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ORIGINAL ARTICLE

Opportunities to engage with proof: the nature of proof tasks in two geometry textbooks and its influence on enacted lessons

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Abstract This paper compares task features and cognitive demand of proof tasks in two US high school geometry textbooks and considers how such differences influence geometry teachers' facilitation of proof and students' engagement with proof tasks during enacted lessons. Data were collected via interviews, task cover sheet-before implementation, task reflection sheet-after implementation, samples of students' work, and classroom observations. Descriptive statistics were used to summarize task features and cognitive demands of proof within textbooks, and a grounded theory approach was used to analyze the enacted lessons. The results revealed variation in the nature of proof tasks within textbooks. Additionally, even though geometry textbooks may have higher-level demand proof tasks, there is no guarantee that such tasks would be assigned, or that the levels of cognitive demand of tasks will be maintained from the written to the enacted curriculum. Factors that can influence how teachers' use textbooks include: beliefs, students' disposition, and assessment. Thus, teachers' actions can limit the extent students engage with proof. This study has implications for unpacking the complexities of students' engagement with proof.

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1 Introduction

The importance of proof in mathematics cannot be overstated (Harel, 2008; Hanna & DeVilliers, 2012). Despite recommendations for proof to be taught across grade levels (Mariotti, 2006; NCTM, 2000), the visibility of proof in high school mathematics textbooks in the United States (US) is limited, with perhaps the exception of geometry courses (Harel & Sowder, 2007; Stylianou, Blanton, & Knuth, 2009; Thompson, Senk, & Johnson, 2012; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki-Landman, 2012). Teaching proof in high school geometry provides opportunities for students to engage in mathematical rigor, as geometry is the first mathematics course in which students are introduced to proving activities (Zaslavsky et al., 2012).

The emphasis placed on proof can differ across geometry textbooks in countries around the world (Chang, & Reiss, 2012; Pepin, Gueudet, & Trouche, 2013). For example, geometry textbooks in the United Kingdom provide little opportunity for students to engage with proof; however, in Japan, deductive reasoning and proof are core facets of geometry textbooks (Fujita & Jones, 2003). Historically, two-column proofs were the standard means to introduce students to proof within US geometry textbooks and were instrumental in changing students' role from solely learners of mathematical proof to doers of mathematical proof (Herbst, 2002).

The way proof is treated is influenced by the context of the learning environment, students and schools (Furinghetti &Morselli, 2011). Researchers found that students are seldom afforded the opportunity to engage deeply with the proving process (Heinze & Reiss, 2004), and a single intervention is not sufficient to develop students' understanding of proof (Jahnke & Wambach, 2013).

Given the importance that textbooks play in affording opportunities to teachers, in this study, we addressed the nature and cognitive demand of proof tasks available in two high school geometry textbooks in the US. We were interested in how such differences may influence teachers' ability to facilitate the learning of proof. Furthermore, we sought to identify potential factors that can contribute to how teachers use tasks (Lin et al., 2012) within the textbook to facilitate how proof is taught. Our overarching research questions are:

- How do two US geometry textbooks present task features, and what are the levels of cognitive demands of the proof tasks in the textbook?
- To what extent do differences among geometry textbooks influence teachers' ability to facilitate students' learning of proof and their engagement with proof tasks?
- What factors influence how teachers use textbook tasks to facilitate students' learning of proof?

2 Literature review

2.1 Features of proof tasks within textbooks (other than geometry)

Researchers who have examined students' opportunities to engage in reasoning and proof in US high school textbooks have examined identifiable features in the narrative and exercise sections within textbooks and have coded proof tasks according to those features (Davis, 2012; Thompson et al., 2012; Otten, Males, and Gilbertson, 2013). For example, Thompson et al. (2012) examined 20 high school Algebra 1, Algebra 2 and Precalculus textbooks on three topics: exponents, logarithms, polynomials expressions, equations and functions. The framework that Thompson et al. used for the narrative section considered the extent to which the textbook provided opportunities to investigate or formulate conjectures and considered the nature of the justification expected, such as *proof*, *transparent pseudo-proofs*, or *no justification*. Davis (2012) examined polynomials in three advanced algebra textbooks, namely *Core Plus Mathematics* Series, *Prentice Hall Algebra 2*, and *Discovering Advanced Algebra*. Davis coded tasks in the exercise sections for *identifying patterns*, *constructing and testing conjectures*, *developing arguments*, *proof building blocks*, *technology*, purposes *of pattern/conjecture/argument*, and the extent to which units and theorems aligned with particular proof schemes. These features of the tasks provide insight regarding what students are expected to do and the kind of proof to which students are exposed.

The frequency of proof tasks can also be used to make inferences about the likelihood of students' engagement. According to Thompson et al. (2012), the opportunities for students to engage in proof related reasoning average 6 % of the textbooks' tasks. Although the percentages of proof tasks are small in textbooks, some textbooks provide more opportunities to engage with proof tasks than others. For example, Davis (2012) found that *Core Plus Mathematics* had more reasoning and proof tasks in the exercise section and fewer in the exposition section compared to the other textbooks examined. Otten et al. (2013) noted that in *Prentice Hall* and *Holt Geometry* textbooks, the introduction to proof chapters, had less than 5 % of tasks devoted to the construction of a proof; and that there were more opportunities to engage in reasoning and proving in the earlier chapters of the textbooks than the later chapters.

The orientation provided by teacher guides or teacher's editions of the textbooks could also influence the experiences that students have with respect to proof. Stylianides (2008) examined the guidance related to proof provided to teachers in geometry, algebra, and number theory units within one middle grades reform textbook series (Connected Mathematics Project 1998/2004). He classified 5 % of the 4,855 tasks in the textbook as proof tasks. He noted that 90 % of those proof tasks provided limited guidance, since only the solutions were provided. The remaining 10 % of proof tasks provided either an explanation of the importance of students' engagement with proof, cautionary notes and suggestions to facilitate students' approaches to proof, or support for teachers' content knowledge pertinent to proof.

