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ORIGINAL ARTICLE

# Conceptualizing inquiry-based education in mathematics

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**Abstract** The terms *inquiry-based learning* and *inquiry-based education* have appeared with increasing frequency in educational policy and curriculum documents related to mathematics and science education over the past decade, indicating a major educational trend. We go back to the origin of inquiry as a pedagogical concept in the work of Dewey (e.g. 1916, 1938) to analyse and discuss its migration to science and mathematics education. For conceptualizing *inquiry-based mathematics education* (IBME) it is important to analyse how this concept resonates with already well-established theoretical frameworks in mathematics education. Six such frameworks are analysed from the perspective of inquiry: the problem-solving tradition, the theory of didactical situations, the realistic mathematics education programme, the mathematical modelling perspective, the anthropological theory of didactics, and the dialogical and critical approach to mathematics education. In an appendix these frameworks are illustrated with paradigmatic examples of teaching activities with inquiry elements. The paper is rounded off with a list of ten concerns for the development and implementation of IBME.

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## 1 Introduction

Inquiry-based pedagogy can be defined loosely as a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work. Such pedagogy can be used more or less frequently in teaching. However, the term inquiry-based education (IBE) implies that it is possible and meaningful to have it as a dominant feature of an educational programme and that the associated form of learning has certain specific qualities. In recent years, this position has received strong support at the political and socio-economical level, especially in Europe, as attested for instance by the report of Rocard et al. (2007). According to the expert panel behind that report, the way science has been taught is an important cause of "the alarming decline in young people's interest for key science studies and mathematics in Europe", and the first report recommendation is: "The reversal of school science-teaching pedagogy from mainly deductive to inquiry-based methods provides the means to increase interest in science." The EU system has followed up by launching several major projects to support the further development and implementation of IBE in mathematics (IBME) and science (IBSE).[1]

In this paper, we do not enter into an analysis of the political and socio-economic background behind this trend. However, from the reports published and the calls and background documents for projects funded, our impression is that the promotion of IBE is seen as connected to the societal need in Europe for adequate qualifications for developing further high technology-based societies, and thereby for facing the competition with other regions of the

[1] Descriptions of and materials from all these projects can be accessed through the web portal http://www.scientix.eu.

world. The rhetoric used shows evident similarities with that found in the late 1950s and 1960s referred to as the "sputnik shock". However, IBME does not gain momentum from a particular trend in mathematics as was the case with structuralism in the 1960s. Nevertheless, for IBSE we find prominent scientists promoting IBSE from a science perspective—see for instance the French project "La main à la pâte" (http://www.fondation-lamap.org) founded by among others the Nobel Prize winner in physics Georges Charpak.

In the current situation, research in mathematics and science education may play a role as an amplifier and as grounds for legitimation of a political agenda for reforming science and mathematics education, but also as a critical voice. Therefore, we see the strong political promotion of IBE as one important reason to analyse more closely the pedagogical and didactical background and implications of this trend. For research it is relevant to pinpoint the mechanisms responsible for the learning outcome of IBME. In this paper we try to conceptualize the notion so as to make it assessable for pedagogical discussions and operational as a tool for didactical design.

As we will see, IBE can be traced back to complex ideas in the research literature. However, in political processes of implementation complex ideas are often reduced and transformed in order to be easy to communicate and manageable to implement. For IBME this transformation is doubled in the sense that IBE emerged in science education and has only quite recently migrated into our field. We consider it thus a real challenge to understand how the existing theories and forms of practice in mathematics education situate with respect to the tradition of IBSE, and how they can contribute to the conceptualization of IBME. This is the main aim of this paper. For that purpose, we first go back to the emergence of inquiry as a pedagogical concept in Dewey's work and how this work found its way into science education. We also compare Dewey's vision of inquiry with the epistemology of Bachelard, whose influence on science and mathematics education cannot be denied (Bachelard 1938). This comparison helps us to elaborate categories for understanding the diversity of concerns and approaches on which IBME can rely, and the possible tensions between these.

