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ORIGINAL ARTICLE

Students discussing mathematics in small-group interactions: opportunities for discursive negotiation processes focused on contentious mathematical issues

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Abstract Small-group discussions involving students and their teacher that focus on meanings constructed during the mathematics lessons or solutions to problems produced in these lessons offer great potential for debate and argument. An analysis of the epistemological nature of knowledge can give deeper insight, to gain a better understanding of the emerging discontinuities in argumentations, negotiations, and clarifications about contentious meaning differences that arise. In most cases mathematical interactions between students and a teacher about contentions are very fragile and seem to be handled more or less directly—by side-stepping to another topic or by resolving via the teacher's authority, for example. Therefore, the maintenance of such negotiation processes in mathematics teaching is a specific challenge for students and the teacher. The type of closure of these processes seems to be related to the emerging maintenance processes. In this paper, small-group discussions are interpretatively analyzed in the three steps "Initiation—Maintenance—Closing" with the focus on fundamental (dialogical) learning.

Keywords Contention · Debate · Dialogical learning · Epistemology · Interaction

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1 Contention in mathematical discourse

In recent years, mathematical education research has paid increased attention to social and interactional contexts and to the importance of communication as the main medium in which (dialogical) learning takes place. Since then, interaction in classrooms has been videotaped, transcribed, and analyzed (e.g., Bauersfeld 1980; Pimm 1989; Steinbring 2005). These classroom recordings allow researchers to carefully reconstruct and reflect in detail on the processes of teaching and learning mathematics. Specific outcomes of some of these research projects include appreciating particular classroom cultures (e.g., Voigt 1998; Wood 1994) and the identification of interactional patterns and routines (e.g., Alrø and Skovsmose 2004; Bauersfeld 1980; Voigt 1995). One interesting outcome is that the interactive negotiation of meaning between the participants is crucial for learning mathematics. Voigt (e.g., 1998) highlights the negotiation of meaning by discussing ambiguous mathematical issues as well as considering the time at which mathematical meaning can be seen as "taken-to-be-shared." A meaning taken-to-be-shared emerges during the process of negotiation. Thus, mathematical meaning is taken as a product of social interaction. Alrø and Skovsmose (2004) identify communicational elements such as "locating," "advocating," and "challenging" in dialogical learning. From an epistemological viewpoint, Steinbring (e.g., 2005) works out the specific mathematical elements that the students refer to in their explanations by using different reference contexts and mathematical concepts. All the analyses emphasize that the negotiation of mathematical meaning in interaction might reveal very different, sometimes exclusive or controversial, interpretations, although "the participants interact *as if* they interpreted the mathematical topic of their discourse in the same way. [...] Even if the participants collaborate without conflict, [...] a person cannot be actually certain that her or his understanding is consistent with those of the other participants" (Voigt 1998, p. 203).

In particular, considering that participants in discourse often interact by apparently speaking about the same thing, it seems to be difficult for the teacher to focus on emerging contentious interpretations and maintaining this focus. In this study, those differing interpretations are highlighted.

Our understanding of contention is based on the following preliminary definition:

A *mathematical contention* is realized:

1 when a statement of one of the participants is *put into question*, *contradicted*, or *challenged* by another (possibly contrary) statement, and
2 when the evolving conversation is based on an interpretation of a mathematical object or phenomenon as subject matter (theme-centered).

Even if the participants do not explicitly contradict each other on a mathematical statement, a negotiation of meaning that refers to different mathematical aspects might be necessary. "In the course of negotiation, the teacher and the students create a network of mathematical meaning taken-to-be-shared. From the observer's point of view, this network of meanings can be called a mathematical 'theme'" (Voigt 1998, p. 204). In this paper, "theme-centered" means that the participants' contributions are dedicated to the interactively evolving mathematical theme. The difficulty of *maintaining* an interaction about contentious points concerns the difficulty of sustaining a continuous "theme-centered" discussion, clarifying mathematical perspectives and relations.

Certainly, the contentions that are the main focus of this paper are not necessarily helpful in students' learning of mathematics. In questioning everything that is said in a non-selecting way, a moment in interaction might arise in which the students do not know which conventions they can take for granted and which are the mathematical issues that have to be discussed. Against the theoretical background that the learning of young students mainly occurs though argumentation (e.g., Krummheuer 2000; Miller 2006), the main purpose of this research is to investigate the nature of the teacher's interventions that support students' learning, focusing on theme-centered contributions worthy of discussion.

To approach the difficulties of maintaining these interactions, small-group discussions between four students and a teacher about contentious interpretations and explanations were held and recorded. The four topics focused either on the construction of multifarious meanings dealing with visual diagrams or on students making sense through observing mathematical relations when dealing with mathematical problems. In both cases, there is a great range of different perspectives taken by the students. The students might try to find out what the teacher would like to hear, or alternatively their interaction might be caused by epistemological issues, or might not have a mathematical context at all.

Fig. 1 Number representation

One example from elementary school is the interpretation of bars within the mathematics classroom.[1] When confronting students with a representation similar to that in Fig. 1 without them knowing the ordinary meaning of the symbols, students can find very different explanations for their meaning. Seeing the bars as the result of counting bar by bar such as in a tally sheet is one possible interpretation. Seeing the lines representing place values could be another—the number of bars indicates which number is incorporated to each place value. A third view could be related to the convention often agreed upon in elementary schools that a bar is a "tens" (referred to the Dienes material[2]: squares represent hundreds, bars tens, and dots ones). The three interpretations lead to different numerical values: 12, 417, and 120. Referring to mathematical objects, all three examples have a mathematical focus. If more than one of these interpretations is verbalized, or if just one is put into question and discussed, the interaction contains a communicative situation with a contentious issue in dialogue.

In classroom talk, the interaction often rushes from one topic to the next. Students' attempts towards one specific interpretation are often ignored or skipped over many times. The dynamic nature and complexity of communication in general might be a possible reason, but so could be the existing classroom culture and the teacher's conversational techniques. However, this paper will show that it is exactly these students' attempts to interpret mathematical objects (including all misconstructions) that have a great potential for learning, just as in the case of the discussion of student errors, whose potential and positive effects have long been identified (e.g., Althof 1999; Swan 2006, p. 80ff.). It is basically assumed that contention is supported to arise by discussing mathematical issues with

[1] A discussion of classroom talk about this diagram can be found in Gellert and Steinbring (2012).

[2] Named after Zoltan Paul Dienes.

multiple meanings or through a variety of possible solutions to mathematical problems. The interaction about these mathematical issues occurs in a cultural context, in which the negotiation of meaning takes place. Emerging regularities might lead to interactional patterns. In the following, firstly an epistemological view on mathematically contentious issues will be presented and then the taken perspective referring to interactional patterns will be described. As a result of the connections between an epistemological view of school mathematics and a socio-interactionist view of interactions, the research setting and questions will be developed, and two examples will be presented and analyzed from an interpretative perspective. Applying the results to the theoretical background will show that it is worth having a deeper look at interactional patterns using students' perspectives constructively.

2 Epistemological issues of mathematical contentions

Mathematics understood as a science of patterns (Devlin 2003) requires an interpretation process by the people dealing with it. The interpretation of mathematical diagrams is a challenge for young students, especially in elementary school—students usually try to find an interpretation according to their own experiences, which might be different from the anticipated one. One assumption is that the mathematical issues in classroom discourse are multiple and varied so that, according to the multiple meanings of the mathematical content, the students' interpretations might differ widely. The outcome of these meanings in classroom interaction might be contentious and primarily there are two possibilities for dealing with this: the first is to be open and curious about the students' constructions of meaning and their solutions; otherwise "[…] the 'openness' is lost if the teacher proceeds as though only one method is presupposed as the correct one" (Shimada 1997, p. 1).

