In the end it turns

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out that mathematical "… cognition and knowing … as active processes that occur directly through the interaction between the cognizing subject and the environment …" has to be situated within a broader environment of mathematical knowledge as an object of interactive teaching–learning processes.

2 Mathematical interaction: how understanding comes about in communication

Students' learning of mathematics in teaching processes is enclosed in language and communication. All sorts of communicative actions—speaking, depicting and gesturing—are taken as the main sources of human interaction and meaningful exchange; but how is understanding realized within this complex communicative setting? What *understanding* of meaningful understanding between teachers and students have we as mathematics educators to elaborate? Could human understanding in interaction imply as an intended ideal situation that all participants' thinking is the same?

When looking at the mathematics classroom and its culture with its planned direct transmission of elements of mathematical knowledge from the teacher to the students, language and communication are mostly seen as technical supports for making understanding immediately possible.[1] The German sociologist Niklas Luhmann has sharply criticized such a view. He first explained the concept of 'trivial machine' and 'non-trivial machine' within his sociological theory of communication: "… trivial machines are those which transform an input impulse according to a certain rule into an output, in a way that whenever one puts in the information or the energy quantum the machine operates and generates a certain result. If one puts in another input the machine again operates and generates, unless it has several functions available, another result …" (Luhmann 2002a, p. 97 ff.). And he continues: "Non-trivial machines … always switch on their own status and put interposed questions in between, Who am I?, What have I just done?, What is my state of mood?, How strong is my interest still?, and so on, before generating an output. A self-referential loop is built in" (Luhmann 2002a, p. 98).

Luhmann takes this distinction between 'trivial' and 'non-trivial machine' not simply as the opposite between negative and positive. He explains that the functioning as a 'trivial machine' in social interaction within certain institutions is expected and required: "… we often want social systems to act as trivial machines. If you imagine a court in which a judge applies the law, he functions as a trivial machine. Always when a certain input is put in, a certain distinction is resulting" (Luhmann 2002a, p. 98). But teaching and learning in school cannot be understood in a way that the students have to function as 'trivial machines': "I have harvested extensive resistance of pedagogues when I told them that they wanted to educate their students as trivial machines, when these had to give correct answers to certain questions. When the answer is wrong, it is wrong, when it is correct, it is correct. When the answer is wrong, the machine has made a mistake, when it is correct, it is good. In this system it is not planned that for instance the student puts the question into question or looks for a creative way out, so considering the aesthetics of mathematical formulae like concrete poetry distributed on a sheet, he or she makes something which only could be explained when knowing his or her actual mood" (Luhmann 2002a, pp. 98–99).

Although in everyday life it seems in many cases necessary that trivial machines function reliably, teaching and learning processes should not be reduced to the functioning of trivial machines. "The learning student should not be seen as a trivial machine, and … the re-setting of the learning process to self-organization is indispensable …" (Luhmann 2002a, p. 106).

Teaching and learning mathematics in classroom interaction cannot function in the way of trivial machines. Mathematical communication in learning processes cannot be strictly regulated as an input–output game with definite, correct answers. Further, communication is in general not a direct mediation of meaning from one to another person: "… neither individuals nor the interaction system of teaching are trivial machines that produce the desired results when the correct input is inserted …" (Luhmann 2002b, p. 157). And Bauersfeld emphasizes that the mathematics classroom constitutes a multifaceted culture:

> … teaching takes place as a culture: its members form a typical 'habitus' by adaptation (Bourdieu) which brings about and makes possible a conflict-free and at the same time adequate action in this culture. (Bauersfeld 2000, pp. 124–125)

The understanding of how students as non-trivial machines learn mathematics in the culture of a mathematics classroom is in accordance with a view on mathematical knowledge that Freudenthal has always emphasized: "Every mathematician knows … that besides ready-made mathematics there exists mathematics as an activity. But this fact is almost never stressed, and nonmathematicians

[1] To be clear: researchers in mathematics education would completely disagree with such a conception of teaching as a transmission of unambiguous mathematical knowledge (see Ernest 2010). But in everyday mathematics teaching the teaching–learning processes of mathematical knowledge often transform to interactions in which mathematical knowledge is intended to be directly transported from the teacher to the students (see the criticisms made by von Glasersfeld 1995, p. 83).

are not at all aware of it" (Freudenthal 1973, p. 114). And he continues: "The opposite of ready-made mathematics is mathematics in statu nascendi. This is what Socrates taught. Today we urge that it be a real birth rather than a stylized one; the pupil himself should re-invent mathematics…. The learning process has to include phases of directed invention, that is, of invention not in the objective but in the subjective sense, seen from the perspective of the student" (Freudenthal 1973, p. 118). This perspective on mathematical knowledge being a process rather than a product in teaching–learning processes is a fundamental assumption taken in the following explanations.

Essential dimensions of the mathematics classroom culture are 'communication/interaction' and 'mathematics'. This seems obvious, but it implies two critical issues: 'communication' does not simply function as a technical discussion and 'mathematical knowledge' is not concretely to be grasped with the senses. But how do interactive processes happen in reality? How is understanding realized in discourses amongst the participants? Luhmann introduces the strict separation between psychic and social processes and in this way denies the possibility of a direct exchange of understanding between persons: "Pedagogy will hardly admit that psychic processes and social processes operate in completely separate manners. However, the individuals' consciousness cannot reach other individuals with its own operations. … But when communication is to come about, a completely different, also closed, also autopoietic system has to become active, as a social system which reproduces communication by communication and does not do anything else than this" (Luhmann 1996, p. 279).

