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ORIGINAL ARTICLE

Knowledge shifts in a probability classroom: a case study coordinating two methodologies

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Abstract Knowledge shifts are essential in the learning process in the mathematics classroom. Our goal in this study is to better understand the mechanisms of such knowledge shifts, and the roles of the individuals (students and teacher) in realizing them. To achieve this goal, we combined two approaches/methodologies that are usually carried out separately: the Abstraction in Context approach with the RBC+C model commonly used for the analysis of processes of constructing knowledge by individuals and small groups of students; and the Documenting Collective Activity approach with its methodology commonly used for establishing normative ways of reasoning in classrooms. This combination revealed that some students functioned as "knowledge agents," meaning that they were active in shifts of knowledge among individuals in a small group, or from one group to another, or from their group to the whole class or within the whole class. The analysis also showed that the teacher adopted the role of an orchestrator of the learning process and assumed responsibility for providing a learning environment that affords argumentation and interaction. This enables normative ways of reasoning to be established and enables students to be active and become knowledge agents.

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1 Introduction

Tracing students' knowledge construction and tracing shifts of the constructed knowledge in a classroom setting are challenges that still need to be achieved (Saxe et al. 2009). This paper is an attempt in that direction, using a sequence of two lessons of a probability course for eighth grade students as a paradigmatic example. We use Abstraction in Context—AiC (Hershkowitz et al. 2001)—to analyze the construction of knowledge by individuals and groups in the mathematics classroom; we use the documenting collective activity—DCA (Rasmussen and Stephan 2008; Stephan and Rasmussen 2002)—approach to analyze whole-class discussions. Thus, the present study combines the theoretical and methodological aspects of two lines of research that are usually carried out separately, namely investigating (a) the construction of knowledge by individual students working as a group in a classroom, and (b) the establishment of normative ways of reasoning. The combination of the two methodologies allows us to follow the evolution of ideas as they flow between individuals, small groups, and the whole class.

Our overall goal is to illuminate the role played by individuals and groups in the class as well as by the class as a whole and the teacher in the knowledge constructing process, and to learn more about shifts of knowledge between the different social settings in a mathematics classroom during the knowledge constructing process.

In order to achieve this goal, we undertook the following analyses:

1. Studying the mathematical knowledge development of individuals and groups with AiC,
2. Studying the mathematical knowledge development of the classroom community with DCA,
3. Initiating a study concerning the role of the teacher and the design of instruction in encouraging and creating a learning environment facilitating the above knowledge development,
4. Integrating (1), (2), and (3) into a coherent description of the learning process.

In doing so, we offer a way to adapt existing methodological tools in order to coordinate analyses of the individual, the group, and the collective in a mathematical classroom. As argued by Prediger et al. (2008), such combining and coordinating of different methodological and theoretical perspectives is a much-needed endeavor that can bring some degree of coherence to research studies in mathematics education. The proposed combined analytic approach is significant in that it offers a new means by which to document the evolution and constitution of mathematical ideas in the classroom and the processes by which these ideas move between individuals, small groups, and the whole class under the facilitation of the teacher.

2 Theoretical and methodological framework

The last two decades have brought an accumulation of research, theory, and methodology regarding knowledge construction in classrooms. The book edited by Cobb and Bauersfeld (1995) provided a basis for this development by exhibiting meaningful research concerning mathematics learning in classrooms via the psychological and the socio-cultural lenses in parallel. The book represents a collective effort to create a theoretical–methodological framework for classroom research in mathematics. The authors use a common corpus of data from a mathematics classroom to elaborate their different perspectives.

The issue of the meaning of mathematical knowledge construction in classrooms is also emphasized and intensively investigated and interpreted in Cobb and his colleagues' further work. A central goal of their work is to study the development of mathematical knowledge and reasoning in the classroom (a) as it is expressed in the emergence of the collective activity of the classroom community, and (b) in cognitive and socio-interactive processes of knowledge construction by individuals and small groups working in the classroom. In their efforts to analyze the collective learning of a mathematics classroom community, Cobb and colleagues (Cobb et al. 2001) focus on the evolution of mathematical practices. For this purpose, they coordinate "a social perspective on communal practices with a psychological perspective on individual students' diverse ways of reasoning as they participate in those practices" (p. 113). They also discuss the notion of taken-as-shared activities of students in the same classroom and explain their approach:

> We speak of normative activities being taken as shared rather than shared, to leave room for the diversity in individual students' ways of participating in these activities. The assertion that a particular activity is taken as shared makes no deterministic claims about the reasoning of the participating students, least of all that their reasoning is identical (p. 119).

These quotes consider practices as communal activities, carried by the reasoning of the different students participating in these practices, where these practices are taken as shared rather than shared. However, as Cobb notes elsewhere, this does not clarify what the meaning of practices taken as shared is, and leaves the notion of mathematical practices largely intuitive (Cobb et al. 2011); hence, it does to some extent neglect the individual students' actual reasoning.

In what follows, we describe the two theoretical–methodological frameworks that, separately, address the analysis of knowledge construction in small groups and the analysis of the related whole-class discussions. The coordination of these two frameworks using a single corpus of data is a focus of the present paper and will allow us to clarify how individual students' reasoning actually impacts their group and the classroom practices, and how individual students shift knowledge from the whole-class discussion into further group work.

2.1 Abstraction in context and the RBC+C model

Abstraction in Context (AiC) is a theoretical framework for investigating processes of constructing and consolidating abstract mathematical knowledge (Schwarz et al. 2009). Abstraction is defined as an activity of vertically reorganizing previous mathematical constructs within mathematics and by mathematical means, interweaving them into a single process of mathematical thinking so as to lead to a construct that is new to the learner. This definition uses the idea of "vertical mathematization" that also lies at the basis of work by Rasmussen and colleagues (e.g., Rasmussen and Stephan 2008; Rasmussen et al. 2005). According to the Realistic Mathematics Education (RME) framework, learners engage in vertical mathematization, that is, the reorganization of knowledge within mathematics itself, finding connections between elements and strategies and applying them for constructing new knowledge (Freudenthal 1991; Treffers and Goffree 1985).

AiC research is standing on the shoulders of others' research, such as that of Voigt (1995), Cobb et al. (2001), and many others and adds by simultaneously stressing four perspectives: a micro perspective, a continuity perspective, a theoretical perspective, and a methodological perspective.

According to AiC, the genesis of an abstraction passes through three stages (Hershkowitz et al. 2001): (1) the arising of the need for a new construct, (2) the emergence of the new construct, and (3) the consolidation of that construct. AiC includes a theoretical/methodological model, according to which the description and analysis of the emergence of a new construct and its consolidation relies on a limited number of epistemic actions: *recognizing*, *building-with*, and *constructing* (RBC).

These epistemic actions are often observable as they are expressed by learners verbally, graphically, or otherwise. Recognizing takes place when the learner recognizes a specific previous knowledge construct as relevant to the problem currently at hand. Building-with is an action comprising the combination of recognized constructs in order to achieve a localized goal, such as the actualization of a strategy or the solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. Constructing consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed by the learner.

Typically, recognizing actions are nested within building-with actions, and recognizing and building-with actions are nested within constructing actions. Moreover, constructing actions are at times nested within more holistic constructing actions. Therefore the model is called the *nested epistemic actions model of abstraction in context*, or simply the *RBC+C model*. The second "C" stands for consolidation. The consolidation of a new construct is evidenced by students' ability to progressively recognize its relevance more readily and to use it more flexibly in further activity.

We distinguish the RBC+C model from the RBC+C methodology. The first is a model of knowledge construction; the methodology is a way of analyzing students' activity using the model. The role of context is central to RBC+C as a model and as a methodology. The historical context includes the student's prior learning experience; the learning context includes classroom features such as computerized tools, the classroom culture orchestrated by the teacher, and the purposefully designed learning materials; and the social context includes interactions with other learners and with the teacher.

In one of the early RBC+C model based studies (Dreyfus et al. 2001) the constructing by each student in a dyad was investigated in parallel with the interactions between the two students. At a later stage (Hershkowitz et al. 2007) the constructing of mathematical knowledge within focus groups in a working mathematics classroom was researched and analyzed by RBC+C, taking into consideration the interaction among the group's members and the flow of knowledge from one student to the other. Abstraction in other social settings and other learning environments has been investigated by several researchers (e.g., Dooley 2007; Kidron 2008; Monaghan and Ozmantar 2006; Stehlíková 2003; Williams 2002; Wood et al. 2006).

The RBC+C methodology usually begins with a thorough a priori analysis of the task sequence in terms of the knowledge elements intended to be constructed when students with a given background work on the tasks, and the relationships between these knowledge elements. These intended knowledge elements are used as milestones during the a posteriori data analysis, whose main aim is to identify the constructs that actually emerge for the students, and the processes by which these constructs emerge. Hence, while students' constructs may correspond to knowledge elements, knowledge elements are theoretical predictions by researchers, whereas constructs emerge in the minds of students. For a detailed description of the methodology, see Dreyfus et al. (in press).

2.2 Documenting collective activity (DCA)

Collective activity is a sociological construct that addresses the constitution of ideas through patterns of interaction. More specifically, collective activity is defined as the emergence of normative ways of reasoning which are developed in a classroom community. Such normative ways of reasoning emerge as learners engage in activities including solving problems, explaining their thinking, and representing their ideas. A mathematical idea or way of reasoning becomes normative when there is empirical evidence that it functions in the classroom as if it is shared. The phrase "function as if shared" means that particular ideas or ways of reasoning are functioning in classroom discourse *as if* everyone in the classroom community is in agreement. The phrase "function as if shared" is similar to "taken as shared," but it is intended to make a stronger connection to the empirical approach used to determine when ideas function in the classroom as if they are mathematical truths. The empirical evidence for determining when ideas function as if shared comes from the use of Toulmin's (1958) model of argumentation (Rasmussen and Stephan 2008; Stephan and Rasmussen 2002).