Several researchers have revealed the potential of contentious classroom situations. In investigating peer groups, for example, Cobb indicated that "contentious relationships, in which the children's expectations for each other are in conflict, can be productive" (Cobb 1995, p. 27). Also for Kamii, situations of conflict (cognitive as well as social) play a significant role in learning (Kamii 1985, p. 27ff.). In Arsac et al. (1992, p. 13ff.), the teacher is asked not to intervene in conflict situations. As far as possible the students have to agree on one solution and prove the validation. The way of dealing with contentious situations in all these studies is seen as crucial for offering students possibilities for learning. But what exactly might be contentious in mathematical classroom interaction?

An epistemological view can help us to look in more detail at what might be mathematically contentious. One example, referring to the representation of numbers, is given in the introduction (Fig. 1). A second example, arising from a problem-solving context, is about the topic "Who can reach the number 50?" (Steinbring 1995). Rows of boxes are given: for example, five connected boxes in a row, one box above these boxes and an extra single box to the right of the row. A *starting number* is written into the first box and a so-called *addition number* into the box above. The sum of the starting and addition numbers goes into the second box, the sum of the number in the second box and the addition number into the third, and so on, until all five boxes contain numbers. Finally, the sum of the numbers in these five boxes, the *target number*, is written into the single box at the end of the row (Figs. 2, 3). One of many possible tasks is: "Try to reach the target number 50 with several different numbers!" (cf. Steinbring 1995, p. 226).

From an initial glance it is hard to imagine that anything within this solution might be contentious. The numbers in the boxes can be confirmed by calculation. Contentious issues might emerge, however, when the teacher asks the students to analyze their different solutions. One possible finding might be the divisibility of the target number by five. This could be explained using the five boxes—the target number divided by five equals the number in the box in the middle. Another finding could be based on the effects of using the starting and addition number illustrated in Fig. 4 in an algebraic way. Students in elementary school often count how many times the starting and the addition number are used without using variables explicitly.

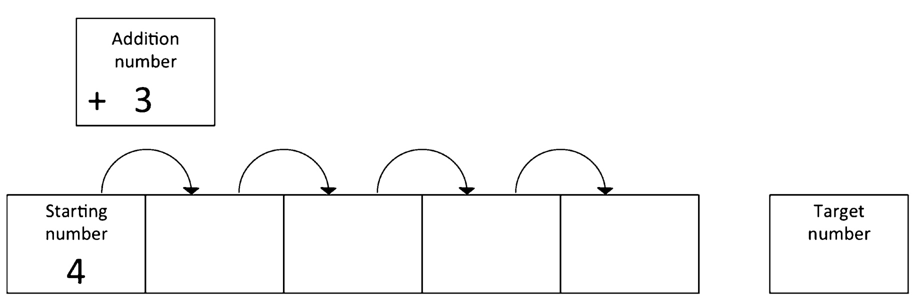

Fig. 2 Addition row with *empty boxes*

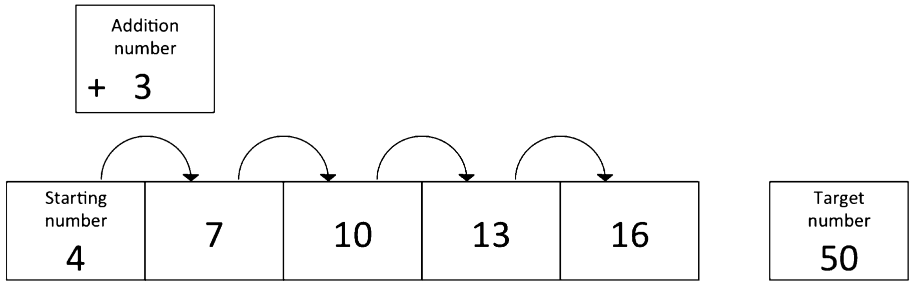

Fig. 3 Addition row with *filled boxes*

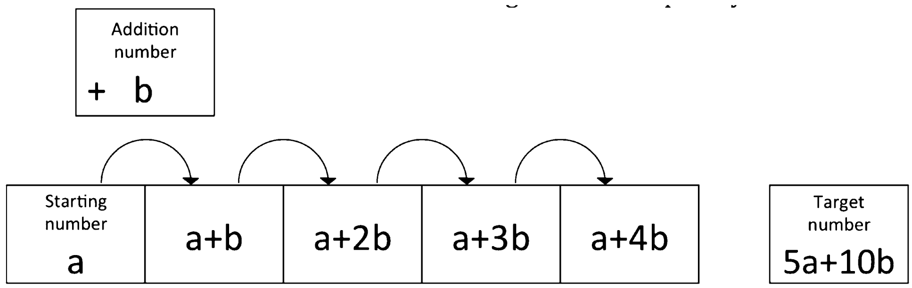

Fig. 4 Algebraic filling of the addition row

Both explanations are possible and might coexist. From our point of view, two such interpretations (or maybe only one of them) relate to epistemological issues that might be contentious if at least one interpretation is put into question, contradicted, or challenged interactively (see introduction).

3 Interaction about mathematics

> When a teacher introduces a new concept to a class, it is common to begin with introductory problems to help students see a need for the new concept. It is also common to construct the new concept on the students' previous learning and to lead them toward a new theory. (Shimada 1997, p. 3)

Shimada refers to frequent, predetermined interactional regularities, which often dominate classroom interaction. Voigt (e.g., 1995) elaborates some of the underlying thematic patterns of interaction. These patterns of interaction minimize the risk of a collapse or disorganization of the interactive processes, which might arise due to different background understandings in the classroom and the negotiation process (cf. Voigt 1995, p. 178). Investigations in different classroom cultures have identified several patterns of interaction. For example, Bauersfeld (e.g., 1980) describes an interactive process in which the teacher asks more and more questions in small steps, leading towards the one and only correct answer he has in mind, until the correct solution is given. Bauersfeld calls this thematic pattern the *funnel pattern*.

Wood (e.g., 1994) contrasts the funnel pattern with the *focusing pattern* of interaction. This focusing interaction pattern "is characterized by an exchange in which the teacher's guiding questions act to focus the joint action. Initially, this pattern appears to be similar to the funnel pattern as its intent is to provide opportunities for learning through joint activity. However, the pattern that emerges is quite different as the teacher's intent in questioning is to focus the attention of the student to the critical aspect of a problem—to pose a question which serves to turn the discussion back to the student leaving him/her with the responsibility for resolving the situation" (Wood 1994, p. 155). Both of these patterns start with an open question, but the way of dealing with the interactive situation that follows is different. Within the funnel pattern the teacher usually does not take time to discuss different kinds of interpretations. He/she is focused on revealing the answer he/she wants to hear. In the focusing pattern, the students are asked for and know they have to give reasons, explanations, and justifications for their interpretations. That implies that the teacher has to be open towards the students' personal interpretations and attempts at reasoning. In such an interaction, the teacher does not intervene in a controlling or regulating way and the children will not immediately take on the teacher's view. With the students explaining and participating actively in the interaction, such episodes might serve to elaborate the epistemology of mathematical knowledge in the social interaction context (see Steinbring 2005, p. 12ff.).

Also, in the data of the presented study the active role of the students and the special role of the teacher are considered important in the interactions. Often an emerging contention breaks down quickly by shifting to the next topic or through the authority of the teacher. The claim of this paper is that maintaining mathematical contentions in interactions entails several difficulties for both teachers and students. Hence, in this study the interactions will be investigated with a focus on the maintenance of contention and the related kinds of closure by analyzing episodes from small-group discussions. Using a qualitative, interpretative research perspective the following questions are considered:

- To what extent is a contentious interaction situation realized?
- On which epistemological issues is the contention based?
- Which conditions lead to a focus on the contention?
- What makes the interactive maintenance of contention so difficult and what kind of closure emerges?