> The concept of the 'autopoietic system' has been introduced by Maturana and Varela (cf. for example Maturana and Varela 1987 (in English: 1992)). It describes self-referential systems, which mean living systems which exist and develop autonomously and which produce and re-produce those elements that are needed for their existence in order to maintain them. … Luhmann has extended the concept of autopoietic systems to systems within society. The essential property of such non-biological systems is a specific kind of operation that only takes place and is re-produced in this system. The main non-biological systems in society are the social and the psychic system. What is the core difference between a social and a psychic system? Psychic systems are based on consciousness and social systems are based on communication. (Steinbring 2009, p. 51)

The elements that are re-produced within biological autopoietic systems are molecules (see Mgombelo and Reid 2015). In social systems the elements that are re-produced are the messages (exchanged by the participants in communication); in psychic systems the re-produced elements are the thoughts of an individual. According to Luhmann, psychic systems are based on consciousness and the elements of this autopoietic and self-reproducing system are thoughts: "Besides social and living systems, psychic systems or consciousness systems are one of three levels of the constitution of autopoietic systems. The operations of consciousness are thoughts that reproduce themselves recursively within a closed network without contact with the environment" (Baraldi et al. 1997, p. 142). The functioning of the processes of re-production of the elements in the social and psychic autopietic systems is based on meaning. [In German, Luhmann speaks of *Sinn* and not of *Bedeutung*. The word *Bedeutung* (meaning) has a more objective property. The word *Sinn* (in a way: sense) is more the kind of meaning an individual has in mind.] "For meaning constituting systems everything has meaning because everything can only be communicated (or thought of) on the basis of meaning. … On the one side one can only observe the world within the medium of meaning. On the other side meaning only realizes oneself in social and psychic systems" (Baraldi et al. 1997, p. 171).

Accordingly, social and psychic systems (or consciousness) systems operate with meaning [*Sinn*] and living systems do not operate in this way. There is a break from the theoretical approaches of Maturana and Varela who would deny the existence of non-biological autopoietic systems:

> There have been proposals suggesting that certain human systems, such as an institution, should be understood as autopoietic (Beer 1980; Zeleny and Pierre 1976). From what I have said I believe that these proposals are category mistakes: they confuse autopoiesis with autonomy. (Varela 1981, p. 15)

On the one side the social and psychic autopoietic systems are strictly separated because the operations of one system cannot directly act on the operations of the other system: "A social system cannot think, a psychic system cannot communicate" (Luhmann 1997, p. 28). On the other side there exists a specific relationship between both systems: "These two kinds of systems are, however, connected to each other in a particularly tight relation and mutually form a 'portion of necessary environment': without the participation of consciousness systems there is no communication, and without the participation in communication there is no development of the consciousness" (Baraldi et al. 1997, p. 86).

According to Luhmann the participants of a social (communicative) system reciprocally exchange by messages (or actions) 'signifiers' that point to information or 'signifieds'. The messenger only can contribute a 'signifier' but the 'signified' intended by the messenger—that only could lead to an understandable 'sign'—remains open and relatively vague, and it can only be generated by the receiver of the message through articulation of a new ‘signifier’. The receiver cannot assign the possible signified strictly to the speaker but he or she has to construct a signified himself/herself in the course of the ongoing social communication.

An example might help to better understand this difficult communicative interplay. The participants articulate alternately messages (speaking, pointing, gestures, actions, …) that might refer to intended information and meanings. For example, the young student Annika states: “I have calculated 8 by 4 that was 12.” The messenger, Annika, provides a message by stating a ‘signifier’ but her intended ‘information’ (or meaning) remains open and is in question. Obviously the correct meaning of her message cannot be the arithmetical equation $8 \div 4 = 12$, which is a ‘wrong’ statement—in the context of trivial machines.

The receiver of the message, the teacher, articulates a further message and in this way a potential understanding should be made possible by producing a new ‘signifier’ taking the role as signifying at the old signifier thus making it a new ‘signified’. The teacher states: “You have said … 8 divided by 4 equals 12.” The receiver of the message, the teacher, cannot strictly attach the possible information to the messenger, Annika. Her message now opens possibilities of an interactive constitution of understanding and meaning. In our example this could be: “You know, the equation $8 \div 4 = 12$ is not correct! Explain! I cannot understand you!”

The generation of joint and reciprocal understanding in mathematical interaction is not directly possible. The personal intended meaning Annika has in her head is not directly transferable to her teacher. Her thoughts, the elements of her psychic autopoietic system, cannot directly operate on the teacher’s thoughts. But, how can joint understanding be realized in communication?

In his book *The Origins of Human Communication* (2008) Michael Tomasello identifies necessary conditions for the achievement of understanding in human communication. Human language “… rests on a nonlinguistic infrastructure of intentional understanding and conceptual common ground, which is in fact logically primary…. If we want to understand human communication … we cannot begin with language. Rather, we must begin with unconventionalized, uncoded communication, and other forms of mental attunement, as foundational. Excellent candidates for this role are humans’ natural gestures such as pointing and pantomiming” (Tomasello 2008, p. 58ff.). Accordingly, essential basic elements for the realization of understanding in communication are ‘a conceptual common ground/a common praxis of actions’ and ‘pointing and symbol gestures’. In our example of Annika and her teacher we will see how their conceptual common ground and their praxis of actions within the elementary arithmetical context of natural numbers and operations of addition, subtraction, multiplication and division together with gestures enables the development of joint understanding.