2.2 Task features of proof tasks within geometry textbooks

The features of proof tasks in geometry include different proof representations, diagrams, and the presence of a context for the problem. These characteristics can have implications on how students engage with proof tasks.

Despite the historical prominence of the two-column proof format in the US, there exist other representations that are visible in textbooks (paragraph proof, flow proof and proof tree) (Cirillo & Herbst, 2010). The two-column proof has been criticized for reducing the excitement of proving and for negatively influencing students' view of deductive reasoning (Schoenfeld, 1986); however, it continues to be the primary representation used in assessment in the US (McCrone & Martin, 2004). Paragraph proof utilizes everyday language and written sentences to convey proof arguments, and yet the format has received criticism by teachers because students who use it often exclude their reasoning and make incorrect deductions (Cirillo, 2008). Thus, the opinion that teachers may have about various proof representations can impact the type of representations encouraged or presented during instruction.

Fill-in-the-blank proofs are partially completed proofs. Students are required to complete the proof by providing missing information. McCrone and Martin (2004) reported students performed better on fill-in-the-blank proof tasks than on complete proofs. As the support mechanisms are reduced, the complexity of constructing a proof increases.

Diagrams and drawings appeal to students' geometric intuition and can be used to gain an understanding of geometrical ideas and to represent geometrical relationships and shapes (Battista, 2007). Laborde (2005) noted that diagrams offer *spatio-graphical* properties that can stimulate students perceptually. The inclusion of diagrams can be perceived as a hindrance to the development of geometrical thinking because they can decrease the opportunity for students to engage in reasoning theoretically (Duval, 1995; Fischbein, 1993; Mariotti, 1995). Herbst (2004) noted, "Perception of a diagram thus supports and at times even replaces the logical machinery of discourse as a source of statements about the object; the system of signs not only points to what can be taken as referent, but also makes some other things invisible" (p. 132). Thus, teachers can ease the complexity of students doing proof by providing students with diagrams.

Contextual proof tasks, which align with human endeavors, may elicit different types of arguments and can potentially increase the likelihood that students will devote more time to proof than abstract tasks (Siu, 2008). However, teachers may be uncertain of means to use proof outside of mathematics (Knuth, 2002), if practical usages of proof are not explicitly stated within the textbook.

2.3 Proof tasks during the enacted lesson

Although teachers may pose proof tasks from textbooks during enacted lessons, the extent to which students engage with such tasks can vary. Teachers may provide excessive guidance during whole class discussions or limited opportunities for students to reflect on possible solutions for tasks. According to Bieda (2010), some teachers present proof-related tasks from their textbooks without altering the problem, while students provide responses during whole-class discussions. McCrone, Martin, Dindyal, and Wallace (2002) also found that teachers followed their textbook rather closely when planning lessons on proof and choosing homework assignments, and used technology or hands-on investigation activities sparingly. Following the textbook too closely can create some problems for teachers. For example, Schoenfeld (1988) found that a teacher's strict adherence to the textbook might have caused students to differentiate between constructive and deductive geometry, to give an unwarranted emphasis to the form of the mathematical argument, and to view doing proof tasks as an activity that can be done quickly. An unbalance in the quality and depth of the tasks in a textbook can limit the experiences that students have, if their teacher follows the textbook closely.

Teachers' practices directly influence students' opportunity to engage with proof tasks. Heinze and Reiss (2004) examined how proof is taught in Germany and noted that teachers generally posed brief questions, and then students responded with short answers. This practice reduced the likelihood that students engaged deeply with proofs. Teachers who provide excessive guidance and engage in dialogue that emphasizes the accuracy of the solution rather than probe for understanding may hinder their students' engagement with proof tasks (Harel & Rabin, 2010). Therefore, in exploring students' opportunity to engage with proof tasks, both the written curriculum and the enacted curriculum need to be examined.

3 Methods

This study draws from data collected for the first author's (Author1) dissertation, in which the second author (Author2) served as major professor. Two US geometry textbooks were chosen, and three chapters in each were examined to classify the proof tasks they contained with respect to the task features and the level of cognitive demand of the tasks. Subsequently, a minimum of six lessons on these chapters were observed in order to look at the enactment of the tasks and the level of student engagement. A textbook analysis was initially conducted and a multiple case study research design was used to analyze enacted lessons.

In this section, we describe how the tasks from these textbooks were coded relative to task features and levels of cognitive demand. We also discuss how we documented the influence that textbooks had on teachers' decisions as they implemented these lessons.

3.1 Theoretical framework

For the purposes of this study, we have followed Stylianides' (2007) definition of proof in school mathematics:

> Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: 1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification; 2. It employs forms of

reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and 3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of the classroom community (p. 291).

The definition adequately addresses proof conveyed via textbooks and the mathematical discourse that occurs during enacted lessons.

Additionally, we utilized Henningsen and Stein's (1997) Mathematical Task Framework (MTF) as a lens to analyze and interpret the data. The MTF describe how mathematical tasks progress from the textbook to enacted lessons. It identifies factors that can potentially influence how tasks are setup and implemented across two dimensions: *task features* and *cognitive demands*. The task features are aspects of the task that are central to students making sense of the problem and reasoning mathematically. These features include mathematical communications, representations and strategies (Henningsen & Stein, 1997). The cognitive demand refers to the process a student utilizes in order to find a solution to an assigned task. MTF consists of four levels: *memorization*, *procedures without connections*, *procedures with connections,* and *doing mathematics*. The first two levels are considered lower-level demands, while the latter are considered higher-level demands. According to Smith and Stein (1998) a *memorization* task involves the reproduction of information previously learned that does not require an algorithm. *Procedures without connections* utilize an algorithm, and require little, if any, cognitive efforts. *Procedures with connections* require some degree of thinking as to how to utilize a procedure and may incorporate more than one representation. A *doing mathematics task* requires a method not known to the student, and there is no set way of solving the problem.