In his seminal work, Toulmin (1958) created a model to describe the structure and function of argumentation. Figure 1 illustrates that the core of an argument consists of three parts: the DATA, the CLAIM, and the WARRANT. In an argument, the speaker makes a CLAIM and presents evidence or DATA to support that CLAIM. Typically, the DATA consist of facts or procedures that lead to the

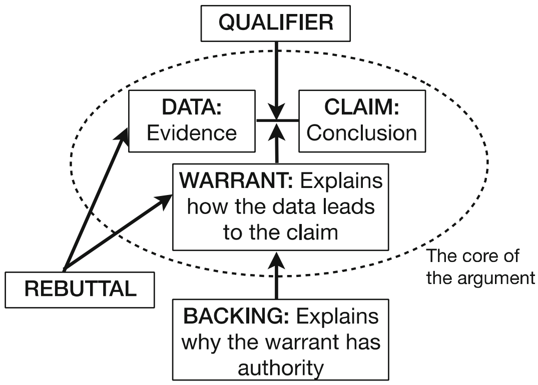


Fig. 1 Toulmin's model of argumentation

conclusion that is made. To further improve the strength of the argument, speakers often provide more clarification that connects the DATA to the CLAIM, which serves as a WARRANT, or a connector between the two. It is not uncommon, however, for rebuttals or qualifiers to arise once a CLAIM, DATA, and WARRANT have been presented. Rebuttals and qualifiers aid in propelling the argument forward. If one disagrees with the CLAIM, he or she may present a rebuttal, or counter-argument that shows disagreement. When this type of challenge is made, often a qualifier is provided, which is a way to provide specific conditions in which the CLAIM is true. Finally, the argumentation may also include a BACKING, which demonstrates why the WARRANT has authority to support the DATA–CLAIM pair.

The DCA methodology begins by using Toulmin's model to create a sequence of argumentation schemes (Fig. 1 is an argumentation scheme) of every whole-class discussion, resulting in an argumentation log across all whole-class discussions. The next step involves taking the argumentation log as data itself and looking across all class sessions to see what mathematical ideas expressed in the arguments become part of the class's normative ways of reasoning (that is, function as if shared). The following two criteria are used to determine when a way of reasoning becomes normative:

Criterion 1: When the BACKINGs and/or WARRANTs for particular CLAIM are initially present but then drop off. That is, criterion 1 is satisfied when the same conclusion gets debated in more than one class period or more than once during the same class period and in subsequent occurrences the BACKING or WARRANTs drop off.

Criterion 2: When certain parts of an argument (the WARRANT, CLAIM, DATA, or BACKING) shift position within subsequent arguments. For example, criterion 2 is satisfied when once-debated conclusions shift function over time and serve as unchallenged DATA or justification for future conclusions.

These two criteria can be thought of as the collective analogue to an individual's process of vertical mathematization, and they provide empirical evidence that particular ideas or ways of reasoning function as if shared, as defined above. The results of a DCA analysis are of high pragmatic value in that the ideas that function as if shared can be used to revise and refine an instructional sequence. Examples of prior work in which a collective analysis has fed back into the refinement of an instructional sequence include that of Rasmussen et al. (2004), Stephan and Akyuz (2012), Stephan et al. (2003), and Whitacre (2012).

2.3 Common features of DCA and RBC+C methodologies

Both methodologies focus on the ways in which abstract mathematical knowledge is constructed and spreads in the classroom. RBC+C focuses on individuals or small groups working in the classroom and DCA focuses on whole-class discussions. In this sense, the two methodologies complement each other in analyzing a sequence of lessons including group work and whole-class discussions and in tracing how knowledge is constructed and becomes normative along this sequence.

The use of both methodologies requires very explicit characteristics and norms. First, they require classrooms in which genuine argumentation is a norm, that is, classrooms in which students are routinely explaining their reasoning, listening to and indicating agreement or disagreement with each other's reasoning, etc. If such norms are not in place, then evidence is unlikely to be found of challenges, rebuttals, and negotiations that lead to ideas where knowledge is constructed and starts functioning as if shared by the whole class. We call such classrooms "inquiry classrooms." Inquiry in this sense is related to reflected acting (Lengnink 2006) and to the PRIMAS project (Artigue and Blomhøj 2013).

Other characteristics of a classroom culture in which DCA and RBC+C methodologies might be enacted together are that the mathematical context should afford inquiry and the emergence of new constructs by vertical mathematization from previous constructs; that the tasks are designed to interweave collaborative work in both small-group work and whole-class discussions; and that the teacher adopts a role that encourages inquiry in the above sense.

In addition, the two methodologies, DCA and RBC+C, have a common methodological feature: namely, they are operational in the sense that they provide means to empirically study abstraction. The epistemic actions in the RBC+C methodology and the two criteria in the DCA methodology are the empirical tools which enable our study.

In this study we add to the above DCA and RBC+C methodologies by combining them so as to enable us to discover new "knowledge relationships" in the classroom.

2.4 Knowledge relationships in a classroom

In a previous study with the same theoretical–methodological framework (Tabach et al. 2014) two trends were observed. First, students in different small groups went through similar but not identical processes of knowledge construction. Indeed, it is on this basis that participants can communicate across groups yet still have differences to debate. Second, the whole-class discussions emerged from the small-groups' work but were also occasions to develop the mathematical ideas beyond what was constructed in the small groups.

By combining the RBC+C analysis of the group-work episodes with the DCA analysis of the whole-class episodes, we were able to identify the links between the constructs that emerged and the ways of reasoning that became normative. These links revealed several instances of uploading and downloading. By *uploading* we mean shifts of knowledge from a group to the whole class; *downloading* refers to shifts in the opposite direction. These analyses also focused our attention on the students who initiated or actively contributed to these shifts, and we characterized these students as knowledge agents. A *knowledge agent* is a student who first initiates an idea within a classroom setting (the whole-class community or small group), and the idea is later expressed by one or more other students. Elaboration of the knowledge agents' roles contributes to articulating the processes related to the uploading and downloading of ideas. For example, if a student in the classroom is the first one to express an idea according to the researchers' observations, and this idea is later expressed by others, then this student is considered to be a knowledge agent. The idea can be shifted from a group to the whole class (uploading), within the whole class, within a group, from a group to another group, or from the whole class to one or several groups (downloading). The shift actions can last from seconds to a few lessons. In all these cases, knowledge agents provide an opportunity for others to construct knowledge related to an idea.

In the present study we elaborate the role and function of knowledge agents and knowledge shifts in classroom. We also pay attention to the role of the teacher in creating an environment that encourages knowledge shifts by affording opportunities for students to function as knowledge agents.

3 Methodology

3.1 The research questions

Our main goal is to identify and understand the processes governing shifts of knowledge in the classroom. Hence, we must use methodologies that can analyze data from the whole-class community on the one hand and from small-group work on the other hand. The above overall goal can be expressed in the following research questions:

1. How can knowledge shifts in inquiry classrooms that include small-group work and whole-class discussions be characterized? These include shifts of knowledge that are downloaded from the whole-class discussion into a group's work or uploaded from a group's work to the whole-class discussion.
2. What are the characteristics and roles of knowledge agents in the classroom?
3. What is the teacher's role in relation to knowledge shifts and knowledge agents in the classroom?

3.2 Research setting

The study presented in this paper was carried out in the framework of a larger research project that involved seven eighth-grade classes that were engaged in learning probability. All lessons in the participating classes were observed, videotaped, transcribed, and analyzed. The camera was focused on a specific small group (henceforth: the focus group) during group work and on whoever spoke during whole-class discussions. The data for the present study were collected in one class of 27 students, with the focus group consisting of three girls: Yael, Noam, and Rachel.

The probability learning unit consisted of a ten-lesson sequence of activities embedded in a learning environment. The sequence was carefully designed to offer opportunities for the collaborative construction of knowledge (see the analysis of the Dial Problem in Sect. 4.2 for an example). The unit included five activities organized as sequences of tasks for investigative group work, for whole-class discussions, and for homework which was mostly done on an individual basis.

The probability unit dealt with concepts and problem-solving aspects of empirical vs. theoretical probability, and one- and two-dimensional sample spaces. The overall construct of sample space, which was investigated in the project, included three hierarchical stages:

A. Sample space in one dimension; for example the sample space for rolling a die or for throwing a thumbtack (does it land on the pin or on the head?). The chances of simple and composite events were presented on a chance bar.
B. Sample space in two dimensions, where the possible simple events are equi-probable and can be counted and organized into a table.
C. Sample space in two dimensions, where the probabilities of the possible simple event are given but are not necessarily equal.

Research studies based on AiC in which Stage B and Stage C were used as the research context have been presented elsewhere (e.g., Hershkowitz et al. 2007; Ron et al. 2010).

3.2.1 A general description of stage A

The present study is focused on stage A of the unit. The topics in stage A (Lessons 1–4) included theoretical probability as a ratio of the number of relevant outcomes to the number of all possible outcomes, as well as some experience with the fact that empirical probability values tend to the theoretical value as the number of trials becomes large.

The main activity in Lessons 1 and 2 focused on Danny and Yael's Problems, which were discussed also at the beginning of Lesson 3. Lesson 4 focused on the Dreidel and Coin Problems. Next we discuss each pair of problems followed by an a priori analysis.

3.2.2 Danny and Yael's Problems

Danny's Problem

> Danny works in a shop after school. At the end of each day, he receives 50 Shekel for his work. One day, the shop owner offers Danny the following deal: Instead of paying you 50 Shekel each day, let's each day roll a die; if the die shows 5, you'll get 200 Shekel, but if it shows a number different from 5, you'll get only 10 Shekel.
>
> a) What do you think? Is it worthwhile for Danny to agree? Write down your conjectures and justify them.
> b) In order to help Danny, each of you roll a die 10 times and write down how many times you got 5.
> c) Add the results of both of you: how many times did you get 5 out of the 20 rolls you rolled together?
> d) In your case, would it be worthwhile for Danny to accept the owner's proposal? Explain your answer.

Yael's Problem

> Yael began working in a sports store for 50 Shekel a day. Yael heard the story from Danny and proposed to the owner of her store the following method of payment. At the end of each day, we'll throw a thumb-tack. If the tack falls on its head, I'll get 100 Shekel and if it falls on its side, I'll get 20 Shekel.
>
> a) What do you think? Is it worthwhile for the owner to agree? Write down your conjectures and justify them.

3.2.3 A priori analysis of Danny and Yael's Problems

According to our analysis of these two tasks (as designers and researchers), Grade 8 students may be expected to construct the following knowledge elements while working on these two problems:

- E_u Uncertainty is inherent in probability problems.
- E_e There are expected probability values for each event. Hence, probability is amenable to mathematical reasoning.
- E_m Results of multiple experiments accumulate to the expected probability value of an event.
- E_{tp} There is no need to experiment for events where a clear theoretical probability can be calculated, for example rolling a die.
- E_{xp} There is a need to experiment for events in which it is not possible to calculate the theoretical probability, for example throwing a tack.

Figure 2 shows the relations among these knowledge elements.

Students worked on the two problems during the first two lessons. In Lesson 1, they worked on Danny's

Problem. They also tallied the results of the die rolls of the entire class on the board. Unfortunately, perhaps because we did not have enough data, the experimental probability appeared to converge to 1/5 rather than 1/6. No student mentioned that it should converge to 1/6. In Lesson 2, the students worked on Yael's Problem, and again students collected data. The experimental probability for the tack to fall on its head was 143/240 or about 60 %. At the beginning of Lesson 3 the class discussed the findings of the two experiments.