Tomasello’s concept of ‘a conceptual common ground/a common praxis of actions’ that is compressed here in the term ‘rational nexus’ is related to Maturana’s ‘domain of explanations’:

> When two or more autopoietic systems interact recurrently … there is a co-ontogenic structural drift that gives rise to an ontogenically established domain of recurrent interactions between them which appears to an observer as a domain of consensual coordinations of actions or distinctions in an environment. This ontogenically established domain of recurrent interactions I call a domain of consensual coordinations of actions or distinctions, or, more generally, a consensual domain of interactions …. (Maturana 1988b, p. 18, 8.ii.a)

And further, there is a connection between Tomasello’s ideas on language and Maturana’s. Maturana sees languaging—using language—as a special kind of consensual co-orientation of actions which occur normally in the interactions of living systems and give rise to a consensual domain—a joint praxis of action:

> There are circumstances in which an observer can see that under the expansion of a consensual domain of co-ordinations of actions there is a recursion in the co-ordinations of actions of the organisms that participate in it. … Due to this, I claim that when this occurs, language happens, and that the phenomenon of language takes place in the flow of consensual co-ordinations of consensual co-ordinations of action between organisms that live together in a co-ontogenic structural drift. (Maturana 1988a, pp. 46–47)

The emergence of understanding in communication is neither directly possible nor transferable from one to another person. Every participant in communication has to construct his/her own and personal understanding (Luhmann). Further, understanding in communication presupposes a conceptual common ground and a joint praxis of actions (Maturana, Tomasello).

Every participant in communication cannot understand the occurring messages of other participants on their own terms but he or she has to try to sensibly embed the heard messages into his/her (individual) rational nexus (or network) of conceptual common views and a comprehensive praxis of actions. A single message cannot be understood directly by itself, or in the way it is communicated. The receiver of a message has to integrate this single piece of information into a possible coherent network or nexus in which the information gets meaning by relations to other elements of this nexus. The example of Annika and her teacher will help to clarify this basic idea.

Example (taken from Fromm and Spiegel 1996, p. 109), 'Annika calculates 60 ÷ 4' (Grade 3):

7 A:	First, I have calculated 8 by 4 that was 12,
8	and then I have the remainders of it, that was 2 with each,
9	and then I have … and that was then again 12
11 T:	Well, I now must interrupt you once.
12	Namely you have said, 8 divided by 4 is 12.
14 A:	Hmm, I mean 8 by 60.
15 T:	8 divided by 60 … does it work?

Taking the perspective of 'trivial machines' Annika produces an incorrect output: '$8 \div 4 = 12$' is 'wrong' and the task '$8 \div 60 =$' conceptual difficulty there' is for elementary teaching a not acceptable—and therefore 'wrong'—mathematical problem. From the perspective of 'non-trivial machines' the interpretative analysis of the further course of interaction between Annika and her teacher offers a reconstruction of how Annika might have produced her own understanding within her personal rational nexus of conceptual views and actions. During the course of interaction the teacher seems to have no real access to Annika's rational nexus.

The qualitative analysis shows and gives evidence that Annika might have reasoned in the following way (here shortly summarized): "10 fits 6 times into 60 [$60 = 6 \cdot 10$]", "8 is in 10 (with remainder 2) [$10 = 8 + 2$]" and $8 \div 4 = 12$ can be reinterpreted as "4 is 2 times in 8 hence (at least) 12 times in 60". Moreover, with the help of $60 = 6 \cdot 10$ Annika calculates how many remainders of 2 ($10 - 8$) have to be considered still: "… the remainders of it, that was 2 with each, and then I have—and that was then again 12."

The interactive understanding emerging between Annika and her teacher cannot be reduced to a direct exchange of 'correct' words. This communication constitutes a "social system" in Luhmann's terms; messages are exchanged, and one message provokes the mediation of another message—the reproduction of the elements of the autopoietic interaction system. The elements of the two psychic systems—the thoughts of Annika's and the teacher's consciousness systems—cannot be directly observed, and Annika's thoughts cannot directly couple with the teacher's thoughts and vice versa. Understanding what Annika and the teacher might think about needs a common conceptual background and a joint praxis—according to Tomasello—for placing the communicated message into a rational network. Understanding the other person requires the one person to reconstruct, from the messages the other person communicates, one's own interpretation by embedding these messages into one's own nexus of conceptual views and actions. Thus, in trying to make sense of the messages, one receives Annika's message "I have calculated 8 by 4 that was 12" as seeming to be a statement without 'correct' meaning within the ordinary context of elementary arithmetic. And the teacher seemed to have put in her nexus of understanding of how to correctly perform the division operation. In the end, with more insight into Annika's complex rational nexus of arithmetical concepts and arithmetical operations this message got meaning and could be understood as an idiosyncratic description of: "8 contains 2 times 4, 8 is at least 6 times a factor in 60, so 4 is (at least) twice often a factor in 60, hence 12 times".

This section has clarified the 'twofold problem of understanding'—the impossibility of a direct mediation of meaning between individuals, and the impossibility of research insights by mere observation—by referring to fundamental concepts of Luhmann's social theory and to Tomasello's ideas of how understanding in communication can come about by constructing a rational nexus of a conceptual background within a joint praxis. Analogously, mathematics education research has developed theoretical perspectives and methods for reconstructing interactive processes of negotiating mathematical meaning and understanding in teaching and learning processes. Maturana characterizes this basic issue in the following way: "… what distinguishes an observer in daily life from an observer as a scientist is the scientist's emotional orientation to explaining his or her consistency in using only the criterion of validation of scientific explanations for the system of explanations that he or she generates in his or her particular domain of explanatory concerns, and his or her commitment to avoid confusing phenomenal domains in his or her generation of scientific explanations" (Maturana 1988a, p. 36).

3 Mathematical interaction: imperceptibly operating nexus of meaning and patterns of classroom communication

For a long time mathematics teaching was unconsciously understood as a clear and simple process of direct transfer of mathematical knowledge from the teacher to the students. Within this teaching–learning process the students were expected to function like 'trivial machines'. There was no real observance of the problem of communication and of how mathematical meaning can be developed in interaction.