3.2 Textbooks

The *McDougal Littell* (Larson et al., 2007) and *Prentice Hall* (Bass et al., 2004) geometry textbooks were selected because they were used by school districts where the teachers were observed. These textbook series have similar organizational structures (Tarr et al., 2010) and are large market shareholders in the US (Weiss, Banilower, McMahon, & Smith, 2001). Both textbooks have a similar lesson structure (such as concepts, worked out examples, and exercises) and ancillary materials. The proof tasks in these textbooks require students to write proofs or complete skeletal proofs (such as fill-in-the-blanks or matching statements to appropriate reasons).

We focused on three chapters with comparable content: Chapter 2 (Reasoning and Proof); Chapter 3 (Perpendicular and Parallel Lines); and Chapter 4 (Congruent Triangles). These chapters were selected because they often provide opportunities for students to prove (Donoghue, 2003; Herbst, 2002b). The teachers who participated in the study noted these were the chapters in which they explicitly sought to develop students' proof skills.

3.3 Teachers

Three teachers (Mrs. Davis, Mrs. Bethel and Mr. Walker—all pseudonyms), in the Midwestern region of the US, were chosen because they had used their textbook for at least 3 years and were familiar with its organizational structure. Mrs. Davis and Mrs. Bethel taught at a large urban school (with a population in excess of 1,800 students) and used *Prentice Hall Geometry*; Mr. Walker taught at a small rural school (population less than 500 students) and used *McDougal Littell Geometry*. Mrs. Davis and Mrs. Bethel's class periods were 88 min per day and Mr. Walker's were 75 min. Author1 observed the teachers for a minimum of six lessons.

To obtain information about teachers' facilitation of proof and students' engagement with the proof tasks, data were collected by means of teachers' artifacts, teachers' interviews, and classroom observations.

Teachers' artifacts were used to document the nature of planned proof tasks, teachers' reflection of proof tasks after the lesson, and teachers' perceptions of students' learning of proof. A modified version of the artifact packet for reasoning and proof developed by Horizon Research, Inc., for the Cases of Reasoning and Proving (CORP) in Secondary Mathematics Project[1] was used. Teachers were asked to provide:

- *Copy of the original task*, *copy of the modified task (if applicable)*, samples of student work (that met/exceeded expectation, demonstrated progress, and struggles), and copies of additional materials (e.g., class notes, homework assignments).
- The *task cover sheet-before implementation*—teachers had to describe the nature of tasks, source of tasks, goal of tasks, and whether students would be engaged in proofs, mathematical arguments, identifying patterns, and making conjectures, etc.

[1] Horizon Research, Inc. developed the artifact packet for the Cases of Reasoning and Proving (CORP) in Secondary Mathematics Project, with funding from the National Science Foundation (Award No. DRL-0732798). CORP seeks to develop curriculum that can be used for professional education that promotes reasoning and proving, and the development of mathematical knowledge needed for teaching.

Table 1 Coding of proof task features

Proof task features	Criteria
Proof representation used	We classified two-column, paragraph, flow, proof-tree (Cirillo & Herbst, 2010), and other. In many instances, the two-column proof representation was presented with the statement adjacent to the reason in parentheses; therefore, although this form was not a standard two-column proof (with a clear line dividing statement and reason), these tasks were still coded as two-column proof. Paragraph proof utilizes everyday language and written sentences to convey a proof. Flow proof seeks to make connections between statements by using arrows, and a proof tree outlines a proof and is considered an "abstract specification of a proof" (Anderson, 1983, p. 193)
Reasoning provided	This notion slightly extends G. J. Stylianides (2008) who used *solution only* as a code, because it considered if both a statement and supportive reasoning were provided for proof tasks, rather than just a response in isolation
Labeled "challenge"	Textbook authors noted, "challenge exercises provide richer skill and application problems to extend students' thinking" (Bass, Charles, Johnson, & Kennedy, 2004, p. T29). Therefore, Author1 counted how many proof tasks were identified as a *challenge* in textbooks
Context of tasks	The context in which the proof was situated was coded as abstract or real world (Siu, 2008)
Inclusion of diagrams, pictures, tables, or figures	The presence of a diagram, figure or drawing was viewed the same way; hence, Author1 determined whether either was present (Battista, 2007)
Fill-in-the-blank	The task required students to complete the proof by providing missing information in a blank (McCrone & Martin, 2004)
Multiple choice	Students had the option to select a correct proof from a list of options

- *Task reflection sheet—after implementation* (TRS-AI)—Teachers reflected on the nature of students' mathematical arguments, conjectures, and what they believed students learned.

Before being observed, teachers were interviewed about their conceptions of proof, the importance of proof in school mathematics, and the frequency of proof and proof-related tasks in their instructional practices. At the end of observed lessons, teachers were asked about specific actions observed during the class: e.g., why a particular feedback was provided for a student's response. The interviews were generally audio recorded and were used to triangulate teachers' written responses submitted via the artifact packets. All recorded interviews were transcribed. In some instances, teachers' comments were not audio recorded, since they were stated during informal conversations, which discussed events that transpired in their teaching.

Author1 observed the same class periods weekly, for a minimum of six lessons per teacher, as a non-participant observer (Creswell, 2008). Another researcher observed one-third of all lessons, to assist with data validation. We used a classroom observation protocol[2] to document background information, context and nature of the lesson, outline of the lesson, classroom culture, use of instructional tool, cognitive demand of the tasks, and proof schemes. Other data were collected by means of field notes, video, and audio recordings.