3.2.4 The Dreidel and Coin Problems

The Dreidel Problem

A Hanukkah dreidel (a four-sided top with one of the letters N, G, H, and P on each side) was spun 100 times. Mark approximately, on the chance bar, the chances of the following events:

A. The dreidel will fall 100 times on the letter N.

B. The dreidel will never fall on the letter N.

C. The dreidel will fall on N between 80–90 times.

D. The dreidel will fall on N between 23–30 times.

The Coin Problem

A coin was flipped 1000 times. Mark approximately, on the chance bar, the chances of the following events:

A. The coin will land 1000 times with heads facing up.

B. The coin will land with heads facing up more than 450 times and less than 550 times.

C. The coin will land with heads facing up more than 850 times and less than 950 times.

D. The coin will fall 1000 times with tails facing up.

3.2.5 A priori analysis of the Dreidel and the Coin Problems

According to our analysis (as designers and researchers) of these two problems, Grade 8 students who have appropriate previous constructs for the five elements E_u, E_e, E_m, E_{tp}, and E_{xp} (see Fig. 2), can be expected to construct the following three knowledge elements while working on the Dreidel and the Coin Problems, because these knowledge elements are inherent in the two problems and are based on the above knowledge elements:

E_{re} The probability of a simple event is different from the probability of a composite event that consists of a repetition of the simple event.

E_d The probability of a composite event which consists of repetitions of the same simple event decreases with each repetition.

E_{ra} The probability of an event falling into a given range of values, which includes the expected value, is high.

Figure 3 is an extension of Fig. 2 and shows the relations among all eight knowledge elements.

4 Analysis of Lesson 3—findings

The lesson started with a whole-class discussion, followed by small-group work, and a final whole-class discussion. We segmented the lesson into episodes. Table 1 summarizes the chronological order of the episodes. Whole-class discussion episodes will be analyzed using DCA and small-group work episodes will be analyzed using RBC+C.

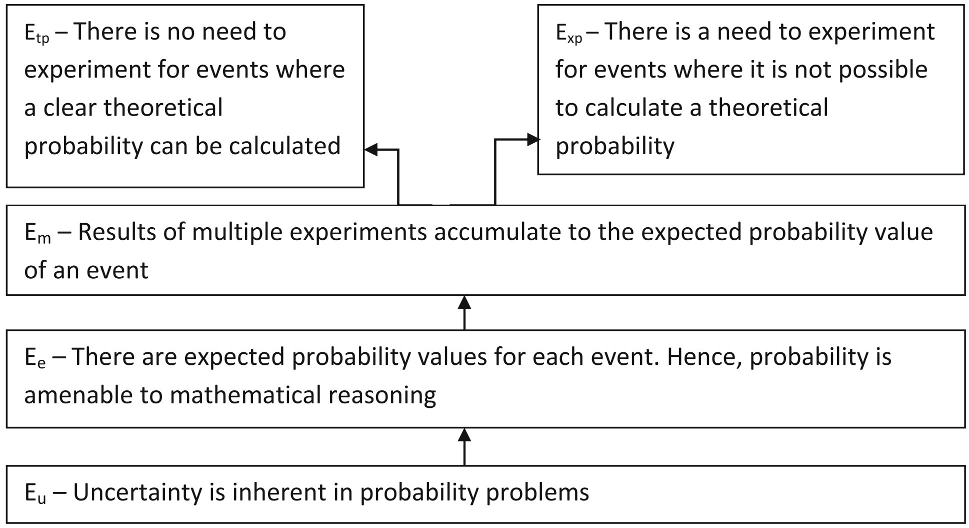


Fig. 2 Constructs related to Danny and Yael's Problems

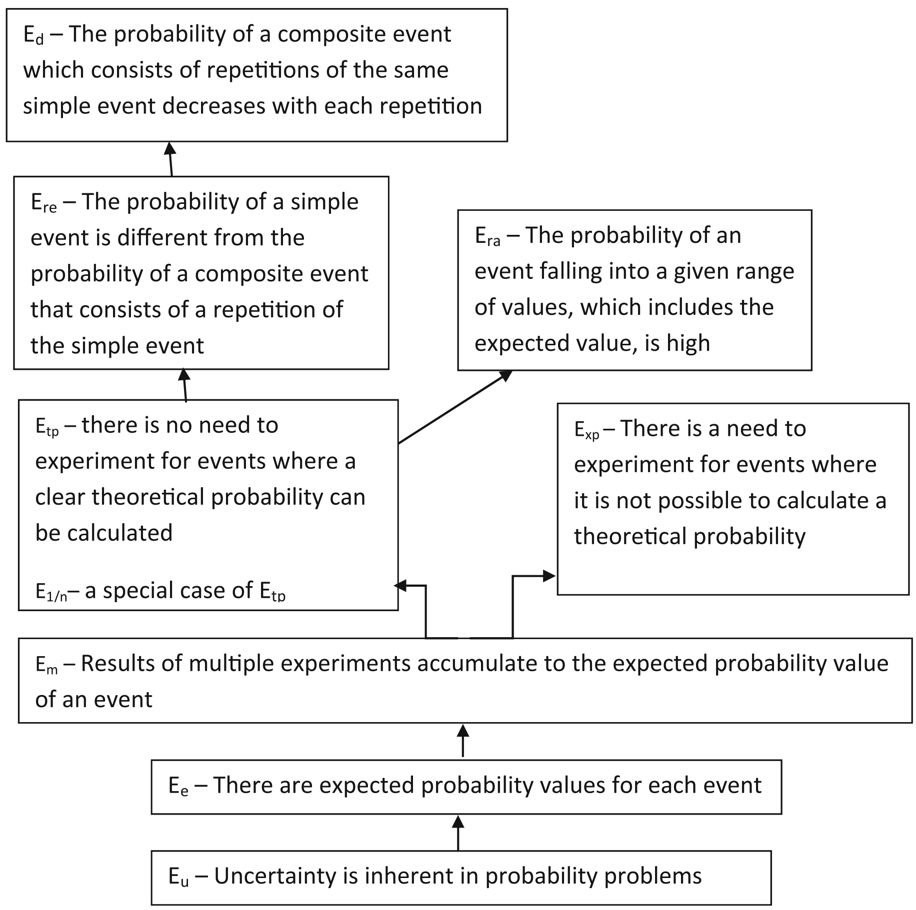


Fig. 3 Constructs related to the Dreidel and Coin Problems

4.1 Whole-class discussion

The lesson started with a whole-class discussion (Episodes 1–4, utterances 1–32) about the comparison of the two deals—the one with Danny and the other with Yael, on which the class had worked in the previous lessons.

Episode 1: Recalling Danny and Yael's Problems

No.	Speaker	Utterance
1	Teacher	I would like to start with probability now, whoever did his homework may remember more, and whoever didn't can look at his worksheets. We had Yael's way, the deal that she tried to organize for herself, what was her deal? Remind me, what was it?
2	Itamar	That she will throw a tack in the air and if it falls on its head she will get 100 shekels and if it falls on its side she will get 20 shekels.
3	Teacher	We had Danny's deal, what was it?
4	Boy	The store owner offered Danny that instead of getting 50 shekels every day, if he rolls a die and gets a "5" he will get 200 shekels.
5	Teacher	Does everyone remember that? Do you know what we are talking about? Did you find it in your notes?
		Now, I would like to ask, according to your opinion, are these two deals similar or different? What do you think about it? What do you think?
6	Itamar	Similar, because both are luck! They are both a deal of luck.
7	Teacher	Itamar says that both deals are similar, because they are both a matter of luck. Yes? Does anyone think differently? What do you think?

The discussion started with a brief (and in the case of Yael's Problem) partial recall of Yael and Danny's Problems (1–4), and an opening question (5) posed by the teacher about the similarity or differences between them. The first argument was given by Itamar (6). Itamar related to the knowledge element E_u (uncertainty, see Fig. 2) that probabilistic situations are uncertain, meaning that luck plays an important role. In terms of Toulmin's model, Itamar's argument is:

ARGUMENT 1

CLAIM1	Yael and Danny's deals are similar
DATA1	Because they are both a deal of luck

The teacher then re-voiced this argument in order to elicit possible other ideas.

Episode 2: Comparing Danny and Yael's deals

Matan (seconded by Yael), objects to the CLAIM of Itamar in Argument 1:

8 Matan	I disagree. Danny has a 1 out of 6 chance, which is a smaller chance than with the tack that has two sides it can fall on.
9 Yael	I think so too!
10 Teacher	Matan says that there is a 1 out of 6 chance because the die has 6 sides, and with Yael there are 2 sides to the tack, and that's where you see the difference between the two deals. Yes? Who else has something to say?

Table 1 Lesson 3—summary of analysis

Setting	Episode number	Title
Whole class	Ep. 1	Recalling Danny and Yael's Problems
	Ep. 2	Comparing Danny and Yael's deals
	Ep. 3	Need experiment?
	Ep. 4	Experiment or not?
Group work	Ep. 5	The Dial Problem
	Ep. 6	The Matchbox Problem
	Ep. 7	The Box with Notes Problem
Whole class	Ep. 8	The Matchbox Problem
	Ep. 9	Inviting other ideas on the Matchbox Problem
	Ep. 10	Closing the discussion on the Matchbox Problem
	Ep. 11	The Box with Notes Problem
	Ep. 12	Back to the Matchbox Problem

Matan relates to the probability of each deal (8) and phrased a different CLAIM, that the two deals are different (Danny has smaller chances…). Matan provided a partial WARRANT concerning Yael's chances, and the teacher re-voiced and completed the argument (10) by providing a WARRANT to the part of his argument concerning Danny's chances:

ARGUMENT 2

CLAIM2	Danny's chances are smaller than Yael's (the deals are different)
DATA2	Danny has 1 out of six chances; Yael's tack has two sides
WARRANT2	Because the die has 6 sides; the tack has two sides it can fall on

We note that while Itamar in DATA1 of Argument 1 related to the uncertainty in probability situations (E_u), Matan objected to CLAIM1 and related in DATA2 of Argument 2 to a different knowledge element (E_e—expected value), namely that each event has an expected probability value. The teacher continued to seek different points of view among the students concerning the similarities of both situations.

Episode 3: Need experiment?

The teacher then explicitly raised a new issue, which relates to the difference between the two tools—die and tack. Do we need an experiment to know what the chances are?