4 Method and participants

The data presented derives from a project called "**Pro**bing and Evaluating **F**ocusing **I**nteraction Strategies in Elementary Mathematics **T**eaching (ProFIT)," which aims at elaborating the teacher's role in small-group discussions with emerging contentions in third and fourth grade (students from 8 to 10 years of age). In contrast to investigations with peer groups, the teacher takes part actively in the interaction process. The students were introduced to the mathematical content by the teacher him/herself in between one and three lessons on each of the four topics. In the previous sections, two tasks have already been presented. The first one is to find different representations for a number written in characters on the board, the other one is "Who can reach the number 50?" Additionally, the topic "number line" (discussed in Gellert and Steinbring 2014) and the topic "addition tables" are chosen. After attending the class, four students and a teacher performed the small-group discussion in another room.

The following two episodes differ in two particular ways. In the first one the contention is initiated by the teacher and the interaction continues without reaching a consensus. In the second one the contention arises between two students and they interactively create a taken-to-be-shared argument.

For the analysis two main foci are set. In the first one the epistemological dimension concerning the mathematical contention is worked out. The *epistemological triangle* developed by Steinbring (e.g., 2005; Fig. 5) is used as the conceptual scheme. For characterizing mathematical signs and symbols, Steinbring connects a new *sign/symbol* to an

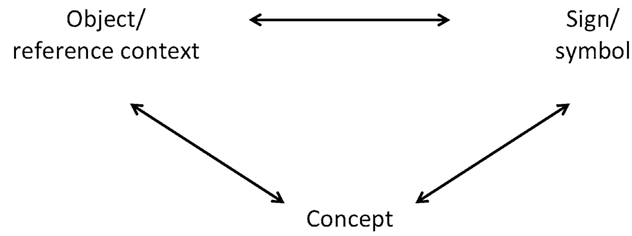

Fig. 5 The epistemological triangle

object/reference context by referring to the semiotic function that the mathematical sign is related to "something else" (cf. Steinbring 2005, p. 21).

Originally, the epistemological triangle was applied to reconstruct very precisely each verbal and gestural act of communication. "The reciprocal actions between the 'points' of the triangle and the necessary structures for the signs/symbols (for example mathematical operations) and the object/reference context (for example diagrams, functional structures, etc.) must be actively produced by the student (in the interaction with others and with the teacher). This active production is always subject to the epistemological constraints" (Steinbring 2005, p. 23).

In this paper, the epistemological triangle serves to model the epistemological nature of the contentious statements. The use of the triangle differs from Steinbring's use insofar as it is not constructed by analyzing the discussion step by step. Instead the students' constructions of meaning are analyzed in a *summarizing way*. This variation is in order to provide further depth and clarity concerning the epistemological conditions of the contentious mathematical features (What comes into question?).

The second focus for analysis is the triad of "initiating," "maintaining," and "closing" discussions about controversial mathematical issues. This triad is used for structuring the transcripts in units of meaning related to the three entities initiation, maintenance, and closure of a theme-centered interaction. In the *initiation* one or more perspectives on a mathematical problem are developed and indicated as in need of clarification. *Maintaining* the interaction about the mathematical issue in question means to have a closer look at the emerging perspectives. For the analysis the researcher's focus lies on how the participants deal with the contentious points—more precisely, which references are used for clarification. Possible references could be the displayed material, previously expressed perspectives, or verbalized background knowledge. An analysis of the maintenance of the interaction implies exploring the nature of the relation between the verbalized perspective and the reference used in the explanation, as well as the consequences on the kind of closure that follows.

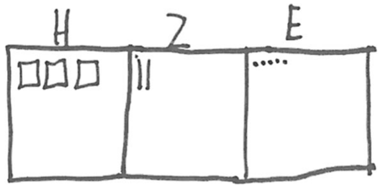

Fig. 6 Student's representation of the number 325

In this paper, the *maintenance* is not understood as an interaction with pre-determined rules of procedure or a code of practice which govern how to speak about a mathematical topic. The underlying dynamics might instead lead to new perspectives (new initiation), possibly without taking the previous perspective into account any more (*closure*). An obstacle to maintaining this theme-centered interaction becomes obvious.

The analyses of the following two episodes are structured in a similar way. In the beginning the contentious point is displayed. Then the transcript of the episode as a whole is presented. In the subsequent interpretative parts the contention is epistemologically analyzed, the triad "Initiation—Maintenance—Closing" is analytically emphasized, and the contentious points are related to each other.

5 Episode 1: twenty or twenty tens?

The first episode originates from a small-group discussion after the first lesson within the research project ProFIT, in which the students had to develop different representations of the number "three hundred twenty-five" (written in characters on the board).

5.1 Choice of the scene—a priori epistemological description of a contentious point

During the interaction, the question arises as to whether the bars in the center box of Fig. 6 mean *twenty* or *twenty tens*. This representation, developed by the students in the class before, causes a controversial discussion.

Some alternative meanings, which might be interpreted from the bars in the representation, are mentioned in the introduction. The potential for contention lies on the one hand in an interpretation of the bars used conventionally as an iconic representation of the Dienes material, "collected" in boxes labeled with "H," "Z," and "E." "H" means hundreds (German: Hunderter), "Z" means tens (German: Zehner), and "E" means ones (German: Einer). The interpretation of the representation can be described as *a classified accumulation of objects in three boxes*: 3 boards of hundreds, 2 bars of tens, and 5 cubes of ones.

On the other hand, the labeling might depict place values in a place value table. In this case, the decimal relation between the different place values is important: the quantity of symbols in one column of the place value table shows how many of the according place values have to be chosen. This interpretation of the representation can be described as a *relational arrangement of units in three columns of the place value table*: three symbols in the hundreds column, two symbols in the tens column and five symbols in the ones column. Obviously, in this second interpretation there might be the conflict that the "additional" characteristics of the symbols have to be abandoned, that is, the iconic meaning connected with the symbols: hundreds board, tens bar, and ones cube.

5.2 The episode

Before the episode starts the children explain why Fig. 6 can only represent the number 325 and no other number by only using the symbols square, bar, and dot.

1a	Teacher	But, as I said, this is not obvious to me. So that […] without the place value table, that's clear. […] I know one square we said equals a hundred, doesn't it? […] A bar you said equals (Student: ten) and a dot equals one. […] (*Teacher writes on the paper*) We agreed on this in the second or third grade. Now, if I see that, ok (*covering the labeling of the boxes in Fig. 6*).
1b	Teacher	But now (*uncovering the labeling of the boxes*) I also have my place value table.
		[…]
2	Kevin	Yes, but when. […] One bar is a ten and then when we, if there is a tens above it, then we know that bars need to be put in there, if there are tens. And therefore a bar is one ten and so on.
3	Teacher	But these are twenty, right? (*pointing on the two bars*).
4	Kevin	Yes, these are twenty.
5	Teacher	But twenty tens. And twenty tens are
6	Kevin	No. Two bars are two, two bars are two bars, which are in the tens' place. Are twenty.
7	Teacher	But these are twenty tens. (*first pointing at the two bars then at the Z for the tens*) twenty times ten don't equal two hundred?
8	Kevin	This is not about multiplication.