3.4 Data analysis

The proof tasks were coded based on the visible features of the tasks and the solutions in the teacher's edition. Most of the features coded were identified in literature relative to teaching, learning, or assessment of proof. Solutions were coded because solutions often used specific proof representations. When a solution was provided, it was highly likely that teachers would steer students towards that particular path.

When tasks (e.g. 1a and 1b) were independent of each other, they were considered to be different tasks, and the two questions were counted separately in the total number of tasks, similar to the practice employed by Thompson et al. (2012). However, tasks where students had to write a proof and label each line in the proof separately (1a, 1b, 1c, etc.) were viewed as dependent statements and were counted as a single task.

Author1 developed and coded proof task features based on the descriptions provided in Table 1.

Additionally, Author1 and a team of four researchers coded all proof tasks in the two textbooks according to the levels of cognitive demands: *memorization, procedures without connections*, *procedures with connections* and *doing mathematics* (Henningsen & Stein, 1997). There was an 89 % inter-rater reliability. Later, we refined the *doing mathematics* proof tasks classification to ensure it reflected a notable degree of academic rigor. A secondary coding of

[2] The observation protocol was adapted from an instrument developed by Horizon Research, Inc. for CORP that documented how teachers use the proof tasks during the lesson.

the tasks with another researcher obtained similar inter-rater reliability. Table 2 depicts how proof tasks were coded for levels of cognitive demand. We counted the frequency of the task features and cognitive demand of proof tasks in these textbooks. Similarly, we used the assigned levels of cognitive demand to code tasks within enacted lessons.

Figure 1 is used to illustrate how a proof task was coded for task features and level of cognitive demand. In this task, statements and reasonings were provided; the task was identified as a challenge, is abstract in nature, includes a diagram, and is not a multiple choice item. A two-column proof representation was used to convey the solution and the level of cognitive demand was *doing mathematics*.

The data used in the analysis included transcriptions of interviews, audio-recorded lessons, observation records, and teachers' artifacts. Using grounded theory (Glaser & Strauss, 2009), we engaged in a constant comparison of the data to identify patterns and actions teachers used to teach proof. The enacted lessons were coded for task features, cognitive demand of enacted tasks, factors influencing the teaching of proof and visible proof schemes. Although teachers were examined individually, cross-case analyses (Patton, 2001; Stake, 2013) were also conducted to identify similarities and differences in patterns observed among the teachers.

4 Results

In this section, we present findings related to the task features and levels of cognitive demands of proof tasks across selected chapters in the textbook. Then, we describe the extent teachers use proof and how the proof tasks from the textbooks influenced teachers' facilitation and students' engagement with proof tasks. We conclude with identifying factors that can influence how teachers use the textbook in the enacted lesson pertinent to proof.

4.1 Task features and cognitive demand of proof tasks within textbooks

Of the 977 tasks in the three chapters examined in *McDougal Littell Geometry*, 128 were proof tasks (tasks designed to have students write proofs or complete skeletal proofs). The chapters devoted to reasoning and proof and congruent triangles had equal numbers of proof tasks (55 each, respectively), whereas the chapter on parallel and perpendicular lines had 18 proof tasks. In contrast, of the 1,066 tasks in the three chapters examined in *Prentice Hall Geometry*, 79 were proof tasks. The chapter for congruent triangles had more proof tasks ($n = 48$) compared to chapters for reasoning and proof ($n = 15$) and parallel and perpendicular lines ($n = 16$). In many instances, the tasks required students to prove theorems known to be true to them. Table 3 illustrates the percentages of task features, proof representations, and levels of cognitive demands of proof tasks in both textbooks.

McDougal Littell Geometry provided more opportunities to prove and included more tasks that require higher levels of cognitive demand than *Prentice Hall Geometry*. Furthermore, *McDougal Littell Geometry* primarily used two-column proof representation, while *Prentice Hall Geometry* used two-column proofs just as frequently as paragraph proofs.

As we discuss later, the variation between the two textbooks in the task features and levels of cognitive demand of proof tasks resulted in differences in teachers' enactment and students' engagement with proof.

4.2 Teachers' use of textbooks during classroom instruction

In the observed classrooms, the proof tasks in the textbooks were the primary source of proof tasks to which students were exposed. Teachers who used *Prentice Hall Geometry* proof tasks posed fewer proof tasks and less rigorous tasks when compared to the teacher who used *McDougal Littell Geometry*. These instructional decisions regarding proofs were dependent on the amount and nature of the tasks in the textbooks. For example, Mr. Walker valued reasoning as emphasized in the textbook and used the textbook as a primary source of all homework assignments. He often encouraged his students to provide justifications for their responses and believed that the textbook also required students to explain their responses:

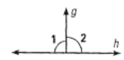
> [In] The book probably at least half the problem will say, what's the answer, explain…. When I look through their homework, their explanations aren't that great, so the homework usually isn't enough to hold them accountable for their explanations. So I've got to kind of press them to do that in class and have them speak up because like I tell them all the time that the reasoning and the explanation is more important than that final numerical answer. (Interview, 11-2-11[3])

Given that Mr. Walker wanted students to provide rich explanations, he was appreciative that many tasks in the textbook required students to explain their responses and often would assign tasks from the textbook that required students to provide their reasoning.

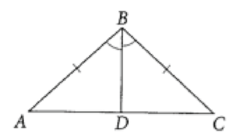
Both Mrs. Davis and Mrs. Bethel, the two other participants in this study, followed their textbooks closely.

[3] The dates are read as month, day, and year, so 11-2-11 means November 2, 2011.