16 Teacher	In both cases we have to experiment in order to know? Yes?
17 Students	Yes!
18 Teacher	In both cases we have to experiment in order to know what's happening?

19 Yael — What do you mean? How much we get?
20 Teacher — Yes, what are the chances?
21 Yael — Here the chance is 1 out of 2 and here a sixth.
22 Teacher — Here you are saying that it's 1 out of 6 and 1 out of 2, like Matan said, do you agree with that?
23 Students — No!
24 Omri — I think that with the tack it's not 1 out of 2, because the head area is wider and so the chances are higher that it will fall on its head.
25 Teacher — Wait a minute, then you are saying that we were wrong about the tack, it's not 1 out of 2 but something else because the head has a bigger area? What do you think?

At first, it seems that the students confirmed that an experiment is needed in both cases (17). Yael rephrased (21), labeled earlier as DATA2. This time, the students objected (23). In fact, Omri (24) refuted part of what was previously labeled as DATA2, namely that the chances of the tack are 1 out of 2. He presented a new CLAIM, which is a refinement of CLAIM2 and DATA2: the chances are higher that it will fall on the tack's head:

ARGUMENT 3

CLAIM3	Tack chances are not 1 to 2
DATA3	The head's area is wider
WARRANT3	So the chances of the tack falling on the head are higher

The teacher continued to encourage the students to think and express their thinking about whether or not there is a one in two chance. The issue raised earlier by the teacher about whether an experiment is needed had not yet been resolved.

Episode 4: Experiment or not?

The following is the first explicit reaction to the teacher's query about the need to experiment (16). We remind the reader that the first two lessons were devoted to experimenting with a die and a tack.

28 Michael — It's not really, experiment won't really help, there can be 12 times or 24 times, so it's not really 1 out of 6.
29 Teacher — So you are saying that to experiment won't really help, so it's not really 1 out of 6. Does anyone else have something to say? Yes, Orly?
30 Orly — I agree with Michael, it's not exactly like that, each one can get something else, it's a matter of luck.
31 Girl — I don't think it's a matter of luck with the tack. It depends on how you throw. (She demonstrates throwing a tack by holding it by its tip and dropping it).
32 Teacher — You are saying that it's a matter of luck, each one has a different chance.

Michael (28) and Orly (30) together raised a new argument that experiments will not help, because "it is a matter of luck" (E_u—uncertainty). This argument is a rebuttal to the previous debate, which we labeled Argument 3, about the chances in the case of the die and tack. Interestingly, however, Michael doubted the chances of the die, which until this point weren't questioned. Another girl in the class provided a rebuttal to Argument 4, and advanced a new CLAIM (31), in favor of the existence of an expected value for the tack's event (E_e), saying that it is a matter of the way one throws the tack. Note that the students' contributions in this case use the same DATA (which is related to E_u—uncertainty) as in Argument 1:

ARGUMENT 4 [rebuttal to Argument 3]

CLAIM4	Each one can get something else
DATA4	It is all about luck

ARGUMENT 5 [rebuttal to Argument 4]

CLAIM5	Tack is not a matter of luck
DATA5	The manner of throwing it matters

At this point in the lesson, the teacher asked the students to move to work on a worksheet composed of a sequence of several problems with the same underlying idea. From the point of view of the DCA analysis, we can say that although several students expressed their ideas, and some of them were raised with respect to ideas expressed by their peers, at this moment in time we do not find ideas which function as if shared by the whole class. Indeed, students need time to express their ideas and to consider the ideas of others before rushing to some consensus, and the DCA analysis affords a method to trace the evolution of ideas over multiple whole-class discussions. Now, the work continued in groups.

4.2 The focus-group work

In this episode of focus-group work, Noam and Yael were working together and Rachel was absent. Their work focused on some further problems of the designed sequence. For each problem we discuss first the problem itself followed by an a priori analysis. Then we present parts of the transcript of the work of the pair, and an RBC+C analysis of the work.

Episode 5: The Dial Problem[1]

[1] The letter A was missing in the drawing provided to the students.

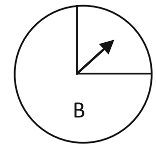
A dial of a spinner was divided into two sections, A and B, as shown in the picture.[2]

The spinner was turned.

Is it possible to determine, without an experiment, the chances that the spinner will stop in region A?

If yes, what are the chances?

If no, what do you think one needs to do to determine the chances?



As we already concluded from Danny's and Yael's Problems, in which students were engaged in Lessons 1 and 2, there are two types of problem situations concerning the construction of the concept of *finding the chances of a given event*. In the first type one can determine the chances from the situation itself (like in the die case, E_{tp}—theoretical probability can be calculated), while for the other type one needs experiments in order to verify the chances (E_{xp}—experiment is needed). The Dial Problem belongs to the first kind, but it is more advanced than the die, since the chances for events A and B are proportional to their unequal areas. The fact that the chances are proportional is a new nuance in the construction of finding an event's chances. Note that this problem deals with a sample space of two unequal probability events like in the Tack Problem. But in the Tack Problem the ratio between the "areas" is unknown, so one needs an experiment for finding the probabilities, whereas in the Dial Problem this ratio is known. Hence, the intended nuance to E_{tp} to be constructed by the Dial Problem is the "translation" of the ratio between the areas into the ratio between probabilities.

40 Yael If the area is a quarter of the dial for each of the … it's half.
41 Noam **No,** there are two parts! This is B and this is A, A is smaller on purpose.
42 Yael No, B is the big one and A the smaller one.
43 Noam Right, that's the idea!
44 Yael So what are the chances?
45 Noam Theoretically there is a higher chance for B because B is bigger but …
46 Yael The chances are a quarter.
47 Noam There is a chance that the spinner … but it's smaller.
48 Yael No! The chances here are 1 out of 4 (reads): "If it's not possible to determine, what in your opinion do you need to do in order to know what the chances are?"
49 Noam You can't determine! Basically because there is a bigger area …
50 Yael Yes you can! You can know that there is a quarter of the clock …
51 Noam There is a bigger chance it will fall on B
52 Yael Well … there is a bigger chance it will fall on B! But, there is a chance it will fall on A, ok?
53 Noam And …?
54 Yael So we answered that!
55 Noam I still don't understand the question!
56 Yael The question is what are the chances that the dial will fall on A?
57 Noam Slim
58 Yael Slim. What are they? The chances? 1 out of what?
59 Noam 1 out of 4.
60 Yael Exactly!

It seems that in utterance 40 Yael recognized her construct (a mathematically inadequate one), mentioned in the previous whole-class discussion (20), that in the case of the tack, which had two options, the chances are 1 out of 2. In the present problem, the dial has *two* areas, A and B, hence "… it's half." Noam (41) objected, taking into account qualitatively the idea that the different areas of A and B affect the probabilities. It might be that Noam recognized the relevance of Omri's argument (24) from the whole-class discussion for the present situation, but it could just as well have been from her sense of the problem at hand. In the first case, we may say that Omri acts as a *knowledge agent* for Noam (but not yet for Yael). His idea is *downloaded to* the group's work by Noam.

Noam in utterances 41–43 succeeded in convincing Yael that they have to consider that the area of B is bigger than the area of A, and that this was done purposefully by the learning units' designers. So, they built with E_{tp} (theoretical probability can be calculated) the idea of the different size areas as a factor in their further considerations concerning the chances, which is a nuance of E_{tp}.

Yael was not satisfied with qualitative considerations only and shows "a need" for a numerical answer (44), while Noam stayed with the qualitative argument. Led by her need, Yael constructed accurately the chances as a ratio proportional to the ratio between the areas of A and the entire circle: 1/4 (46, 48: "Chances for A are 1 out of 4").

Noam did not construct the quantitative proportional probability and stayed in the qualitative level for quite a long time (45, 47, 49, 51, and even 57). Yael, however, seemed eager "to shift" her idea to her peer Noam (50, 52, 56, 58). In these utterances she guided Noam to see that her qualitative considerations did not fully answer the Dial Problem (56, 58), until Noam confirmed (59) that the chances are 1 out of 4 and Yael celebrated (60): "exactly." We may conclude that the shared knowledge which was constructed by the dyad is the nuance of E_{tp}, namely the proportional idea.

This episode is meaningful concerning our analyses for the following reasons:

- It demonstrates a process of constructing abstract knowledge.
- The process is carried out cooperatively by two students.
- The process might be initiated by one of the students, who acts as *knowledge agent* by recognizing an idea raised in the whole-class discussion, which took place before the dyad's work. The student (Noam 41) *shifts* this idea into the dyad's work.
- The other student (Yael) first constructs the new knowledge.
- Yael acts in order to *shift* her new construct to her peer (Noam). We may conclude that Yael is a *knowledge agent* of this new construct.
- We do not have clear evidence that Noam constructed the new construct in this episode; her confirmation (59) is too short for this. So, we don't know if Yael succeeded meaningfully as a *knowledge agent*.

Episode 6: The Matchbox Problem

Is it possible to determine without experimenting what the chances are that we will take out a defective match from a matchbox? If yes, what are the chances? If not, what do you think one needs to do to determine the chances?

This problem belongs to the type in which an experiment is needed in order to determine the chances (E_{xp}—experiment is needed). This problem is more advanced than the basic problems of this type (the tack), in the sense that here the experiment is more complex—first, one needs to take many boxes, and count the number of matches and the number of defective matches in each box separately, and then average the chances of taking out a defective match. The expected knowledge element in this case is a nuance of E_{xp}, related to the idea of averaging the information of individual boxes.

95 Noam	(reads the next paragraph) 'Is it possible to determine without experimenting what the chances are that we will take out a defective match from a matchbox? If yes, what is it?' You can't know! Unless … you have to experiment! You can't know! You need to experiment! I'm writing "You need to experiment!"
96 Yael	You don't have to!
97 Noam	Of course you do!
98 Yael	What the chances are of taking out a defective match from a matchbox?
99 Yael	It's 1 out of the number of matches in the box.
100 Noam	Right, so you take many boxes, how many, if, in the box …
101 Yael	Noam, it depends on how long you have been using the box, if you used it once then maybe it will be less …
102 Noam	No! If it's defective! You have to take many boxes and see in each one if there is … if there are let's say 50 matches in each box and 1 is defective so it says on the box 1 out of 50, so you have to experiment!
103 Yael	So it's 1 per the number of matches in each box
104 Noam	Not 1, there may be 2 defective matches in the box
105 Yael	But what are the chances?
106 Noam	But with 2 defective ones?
107 Yael	But Noam, you are speculating … you can say 50 out of 50
108 Noam	But you can't say 1 out of 50? Out of … whatever! What is the probability? It's not correct what you are saying!
109 Yael	What isn't correct?
110 Noam	Because just like you can't say 2 out of the matches because you don't know that it's 2 or that it's 1
111 Yael	(writes) "can't determine without experimenting"
112 Noam	We can, if we experiment

113 Noam Ok, so what is the probability? It's, we have to write that we won't know until we experiment …
114 Noam Let's write at the bottom, that we need a few boxes, suggest an experiment (dictates: "we need to take a few boxes of matches and see out of them …" [Dictate together])
115 Noam No, wait! How many matches does the box contain, and see how many defective matches are in it … [Dictate together].
116 Yael (continues to dictate) "then, check how many defective matches are in the box" [Dictate together].
117 Noam Then we will write "the probability is the number of defective matches in the …" [Dictate together].
118 Yael Forget it.
119 Noam Let's write it down.
120 Yael We can't know.
121 Noam We can!
122 Yael Noam, each box will come out differently.
123 Noam So it's average, not precise!
124a Yael Noam, leave it (moves to the next problem).