The research group around Heinrich Bauersfeld at the IDM (Bielefeld) promoted the so-called 'interpretative classroom research' as a corrective for a kind of mathematics teaching as the 'mediation of an inconvertible subject matter':

> … the 'turn to everyday life'… with its criticism of 'holiday didactics'… contained the claim of assigning a greater meaning than before to the features of everyday instruction. In ethnographic observations of instruction and interpretative studies, one saw a corrective for conceptions of instruction which emerge at the didactical desk; one was disillusioned by the effects of the school reforms (see among others the 'New Mathematics') and wanted to understand better the surprising stability of everyday instruction, its own progress and its traditions. At the same time, there was the hope of being able to better connect with the experience and the problem awareness of the practitioners through softer methods of empirical research. (Voigt 1996, p. 384)

The interpretative mathematical interaction research paradigm emphasized two perspectives that had been disregarded in the German-speaking mathematics education community until then (Voigt 1994, p. 74):

- An individual psychological perspective emphasizing the autonomy of the learner and his/her cognitive development
- A collectivistic perspective that understands mathematics learning as a socialization of the (young) students in a given mathematical culture.

Within the context of the 'individual psychological perspective' one theoretical construct is the 'subjective domain of experience': "The learning students (the 'subjects') make experiences within a certain domain, for example by performing activities. In social interaction with others … these activities obtain sense for the learners, they recognize what meaning these activities have. … These meanings are tightly connected with the perceptible means, the material and the introductory examples" (Hasemann 2003, p. 50). Another example is the construct of 'frame'. A subject directly takes within a social situation a personal 'frame' as a horizon of sense-making; it is an individual view a person takes, for spontaneously interpreting a (new) social situation. Frames are rarely taken consciously; in most cases they are activated in social interaction on the basis of already lived or similar experienced situations (Krummheuer 1984).

These two examples show that meaning and understanding in mathematical interaction cannot be directly transferred but must be seen as a personal construction within an (individual) rational nexus, not a 'correct output triggered by a precise input'.

'Patterns of mathematical communication' are a central field of interpretative research within the 'collectivistic perspective'. The reality of classroom interactions is much too complex to be modelled entirely and exhaustively as reciprocal effects of variables. But: "… the putative chaos of teaching … [could] emerge as a relatively well ordered event. Order in teaching here is not understood as a controlled network of intervening variables—but as a formation induced by the actors in the teaching process. The hidden regularities, the patterns and routines of interaction, permit the participants to behave in an ordered manner …" (Voigt 1984, p. 46).

The example of the 'funnel pattern' of the 'narrowing down of actions in view of the answer expectation' (Bauersfeld 1978) shows how communicative features emerge in small-step and question-based mathematics teaching. The essential dimensions of the funnel pattern are (cf. Bauersfeld 1978):

1. The student does not recognize the mathematical operation or conclusion.
2. The teacher intervenes with a short question. (No or wrong student's answer.)
3. The teacher continues to strive for an insightful student's answer. (No or wrong student's answer.)
4. Further absence of the expected answer leads to a narrowing to the mere recitation of the expected answer.
5. The interaction process stops as soon as the expected answer falls.

As an illustration, the following short episode shows elements of the funnel pattern (Interviewer, Julia, Johanna):

134 I:	… Ehm, why do you want to <u>add</u> them? … We have nearly almost solved it … hm?
135 Ju:	You, say it!
136 Jo:	I don't understand you …
162 I:	… Well, and we still have these six kids who raised their hands on both questions. What is to be done with them? You have, you always wanted … to <u>add</u> them …
179 I:	… <u>How else</u> could you probably calculate this?
193 Ju:	And then 6 less

The contributions of the interviewer (134: 'to <u>add</u> them', 162: 'you always wanted … to <u>add</u> them', 179: '<u>How else</u> … calculate') enforce in a way of small interaction steps the "the mere recitation of the expected answer" (dimension 4) and in the end the "the expected answer falls" (dimension 5).

The functioning of this extreme form of a funnel pattern is based on the discursive scaffolding. The interviewer's accentuation of 'to <u>add</u>' and '<u>How else</u> … calculate' suggest to the students to try out the opposite of 'add', namely '6 less' or 'to subtract', which is then accepted by the interviewer as correct. The functioning of the funnel pattern sometimes is so perfect that the participants of the mathematical communication do not have to know nor to understand the mathematical problem that was posed in the beginning. A correct student's answer can be forced simply by rhetorical means and by strong suggestions and signal words.

The mathematical problem in this example was the following: "In a first grade class with 20 children the teacher asks the children who of you have sisters and/or brothers. When asking the question: 'Who of you have a brother?' 8 children raise their hands. When asking the question: 'Who of you have a sister?' 9 children raise their hands. And when asking the question: 'Who of you have a brother and a sister?' 6 children raise their hands. Question: How many children have no siblings at all?"

The idea to solve the problem is as follows: You add the number of children raising their hand on the brother-question, 8, and the number answering the sister-question, 9, which makes 17. Now, 6 children raised their hands on the brother&sister-question, denoting that these 6 children have also raised their hands on each of the two first questions, therefore two times. Consequently this number 6 has to be subtracted—and not to be added, as the two girls proposed—which leads to $17 - 6 = 11$, the number of children having siblings, and $20 - 11$ gives the number of children in this class, 9, having no siblings at all.