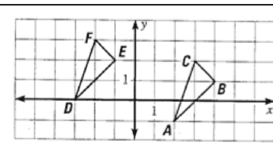
Table 2 Coding of levels of cognitive demands of proof tasks

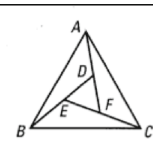
<table>
<tr><th>Levels of Cognitive Demand</th><th>Examples of Tasks</th></tr>
<tr><td>Memorization tasks require students to restate postulates, theorems and rules. Students recall definitions or fill-in-the-blanks using a list of memorized statements, as observed in the prescribed notes (that are on the board or class examples)</td><td>EXAMPLE 2 on p. 191 for Exs. 31–34
31. PROVING THEOREM 3.8 Copy and complete the proof that if two lines intersect to form a linear pair of congruent angles, then the lines are perpendicular.
GIVEN ▶ $\angle 1$ and $\angle 2$ are a linear pair. $\angle 1 \cong \angle 2$
PROVE ▶ $g \perp h$


STATEMENTS / REASONS
1. $\angle 1$ and $\angle 2$ are a linear pair. — 1. Given
2. $\angle 1$ and $\angle 2$ are supplementary. — 2. ?
3. ? — 3. Definition of supplementary angles
4. $\angle 1 \cong \angle 2$ — 4. Given
5. $m\angle 1 = m\angle 2$ — 5. ?
6. $m\angle 1 + m\angle 1 = 180°$ — 6. Substitution Property of Equality
7. $2(m\angle 1) = 180°$ — 7. Combine like terms.
8. $m\angle 1 = 90°$ — 8. ?
9. ? — 9. Definition of a right angle
10. $g \perp h$ — 10. ?

From Section 3.6 Question 31 in McDougal Littell Geometry (Larson et al., 2007 p. 196)</td></tr>
<tr><td>Procedures without connections tasks require limited cognitive effort and are not ambiguous. Such tasks are similar to examples provided in the chapter. These tasks may include writing the converse, and conditional statements, solving equations and writing the appropriate reason, coding triangles, or writing a two-column proof using a different proof representation</td><td>24. Use the plan to write a paragraph proof. See left.
Given: $\overline{BA} \cong \overline{BC}$, $\overline{BD}$ bisects $\angle ABC$.
Prove: $\overline{BD} \perp \overline{AC}$, $\overline{BD}$ bisects $\overline{AC}$.


Plan: To show $\overline{BD} \perp \overline{AC}$, you can show that $\angle BDA \cong \angle BDC$ and use the fact that congruent supplementary angles are right angles. To show that $\overline{BD}$ bisects $\overline{AC}$, you can show that $\overline{AD} \cong \overline{CD}$. The desired congruent angles and segments are corresponding parts of $\triangle ABD$ and $\triangle CBD$. So, first show that $\triangle ABD \cong \triangle CBD$.

From Section 4.4 Question 24 in Prentice Hall Geometry (Bass et al., 2004 p. 207)</td></tr>
<tr><td>Procedures with connections tasks required some degree of thinking while utilizing a procedure, and may incorporate more than one representation. For such tasks, students may be required to organize shuffled proof statements and reasons to make logical proof arguments or write proof plans</td><td>42. VERIFYING CONGRUENCE Show that $\triangle ABC$ and $\triangle DEF$ are right triangles and use the HL Congruence Theorem to verify that $\triangle DEF$ is a congruence transformation of $\triangle ABC$.


From Section 4.8 Question 42 in McDougal Littell Geometry (Larson et al., 2007 p. 278)</td></tr>
<tr><td>Doing Mathematics proof tasks require students to write a complete proof that were not similar to previous tasks and examples or were not algorithmic. The tasks may change the context or utilize a different representation from which students are accustomed</td><td>49. PROOF Write a proof.
GIVEN ▶ $\triangle ABC$ is equilateral, $\angle CAD \cong \angle ABE \cong \angle BCF$.
PROVE ▶ $\triangle DEF$ is equilateral.


From Section 4.7 Question 49 in McDougal Littell Geometry (Larson et al., 2007 p. 270)</td></tr>
</table>

Table 3 Percentages of proof task features, representations and levels of cognitive demand in *McDougal Littell Geometry* and *Prentice Hall Geometry* Chapters 2–4

Task features, proof representations and levels of cognitive demand of proof tasks	*McDougal Littell geometry* $N = 128$	*Prentice Hall geometry* $N = 79$
Task features		
Answers		
Answer with reasoning	88	52
Answer only	12	48
Labeled as *challenge*	12	28
Solution strategies		
One solution strategy	100	100
Multiple solution strategies	0	0
Context		
Real world context	5	1
Abstract context	95	99
Diagrams, pictures, tables, or figures provided	64	87
Fill-in-the-blank	10	47
Multiple Choice	1	0
Proof representations		
Flow	4	18
Paragraph	22	41
Two-column	73	41
Other	2	1
Levels of cognitive demand		
Lower-level demands (memorization)	10	47
Lower-level demands (procedures without connections)	25	11
Higher-level demands (procedures with connections)	50	29
Higher-level demands (doing mathematics)	15	13

Mrs. Davis used the textbook without adaptation. She followed the order in which content was presented and used the tasks that were included. Mrs. Davis stated, "How the book lays out proof is how we model proofs on our test and on our homework" (Interview, 9-23-11). Mrs. Bethel noted "the geometry textbooks actually set you up with something that you know so that you feel more comfortable when you are actually doing those proofs" (Interview, 9-6-11). Therefore, it seems that teachers chose to follow their textbooks, among other reasons, because of students' comfort level and the materials build on ideas previously learned.