The above discussion did not conclude with an agreement between the peers. On the whole, Yael seemed impatient (possibly because she felt she was not on the right path). It seemed that she did not go deeply into the meaning of the problem and just aimed to provide an answer and move on to the next problem quickly (e.g., 118, 124).

Noam was quite clear from the very beginning. She recognized the knowledge element E_{xp} (experiment is needed), that for getting the chances in this problem there is a need for experimenting (95). She even built with this knowledge element an algorithm for doing the experiment: first, she suggested that they should take many boxes (100), and second, she explicitly stated that they have to calculate the average of the number of defective matches over all boxes. Finally, she argued that the ratio between this average and the average number of matches in the box would yield the chances of picking a defective match (117).

In between the two students' different ways of relating to this problem there were some "deaf interactions": in utterances 96 and 97 they each objected by raising primitive opposing CLAIMs without any DATA or WARRANTs to support them. Then Yael recognized a previous construct (similar to the construct in the Die Problem) that the probability always has to be expressed as 1 out of something and wrongly tried to build with this construct in the Matchbox Problem (99, 103).

Yael also considered everyday issues, like the habit of using a match and either inserting it back into the box, or discarding it (101). Hence the situation is changed (larger number of defective matches or smaller total numbers of matches). This idea was not the designers' intention, but yet she interpreted the situation in its everyday context.

It is very interesting to follow how Noam constructed her insights and how Yael first follows her (116), and then stopped her quite aggressively (118, 120, 122, and 124). We may conclude that Yael and Noam recognized that there is a need to do an experiment in this situation, and that a decent way to express the chances is of the form: *one out of* … (99, 100).

But, while Noam continued to elaborate her idea and constructed the idea of experimenting with many boxes and calculating the average (123), Yael did not follow her and rather tried to block Noam's idea. In spite of Noam's additional attempts to convince Yael (123), Yael was very insistent and Noam could not go on and argue with her. They moved to the next problem.

We have here a quite interesting process of construction when the two students in a dyad construct together (with a great deal of argumentation and objection) up to the point of awareness of the need to experiment, and the understanding that the situation of each box might be different. But, while Noam went on to construct a complete structure of an appropriate experiment, Yael refused to follow her. We may verify that Noam in this case raised two ideas: the first is that an experiment is needed (95, 97, 100, 102, 112, and 113); the second is the idea of averaging the findings from many boxes (100, 102, 108, 110, 114, 115, and 123).

Episode 7: The Box with Notes Problem

Ten notes, each with a different name written on it, are inside a box. One of the names is Ruth.

Can you determine, without an experiment, what are the chances of taking out a note with Ruth's name on it?

If yes, what are the chances?

If not, what do you think one needs to do to determine the chances?

The problem is clearly a situation where the chances can be calculated and there is no need to have an experiment in order to hypothesize the chances. It can thus be dealt with by use of the knowledge construct E_{tp}.

124b Yael	"in a box there are 10 notes with different names, on one it is written 'Ruth'."
125 Noam	Can't be determined.
126 Yael	Of course you can!
127 Noam	1 out of 10.
128 Yael	We said probability … one note "Ruth" out of 10.
129 Noam	Pass the eraser.
130 Yael	(dictates) "the probability is 1 out of 10."

The girls agreed that the probability is known and it is 1 out of 10. We can see that the girls consolidated their previous construction from Danny's Problem and the Dial Problem, that in these problems one can calculate the probability out of the situation itself. It is worth noting that they are expressing it immediately (127, 130). Hence, this is evidence that calculating the probability in such a situation was consolidated.

4.3 Whole-class discussion following the group work

At this point the teacher stopped the work of the groups and a whole-class discussion took place until the end of the lesson. The next five episodes are all taken from this discussion.

Episode 8: The Matchbox Problem

When the class reconvened, the teacher drew the students' attention to the issue which was at the heart of the tasks on the worksheet, and is a direct continuation of the discussion before the groups' work, namely do we need an experiment (see Episode 3, utterance 16 above). Next the teacher focused the discussion on the Matchbox Problem.

134 Teacher	I would like to look at the Matchbox Problem: "Can you determine without experimenting what the chances are that we will take out a defective match from the box?" Who wants to give us his answer?
135 Noam	In my opinion you need to experiment!
136 Teacher	Why?
137 Noam	I don't know. I can suggest an experiment.
138 Teacher	Friends, listen, you need to express your opinion on what they said.
139 Yael	[addressing Noam] Why can't you say why you need an experiment, you can't know how many matches there are in the box.
140 Teacher	Let's say I can reveal to you that there are 45 matches in the box.
141 Yael	And inside you have to [check].
142 Noam	[you need to take some] matchboxes, you need to see how many matches are in each box, and how many of them are defective.
143 Teacher	Let's say we know that information, what do I do with it?
144 Noam	So …
145 Yael	So I do the average.
146 Teacher	What average?
147 Yael	Of the defective matches in each box.
148 Teacher	And how is that going to help us know what the probability is that we take out a defective match?
149 Noam	Let's say we have 2 defective matches in a box with 50 matches, so it's 2 divided by 50.
150 Teacher	2 to 50, what do you think? Don't repeat it, Matan listen! Explain again!
151 Yael	We are saying that you can't do it without an experiment. You can't know how many defective matches there are because we don't know how many matches are in the box and we don't know either … We can't speculate how many defective matches there are. We wrote that we need to take a number of matchboxes and see how many matches they contain, then count how many out of them are defective and do an average of how many defective matches in each box. If we got 3 then it's 3 divided by 50.

Three speakers took part in the whole-class discussion (134–151)—the teacher, Yael, and Noam. The teacher mainly questioned the CLAIMs raised by the students, in order to get a clearer picture of their argument. Their main CLAIM is that an experiment is needed (Noam 135), and in fact all the next utterances (136–150) express the essence of the needed experiment, and in 151 Yael repeated the whole argument again in the name of both students ("we are saying …"). Comparing to the group-work episode on this problem, where Noam acted as a *knowledge agent*, and we even did not have evidence that Yael followed her, here they act as a single *knowledge agent*, meaning that they pass to the whole class their idea of the need for an experiment, as well as an explanation of why an experiment is needed and an algorithm for how to do it:

ARGUMENT 6

CLAIM6	An experiment is needed to tell you the chances
DATA6 (partial)	You can't know how many matches there are in the box

WARRANT6	ARGUMENT 7 (see below)

ARGUMENT 7

CLAIM7	Here is a proposal for the experiment
DATA7	Use the average number of matches and of defective matches in the collection of matchboxes
WARRANT7	Take a number of matchboxes and see how many matches it contains, then count how many out of them are defective and do an average of how many defective matches in each box. If we got three then it's 3 divided by 50

Episode 9: Inviting other ideas on the Matchbox Problem

The teacher's next move (152) was to invite other students to take part and express their thinking on the problem, in light of what was presented by Yael and Noam. Guy, Itamar, Chen, and Matan all expressed their ideas in relation to the experiment suggested by Yael and Noam. In their reactions (153–160), they did not challenge the experiment itself (Yael, 151).

152 Teacher	What do you think, Guy? Chen, you need to listen to what Guy says and relate to it.
153 Guy	I am not sure that is correct, that you can't know without experimenting, but on the other hand, also with experimenting you can't know for sure, all in all, its probability, because in each box there is a number of defective matches. There isn't the same amount in each box.
154 Itamar	That's right, but you can still make the definition that 1 divided by 50 will be defective … it won't necessarily be correct …
155 Guy	That's right, we can speculate, but you can't know for sure.
156 Itamar	Right, I know …
157 Teacher	Ok. Chen needs to respond to that.
158 Chen	I say that it's correct, because even if you define that it's 1 out of 50 matches is defective, it could be that in your next box you'll have 5 or 7 or …
159 Teacher	Anyone else? Any comments?
160 Matan	That's correct. That's randomness! Because why? Because there is no regularity, it's not like they put 1 defective match per every 10 matches, it's random.

Argument 8, summarized below, was made by Guy (153), Itamar (154), and Chen (158):

ARGUMENT 8

CLAIM8	You can conjecture the chances but you do not know for sure
DATA8	Boxes may have different number of defective matches
WARRANT8	Because there is no regularity

This argument stands at the core of probability: there are cases in which one can pre-determine the chances, like the die; in other cases an experiment will provide the theoretical probability. However, in both cases, you cannot know what will happen in the next event.

Episode 10: Closing the discussion on the Matchbox Problem

The whole-class discussion on this problem goes on, with two more students contributing to the discussion—Michael and Or.

161 Teacher	So it means there is no regularity.
162 Michael	You can't know for certain. But you just said that because of that you can't experiment, I say that even experimenting won't help.
163 Teacher	So you are saying that nothing will help.
164 Guy	Will help! (shouts) in any case you can't know for certain.
165 Michael	You can't experiment because there could be in these boxes 2 defective matches and in other 20 defective matches. It's not like the machine breaks every 5 min and you get a defective match.
166 Teacher	What do you have to say, Or?
167 Or	An experiment is just a conjecture.
168a Teacher	To sum up what you all said: everyone agrees that if we don't do anything we can't know anything. Some say that if we experiment we can at least speculate and some say that even if we experiment it won't help, we won't even be able to speculate. Did I summarize everyone's opinion? Ok.