## 2 Historical origin and theoretical roots of IBE

The emergence of IBE is generally attributed to the American philosopher and educator John Dewey (1859–1952). However, the sources of the educational philosophy underlying IBE, namely that education should be for all, stimulate students' interest for learning and cultivate their autonomy, aim at the formation of human beings able to play an active role in the development of societies, and reject traditional teaching practices focusing on instruction and drill, can be found in earlier texts and realizations. We mention here a few of the philosophers to whom Dewey referred: von Humboldt (1767–1835) and Pestalozzi (1746–1827), himself inspired by Rousseau (1712–1778) and his followers Fröbel (1782–1852) and Herbart (1776–1841). All these educational philosophers, and many others, expressed and contributed to a shift in epistemology from seeing knowledge given as faith to knowledge based on thinking, reflection, experimentation and science. Knowledge became something developed by and belonging to mankind, and education became imperative for bringing forward the new modern citizen. This epistemology made it possible to place the learning subject in the centre of the learning process, and ask questions about what types of activities the learner should engage with in order to acquire and develop knowledge which would be useful for solving real-life problems and for making sense of the world.

In this paper, concerning the historical roots for *inquiry*, we focus on Dewey's contribution, because he developed the concept and in particular that of *reflective inquiry* to form a basis for a pedagogical practice, which he also tried to implement in an experimental school project. However, we see Dewey as a representative of a much broader trend in educational philosophy with strong roots also in the concept of liberal education (or *Allgemeinbildung* in German), which around the turn of the twentieth century had strong influence on educational reforms in Europe (Kaiser 2002, pp. 245–247). One of the main challenges for general liberal education as a philosophy was and is to define the general elements or qualities in the sciences which should be put in focus in the school curriculum. "General" here refers to elements that would be of value for living a normal life in a particular culture and society. Thus, IBE can be seen as a possible answer to this challenge. By focusing on inquiry in science and mathematics the students can develop relevant subject knowledge through working with problem situations and at the same time develop general attitudes and habit for inquiry across disciplines.

For Dewey (1938), inquiry is the basis both of discovery and learning. He defines it as "the controlled or directed transformation of an indeterminate situation into one that is as determinate in its constituent distinctions and relations as to convert the elements of the original situation into a unified whole" (p. 108). Here a situation is conceived as the set of interactions between an organism, an individual and its environment, and inquiry as a general process not reserved to scientific activity; most of Dewey's examples refer in fact to daily life or professional activities. The inquiry process develops as interplay between known and unknown in situations where some individual or group of individuals is faced with a challenge. This supposes that some part of the unknown exists in a situation and is being recognized as challenging or simply intriguing; but inquiry can develop only because this part of the unknown can be approached with what is already known, because data and references can suggest hypotheses and inferences. Already these basic ideas about inquiry constitute a major didactical challenge when implemented in educational practice.

Dewey is a pragmatic philosopher trying to overcome the dualism between thinking and action without falling into the trap of empiricism. Hiebert et al. (1996, p. 14) encapsulate this nicely:

> Dewey placed great faith in scientific (and ordinary) methods of solving problems. He referred to the methods by several names including the "experimental practice of knowing" (1929) and "reflective inquiry" (1933). He believed *reflective inquiry* was the key to moving beyond the distinction between knowing and doing, thereby providing a new way of viewing human behaviour.

Dewey sees learning as an adaptive process in which experience is the driver for creating connections between sensations and ideas, through a controlled and reflective process, labelled *reflective inquiry*. The organization of students' experience and the development of general habits of mind for learning through reflective inquiry is thus an essential function of education (Dewey 1938). He implemented this vision of education at a laboratory-school created at the University of Chicago. The importance attached there to action and hands-on activities made "learning by doing" a slogan often used to encapsulate Dewey's vision, which created some misunderstanding. Learning in Dewey's philosophy results from action, but action incorporated in a process of reflective inquiry, and the ultimate value of action lies in the generality that potentially emerges from it. Not all experiences are "genuinely or equally educative". Experiences may even be mis-educative if "they have the effect of arresting or distorting the growth of further experience" or if they are "so disconnected from one another that, while each is agreeable and even exciting in itself, they are not linked cumulatively to one another" (Dewey 1938, pp. 25–26). Therefore, it is essential for the teacher to select appropriate experiences, to guide students' reflections on these experiences so that their educational potential actually emerges, and to organize inquiry activities so that knowledge, in particular subject matter knowledge, progressively accumulates. However, this part of the teaching process is not really theorized in Dewey's philosophy. He put more emphasis on the necessity to organize school activities around real-life situations and through these to link school and out-of-school activities. Such links are expected to increase students' motivation and to allow them to put their out-of-school experience and knowledge at the service of school experiences and inquiries. They have also another essential purpose: the ambition of having education serving the cause of democracy, well explained in Dewey (1916). This we consider to be of actual importance to IBE today, not least because this dimension seems to be neglected or vanishing in many IBE projects.