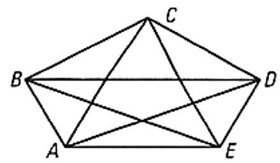
Unlike the other teachers, Mr. Walker was not satisfied with the content of the textbook. He indicated that his textbook had several limitations: it did not provide sufficient problems to practice proofs, it did not address the need to connect mathematical ideas, and the order in which content was presented could be more logical. To address these limitations, Mr. Walker supplemented his textbook with additional proof tasks. For example, in reference to Sect. 4.2 ("Apply congruence and triangles") Mr. Walker remarked, "… If I look in this section in the book, there's one—there's two— proofs of how we want them to be thinking about [it]…. I need more than that; it's not good enough" (Interview, 11-3-11). Therefore, his perception of the textbook influenced the extent he used it or alternative resources.

4.3 Textbook influence on the facilitation of proof tasks and students' engagement

The textbook influenced how proof tasks were facilitated in the classroom. Mr. Walker used proof tasks from *McDougal Littell Geometry* during enacted lessons, likewise Mrs. Bethel and Mrs. Davis used proof tasks from *Prentice Hall Geometry.* The teachers posed approximately half of the proof tasks within the three chapters for students to complete as practice exercises. Of the proof tasks assigned, the task features and level of cognitive demand frequencies aligned with the percentages observed in the textbook analysis. The teachers' instructional practices for proof aligned with the nature of proof tasks within the textbook. For instance, Mr. Walker posed more proof tasks, required students to write complete proofs more frequently, and illustrated less variety of proof representations, when compared to Mrs. Bethel and Mrs. Davis. In all of the classes, students were exposed to proof tasks that were situated strictly in abstract context. Additionally, teachers generally avoided challenging tasks. For example, these teachers seldom assigned proof tasks labeled as *challenge* since they perceived that such tasks were too difficult for their students. Hence, the inherent differences among the features of proof in the textbooks influenced the nature of the enacted lessons pertinent to proof. According to our observations, teachers adhered to the pedagogical suggestions and implemented proof tasks as presented in the teacher's edition of the textbook. Teachers who used the *Prentice Hall Geometry* textbook posed more fill-in-the-blank proof tasks than the teacher who used *McDougal Littell Geometry*.

To provide insight as to how the textbook influenced teachers' facilitation and students' engagement of proof tasks, we describe task features teachers emphasized and valued and the cognitive demand of tasks during enacted lessons.

40. CHALLENGE In the diagram of pentagon $ABCDE$, $\overline{AB} \parallel \overline{EC}$, $\overline{AC} \parallel \overline{ED}$, $\overline{AB} \cong \overline{ED}$, and $\overline{AC} \cong \overline{EC}$. Write a proof that shows $\overline{AD} \cong \overline{EB}$. See margin.



Solution within the Teacher's Edition

40. Statements	Reasons
1. $\overline{AB} \parallel \overline{EC}$, $\overline{AC} \parallel \overline{ED}$, $\overline{AB} \cong \overline{ED}$, $\overline{AC} \cong \overline{EC}$	1. Given
2. $\angle DEC \cong \angle ECA$, $\angle ECA \cong \angle BAC$	2. Alternate Interior Angles Theorem
3. $\angle DEC \cong \angle BAC$	3. Transitive Property of Angle Congruence
4. $\triangle DEC \cong \triangle BAC$	4. SAS
5. $\overline{BC} \cong \overline{CD}$, $\angle BCA \cong \angle DCE$	5. Corr. parts of $\cong$ $\triangle$s are $\cong$.
6. $m\angle BCA = m\angle DCE$	6. Definition of congruent angles
7. $m\angle BCA + m\angle ACE = m\angle DCE + m\angle ACE$	7. Addition Property of Equality
8. $m\angle BCE = m\angle DCA$	8. Angle Addition Postulate
9. $\angle BCE \cong \angle DCA$	9. Definition of congruent angles
10. $\triangle BCE \cong \triangle DCA$	10. SAS
11. $\overline{AD} \cong \overline{EB}$	11. Corr. parts of $\cong$ $\triangle$s are $\cong$.

Fig. 1 Proof task from Sect. 4.6 Question 40 in *McDougal Littell Geometry* (Larson et al., 2007 p. 263)

4.3.1 Proof task features and levels of cognitive demand emphasized in whole class discussions

To support students' use of pictorial representations, teachers required students to practice the process in a ritualistic manner that was often devoid of context and did not develop students' conceptualization of an accurate construction of a proof. For example, Mrs. Davis encouraged students to place markings on the congruent triangles, code the triangles for congruent sides and angles, and conclude with a congruent statement. Whenever the students and Mrs. Davis talked about the proofs, "tick" and "swoosh" were used to refer to marking on the diagrams. She said:

> You can never swoosh or tick what you're trying to prove; if you tick or swoosh what you're trying to prove, this reason is going to be wrong. Prove the triangles are congruent first. So you code what you're given, then you code anything else you know, vertical, reflexive, alternate interior with parallel lines, the word mid-point or the word bisect. (Observation, 10-14- 11)

Mr. Walker also encouraged students to use the information given in the diagram or in text form. He told students, "So whenever we're given information, we're going to use that information somehow, okay" (Observation, 10-6-11). The markings on the diagram were emphasized more than the correctness of students' arguments. As a result, students often ensured they placed markings on the diagram, rather than focusing their attention on writing logical ideas.

Mrs. Davis and Mrs. Bethel acknowledged that they used fill-in-the-blank proof frequently because students were not familiar with proof. Mrs. Davis noted, "… a lot of students can't construct a proof from beginning to end on their own, so we will have fill-in-the-blank, two-column…" (Interview, 9-2-11). Likewise Mrs. Bethel admitted, "We teach the basics of proof and do so with the idea that students can 'master' proofs if given a 'skeleton' and a word bank to fill-in-the-blanks" (TRS-AI, 9-22-11).