5.3 Epistemological analysis

By examining the opposing points of view in more detail, both interpretations of the representation will be analyzed using the epistemological triangle in a summarizing way. The *sign/symbol* to be explained in this episode is the representation in Fig. 6, which was agreed by the class as representing 325. This is the point that the teacher questions. She expands the sign/symbol by the claim that twenty times ten equals two hundred. In her case, the convention used in earlier school grades (squares, bars, and dots as hundreds, tens, and ones) and the reading of the caption and the frames as a place value table constitute the *object/reference context*. Through this interpretation, the contentious change in the value of the two bars emerges.

In her reasoning, the teacher refers to the *place value concept* of decimal numbers. Numbers (usually digits but here multiples of ten for each bar), located in a place value table, are multiplied with its place value. In this case both bars result the numerical value of twenty. Multiplied with the place value it equals the number 200 (Fig. 7).

However, for Kevin the displayed representation means 325 and “This is not about multiplication” (line 8). In order to justify his point of view, he refers to the Z and the box, in which “bars need to be put in” (line 2) as *object/reference context*.

Kevin does not use the place value *concept* of the decimal system in his interpretation, but the mathematical activity of putting the symbols in an order (Fig. 8). The caption above the framing boxes is used by Kevin as a visual marker, through which an order of the symbols square, bar, and dot in the boxes according to their values becomes possible. To justify his point of view he does not need an arithmetical operation. Therefore, he considers the use of multiplication as inappropriate.

5.4 Analysis of the triad “Initiation—Maintenance—Closing”

By entering in a discussion about a contentious mathematical feature in interaction, the transcript can be divided into the triad: initiation, maintenance, and closing.

5.4.1 Initiation

By recapitulating the meaning of the square, bar, and dot in the lesson the teacher refers to a conventional interpretation known and accepted by all the students (line 1a). The teacher’s statement in line 1b is different: here, the teacher comments on the representation by provokingly interpreting the caption and frame boxes as a place value table, initiated by the restricting conjunction “but”—a potential contention arises and is *initiated* by the teacher.

5.4.2 Maintenance

Kevin does not contradict the teacher; instead he takes an opposite position in line 2. He introduces his argument with the words “yes, but when,” and before he verbalizes his counter-position, he confirms the teacher’s statement that a bar is a tens. So, in this case, there is an agreement between the interlocutors. In the next sentence he declares that the meaning of the two bars is twenty, because of the caption above the framing box “if there is a tens above it, then we know that bars need to be put in there”

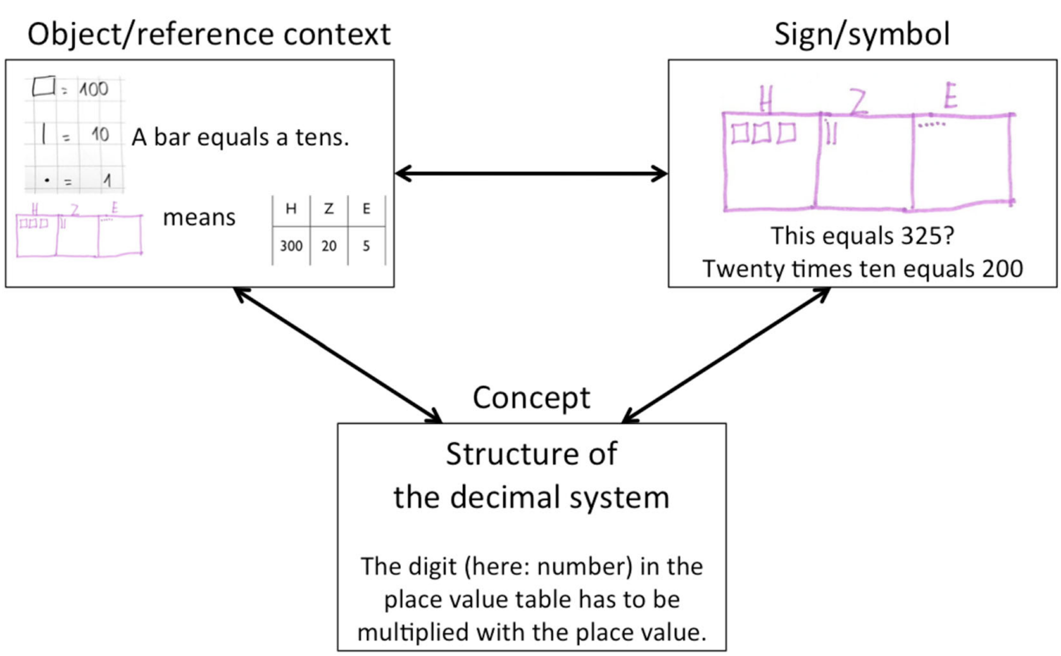

Fig. 7 Summarizing epistemological triangle of the teacher’s view

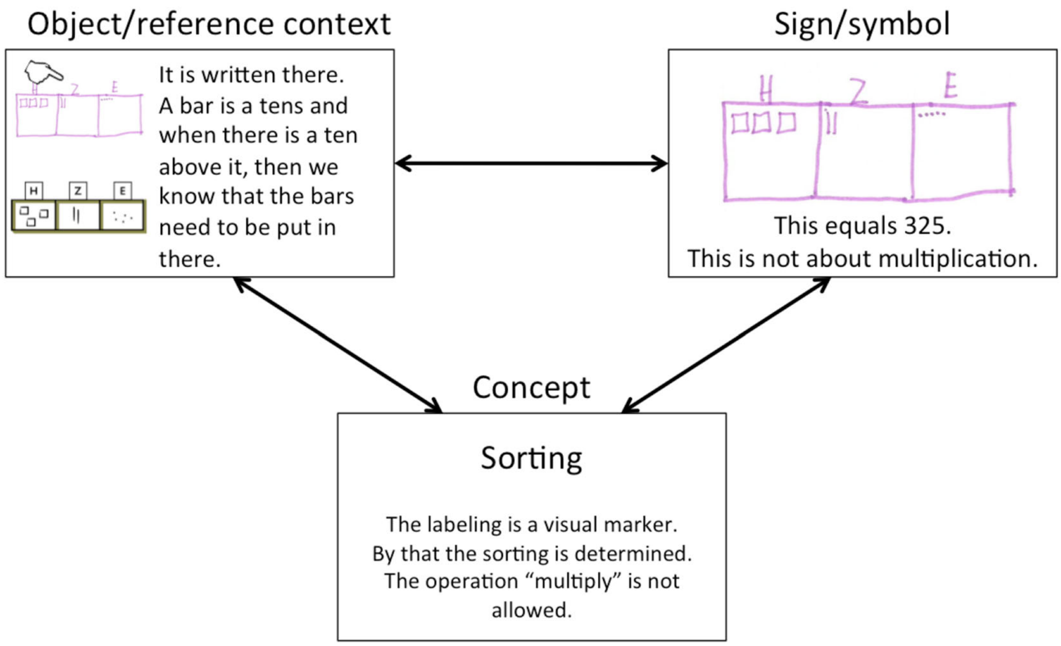

Fig. 8 Summarizing epistemological triangle of Kevin's view

(cf. Fig. 8). With this reasoning the contention is realized in the interaction.

After an agreement about the meaning of the two bars as twenty, the dissent points are sharpened in line 5 and line 6. The teacher claims that the two bars have to be interpreted as twenty tens because they are placed in the tens column of the place value table. Kevin negates this at once. In this moment he takes another view on the material: "[…] two bars are two bars, which are in the tens' place. Are twenty" (line 6). The contention is *maintained* and provokes an explicit contradiction by Kevin ("no," line 6). His justification refers exclusively to the two bars in a particular place. The teacher maintains her interpretation of the bars as twenty tens (cf. Fig. 7) and expands her argument bringing the multiplicative handling with the place value table into account (line 7). In this case the dissent views are *maintained*, even if this time the teacher's argumentation has changed mathematically.