The mathematical problem posed in this interview is a demanding one for first grade students. It has a basic complexity that cannot be overcome simply by using direct strategies of operating with numbers in analogy to concrete actions in the real world. Why calculate -6 and not $+6$? To give a mathematical justification needs an elementary model that offers a mathematical structure of its own and is not directly to be found in the situated example of the classroom with 20 children. Surprisingly, the communication pattern of the funnel in the end leads to the expected answer the interviewer had in mind and looked for and the young students did not need to have an adequate mathematical understanding of the problem or the solution.

Without patterns and routines, everyday social situations could not be managed. All sorts of interactions need secure patterns based on former experiences. When communicating with other participants not every social exchange can be completely new nor rationally reconstructed at that specific moment. In communication, such routines and patterns support the "rational nexus of common conceptual views and actions" (Tomasello 2008) that is essential for realizing understanding between participants in social interaction.

Interaction patterns in mathematics teaching that might be helpful otherwise can degenerate, as for instance the extreme funnel pattern shows. Dimension 4 of the funnel pattern can disconnect the insightful mathematical problem completely from the discursive negotiation of the accepted 'solution': the 'correct' answer falls and it no longer has any link to the problem it should answer. The interaction pattern no longer supports the "rational nexus of common conceptual views and actions" for understanding the mathematical problem but reduces the students to 'trivial machines': "What happens when non-trivial systems … are exposed to trivialization? By self-socialization they adjust to it. … They learn to cope with it. They build in a reflexion circuit that clarifies conditions under which it is recommendable to behave like a trivial system" (Luhmann 2002b, p. 57). Small-step question–answer communication is widespread in mathematics classrooms and this strongly supports patterns that can lead to trivialization: "The teacher … puts a question even though he knows already the answer. In everyday social life this is unusual, and in case it becomes obvious it is embarrassing. In school this is a standardized procedure for the control of trivialization" (Luhmann 2002b, p. 78).

Patterns of communication in mathematics classrooms might play an ambiguous role. On the one side one needs, as always in communication patterns and routines, elements of a rational nexus of common concepts and actions for facilitating meaning and understanding in interaction. But the students' understanding of mathematics could be narrowed down and finally reduced to the mere recitation of the 'correct' answer that could be completely separated from an insight into the mathematical problem the teaching–learning process started with.

4 Mathematical interaction: epistemological dislocations in the course of mathematical knowledge development

The object of mathematical interaction and meaning-making is mathematical knowledge. This object depends on particular epistemological conditions. This object of mathematical discourse cannot be directly seen with the eyes. In some way it has to be imagined.

Jahnke and Otte assert that mathematics represents 'theoretical knowledge' and this has severe consequences for mathematics education as a scientific discipline: "For didactics … it is obvious that the didactical problem in its deeper sense, that is in the sense that it is necessary to work on it scientifically, is constituted by the very fact that concepts will reflect relationships, and not things" (Jahnke and Otte 1981, p. 77). This perspective implies a specific understanding of mathematical teaching–learning processes according to which students have to be active themselves and the production of the essential mathematical relationships is a requirement that in the end the learners themselves have to perform.

A fundamental demand in the mathematics learning of students is to understand and use semiotic representations,

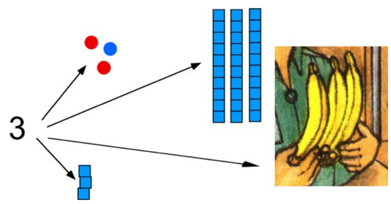

Fig. 1 "Anything can be three!"

visual diagrams and signs as carriers of mathematical 'relationships and structures'. At the same time interpretative mathematics education research is faced with the problem of reconstructing coherent and theoretically based understanding of 'invisible' mathematical knowledge in processes of interaction.

Ultimately, mathematical knowledge is 'invisible' knowledge:

> We do not have any perceptive or instrumental access to mathematical objects, even the most elementary… we cannot see them, study them through a microscope or take a picture of them. The only way of gaining access to them is using signs, words or symbols, expressions or drawings. But, at the same time, mathematical objects must not be confused with the used semiotic representations. This conflicting requirement makes the specific core of mathematical knowledge. (Duval 2000, p. 61)

The semiotic representations—in elementary teaching even concrete material—have not to be taken themselves as the mathematical knowledge. They are carriers of mathematical 'relationships and structures'; they can point to the 'invisible' mathematical knowledge. Some examples from elementary mathematics teaching will give some better insight.

Example: Are numbers 'hidden' in the learning material? What is the 'three'? (see Fig. 1).

The number 'three' can be represented using different concrete objects or more abstract material. But the material and objects themselves cannot 'define' the number three; one has already to know the number 'three' to be able to use the material for representing the 'three'.

Reuben Hersh has clearly contrasted the empirical with the theoretical nature of natural numbers:

> The fact that I have five fingers on my left hand is an empirical observation. 'Five' in that usage is an adjective. There is no conceptual difficulty there, any more than in saying my fingers are long or short. But five in pure mathematics is less than the big number I just defined [((2 to a very high power) raised to a very high power) raised to a very high power], and is relatively prime to it, and so on. It possesses an endless list of properties and relationships, not only in &&&, but also in , in , and beyond. It's part of an abstract theory (Hersh 1998).