Although these three teachers used other representations of proof, they expressed a preference towards two-column proofs. For example, Mr. Walker acknowledged that of the time he allocated to teaching proof, he spent more time working on two-column proofs. He was aware that word choices and sentence structure can potentially change the meaning of a proof, and he felt that two-column proofs provided a clear and precise structure to progress from the given facts to what needs to be proved:

> I spend a lot more time on two-column proof than I do the paragraph proofs…. I guess I feel like I don't want my mistake in grammar in English to disrupt a proof, so…. I think that kind of structure helps see a sequence better, a sequence of thoughts. (Interview, 8-23-11)

Mrs. Davis held a similar view. She explained that although the geometry team at her school taught multiple representations of proofs (two-column, paragraph, and flow), the team prefers the two-column proofs to foster students' development of proofs. They believed that two-column proofs are a practical way for students to learn how to prove and were confident that over time students would be able to construct complete proofs. With two-column proofs, students can be introduced to proofs by completing skeletal proof arguments. Mrs. Davis said:

> We teach two-column proof, paragraph proof and flow proof. And we would like them to be able to construct it ideally, construct proof on their own from beginning to end, and sometimes they can do that with the two-column proof… (Interview, 10-2-11)

The use of the two-column proof representation was observed in all lessons devoted to writing proof in its entirety. In fact, when teachers exposed students to paragraph proof, they would often begin with a two-column proof representation and convert the argument into a paragraph. Teachers' decisions to utilize two-column proofs more often, even in chapters that had more paragraph than two-column proofs were due in part to their belief that two-column proof was an effective means to teach proof.

All three teachers admitted that most students had little, if any, experience doing proofs prior to this geometry class, so they normally posed low-level tasks so that students were comfortable, while developing the notion of what it means to prove. Based on the observation notes, teachers reduced the tasks by doing the proof for students during the whole class discussions, even when teachers posed cognitively demanding proof tasks. For example, Mr. Walker posed and completed a *procedures with connections* proof task relative to congruent triangles; he asked students to identify criteria for congruency based on a list on the board (Observation, 11-10-11). Thus, the cognitive rigor of the task for students was reduced.

Mr. Walker suggested that the procedural nature of proof in geometry reduces the potential value of proof. He acknowledged that students do not necessarily learn new things by doing proofs as much as he would like that to happen. Mr. Walker said,

> …[Proof] can become a little more procedural especially in geometry where there's you know freshmen, sophomores that haven't quite developed yet …so they don't really think about it as, okay, I'm trying to learn something new; they just want this process to go through, so I think it loses some of its value. (Interview, 8-23-11)

Similarly, Mrs. Davis said, "It's not a high-level proof that we are teaching" (Interview, 9-2-11). The reason for assigning tasks of low-level cognitive demand, or reducing the level of cognitive demand of a rich task was because students had limited experiences doing proofs. Mrs. Davis stated, "I don't think they had enough experience with proof… because this is, a lot of the times, this is the first time they've ever seen proofs…" (Interview, 9-2-11). Mrs. Davis was aware that students were new to the notion of doing proofs. So she sought to introduce proof in an elementary fashion so that students could provide appropriate reasoning for each mathematical step involved in the proof. Like Mrs. Davis, Mrs. Bethel admitted, "We teach the basics of proof and do so with the idea that students can 'master' proofs if given a 'skeleton' and a word bank to fill-in-the-blanks" (TRSAI, 9-22-11). She described the proofs taught as "very basic proof, very obvious proofs, and I would say consistently less than 10 steps, never anything that's complicated…and we guide them an awful lot at this stage" (Interview, 9-6-11). Mrs. Bethel acknowledged that she wanted her students to be successful in doing proofs, but because of their lack of readiness, the tasks she posed usually had a low-level cognitive demand. She chose proofs that were short and provided students with a word bank in order to increase students' success in writing proofs or completing skeletal proofs.

In sum, these three teachers opted for assigning proof tasks with low-level of cognitive demand during whole class discussions, with the expectation that these tasks would be more accessible for students. Given that students were inexperienced with proofs, teachers thought that the low-level tasks provided students the opportunity to learn various proof representations in a context that was simple and supported.

4.3.2 Students' engagement with proof tasks

Based on classroom observational data, there existed differences between students' engagement with proof tasks, depending on the textbook they used, when they worked on problems in their groups or independently. For example, Mrs. Davis assigned the following task from the textbook: "Can you prove the triangles congruent? If so, identify the theorem that can be used" (Observation, 10-6-11). A student constructed a two column proof which included the given, supporting reasoning and subsequently concluded that the triangles were congruent by Side-Side-Side (SSS). He also asked his group members to explain their answers. His peers responded that information is not needed; you only need to select a reason. When Mrs. Davis saw what the student did, she noted, "You're doing, like, way more than you need to do. But you did it correctly". The teacher noted that writing proofs occurs later in the textbook. Based on the teacher's remark, the student stopped writing his proof and sought only to identify a theorem. The student exhibited the potential to write a proof and a disposition of inquiry, and yet he was discouraged from doing so because it did not correspond to the textbook's lesson structure. A student in the group remarked that the proofs from the textbook "are repetitious, …you just have to practice the problem…it's not hard." The remark suggests that the student was aware that the proof assigned required little cognitive rigor and knew that mastery of proving can be achieved by recognizing key terms. The students' private conversation suggests that the low level of rigor of proof tasks influenced their procedural engagement with proofs.

In Mr. Walker's classroom, he often allocated a great portion of classroom time for students to write complete proofs individually and with their peers at the table. Thus, students had the opportunity to engage with tasks of higher level of cognitive demand. Mr. Walker readily encouraged students to explain their reasoning and consider the sequence of ideas from the given to the concluding statement. For example, a student recognized that his solution was correct even though it differed from Mr. Walker's solution. The student explained the similarity of his argument to Mr. Walker's and noted postulates could be used to reduce the number of lines in Mr. Walker's proof (Observation, 11-10-11). Nevertheless, the opportunity to consider alternative ways to construct proofs was not afforded when fill-in-the blank proof tasks were assigned.