5.4.3 Closing

The contention, whether both bars could be interpreted as twenty or as twenty tens, breaks down without resolution. The new emerging contentious point, whether the center column can be interpreted as twenty times ten, briefly becomes the main point of interaction. Kevin's fast reaction, "This is not about multiplication" (line 8) leads to the *closing*: Kevin does not accept the teacher's argument by means of the multiplicative character of the place value table, the teacher does not pick up Kevin's argument that the description indicates what is symbolized by the two bars, and nobody tries to hold up the debate until a consensus is reached.

6 Episode 2: fifty-five or fifty-six?

The second episode originates from a small-group discussion after the first lesson about the task "Who can reach the number 50?"

6.1 Choice of the scene—a priori epistemological description of a contentious point

In the sequence below the contentious point emerges between two students and is about two number rows. The students were confronted with the example presented in Fig. 9 developed in class before, and were asked to explore the distinctive features and the relationships between the numbers.

Kilian suggests swapping over the starting number and the addition number. After a short hesitation, all participants agree that the target number has to be bigger than 50. Lisa and Kilian fill a new row with the changed numbers and speculate about the target number being 50 or 56. Finally they agree on 55 as the correct result (Fig. 10).

The contentious point concerns the differences between the diverging boxes in the rows. In their discussion, the students concentrate on the difference of five between the target boxes and where they can find this difference in the other five boxes of the row. Therefore, three questions

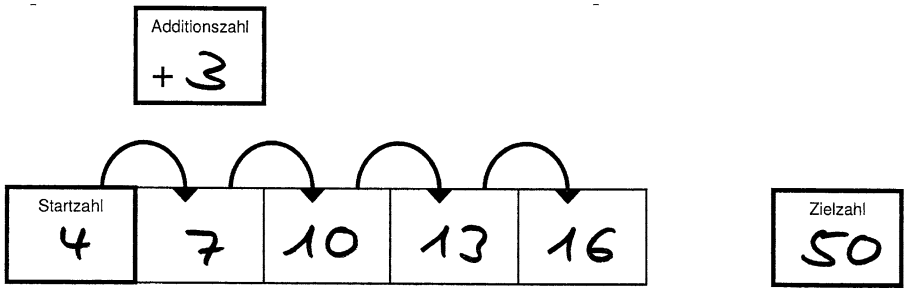

Fig. 9 Addition row created in class

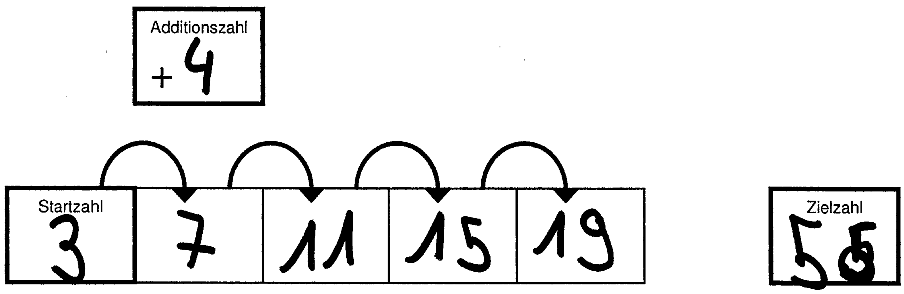

Fig. 10 Newly created addition row during the small-group discussion

are generated: Does the difference of five exist because there are five boxes in the row? Or where else does the difference of five come from? Or is 56 the correct target number with a difference of six between the rows?

6.2 The episode

1	Falk	Surely because here there was the four (*points at the addition number of Fig.* 9) and there were five boxes more (*points at the starting number of Fig.* 9 *and moves his finger to the right across the other boxes*), so five more there *(points at the target number*).
		[…]
2	Teacher	Okay. Now we have five as a difference here […]. You ventured a guess why it could be five more (*points at the target numbers of Figs.* 9, 10). Could you repeat your guess once more?
3a	Falk	[…] Four (*points at the addition number of Fig.* 10) is bigger than three (*points in the direction of Fig.* 9), that's why there are five boxes with one more (*points firstly at the starting-number and then he moves his finger from the left to the right along the boxes of Fig.* 10), so five boxes is o[ne] bigger than the number (*points at Fig.* 9).
3b	Falk	But, well, only this box (*points firstly at the second box of Fig.* 10 *and then at the second box of Fig.* 9) is the same. So three plus four and four plus three equals seven. *(3 s)* But there (*points in direction of Fig.* 10) are technically only four boxes, where there is plus one any more, […] There has to be two or so somewhere else. […] For example there is a nineteen and there is a seventeen (*points at both last boxes*).
4	Teacher	Mmh, where do they come from? The plus one?
5	Falk	There […] is the nineteen, there is the sixteen with difference three (*points at the last box of Fig.* 10 *and then in the direction of the last box of Fig.* 9). There is a difference of two (*points at both penultimate boxes*) and there is a difference of one (*points at both boxes in the center*). So five, because three plus two, er no, difference six? (2 s)
6	Kilian	There is no […] six *(points at the ones digit of the target number of Fig.* 10*)*, there is a five.
7	Teacher	[…] Say it again. You looked at the difference here (*points at both fifth boxes*) […]
8	Falk	Well nineteen (*points at the fifth box of Fig.* 10) and six (*points in direction of Fig.* 9) (Teacher: sixteen), nineteen is three more than sixteen, that's right. And fifteen is two more than thirteen. That is five. Then actually there should be (Kilian: I know why), […] fifty-six
		#Maybe we miscalculated here.
9	Kilian	#I know, I know, I know why.

In the following there is a dispute about who is allowed to speak—Falk or Kilian. The teacher uses her authority by asking Falk to continue. Falk repeats the differences between the last two boxes and the teacher notes arrows and summands. Falk continues:

10	Falk	And together it already makes plus five, then there should be 56, because here (*points at the box in the center of Fig.* 9 *and then in the direction of Fig.* 10) there is a plus one, too (*Teacher writes* +*1 between the center boxes and draws the arrow*, *Fig.* 11).
11	Kilian	Now I have to intervene!

12	Falk	Kilian!
13	Kilian	Look at those two (*points at both first boxes*), there is one less! […] one has to be subtracted from those (*points at the summands written down by the teacher*)
14	Falk	But we weren't talking about that yet […].
15	Kilian	Here you have plus zero (*points at both second boxes*) and here minus one already! (*points at the first boxes*).
16	Teacher	Minus one (*writes* −1 *and the arrow between both first boxes*).
17	Kilian	Because obviously four is […] bigger than three!
18	Falk	I know, Kilian! But we weren't talking about that yet!
19	Kilian	Yes. […] if we add them now those three would be six *(points from the left to the right at the last three summands of Fig.* 12*)*, but here (*points at the* −1) there is already one, this one minus. […] Here, we have to subtract one to get the result. […] I didn't think about that before. That's the reason. […]
20	Teacher	Well, now, did we or didn't we miscalculate then?
21	Fa, Li, Ki	No.
22	Kilian	No. […] We only […] thought, the sum has to be 56 (*points at the target number of Fig.* 10), but we didn't look at those two numbers (*points at both first and both second boxes*), and because this is one bigger (*points at both first boxes*), you have to subtract one. […] Then there are only five left […].

6.3 Epistemological analysis

In this episode there are three main interpretations, which will be summarized by the use of the epistemological triangle. In the first lines, Falk presents different interpretations to justify the difference between the target numbers. Specifically, this difference of five between the target numbers is the first *sign/symbol* in question.