The CLAIM made by Michael in utterances 162 and 165 is a rebuttal to Argument 8, and is consistent with his previous contributions to the whole-class discussion, which were labeled as Argument 4, about the meaningless of experimenting on the probability of a given situation (E_u—uncertainty). We might conclude that Michael did not construct the element E_e (there are expected values for each event). That is to say that he still did not understand that in some cases there is an expected theoretical value for an event. Michael's argument is summarized below as Argument 9:

ARGUMENT 9

CLAIM9	Experimenting won't help
DATA9	There could be 2 defective matches in these boxes and 20 in another
WARRANT9	No regular pattern

Guy (164) and Or (167) objected to Michael's argument, by relating to an experiment as a way to conjecture probability (E_{xp}—experiment is needed). The teacher (168a) summarized the two points of view, and moved on to the Box with Notes Problem.

Episode 11: The Box with Notes Problem

168b Teacher	On the other hand what do you think about the Box with Notes Problem, a box with 10 notes, on each one is written a different name and on one is written "Ruth." And the question is if you can determine without experimenting, what is the probability that you will draw the note with "Ruth" written on it? Noa what do you say?
169 Noa	Yes, you can, 1 out of 10.
170 Teacher	What do you think?
171 Students	She's correct!
172 Teacher	She's correct, we are in consensus? Without an experiment?
173 Students	We have all the facts!

The very short and consensual discussion on the Box with Notes Problem is in contrast to the lengthy and complex discussion on the Matchbox Problem. Noa made two CLAIMS: one that the probability is 1 out of 10 (CLAIM 10), and the other that no experiment is needed (CLAIM 11). Noa did not provide any justification for the first claim, nor did the teacher push for any. The teacher did, however, implicitly push for justification for the second claim by raising the question, "Without an experiment?"

ARGUMENT 10

CLAIM10	1 out of 10

ARGUMENT 11

CLAIM11	Don't need to experiment
DATA11	We have all the facts

Argument 10 is interesting when juxtaposed with Argument 2. Recall that in Argument 2 there was a justification (given by a student and elaborated on by the teacher) for why the probability in the die situation is less than in the tack situation. In Argument 10 the claim of 1 out of 10 was made for a situation where a clear theoretical probability can be calculated, yet no justification (neither DATA nor WARRANT) was provided. The dropping off of any justification is related to Criterion 1 of the DCA approach for empirically determining when particular ideas function as mathematical truths. To clarify, in Argument 2 the student stated the reason why he thought the die situation had a smaller probability than the tack situation (i.e., there were two sides of the tack) and the teacher completed the justification to include the fact that there are six sides to a die. In Argument 10, however, neither the teacher nor the students felt the need to provide either DATA or WARRANT for the claim that the probability of the notes situation is 1 out of 10. Thus, we can say at this point that the idea that the theoretical probability for events in which one has "all the facts" can be calculated *functioned as if shared*. What constitutes a situation in which one does or does not have all the facts, however, was still under development.

Episode 12: Back to the Matchbox Problem

The students' claim "We have all the facts" (173) triggered the teacher to go back and raise again the Matchbox Problem, which ended in a disagreement.

174 Teacher	We have all the facts and with the matches some facts were missing?
175 Students	Yes!
176 Or	We didn't know the number of matches
177 Guy	We didn't know the number of defective matches in each box, we are missing a lot
178 Teacher	I understand in any case there is a difference between the Matchbox Problem and the Notes Problem!
179 Michael	I didn't understand the difference
180 Teacher	If the box has 10 notes, on one of which is written "Ruth," now, you close your eyes and pick out one note, you open it … what are the chances that "Ruth" is written on it?
181 Michael	You can't know!
182 Students	1 out of 10.
183 Teacher	You can't know what that chances are that you pick "Ruth"?
184 Michael	According to what I checked, that the chances are 1 out of 10, I cannot say that it was correct, maybe I took 20 notes and still I won't get "Ruth."
185 Teacher	There are only 10 notes.
186 Michael	(smiles) I took, I mean, what do you mean?
187 Teacher	We won't go into it, I respect your opinion.
188 Matan	That if you take out 10 notes you will for sure take out "Ruth."
189 Michael	What do you mean? That you took out the note and didn't put it back?

Indeed, in utterances 174–177 students gave reasons for why an experiment was needed for the Matchbox Problem:

ARGUMENT 12

CLAIM12 With the matches some facts are missing (an experiment is needed)

DATA12 We didn't know the number of matches and we didn't know the number of defective matches in each box

The teacher then tried to sum up the difference between the Matchbox Problem and the Box with Notes Problem (178):

ARGUMENT 13

CLAIM13 There is a difference between the Matchbox Problem and the Notes Problem

DATA13 We have all the facts [in the Notes Problem] and with the Matches some facts were missing

If we follow WARRANT2, in which the theoretical probability is used as justification, we can see that in Argument 10 it becomes a CLAIM (without any need for justification). Whereas in CLAIM12, the opposite claim is expressed: with the matches some facts are missing (an experiment is needed). In DATA13 it appears again as fact without justification. We can thus say that WARRANT2 *functioned as if shared* in the class.

However, here Michael (179, 181, 184) refuted CLAIM13 that there is a difference between the Matchbox Problem and the Notes Problem. He objects to the teacher's and some students' claims. He was still claiming this on the basis of E_u, meaning that, generally, uncertainty is inherent in all probability problems. He did not construct E_e—expected value. This individual difference highlights an essential theoretical stance underlying what we mean when we say that an idea or way of reasoning functions as if shared. In the analysis of Argument 10, we determined that the idea that the probability of events such as die rolling and coin flipping were theoretically determinable functioned as if shared. This does not mean that all students have identical conceptions, as we see quite clearly in the case of Michael.

4.4 Concluding remarks on Lesson 3

Looking throughout Lesson 3, we can see that the teacher consistently fosters argumentation in the whole-class discussion as a way to foster knowledge shifts. The teacher's moves work together to constitute a broader conceptual category which we refer to as *consensus-building*. Consensus-building as a broad teacher move construct subsumes: eliciting additional ideas, seeking comparison across ideas, re-voicing student contributions, seeking justifications, and legitimating students' ideas. These teacher moves are also complementary; they resonate and work together to promote mathematical progress and make thinking more public and accessible.

In addition, we can say that during the two whole-class discussions which took place in the third lesson, many students raised their ideas. We also were able to document, using a variation of Criterion 1, that determining directly the probability of events such as die rolling and coin flipping functions as if shared, and yet two students did not share this idea (see Michael and Or in Episodes 10 and 12).

We also show an idea constructed by two students in group work, which was shifted by the two of them as one knowledge agent to the whole-class discussion that followed. Yet, we do not have clear evidence that this idea was really uploaded to the whole class. We remind the reader that *uploading* means a shift of knowledge from a group to the whole class. In the following lesson (Lesson 4), we will show processes of downloading from the whole class to a small group.

5 Analysis of Lesson 4—findings

This lesson started with a whole-class discussion followed by work in small groups. We documented the whole-class discussion, as well as the same focus group as in the previous lesson. In this lesson the concept of chance bar appeared. This concept was introduced as part of the probability learning unit because it is used as a prerequisite to the area model later on in the unit. However, in the problems relevant here (the Dreidel Problem and the Coin Problem), the chance bar has a qualitative meaning only. Table 2 summarizes the chronological order of the episodes along Lesson 4.

The whole-class discussion focused on the Dreidel Problem and the focus-group discussion focused on the Coin Problem. Next we present an analysis of episodes from the whole-class discussion followed by an analysis of the focus-group discussion.

Table 2 Lesson 4—summary of analysis

Setting	Episode number	Title
Whole class	Ep. 1	Marking the chance bar for event A of the Dreidel Problem
	Ep. 2	Rebuttal of Itamar's second claim
	Ep. 3	Final discussion of event A
	Ep. 4	Event D, Dreidel Problem
Group work	Ep. 5	Event B, Coin Problem

5.1 Whole-class discussion

The students were given homework between Lessons 3 and 4, in which they were supposed to become familiar with the chance bar. However, many students did not prepare their homework. Hence at the beginning of Lesson 4, the teacher initiated a whole-class discussion about the chance bar and, for the first time, the students were asked to mark a composite event on the chance bar.

Episode 1: Marking the chance bar for event A of the Dreidel Problem

46 Teacher	(Reads the question) “A Hanukkah dreidel (with four letters N, G, H, and P) was spun 100 times.” You know what I am talking about? Remember? Guy, are you with us? “Mark approximately, on the chance bar, the chances of the following events.” The first event, A, says, “the dreidel will fall 100 times on the letter N”.
47 Teacher	Do you understand event A? We spun the dreidel 100 times and <u>all the 100</u> times that we spun the dreidel, it fell on N. (To Itamar) Come mark it, you mark and we will relate.
48 Itamar	It should be … [Stands and ponders near the board].
49 Teacher	You need to write the letter A We can always erase it. That’s easy. Write the letter A where it fits in this event: we spun 100 times, 100 times it fell on the letter “N”.
50 Itamar	[Itamar marks the letter A near 1/4].
51 Teacher	[To the whole class] What do you think? Would you have marked the same place? Someplace else?
52 Itamar	It is supposed to be impossible.
53 Teacher	Hold on, hold on, you marked, and now sit down. Now we will see what the others think, let’s hear. Yes!

To begin the discussion of the Dreidel Problem, the teacher clarified the task and then invited Itamar to come to the front of the class to mark the *expected value* (E_e) of the probability of event A on the chance bar that was on the board. Itamar marked the letter A near the 1/4 location on the chance bar. The marking of the chance bar near the 1/4 location functions as the CLAIM for Itamar’s argument. It seemed that at this point Itamar relied on the probability of the simple event: a dreidel was spun; what are the chances that it will fall on N? At this point the teacher did not ask Itamar to provide a justification (i.e., DATA or WARRANT) for his CLAIM, but instead turned to the class and asked if they would have marked it at the same place or someplace else (51). This teacher move prompted Itamar to put forth a different CLAIM (again without DATA) that the chance of getting 100 N’s is impossible (52).

Thus far, a Toulmin analysis reveals that two CLAIMs had been made, each without any DATA or WARRANT. Itamar’s first CLAIM was incorrect while his second CLAIM was closer to the correct response. Nonetheless, the teacher did not evaluate either of Itamar’s CLAIMs but instead elicited other ideas about where to record the chances of getting 100 N’s with 100 spins (51, 53). As the discussion continued, we see the effectiveness of the teacher’s move to elicit other student ideas for where to mark the letter A.

Episode 2: Rebuttal of Itamar’s second claim

54	Matan	There is a chance that it will happen.
55	Teacher	Would you have marked the letter A in the same place?
56	Matan	No, it is closer to zero.
57	Teacher	It is closer to zero, because …
58	Matan	Because there is a slighter chance that it will happen.
59	Teacher	a slighter chance that …
60	Student	In between 1 and 4 not 1 out of 4.
61	Student	1 out of 100.