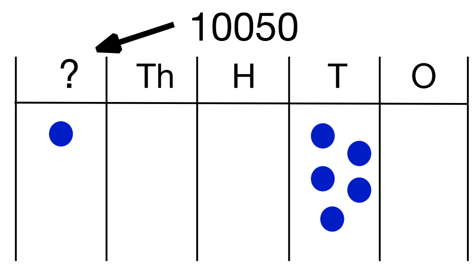

Fig. 2 Chips in the position table representing 10050

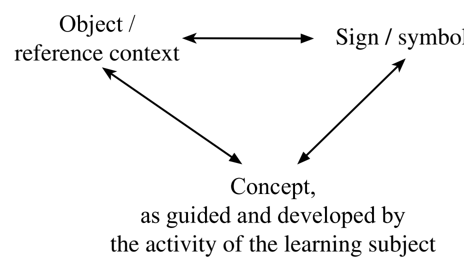

Fig. 3 The epistemological triangle

The use of coloured chips within the position table even more strongly demonstrates that numbers have to be understood via the abstract relation they form with each other (see Fig. 2). Here students are faced with the following epistemological problems: Where is the zero in the position table when using coloured chips? When already knowing the abbreviations—O for ones, T for tens, H for hundreds and Th for thousands—what is the meaning of the highest position (with the question mark)?

The epistemological problem behind the question 'What is the number 10050?' cannot be solved by searching for concrete properties of this specific number or asking what is the correct name of this number. Is the number in the highest column of the position table perhaps one million, as one student supposed? The only and crucial characteristic is the relationship that this new number has with its neighbours: it is ten times as much as its 'right' neighbour and one tenth of the value of its 'left' neighbour.

This epistemological characterization of 'invisible' mathematical knowledge can be modelled with the epistemological triangle (Fig. 3):

> The epistemological triangle represents a theoretical instrument for tackling this problem that one requires signs and symbols for mathematical knowledge, but that these signs and symbols themselves are not the knowledge.
>
> Mathematical knowledge cannot be reduced to signs and symbols. The connection between the signs to

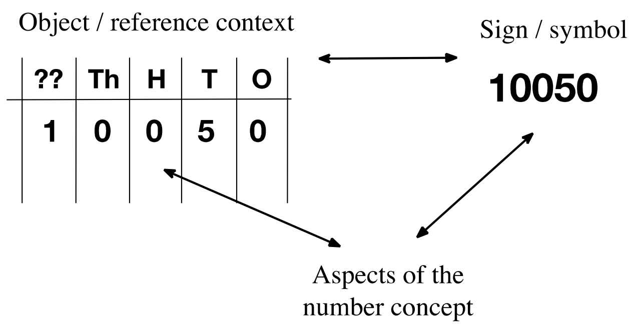

Fig. 4 10050 in the position table

> code the knowledge and the reference contexts to establish the meaning of this knowledge can be represented in the epistemological triangle. … The relations between the corner points of this triangle are not defined explicitly; they form a balanced system, that reciprocally supports itself. In the ongoing development of the knowledge, the interpretations of the sign systems and the chosen according reference contexts will be modified or generalized by the epistemological subject or the learner. (Steinbring 2005, p. 92ff.)

The teacher asks what comes then, after a thousand. The students answer thousand and fifty! How is this number written with numerals, the teacher asks. Svenja proposed writing: 10050. This is now questioned: What number is given by Svenja's proposal? This new symbol has to be explained. Using the epistemological triangle one can characterize the epistemological requirement as follows.

The reference context that is taken here (see Fig. 4) as a rather familiar background that could serve to explain this new symbol is the position table. But in this position table the very column in question that would enable an explanation itself is new. The only possible solution is to search for the internal relationships this 'empty' column has with its neighbours. The internal decimal structure—and not concrete properties of definite denotations—is the epistemological basis for the existence of the number of this column.

The transition and tension between a thing-like concrete and a symbolic relational understanding of the existence of mathematical knowledge leads to a fundamental epistemological problem of learning mathematics: the common conceptual background and the joint praxis of actions—that are the basis for developing understanding in interaction—are subjected to unexpected dislocations and severe changes of the development of mathematical knowledge. What has once been a 'self-evident' fundamental understanding can be destroyed later by modifications to new essential structures and more general relations of the new knowledge in question.

Further examples of mathematical knowledge in school are for instance 'zero' and 'negative numbers'. A common understanding of zero is for students often to say: zero means nothing. The spontaneous empirical reference for explaining the mathematical '0' is a special aspect of reality, namely the absence of something, nothing. Later the meaning of '0' changes drastically and the new understanding requires a completely new, symbolically structured reference context. '0' receives a symbolic-relational meaning within the decimal system. Zero ('0') reaches a new epistemological status. It becomes "… a sign for the absence of signs" (Rotman 1987, p. 57).

'Negative numbers' are a further example for school mathematical knowledge that contains epistemological difficulties for understanding their existence. How can or do negative numbers exist? In history the development of an adequate understanding required the overcoming of epistemological obstacles: "… one remained attached to a 'concrete position', i.e. one tried to confer somehow 'concrete sense' to these numbers and their operations" (Hefendehl-Hebeker 1989, p. 7). The interpretation of negative numbers as 'true' mathematical concepts required overcoming the obstacles by a change of perspective: the existence of the new numbers with their operations no longer rested on empirical objects and concrete operations but have been conceived as an existence that was provided by the internal structure of the system of negative numbers itself together with their operations (Hefendehl-Hebeker 1989).

The tradition of everyday mathematics teaching has provided a number of strategies for circumventing epistemological dislocations. The conceptual mathematical meanings are losing more and more weight compared with a growing emphasis on recipes, procedures and algorithms. And in the course of interaction between teacher and students, routinized communication patterns emerge for making possible 'mutual understanding' and in this way 'negotiating the fabrication of correct answers'.