Summing up, Dewey's educational philosophy resonates with the current discourse about mathematics and science education in Europe as described in the introduction, and it is not surprising that in the philosophy of education there seems to be renewed interest in his work (see Hickman and Spadafora 2009). Dewey offers substantial elements supporting the aim of this paper, and for questioning the reduced vision of IBE most often proposed in policy and curricular documents. Especially we retain:

- the central role given to *reflective inquiry* processes in learning,
- a vision of inquiry as a process which incorporates the determination of the object or problem to be inquired into, mixes induction and deduction, and is reflective in nature,
- a vision of inquiry as a general process concerning daily life and professional practice as well as scientific activities, which is not substantially different in its various contexts,
- the value attached to the pragmatic efficiency of knowledge put into action,
- the importance attached to real-life situations, hands-on activities, and the learners' experience in the implementation of IBE,
- the vision that IBE should develop in a student the habits of minds underlying the inquiry process as a way of being that promotes learning,
- the ambition of having IBE serve the cause of democracy.

Before leaving history, we contrast Dewey's and Bachelard's positions, especially referring to Fabre (2009). As pointed out by Fabre, the pragmatic philosophy of Dewey is in some sense incommensurable with the rationalist philosophy of Bachelard, and thus undertaking such a comparison can appear a strange idea. However, both philosophers share the vision that knowledge is never immediate but results from constructions, that it is an answer to problems which themselves must be constructed, what Fabre captures by saying that: "Bachelard as well as Dewey define an epistemology of problematization and not just an epistemology of problem solving" (p. 112, our translation). For both of them, too, knowledge must be functional and Bachelard defines his philosophy of sciences as a pragmatism of second level, while Dewey insists on the fact that inquiry does not end with "it works", but one must also know *why* "it works", which leads to new questions and conjectures.

After noticing these convergences, Fabre comes to the differences. Two of these are of particular interest for us. Against Dewey's continuist position on the nature of inquiry, considering that scientific, professional and daily life inquiries do not differ in essence, the position of Bachelard is that of a necessary rupture between common sense and scientific thinking. This rupture is captured by the notion of epistemological obstacle that has proven its relevance for understanding learning difficulties in sciences and mathematics (Brousseau 1997). Another and not independent difference resides in their respective vision of genericity. For Dewey, as stressed above, the process of inquiry is a generic process even if its objects and techniques vary from one domain to another. For Bachelard, there is no such generic process. Genericity is progressively built, first through local connections between specific methods and concepts proper to a domain of knowledge, then through structural and functional correspondences between domains at a more regional level.

These differences are not without educational consequences. Dewey's philosophy leads to a practice of teaching based on projects closely linked to students' life and interests, and to the development of inquiry habits of minds considered as generic. Bachelard's philosophy leads to the careful organization of students' experience and inquiries to allow them to face the limitation of common sense, overcome the epistemological obstacles inherent in the progression of scientific knowledge regarding specific areas and concepts, and progressively structure and connect their knowledge, incorporating local constructs into more regional perspectives. We find these epistemological differences of particular importance for IBME.