In the following representation using the epistemological triangle a possible interpretation of Falk's statements in lines 1–3a is noted: in his argumentation he refers to the addition number which has increased by one and to the number of boxes in the row as *object/reference context*. In his comment (line 3a) "that's why there are five boxes with one more, so five boxes is o[ne] bigger than the number," he justifies the difference of five in the target numbers by mentally adding one to each of the numbers in the five boxes. By relating to the addition numbers "Four is bigger than three (*points at the addition number*)" he possibly transferred the difference of one between the addition numbers to each of the five boxes, without observing the numbers in detail.

In his statements he manifests indirectly the known *concept of consistency of the sum/difference*, to which he creates specific conditions: if there exists a difference of five between the target numbers, the difference of five will have to be subdivided in ones because there are five boxes in a row. Then, one has to be added to each box, so that each box has a difference of one to the corresponding box in the other row (Fig. 13).

In line 3b Falk notices the equally filled boxes in the second position (both are 7). He changes his concept by saying that, if there is one box with a difference of zero, there has to be another box with a difference of two. Once again his example is not visible in the material. Instead of referring to the given material, he constructs two new rows mentally ("For example there is nineteen and there a seventeen"). His pointing gesture indicates that he notices the difference of three between the numbers in the boxes. In line 5 he verbalizes this difference and adds the differences of two and one in the penultimate box and the box in the center. He sums up the three differences to a difference of six.

In line 6, Kilian remarks that there is a difference of five between the two target numbers. At this moment it is not clear whether he accepts Falk's strategy. Due to the use of the difference between the target numbers, it can be presumed that he also accepts the concept of the constancy of the sum/difference. In line 7 one cannot find any indication as to whether the teacher ignores Kilian's comment on purpose or only did not catch it. Nevertheless, the effect is that the focus of the interaction rests on Falk's interpretation for the moment. Falk verbalizes his point of view in more detail. The *sign/symbol*, which needs to be clarified, is the difference of six. He elaborates this difference by means of the difference between the numbers of the last three boxes as *object/reference context*. Subsequently he creates a further *sign/symbol*: the target number 56. The mentally changed target number shifts the first *sign/symbol* to a *reference context*. Through this he seems to use the concept of the constancy of sum/difference once again. With the help of the material presented and the added summands by the teacher (Fig. 11) he justifies the difference of six (line 8; Fig. 14).

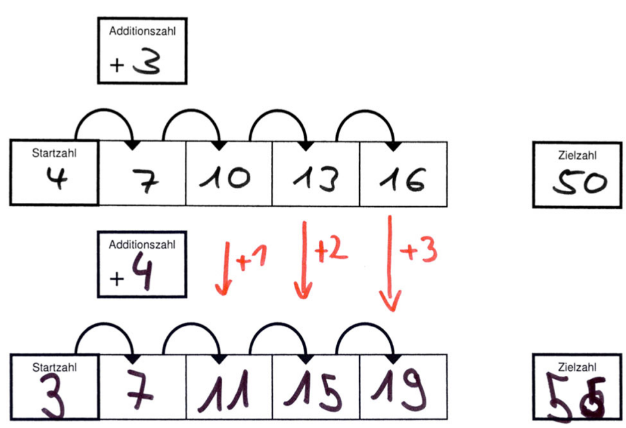

Fig. 11 Completed visualization of differences

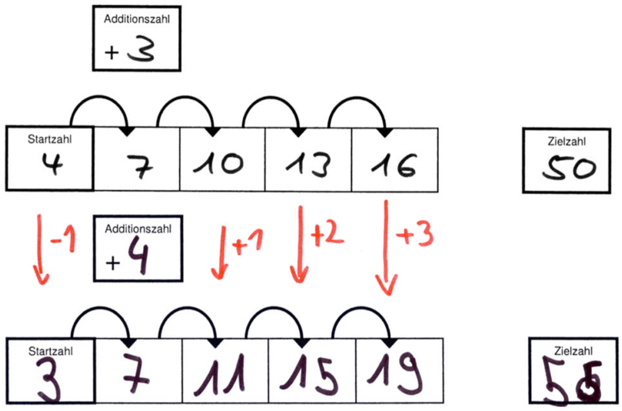

Fig. 12 Summarizing epistemological triangle of Falk's first view

The following intervention by Kilian (line 11ff.) leads to an interactive situation in which Kilian can present, explain, and justify his interpretation. With another summarizing epistemological triangle his construction of meaning is analyzed. The *sign/symbol* is the difference between the numbers in the two rows and between the two target numbers. In addition to Falk's interpretation of the last three boxes he also takes the two first boxes into consideration. The difference of the "one minus" seems to constitute his *object/reference* context. He then deduces the difference of five by summing up all differences between the boxes—also considering "this one minus" (Fig. 12). By this, the coherence between the differences of numbers in the boxes and the difference between the target numbers is secured (Fig. 15).

6.4 Analysis of the triad "Initiation—Maintenance—Closing"

6.4.1 Initiation

In this episode Falk offers an explanation for the difference of five and draws the first conclusions. Noticing the seven in each second box, the first discrepancy arises, which could possibly lead to a contention. Falk is astonished about the lack of the difference in the second box in contrast to his interpretation analyzed in the first epistemological triangle (Fig. 13). But instead of maintaining the contrast, Falk changes the conditions for realizing the difference of five by changing the specifications for the regularities of the consistency of difference. As a result the contentious point is resolved for Falk if there is a difference of two in two other boxes. In his chosen example with 19 and 17 a new point of contention comes up: while talking he points to the last boxes and notices that in the first case there is a 16, not a 17, and therefore a difference of three. The difference of six—calculated by means of the last three boxes and visualized by the teacher (Fig. 11)—is in contrast to the difference of five between the target numbers (Fig. 14). Kilian indicates the discrepancy at once: "There is no six, there is a five" (line 6).

While Falk can resolve the first two potential contentions by reinterpreting and changing meanings, Kilian joins the interaction and expresses his disagreement. The contention is realized, and is *initiated* by the students.