5 Mathematical interaction: communication, epistemological constraints and enactivism

Learning mathematics is an activity that is a component of a complexity of communicational and epistemological conditions and constraints. Students interact with mathematical problems and at the same time speak with the teacher and with other students about mathematics. The teacher interacts with students doing mathematics. And researchers observe teachers and students working and they interact with this environment. With regard to the enactivist position one can state that the environment the student is directly interacting with when doing mathematics in the culture of the classroom is subjected to many factors and fundamental restrictions. The two essential dimensions affecting the '*mathematical* environment' are restrictions in

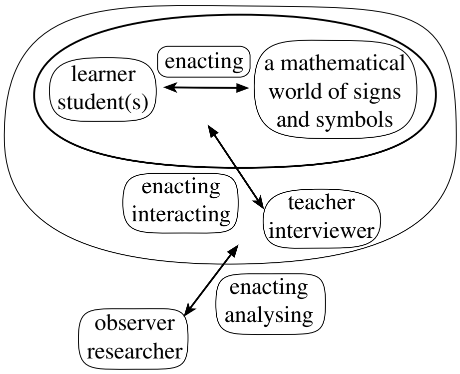

Fig. 5 Complexity of the interplay between autopoietic systems

communication—a direct transport of meaning is not possible in communication—and in the *mathematical knowledge*—mathematics is abstract, incarnated in structures and not directly accessible by senses. Sections 3 and 4 have elaborated these constraints and the examples presented served for further clarification.

The identification of essential issues of the autopoiesis of communication and of the specific epistemological conditions of mathematical knowledge in the preceding sections has brought us to the following overall view (see Fig. 5). The participants—students, teacher (interviewer) and researcher (observer)—represent as individuals psychic systems that re-produce thoughts. The student thinks about the mathematical world of signs and symbols, the teacher thinks about the student's way of enacting with mathematics and the researcher thinks about the interaction between teacher and student(s). All thoughts—the elements of the three autopoietic psychic systems—cannot directly couple with thoughts of other psychic systems (Luhmann). The interaction between teacher and student(s) represents a social system, and the interaction between researcher and the social system of teacher and student(s) represents another social system; only via communication in social systems is there a means of indirectly reconstructing what might have been thought about in the frame of psychic systems.

The student(s) interact with a world of mathematical signs and symbols and this activity is situated in teaching–learning processes with the teacher or in an interview communication with an interviewer. Further, a researcher might observe this teacher–learner interaction, document it and go for a theory-based interpretative analysis.

From an enactivist perspective one could formulate the precondition that the engagement of an agent or of the (learning) subject with the world (of mathematical signs and symbols) is not simply a receipt of the elements of a pregiven world or the individual construction of a true representation of the world: "The insight here is that the world is not something 'that is given to us but something we engage in by moving, touching, breathing, and eating' (Varela 1999, p. 8). As we live, we literally create our world, and, in turn, our identities are created by interaction with the world and other beings" (Brown and Coles 2012, p. 221). How can the subject know about the world? As stated above (Sect. 1): "Cognition and knowing are explained within enactivist theory as active processes that occur directly through the interaction between the cognizing subject and the environment, rather than as a construction of representations of the environment by the cognizing subject" (Goodchild 2014).

In our setting (see Fig. 5) we have three worlds: the world of mathematical signs and symbols (1) for the learning students; the world of the student engaging in mathematics (2) for the teacher; and the world of teacher–learner interaction about mathematical knowledge (3) for the researcher. All three worlds are autopoietic systems (see Sect. 2). According to Luhmann we have to be aware that the psychic systems of the persons involved—students, teacher/interviewer, researcher—also must be seen as autopoietic systems. The world of mathematical signs and symbols is constructed by reciprocal and internal relationships and not by references to empirical properties of the material world. To that extent it can be understood as an autopoietic system with internal reproductions of new signs and symbols. Further, the communicative interactions between psychic systems themselves function as autopoietic systems (Luhmann 1996).

As already stated in Sect. 2, autopoietic systems cannot directly communicate with each other. The interaction between autopoietic systems—that are finally closed—can be understood as a process of co-emergence in which the one system takes the role of the environment for the other and vice versa. Both systems bring forth new structures by structural coupling: "From an enactivist perspective learning is seen as a process of restructuring that is *triggered* by interaction that occurs within the complex dynamic system of coupling (*structural coupling*) between person and environment" (Goodchild 2014).

An example from an interview study will provide more insight into the peculiarities of the mathematical world of signs, symbols and diagrams. This example stems from an interpretative research study (Anke Steenpaß, see Steenpaß and Steinbring 2013). The young students in this study have been asked in the course of an interview the following question: "Which one of the four cards with tasks fits the best for the number line?" (see Fig. 6).

At a certain moment Sonja decided for the task '12 + 7' (see Fig. 7) and she explained by bordering three areas containing some scaling bars (these three areas are additionally highlighted in Fig. 7 by ellipses). Sonja's frame (Krummheuer 1984) for interpreting why '12 + 7' fits to this diagram can be summarized as follows. Sonja knows that the observable elements in the number line, the scaling bars, are symbolizing numbers. But she takes these bars

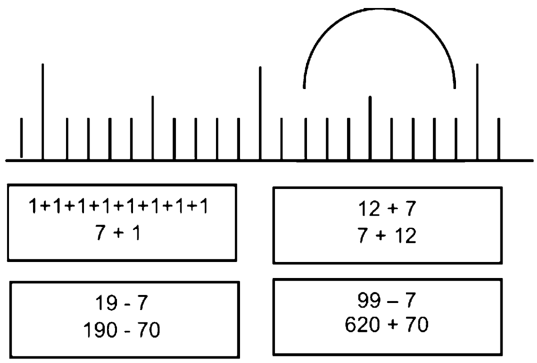

Fig. 6 Which task fits the best?