## 3 IBE in a science perspective

Since Dewey, despite criticisms, IBE ideas have progressively found their way into science education, often under the umbrella of movements such as discovery learning, active learning, open learning, and in reaction to the observed failure of traditional forms of instruction. An important step in legitimizing IBE was the publication of the National Science Education Standards (NSES; NRC 1996) in 1996 in the USA, which called for students to do and know about scientific inquiry. According to NSES:

> Inquiry is a multifaceted activity that involves making observations; posing questions; examining books and other sources of information to see what is already known; planning investigations; reviewing what is already known in light of experimental evidence; using tools to gather, analyze, and interpret data; proposing answers, explanations and predictions; and communicating the results. Inquiry requires identification of assumptions, use of critical and logical thinking, and consideration of alternative explanations and scientific inquiry refers to the diverse ways in which scientists study the natural world and propose explanations based on the evidence derived from their work. (NRC 1996, p. 23)

Texts presenting IBSE start from this or similar generic definitions and emphasize the fact that IBE allows students to behave as scientists. In fact, the science education literature provides evidence for the existence of a wide spectrum of inquiry-based teaching practices, according to the responsibilities given to the students in the inquiry process, and its degree of openness (Barrow 2006). For instance, the revised version of NSES separates inquiry into full and partial inquiries based on the inclusion of the following five essential features (NRC 2000, p. 27):

- students create their own scientifically oriented questions
- students give priority to evidence in responding to questions
- students formulate explanations from evidence
- students connect explanations to scientific knowledge
- students communicate and justify explanations

These five essentials are also part of the foundation for several of the ongoing European projects for professional development of teachers. In the PRIMAS project (http://www.primas-project.eu, 2011) they are embedded in a broader picture capturing what could be meant by an *inquiry-based teaching practice* in science and mathematics; see Fig. 1 and Maaß and Doorman (2013).

Regarding the process of inquiry itself, different descriptions coexist in IBSE, obviously influenced by experimental standards in science; but, as for Dewey, the formulation of questions plays an important role, as well as does the dialectic interplay between inductive and deductive phases. Even when the description has a linear or algorithmic structure, over-simplifying the complexity of the inquiry process, it is generally mentioned that the process of scientific inquiry cannot be reduced to a linear process. Rather, it involves several cycles with revision of previous steps and more or less complex interactions between its different components. Emphasis is put, as in the quotation above, on the relationships that education aims to establish with scientific knowledge, and also on the possible distance between it and students' initial ideas and explanations, well evidenced by research.

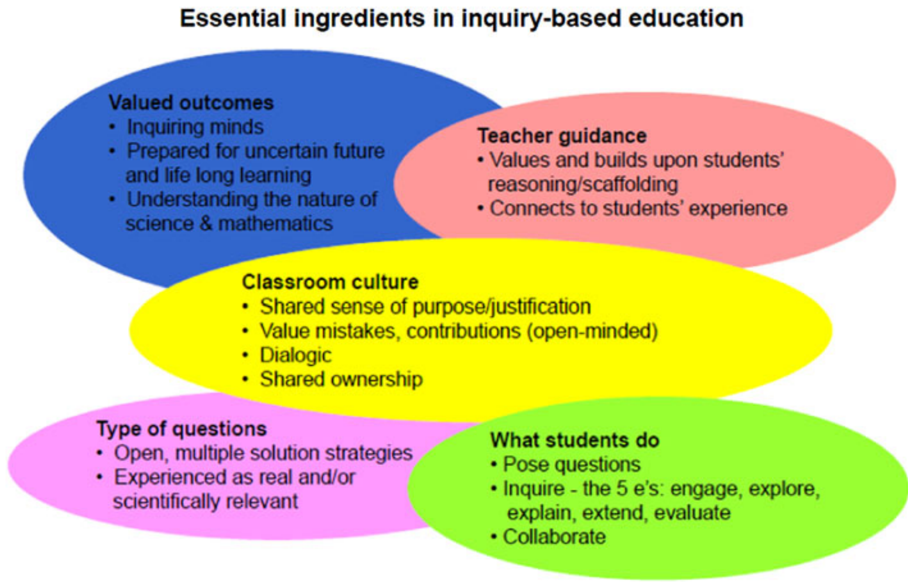


**Fig. 1** The working definition of inquiry-based education in the PRIMAS project (http://www.primas-project.eu)

This is, for instance, the case in a recent document developed within Fibonacci, one of the European projects mentioned above (Harlen 2012). After presenting the nature of scientific inquiry in a way similar to that quoted above, IBSE is described as the process "of building understanding through collecting evidence to test possible explanations and the ideas behind them in a scientific manner" (see Fig. 2).