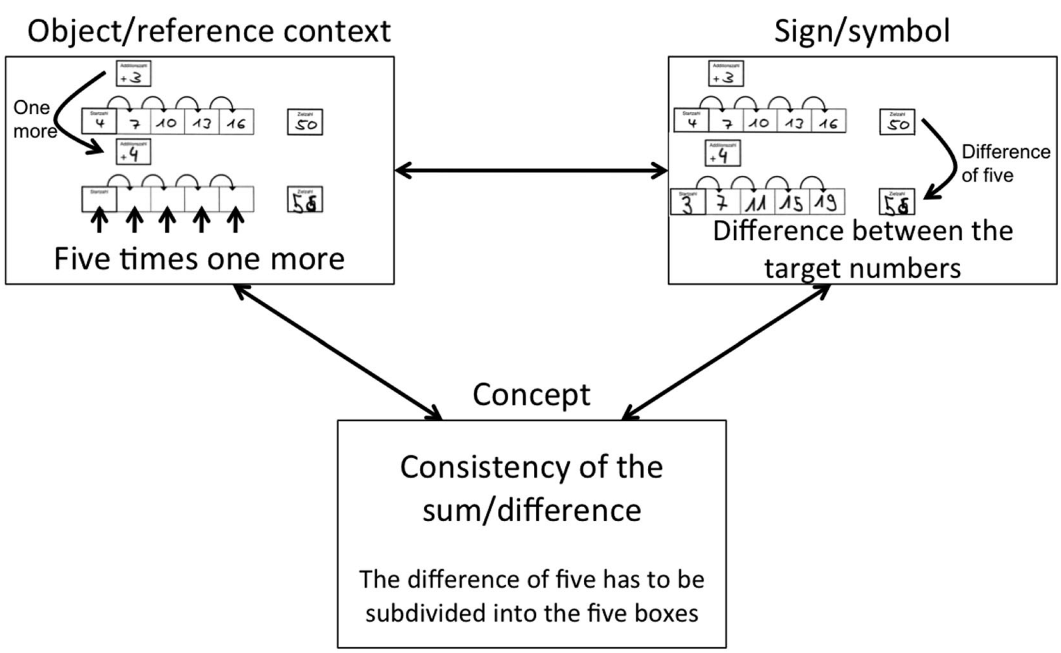

Fig. 13 Visualization of the last three differences between the two number rows

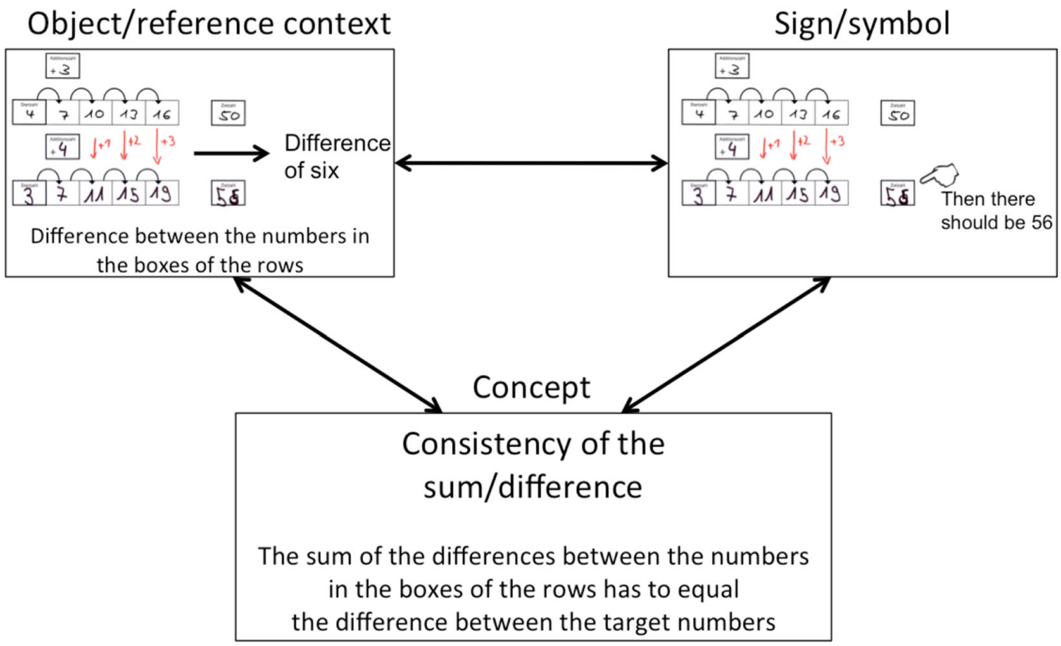

Fig. 14 Summarizing epistemological triangle of Falk's second view

6.4.2 Maintenance

There is still a dissent in Falk's interpretation. While Kilian supports the target number 55 in line 6, Falk puts this target number into question: "then actually there should be fifty-six" (line 8). The difference of six between the boxes might also lead to a difference of six between the target numbers. Several times, Kilian makes efforts to obtain the right to speak, possibly in order to resolve the contention: "I know why" (line 9).

In line 13, Kilian's argumentation leads in a different direction. He does not begin with the sum of the differences between the last three boxes and conclude that the target number has to be 56 like Falk. Instead Kilian takes the target number 55 as starting point and points out how the difference of five can be subdivided according to the numbers in the boxes (lines 13, 15, 17, 19).

In line 18, Falk agrees with Kilian's statement semantically ("I know"), but he has doubts about the legitimacy of Kilian's data. With his comment it becomes contentious as to whether Kilian is allowed to use the whole rows including the two first boxes in the moment he takes over the right to speak. But a restriction like that does not exist during the whole conversation. The discussion is *maintained* with changing contentious points.

6.4.3 Closing

The teacher's question concerning a miscalculation introduces the closing of the contention (line 20). All participants agree on 55 as the correct target number. The difference of five is not only found between the target numbers, but also in the differences between the numbers in the boxes of the row when summed up. As a conclusion, Kilian summarizes his justification of the difference of five once again (line 22). Since nobody dissents, the conclusion can be seen as jointly accepted and taken as shared. The discussion *closes* with a consensus.

7 Discussion

The two episodes show an interesting contrast between two revealed potential contentions and their consequences for the ongoing interaction. In the first episode, the teacher intensely tries to provoke and to maintain a contention by consciously interpreting a number representation differently from the student's intended interpretation when drawing a representation of the number 325 in class. In contrast, in the second episode the contentious point arises between two students in interaction and finally leads to a consensus between the interlocutors. In both episodes the outcome of the contention was deliberated and the teacher tried to maintain the discussion about the contentious point, but the interactive situations evolved completely differently. These observations give rise to the perhaps most persistent and important questions that emerge from this study: What are the emerging differences in realizing a contention in interaction? What is the teacher's role? What role does this kind of interaction play in the learning of mathematics?

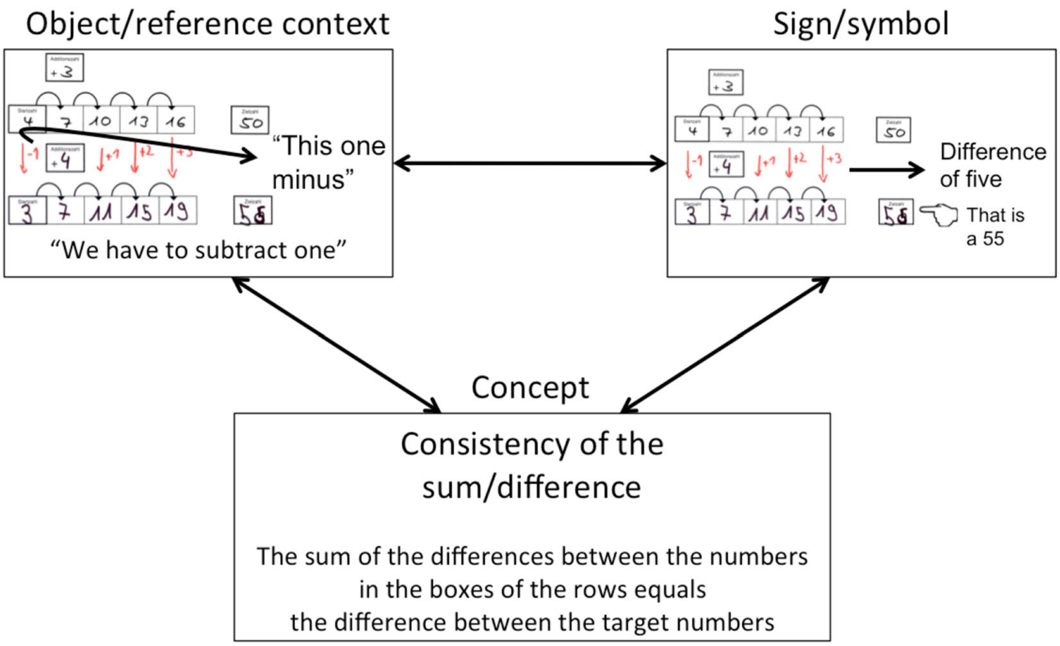

Fig. 15 Summarizing epistemological triangle of Kilian's view

7.1 The role of the mathematical task in interaction

First of all, in the two episodes the tasks that lead to the contention are very different. In the first case, the construction of meaning of ambiguous mathematical representations is paramount. Underlying (mathematical and in classroom agreed) *conventions* as well as *deductions* emerging from the construction of meaning predominate discussions such as the one in the first episode. In this episode the disagreement is based on different presumptions and Kevin interprets the elements of the representation differently from the teacher. The contention analyzed shows confusion about the *meaning of a sign*: the border around the two bars and its effect on the meaning of the bars. Also if it would have been a place value table, the use of the table in combination with the twenty in the tens place is contrarily discussed in mathematics educational research (Meyerhöfer 2013; Richter et al. 2012).