Matan rejected Itamar’s second claim and put forth a new claim that “there is a chance that it will happen” (54). As illustrated in this short excerpt, various teacher moves supported the production of students’ argumentation. For example, in utterance 55 the teacher clarified Matan’s idea in relation to Itamar’s idea. In utterance 57 she sought to bring out the reasoning for Matan’s claim (in terms of a Toulmin analysis, she requested data supporting Matan’s claim). In 59 she re-voiced part of Matan’s argument, which likely had the effect of allowing others in the class to engage with Matan’s idea. As we saw earlier in Lesson 3, these teacher moves (eliciting other ideas, seeking comparison of ideas, re-voicing, and seeking justification) represent the type of teacher activity that constitutes consensus-building.

Episode 3: Final discussion of event A

62 Teacher	Pay attention to the situation: you have a dreidel, you spin it 100 times, and <u>all the 100</u> times it falls on the letter “N”.
63 Itamar	That’s correct!
64 Teacher	All the 100 times, you marked that it is approximately a quarter of the cases.
65 Itamar	That’s right, but, it’s true that it is almost impossible, but, ah, relatively to the letter N, there are 4 letters.

66 Teacher	That is correct!
67 Itamar	So N according to this, is one letter!
68 Teacher	What do you have to respond to that? (Repeats Itamar's argument there are four letters but N is just one letter.) Guy, how would you answer him?
69 Guy	There is, like, each time that you spin there is, like, 4 letters it can fall on, so each time it divides again by 4, the chance (Teacher: yes), and the chance decreases, it decreases each time that you spin that it will fall again on the same letter.
70 Teacher	So you are supporting him?! You are saying that the mark is correct!
71 Guy	No, you have to lower it.
72 Teacher	Because …?
73 Guy	Because each time the probability is much smaller, when you spin twice and it falls on the same letter—the probability decreases.
74 Teacher	(Turns to Itamar) do you understand this point? We didn't mark what the chances are that if we have a dreidel with 4 letters it will fall on N, we marked … I will describe the event again—the dreidel will fall <u>all 100</u> times on the letter N. Do you see the difference between these two things?
74a Students	Yes!
75 Teacher	[To the class community] Event A is clear to you? Any comments referring to this event?
76 Michael	I agree with Guy, but still …
77 Teacher	Where would you have marked the letter A?
78 Itamar	Really close to the zero.
79 Teacher	[Marks it really close to the zero] Is it better suitable here?
80 Itamar	Not that close!
81 Teacher	[Moves the mark a little to the right] Is it better here?
82 Students	Agree!
83 Michael	But still we can't be certain!
84 Teacher	That's right, we are saying what the chances are, we don't know anything for certain.

In this excerpt the teacher continued her efforts at consensus-building. These efforts had a considerable pay-off regarding how one can use the chance bar to record the chances of such composite events. For example, in utterance 68, the teacher re-voiced Itamar's statement that there are four letters and N is just one of the four letters, and then elicited others' thoughts about Itamar's statement. We see this teacher move of eliciting reactions to Itamar's statement as setting the stage for bringing out the difference between the chance of such composite events and simple events and how one may use the chance bar to record these chances.

Itamar, as a main player, presented the main potential confusion between the chances of a simple event and a composite event, which is constituted of a repetition of the simple event (E_{re}). He became aware of the fact that the chance of the simple event is 1/4, and that the chance of the composite event is very close to zero. Guy took over and explained the process by which the chance of the composite event becomes smaller each time (69, 73). Guy succeeded in convincing Itamar (78). This provides evidence that Guy was a knowledge agent for Itamar in the whole-class setting. It is worth noting that for Guy to be a knowledge agent, an adequate partner is needed. Itamar, who is going through the aforementioned dilemma, is mathematically mature enough to follow Guy's idea. Having said that, it is clear that not all students constructed this concept. For example, Michael in utterances 76 and 83 was stuck with the idea that "still we can't be sure" (83).

ARGUMENT 14

CLAIM14	Each time you spin, the chances divide by four
DATA14	There are four letters it can fall on
WARRANT14	Each time you spin there are four letters that it could fall on

Rather than accepting Guy's essentially correct argument, the teacher continued her efforts at consensus-building by asking Guy if he agreed with Itamar's mark on the board, which is at 1/4. She did not let correct arguments such as Guy's stand unchallenged. Instead, she required students to think critically across arguments. Correctness therefore came from the strength of argument rather than from the proclamation by an authority.

In utterance 74 the teacher made explicit the difference between how one can use the chance bar to record events: "We didn't mark what the chances are that if we have a dreidel with 4 letters it will fall on N (E_e—expected value), we marked … I will describe the event again—the dreidel will fall <u>all 100</u> times on the letter N (E_{re}—probability of repeated event is different from the simple event)." Itamar, together with many other students, indicated agreement with how one can use the chance bar to record not just single events, but composite events of repetition.

When Matan wanted to mark the chance bar not at the location for a simple event, but rather at the location for the chance of such composite events (56), the teacher requested justification for this way of using the chance bar (57). In other words, justification was required. In subsequent episodes involving these composite events, our analysis of whole-class discussion reveals that no-one questioned or challenged claims that involved using the chance bar this way. Thus, via criterion 1, we found empirical evidence that the classroom community accepted the idea that the chance bar can be used to record the chance of composite events rather than just a simple event. This way of notating the chance bar *functioned as if shared*.

Did this whole-class discussion, and in particular the discussion among Itamar, Guy, and the teacher, have an effect on the students in the focus group? In other words, did the students in the focus group construct knowledge elements E_{re} (the probability of repeated event is different from the simple event) and E_d (the probability of repeated composite event)? In order to investigate this, we will have to follow the focus-group discussion further on in the lesson. First, however, we examine a second whole-class discussion, regarding part D of the Dreidel Problem, which occurred before the focus-group discussion.

Episode 4: Event D, Dreidel Problem

Due to space constraints only a portion of the episode is presented. In the full episode there were 46 utterances, 22 of which were the teacher's utterances.

128 Teacher	Let's look at D. D says, a dreidel is spun 100 times it will fall on N between 20 to 30 times. What do you think, we will spin the top 100 times, how many times will it fall on N? [To Eliana] Come, you haven't marked yet. (Eliana approaches the board and marks on the chance bar close to the middle)
129 Teacher	Adin, what is your opinion, what do you say?
130 Adin	I think that it is approximately 30 %
131 Teacher	That means that you agree with what Eliana is suggesting, explain why!
132 Adin	It has more of a chance …
133 Teacher	So if it has more of a chance you are marking it on the 30, more chance for what?
134 Adin	More of a chance than A, B and C. There is a higher chance that it will happen; it is closer to the middle
135 Teacher	So if there is a higher chance you are marking it close to what? Does anyone feel different, want to support or oppose?… What do you think, Guy?
136 Guy	I think it is much higher (Teacher: how high?) 80 %, because in fact there are 4 sides to the top [DATA], right? And the chances that it will fall on one of them is 25 % and you said that it will fall between 20 to 30, so …
137 Yael	That means that it is 25 % not 80 %.
…	.
141 Omri	What I am trying to see is if I understood Guy: what he is trying to say is that there is a 1 out of 4 chance, that means that it is a very high percentage that it will be between 20 to 30.
…	.
144 Teacher	Can you explain again why you are supporting Guy?
145 Omri	What he's saying (Guy) is that every time you spin there is a 1 out of 4 chance that it will fall on N, meaning, 25 % now out of 100 is approximately the number of times it will fall on the N, because it is 1/4 out of four.
146 Teacher	What do you think? You are nodding yes (turns to Rachel), who do you agree with?
147 Rachel	With Guy.
…	.
158 Matan	I think it is 65 %, because it can be more or less, there is a chance that it will come out and a chance that it won't.
159 Teacher	But you think it is more than half but a bit lower?
160 Matan	Yes!
161 Yael	I am still not sure, Guy succeeds to convince me. But at first I thought it was half, but still …
162 Teacher	You are still not convinced.
163 Yael	I am not sure.
164 Teacher	Let me ask you this, let's say that you spin the top 100 times and count how many times it will fall on N, what result would you expect?
…	.
169 Yael	25!
170 Teacher	That means that you are expecting an answer between 20 and 30, that is what we are expecting will happen. So if it is what we are expecting that will happen so the chance is close to 1.

This episode began with Eliana and Adin sharing their thinking about where to mark the chance bar. Their ideas came mostly in the form of claims without much elaboration of the data that supported their claims. In utterance 136 Guy expressed for the first time in this episode a detailed argument concerning the answer to question D.

ARGUMENT 15

CLAIM15	The probability is much higher (80 %)
DATA15	There are four sides to the top

WARRANT15	The chances that it will fall on one of them is 25 %

Omri in utterances 141 and 145 followed Guy in different words, and additional students were joining (Rachel 147; Yael 161 with some doubts). In fact the teacher herself in utterance 170 closed the discussion related to Guy's argument. Thus we can identify Guy as a *knowledge agent* in this episode. We argue that Guy's role as a *knowledge agent* was contingent upon the teacher's consensus-building efforts that elicited, compared, and clarified student reasoning. Moreover, our coding of the full whole-class episode reveals nine different arguments that were generated by the students. Only at the end of the episode (utterance 170) did the teacher add, mathematically speaking, to these arguments. In total, we identified nine utterances or portions of a utterance in which the teacher elicited students' reasoning (as in 131), five utterances or portions thereof in which the teacher sought comparison of student reasoning (as in 135), and eight utterances or portions thereof in which the teacher clarified or re-voiced student reasoning (as in 170).

5.2 Focus-group episodes in discussing the Coin Problem

In this lesson all three focus-group students (Yael, Rachel, and Noam) were present. They discussed the Coin Problem, in which Tasks A and B fit Tasks A and D, respectively, of the Dreidel Problem (see Episodes 1–4 in Lesson 4, of the whole-class discussion, above). Unfortunately, while the focus group discussed Task A, a teacher-observer interfered in their discussion by asking them questions concerning Task A, so we cannot regard the discussion recorded as belonging organically to the group. Hence, we do not report the group's discussion of Task A of the Coin Problem, nor do we conduct an analysis of this discussion.

Episode 5: Event B, Coin Problem

227 Yael	(reads event B) "Will land with heads facing up more than 450 times and less than 550 times." It's logical.
228 Noam	It's at half!
229 Yael	No, there are much higher chances (marks B close to 1 on the chance bar) …
230 Rachel	That's right, she's correct (supports Yael).
231 Yael	That's what Guy just explained.
232 Noam	Right.