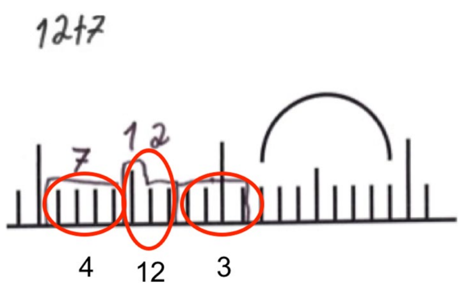

Fig. 7 Sonja's choice: '12 + 7'

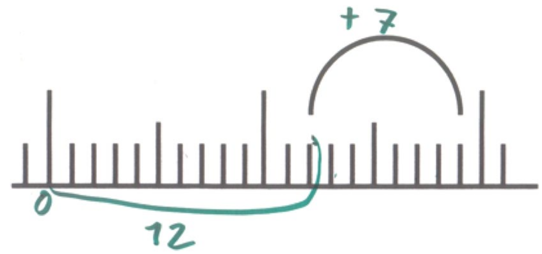

Fig. 8 Anne's choice: '12 + 7'

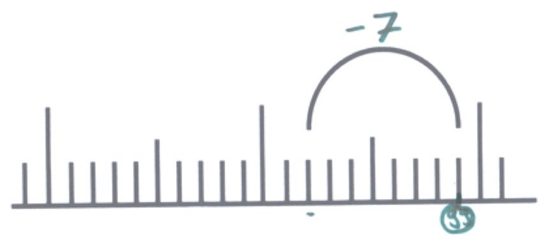

Fig. 9 Anne's choice: '99 − 7'

as if they were objects like concrete things with specific properties. Every 'short' bar means '1' and every 'long' bar (of middle or long length) means '10'. Consequently Sonja collected one 'long' and two 'short' bars for giving '12', then four 'short' bars and separately three 'short' bars giving together 7, and that makes the task '12 + 7' (for a more careful and detailed analysis see Steenpaß and Steinbring 2013). For Sonja in this case the elements of the mathematical world of signs, numbers and diagrams are framed as if they were empiric things with specific attributes that characterize them. Sonja enacts with the mathematical world in a very idiosyncratic manner, thus structuring the diagram as a carrier for numbers by bars that have to be collected.

The interpretation made by the student Anne in the same research study (Anke Steenpaß) looks quite different. She chooses the tasks '12 + 7' (see Fig. 8) and '99 − 7'(see Fig. 9).

Anne frames her explanation and her inscription in the diagram by focusing on the relation and structure between the elements (the scaling bars) and not searching for individual, concrete attributes of these elements. For instance, the '12' is seen as a 'difference' between the first 'long' bar (0) and the second 'short' bar right from the second 'long' bar (Fig. 8). And the '99' is directly left from '100' that is positioned at the third 'long' bar (Fig. 9). The meaning of these graphical elements—used as symbols referring to something else—is given by its position within a structured, systemic network. The single elements do not have meaning in themselves but only as particles within a system of many elements. In this example Anne enacts with the mathematical world of signs, numbers and diagrams by bringing the relational structure of the graphical elements of the number line to the front.

The example of Sonja and Anne shows different ways of how in students' learning of mathematics the enactments with the world of signs, symbols and diagrams are restructured. Sonja framed her world as if it was a world of pseudo-things having specific concrete attributes. Anne framed her world as a structured system wherein the elements receive their meaning by manifold relations with the other elements of the system. In mathematics learning this duality between a 'world of things' and a 'world of relations' is a great challenge. The interpretative analysis of both students' individual framing of the number line offered an interesting and productive research insight and by no means was there any intention to positively or negatively evaluate the students' mathematical competence.

The interactive learning of mathematical knowledge can be understood as a discursive interplay of 'hypostasis' ('conversion to a thing' or 'objectification') and 'theorization' ('conversion to relations and structures'). Such a development might begin with chips used as things (for representing numbers). It then can lead to relations and structures between things (chips, arranged in patterns). And sub-structures can be converted to theoretical objects between which new relations and structures can emerge.

On the one side, the philosopher Ernst Cassirer considers the necessity of a conversion from 'objects' to 'relations':

"… the theoretical concept in the strict sense of the word does not content itself with surveying the world of objects and simply reflecting its order. Here the comprehension, the 'synopsis' of the manifold is not simply imposed upon thought by objects, but must be created by independent activities of thought, in accordance with its own norms and criteria (Cassirer 1957, p. 284).

On the other side, human communication cannot deal with 'pure' structures and relations—human communication has to make use of communicative means that have developed in mankind within a certain environment: "All linguistic representation clings to the world of intuition and returns to it. … Even where language progresses to its highest, specifically intellectual achievements—even where, instead of naming things or attributes, occurrences or actions, it designates pure relations—this purely significative act does not, by and large, surpass certain limits of concrete, intuitive representation" (Cassirer 1957, p. 450).

The seemingly concrete things and objects—the concrete learning materials in elementary mathematics teaching or every mathematical diagram, as for instance the number line—gain a new existence and a modified meaning within a mathematical structure. Enacting as a learner of mathematics with the world of mathematical signs, symbols and diagrams requires a permanent conceptual change and modified interpretation of the meaning of the elements of this world: the meaning alternates between objects, structures between objects, new theoretical objects of converted structures, new structures between converted theoretical objects, and so on.

The students' interaction with the '*mathematical* environment' is contained in a communicative complexity. From an enactivist position, in autopoietic systems "… we literally create our world, and, in turn, our identities are created by interaction with the world and other beings" (Brown and Coles 2012, p. 221). The theoretical discussion in this paper has elaborated specific constraints for the creation of our *mathematical* world by interaction with the world and other beings; communication with other beings is a highly complex interplay between two separated autopoietic systems and the elements of the mathematical world cannot be perceived directly by our senses but consist of 'invisible structures' and relations.

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