In the second episode the task deals with a problem solution. A mathematical result can be verified by controlling the used data and/or recalculating. So, in this case, the contention is based on *differences in the used methods*. Disagreements about certain observations can be verified or refuted by *controlling the used data* or by *calculating*. In the episode, Falk refutes his first claim (Fig. 13) by noticing that in the second box there is no difference of one. His first construction of meaning is not possible because everybody can see that in both second boxes there is a seven (e.g., Fig. 11). In Falk's second interpretation (Fig. 14) everybody can calculate the difference of six between the last three boxes, which is in contrast to the difference of five between the target numbers. But even Falk himself has no counter-argument against Kilian's objection that they have to take the differences between all five boxes, including the first box, into account. The data that Falk refers to is incomplete, as discovered by Kilian. Hence, Kilian's interpretation (Fig. 15) leads to a consensus because all boxes are taken into account *and* computing both differences leads to the same difference of five.

7.2 The role of the students in interaction

Confronting elementary school students with interpretational tasks is quite challenging for them. In general they have to manage the coordination of different perspectives in interaction, which is a difficult social competence. According to Selman (2003), the students have to challenge two reciprocal core competencies, which underlie a steady development, especially in the early age of elementary school students:

> The first is the capacity to be aware of one's own point of view, to know where it comes from, and to be able to express it or keep it in private. The second is the capacity to take, and to keep in mind, the point of view of another person, group, or even society as a whole. Developing each of these competencies is easier said than done. Putting them to work together is even more challenging. (Selman 2003, p. 7)

In his study Selman works out that the social perspectives children take between the ages of six to sixteen years are confronted with "practice-based dilemmas," and he generates developmental levels for analysis. The students participating in our study are 8–10 years old—the “upper elementary” level, referring to Selman (2003, p. 21). At this age they might be able to understand the view of other pupils on their own (subjective) perspective (second-person and subjective level) as well as their probably distinct perspectives (first-person and subjective level). Despite all the criticism dealing with developmental levels, Selman focuses on the difficulty of taking the perspective of other people. It is exactly this shift of perspective that might be the link between the presented epistemological analyses and causes the mathematical perspective. In normal lessons of mathematics, students are confronted with ambiguous mathematical signs and symbols as well as with very different mathematical interpretations of the other students and teachers. So they are regularly expected to understand different perspectives within a classroom culture in which the students are invited to present their own viewpoints (such as in the focusing pattern).

In the first episode, Kevin verbalizes his own perspective and realizes that the teacher has another one. But he possibly does not understand the teacher’s interpretation of the representation. This assertion provides one possible explanation for Kevin’s resistance towards the teacher’s interpretation throughout the first episode. Another interpretation might be that he negates the data of referring to the place value table or to the multiplication. This interpretation is based on a specific understanding of which conventions and deductions might serve in this case (see above). On the other hand, the teacher does not ask what Kevin is talking about. There is no evidence for a deeper interest in Kevin’s interpretation. Instead, during the whole situation the teacher seems to try to convince Kevin of her point of view.

In the second episode Kilian questions Falk’s explanation for the difference of five and later six between the number rows. The analyses show that he understands Falk’s perspective, adopts it, and gives an expanded explanation taking all influencing factors into account. In the interaction, Falk also shows that he agrees with Kilian’s interpretation concerning the mathematical content (even if not the timing of Kilian’s objection).

7.3 The role of the teacher in interaction

The teacher has a difficult role. Previously encouraged to focus on contentious mathematical issues and developing the students’ ability to engage in mathematical argumentation, the dynamics of the interactive reality often reveal a different procedure. Analogous to the experiment in Arsac et al. (1992), the observed teachers’ behavior did not comply with the planned and discussed scenario. In the first episode, the teacher’s provocation results in a rigorous way in making the teacher’s point of view accessible for the students. Despite several attempts by the teacher, it remains unclear whether Kevin does not get what the teacher is saying or he gets it, but does not adopt it. Not even the shift of contention to the use of multiplication leads to an argumentation about the two perspectives. According to the maintenance, a modulation of the different points of view, from Kevin, being strong enough to defend his point of view, and the teacher, insisting on her point of view, seems impossible in this moment.

In the second episode, the emerging contention between two students gives the teacher the opportunity to concentrate on focusing on the different interpretations without herself being the creator of one of these interpretations. In this case the teacher restrains Kilian in his first attempts to interrupt and present his own explanation. Instead, Falk has to clarify his point of view in a more detailed way. But what would have happened if the teacher had not intervened and restrained Kilian? On many other occasions during the project, situations like this break down at once without clarifying the perspectives. Here, the teacher’s intervention leads to more detailed explanations from Falk with changing reference contexts and to a comprehensive explication by Kilian.

8 Final remarks

When investigating the nature of a teacher’s interventions when bringing contentious issues into the focus of interactions, the outcome of the two cases considered emerges very differently. The interaction is based on internal rules, which might end the theme-centered discussion at any time. What is the difficulty of maintaining points of contention in interaction and why does maintenance break down so often? This paper’s aim is to point out that the maintenance of mathematical contentions in interaction entails several difficulties for teachers and students, which the examples of the two episodes show.

In the first episode, it is difficult to maintain the contention because the teacher wants to open the student’s view to a different perspective, which the student negates. Without considering Kevin’s reasons for not accepting the teacher’s objections, a coordination of the dissent or a consensus becomes difficult. If the teacher does not want to continue to constitute a transmitting situation, this can lead to the funnel pattern. The teacher’s focusing strategy is homing in on her own point of view, but she does not give any attention to the meaning constructed by Kevin. That is why the interactive situation has to break down at this point. Accordingly, there seems to be a relation between the interactive situation during the maintenance and the kind of closure revealed: if an argumentative discussion cannot be constituted during the maintenance, leading to a better clarification of previously expressed perspectives, then it is difficult to reach a consensus or a more conscious view of the differences between the emerging perspectives.

In contrast to the first episode, in the second a consensus becomes more likely. The constituted perspectives are explained, justified, and also interactively linked to each other, highlighting the differences (especially by taking the first box into account). Here, the teacher first prevents Kilian from verbalizing his perspective until Falk has clarified his perspective. The interaction reveals a situation in which the students jointly try to find solutions for a problem in question: "If we add them now […] we have to subtract one to get the result. I didn't think about that before" (line 19). In this case the closure seems to be linked to the kind of maintenance as well: a deeper clarity regarding the verbalized perspectives might increase the prospect of success in reaching a consensus or getting a deeper insight in the differences between the perspectives. This again might lead to a deeper insight into mathematical ideas and practices: considering the given conditions and deducing perspectives in a reasoning sort of way are important mathematical activities in the learning process.

While in the second case the focusing is characterized by a repetitive opening of the interaction to the problem as a whole, the interaction in the first episode is restricted to elements of the representation (the bars and the Z). The processing logic in the two cases is quite different, though with neither resulting in a funnel pattern. In the first case, the process seems to be linear (analogous to the funnel pattern), whereas the second one is characterized by breaking up this linearity in favor of a circular rethinking of the contentious issue, and emerges as far more productive. Even though the two episodes give a wide scope for analysis, more investigations of contentions for the purpose of learning situations and their implementation in the mathematics classroom are necessary.

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