The discussion of event B took only 6 utterances and resulted in agreement between all three students. Yael read the question and immediately recognized the claim in the question as "logical" (227). While Noam claimed that the chance is half, Yael objected and marked the chances on the chance bar near one. Rachel backed Yael, with no explanation, and Yael provided support—"that's what Guy just explained" (231). Noam responded "right" (232). Here we have evidence that Yael (and possibly the other students in the focus group as well) consolidated the intended knowledge element, after constructing it during the preceding whole-class discussion. The evidence for constructing it during the whole-class discussion is strong for Guy and others but only weak for the focus-group students (Rachel in utterance 147 and Yael in utterances 137 and 161). However, the present focus-group discussion shows that at least Yael did construct it then, and that she consolidated it during the present focus-group discussion. We interpret Yael's utterances 229 and 231 as showing consolidation because she used the construct with immediacy and flexibility, adapting it to a new context, which is characteristic of consolidation. Here we have also explicit evidence from Yael (231) that Guy was *a knowledge agent* for Yael, and that Guy's ideas were *downloaded* into the focus-group discussion. The fact that this group used a way of reasoning from a previous argument as DATA for a new CLAIM also has strong connections to criterion 2 for determining when ideas function as if shared.

6 Discussion

As we wrote in Sect. 3.1, our main goal in this work was to identify and understand the processes governing shifts of knowledge in an inquiry classroom. To remind the reader, we focus on a mathematical classroom in which the students are working as a whole-class community as well as in small-groups settings within the same lesson. The students in both class settings work on purposefully designed sequences of tasks intended to afford the emergence of abstract mathematical thinking via discussion.

The need for empirical realization of our goal pushed us towards coordination of both methodologies: DCA, for analyzing whole-class discussions, and RBC+C, for analyzing groups' work. The coordination of the findings emerging from the analyses according to the two methodologies allowed us to study knowledge shifts in the classroom as one continuum, and to trace students who have a crucial role in knowledge shifts in the classroom.

In our case, the empirical phenomenon to be understood was the shifting of knowledge between different settings in an inquiry classroom. The two given theoretical frameworks and associated methodologies described different but closely related aspects of the classroom learning process. This was a case where two separate parts of classroom activity had to be considered together and coordinated, and, for this purpose, a coordination of the two theoretical frameworks and their methodologies was in order. In this stage, we do not aim at a single coherent complete theory but at the analysis of the concrete empirical phenomenon of knowledge shifts. For this purpose, we fitted elements of AiC, in particular the elements of knowledge, with the Toulmin model as used in DCA, and were able to see, among others, how these elements of knowledge took different positions in the Toulmin scheme at different times, thus giving evidence for becoming normative in the classroom. We would like to point out that some elements of the two theories/methodologies became shared along the study: (1) the a priori analysis of the expected learning in the form of knowledge elements served the two empirical analyses; (2) the role of individuals as knowledge agents in shifting knowledge was investigated in the group and in the whole-class settings. In this sense, the components we used were complementary, hence leading to a fruitful use of the networking strategy of coordinating, as predicted by Prediger et al. (2008).

There are advantages to making use of the RBC+C and DCA analyses in the same lessons. The RBC+C analysis has a particular focus on cognitive constructing while the DCA analysis examines how ideas function at the collective classroom level. These are complementary foci and their coordination allows for tracing the growth of ideas from small groups to classroom community and vice versa. Both methodologies adopt Freudenthal's (1991) point of view concerning mathematics as a human activity situated in the class and is inseparable from the cognitive constructions that students make. On the other hand, the RBC+C analysis is limited in what it can reveal about the mathematical truths that function as if shared in whole-class discussion and the DCA analysis is limited in the extent to which one can posit individual ownership or acquisition of ideas. These respective limitations are mitigated by the coordination of the two approaches.

Other complementary distinctions include: RBC+C is more micro and DCA is more macro; RBC+C frames more in cognitive terms, while DCA frames more in activity terms. This also plays out in the a priori analyses. As each methodology is used sequentially for different social settings along the same lessons, the a priori analysis served both analyses and the coordination between them. We tried to demonstrate this useful complementary research, using the data and analyses above. By doing this we offer a way to adapt existing methodological tools in order to coordinate analyses of the individual, the group, and the collective in an inquiry mathematics classroom.

For example in Lesson 4, when a student wanted to mark the chance bar not at the location for a simple event, but rather at the location for the chance of composite events in which a simple event repetition occurs, the teacher requested justification for this way of using the chance bar (57). In other words, justification was required. In subsequent episodes of these composite events, our analysis of whole-class discussion reveals that no-one questioned or challenged claims that involved using the chance bar to record the chance of such composite events. Thus, via criterion 1 of the DCA methodology, we found empirical evidence that the chance bar can be used to record the chance of composite events rather than just a simple event. This way of marking the chance bar functions as if shared. Looking through the RBC+C methodological lens, and the a priori analysis, we can conclude that E_{re} (the probability of a simple event is different from the probability of a composite event that consists of a repetition of the simple event) and E_d (the probability of a composite event which consists of repetition of the same simple event decreases with each repetition) (Fig. 3) were constructed by many students in the class.

The proposed combined analytic approach is significant in that it offers a new means by which to document the evolution of mathematical ideas in the classroom, the processes by which ideas move between individuals, small groups, and the whole class, and the role of the teacher in these processes. This combined analytic approach enabled us to identify shifts of knowledge in the classroom and the students who are main players in these shifts—the knowledge agents.

In the following, we refer to the research questions (see Sect. 3.1):

1. How can knowledge shifts in inquiry classrooms that include small-group work and whole-class discussions be characterized? These include shifts of knowledge that are downloaded from the whole-class discussion into a group's work or uploaded from a group's work to the whole-class discussion.

A previous study in the AiC framework focused on constructing knowledge in small groups working in a classroom. The researchers summarized:

> Although we showed that knowledge was highly diverse and subjective, we could define the shared knowledge of the ensemble in an analytical and objective way [The RBC+C methodology]. The fact that students in the ensemble kept on working collaboratively to construct knowledge and to build-with it in further activities, allowed us to identify this shared knowledge not only *ad hoc* but also post hoc, that is, through the observation of consolidation of the shared knowledge. It may have been the existence of this shared knowledge that enabled the students to continue to interact productively in a succession of learning activities (Hershkowitz et al. 2007, p. 65).

This paragraph expresses the interest which researchers in the AiC framework had in the ways of constructing *shared knowledge* (knowledge which enables the students in the group to continue constructing knowledge together on the same subject). Because the researchers used only the methodology of RBC+C, they were able to trace the flow of knowledge including shifts of knowledge only within the group itself.

In a similar way researchers in the DCA framework traced the macro ways in which knowledge and practices *function as if shared* in the whole-class community. We may say that the DCA research work traced macro shifts in the whole class but did not trace micro knowledge shifts between individuals in the working groups.

In the present work, the coordination of the DCA and RBC+C methodologies enabled us to trace shifts of ideas starting from individual/s in the whole-class discussion and downloaded to the small group or vice versa: initiated by individual/s in a small group and uploaded to the whole class. Here as well, we are aware of the existence of similarities and differences that shifting knowledge might have when it is expressed by different individuals while shifting in the classroom.

2. What are the characteristics and roles of knowledge agents in the classroom?

The proposed combined analytic approach is significant in that it offers a new means by which to document the evolution and the shifts of mathematical ideas in the classroom, and the main roles individuals play in these processes. As we defined above, a knowledge agent is a student who, according to the researcher's observations, first initiates an idea within one classroom setting (the whole-class community or a small group), and the idea is later expressed by others in the same or another classroom setting. This means that in addition to the students who act as knowledge agents, there are other students who are qualified enough or had adequate ability to be inspired by the new idea and to appropriate it. We call the raising of a new idea and its appropriation by another student a *shift of knowledge*. The actions between the knowledge agent and the students who appropriate their new ideas create the shifts of ideas in the classroom, whether from the group to the class community setting (uploading), or vice versa (downloading) or from one group to the other or within a group, or within the whole-class discussion. Occasionally, the roles of the students who "carry" the knowledge shifts are not well delineated; for example, in Episode 6 (Lesson 3), Noam was the knowledge agent who raised the idea of experimenting in the Matchbox Problem and "shifted" it to Yael in the group. But then in Episode 8 (Lesson 3), Noam and Yael together acted as a single knowledge agent, uploading the idea of experimenting to the whole-class setting. As another example, in the whole-class setting, we have Guy as knowledge agent in Episode 3 (Lesson 4). He raised the idea of the probability of the repeated dreidel-spinning composite event and "shifted" it to Omri. In Episode 5 (Lesson 4), Yael downloaded Guy's idea to her group (in spite of the doubts she had expressed in the whole-class discussion).

The personal characteristics of students who may act as knowledge agents require separate research: are they the most able students and/or those who have confidence? curiosity? verbalization ability? Other characteristics?

3. What is the teacher's role in relation to knowledge shifts and knowledge agents in the classroom?

In an inquiry classroom the role of the teacher is compound and has many faces: the teacher orchestrated the classroom dynamics by balancing between the whole class and group work in terms of time and tasks. She kept an equilibrium between the need of teaching a certain content on one hand, and the strategy of affording opportunities for students to construct their knowledge in argumentative autonomic ways on the other. In this work, we mostly highlighted the role of the teacher in consensus-building, and left the other aspects of the teacher as orchestrator of the class learning processes for further analyses. In our analysis of the above episodes we pointed to two aspects of what makes up consensus-building. In particular, the teacher (a) elicits multiple students to originate their ideas, to give conclusions or to give justifications (provide DATA and/or WARRANTs), and (b) is very intentional in her efforts to make clear whether or not a student agrees or disagrees with another student. These are two interrelated teacher's moves that support and promote consensus-building and encourage knowledge shifts carried out by knowledge agents.

The next step in our research might be a long-term study, in which a longer period of learning is analyzed via the two methodologies, so that the coordination between the RBC+C and the DCA will become a useful tool in additional contexts. This subsequent work will take up more directly the role of the teacher as well as tasks in shaping and sustaining inquiry classrooms in which students learn mathematics with understanding. Prior research has examined various teacher discursive moves (e.g., Chapin and O'Connor 2007; Stein et al. 2008) that are believed to support student learning. However, as O'Connor et al. (in press) point out, little strong empirical evidence for the link between high quality classroom discourse and student learning exists. Working with data from longer time spans, our future research seeks to contribute to the small, but extremely important, body of research that links the interplay between individual, small-group, and whole-class mathematical progress. Finally, our focus on knowledge agents and their role in shifting knowledge in an inquiry classroom will provide an entry point to investigate the effects of inquiry classrooms and differing student roles within these classrooms on student attitudes, confidence, and beliefs (Boaler 1998).

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