Although teachers might have posed potentially cognitively demanding tasks, during whole-class instruction, the rigorous nature of proof tasks was usually reduced. In many instances, tasks that were originally classified as having a higher-level of cognitive demand (procedures with connections) were enacted as *procedures without connections*. None of the teachers posed tasks that reflected *doing mathematics.*

4.4 Factors that influences how teachers enact lessons pertinent to proof

Even though teachers use textbooks, there are internal (students' disposition) and external (assessment) factors that can contribute to how lessons on proof are enacted.

Teachers reported that their students viewed proof as overwhelming and displayed a negative disposition towards it, which was observed during classroom visits. In some instances, students did not attempt proofs. For example, Mrs. Davis knew that students found it difficult to explain their actions when doing proofs. She said:

> Well they have a hard time explaining why they did something. A lot of time students don't like to show their work and how did they get to that answer…. They are not used to thinking and showing why they did something and that's a problem that we encountered when we are doing proofs… (Interview, 9-2-11)

Because students had difficulty supporting their reasons, the use of a word bank made proofs more manageable for students during classroom instruction. For instance, Mrs. Davis stated, "I was getting incomplete homework and students who refused to do the proof portion" (TRSAI, 9-13-11). To curb this problem, she posed tasks that required either filling in the blanks or tasks that could be completed if students used memorized theorems and postulates.

Additionally, teachers indicated that the type of assessments used to measure students' performance influenced how they provided proof instruction. Specifically, they said that because the chapter exams and the state mandated end of course exam[4] required students to match statements with reasons or complete a proof by filling in missing statements, they emphasized these types of tasks in their instruction. Mrs. Davis said:

[4] This is a standardized exam administered by the state, which assesses the state geometry curriculum.

> The state test, the math tests … don't have a performance event, so the students wouldn't have to create a proof completely on their own. It would be something, what step [is] missing or what step is incorrect in the proof… So that probably influences the fact that we don't have them create entire proof on their own because we want them to be familiar with how… they would be assessed on a state test. (Interview, 9-23-11)

Considering that proof is seldom assessed, it can potentially be devalued during instruction (Wilson, 1994). Mr. Walker said, "… I feel like if we're explaining our reasoning, then we're going to build our understanding. So to me, doing the proofs in class will indirectly help us on testing" (Interview, 11-3-2011). Therefore, because Mr. Walker believed proofs could be beneficial to students, he chose to assign proof tasks from the textbook, which required students to construct complete proofs, despite the fact that such tasks are rarely used during assessment.

5 Discussion

The findings indicate that these two textbooks had differences between the task features and levels of cognitive demand of proof tasks. *McDougal Littell Geometry* had more proof tasks with a greater degree of rigor, utilized two-column proof representation more frequently, and used less diagrams for proof tasks when compared to *Prentice Hall Geometry*. Both textbooks generally situated proof tasks within an abstract context.

Despite textbooks' inclusion of proof tasks of various types of proof representations and levels of cognitive demand, we found that teachers emphasized the two-column proof representation and generally required little rigor in the proofs their students did. In some instances, teachers reduced the cognitive demand of tasks during observed lessons. Thus, the nature of proof tasks can change from the written to the enacted curriculum.

For the three chapters examined, the small percentage of proof tasks in these geometry textbooks is comparable to the low frequency of proof tasks in non-geometrical courses reported by Thompson et al. (2012). This suggests that students have few opportunities to engage in proof in US secondary mathematics courses. To address the limitation of small numbers of proof tasks in textbooks, additional resources and strategic actions by teachers are needed to increase students' opportunities to prove. Teachers need to plan carefully and, if necessary, modify tasks (Thompson, 2012) to increase opportunity for students to engage in the construction of viable arguments. Additionally, increasing the number of proof tasks in assessment can be a catalyst to increase the importance of proof, because teachers value what gets graded (Wilson, 1994).

Seeing that teachers emphasized different features of proof tasks, such as diagrams and different proof representations, by encouraging students to mark on diagrams or use two-column proofs, it is likely that these inherent features can influence the depth and creativity of proofs students produce. Therefore, future studies should seek to examine the nature of the proofs written by students and their relation with diagrams or particular representations of proof.

We documented how the cognitive demands of tasks can be reduced or enhanced based on teachers' actions, which may be influenced by their beliefs and students' disposition (Boston & Smith, 2009; Herbst, 2006) or external factors. For example, we found that geometry teachers provided excessive guidance to students, therefore reducing the cognitive demand of tasks. To encourage appropriate practices for the teaching of proof (Furinghetti & Morselli, 2011), the value of proof needs to be emphasized and explicated. Future studies should examine the extent to which professional development initiatives influence teachers' orientation relative to the selection and subsequent enactment of higher-level proof tasks.

6 Conclusion

In closing, this study shows the usefulness of examining textbooks' content and enacted lessons in order to learn what opportunities students have to prove geometric propositions. We found that students' opportunities to engage with proof can differ based on the textbooks they use and on their teachers' instructional decisions. Hence, if the textbooks have few tasks where a rigorous proof is required, it is highly unlikely that students will ever have the opportunity to engage with such tasks.

Textbook publishers should consider ways to increase proof tasks and reduce the number of tasks that require simply filling in blanks, and professional development is needed to promote effective ways to engage students during whole class discussions. The relationship between the textbook, internal and external factors, and the enacted lessons cannot be ignored because it influences the nature of what and how students learn to prove.

We are aware that different factors influence teachers' decisions, but we are convinced that looking at the tasks available to teachers is particularly productive. International studies that compare how students in different countries are afforded opportunities to prove, and which examine the nature of the association of textbooks and teachers' decision-making and instructional practices, would make an important contribution to our knowledge of these important matters.

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