Without going into the details of the comments accompanying the progressive building of this diagram in the document, we point out three important aspects. The first is the cyclic nature of the process. The second is the importance attached to the generation of "bigger ideas" (see also Harlen 2010):

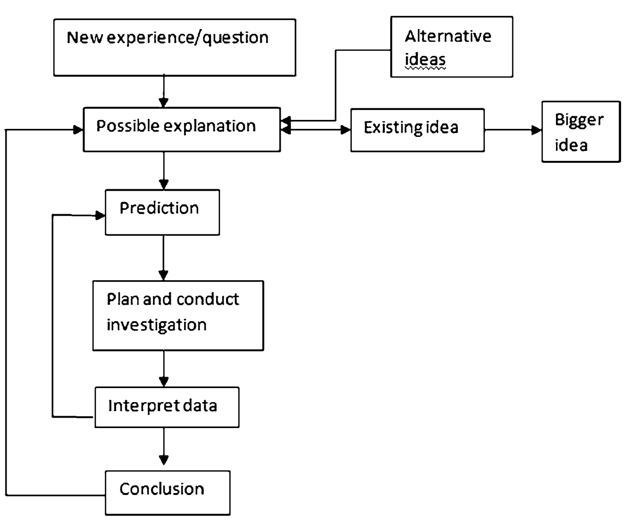


**Fig. 2** A route diagram aiming at capturing the inquiry process in the Fibonacci project (Harlen 2012, p. 5)

> Modelling the building of understanding in this way offers a view of how smaller ideas (ones which apply to particular observations or experiences) are progressively developed into big ideas (ones that apply to a range of related objects or phenomena). (Harlen, 2012, p. 5)

The third is the acknowledgement of possible obstacles in students' existing ideas:

> In doing so, it is important to acknowledge, and to start from, the ideas the students already have, for if these are just put aside the students will still hold onto them because these are the ones that they worked out for themselves and make sense to them. They must be given opportunities to see for themselves which ideas are more consistent with evidence. (ibid. p. 5)

Even if the document does not make any explicit reference to Bachelard's epistemology, it resonates with his ideas, and for instance the importance given to the progressive elaboration of "big ideas" can be interpreted as a way of expressing the movement from local to regional levels in the progression of scientific knowledge. In the same document, it is also acknowledged "that inquiry is only one of a range of ways of learning and teaching involved in science education", but a particularly important one. This raises issues which do not seem yet fully resolved. When is IBSE especially appropriate and what are its exact limits? In the continuum which seems to be offered by the different forms of IBSE, from partial to full forms, are there minimal conditions that educational IBSE practices must obey?

Moreover, coming back to the comparison between Dewey's and Bachelard's perspectives, and the tension it reveals between different possible visions of the relationship between the generic and the specific, between common sense and scientific approaches, the IBSE discourse is not fully clear, giving the perhaps fallacious impression that these tensions can be easily managed. We will come back to this point in the next section dealing with mathematics education.

In fact, in the initial documents presenting European projects we found neither sound conceptualization of IBE nor clarification of the relationships between IBSE and IBME. This situation reinforced our conviction that a reflection as developed in this paper, going back to the historical sources of IBE and analysing the progressive evolution of ideas, might be pertinent.

## 4 The migration towards mathematics

IBE is not traditionally used in mathematics education and its recent appearance seems to have been fostered by the proliferation of projects addressing both mathematics and science education. The adoption of IBE terminology reflects the view increasingly shared that mathematics and science education are closely connected fields, and that mathematics, whatever its specificity, is not a purely deductive science but has also a strong experimental component similarly to natural and life sciences. However, this terminology does not enter an empty space. For several decades, research approaches and teaching practices have been developed for theorizing and promoting mathematics learning with understanding. These theories necessarily impact the way mathematics education endorses IBE today. Without trying to be exhaustive, in this section we evoke some of these, which offer substantial and complementary tools for conceptualizing IBME.

In the attempts made to connect mathematics education to IBE, reference is often made to problem solving (PS), in which there is a long tradition of research and practice in our field going back to the seminal work of Polya. So the PS tradition is relevant for our analysis. But, beyond this tradition, most theoretical frameworks give an essential role to the solving of problems. This is the case for instance for two main theoretical frameworks which emerged in the 1970s: the theory of didactical situations (TDS) initiated by Guy Brousseau; and realistic mathematics education (RME) initiated by Hans Freudenthal (1905–1990). These theories have a lot to offer to the conceptualization of IBME and their affordances are substantially different from those offered by the problem-solving tradition. Considering the importance attached in IBE to the connection with daily life or institutional and societal concerns, it seems also important to consider approaches paying specific attention to this dimension, such as the modelling perspective and the anthropological theory of didactics (ATD). In IBE, moreover, particular attention must be paid to the delicate role of the teacher in supporting and guiding the development of productive inquiry and on how forms of teacher–student(s) interaction contribute to the negotiation of meaning. For addressing this dimension, we rely on the dialogic perspective developed by Alrø and Skovsmose (2002) who also emphasize the critical potential of inquiry. In making such a selection we provide a very partial view of what mathematics education research has to offer for conceptualizing IBME, but in our opinion, a view rich and diverse enough for supporting productive reflections. In an appendix published on the *ZDM* website, we present and discuss examples which illustrate how each of these theoretical frameworks offers particular perspectives on the conceptualization and implementation of IBME.

### 4.1 The problem-solving (PS) tradition

As evidenced by the well-known survey by Schoenfeld (1992a), PS research is primarily interested in the identification and development of the competences and habits of mind which allow students to become successful problem solvers, able to efficiently face non-routine and challenging problems in and with mathematics. In this tradition, the development of PS competences is often considered as a goal per se, not necessarily integrated with the teaching and learning of specific concepts and techniques, and the accent is put on metacognition and heuristics for PS. Connections with inquiry-based learning (IBL) can easily be made. Students facing non-routine problems have to develop their own strategies and techniques; they have to explore, conjecture, experiment and evaluate; they are given substantial mathematical responsibilities and generally encouraged to generate questions themselves and to envisage possible generalizations of the results they obtain; PS competences and metacognitive skills can be interpreted in terms of inquiry habits of minds and related to "the five e's" attached to inquiry in Fig. 1. More specifically, the emphasis put on reflections on methods in Dewey's philosophy (1938) leads quite naturally to a focus on problem solving and related meta-reflection on methods and problem-solving attitudes in mathematics teaching. Knowledge accumulated in this PS tradition is thus of high interest for IBME (see also Schoenfeld 1992b).

PS research is, however, multiform and, in the context of this paper, we find it particularly relevant to mention Hiebert et al. (1996, p. 12) who explicitly refer to Dewey and his concept of reflective inquiry:

> We argue that reform in curriculum and instruction should be based on allowing students to problematize the subject. Rather than mastering skills and applying

> them, students should be engaged in resolving problems. In mathematics this principle fits under the umbrella of problem solving, but our interpretation is different from many problem-solving approaches…. We then propose our alternative principle by building on John Dewey's idea of *reflective inquiry*, argue that such an approach would facilitate students' understanding and compare our proposal with other views on the role of problem solving in the curriculum.

This vision is in line with what Cai (2010) describes as the teaching-through-problem-solving approach where the development of PS competences is not separated from domain-specific objectives in the curriculum. In this approach, according to Cai, teaching starts with exploring a problem likely to allow students to learn and understand important aspects of a mathematical concept or idea. The problems tend to be open-ended and allow for multiple correct answers and multiple solution approaches. After individual attempts under teacher guidance, emphasis is put on the collective sharing and discussion of different attempts and solutions allowing students to discover and discuss alternative approaches and solutions, and to clarify their own ideas. Finally, "teachers make concise summaries and lead students to understand key aspects of the concept based on the problem and its multiple solutions". The mathematical richness of the problems addressed, their relations to mathematical ideas and concepts, and the possibility for the students to engage in the problem-solving activities with their own motives and to obtain results, which can lead to new problems, become important issues in such an approach.

However, when adopting a teaching-through-problem-solving perspective, mathematics education also has a lot to offer through theoretical constructs that have progressively matured since the early 1970s, and we consider two of them below.

## 4.2 Theory of didactical situations

TDS makes us enter into a theoretical framework close to Bachelard's perspective. Brousseau was indeed at the origin of the incorporation of the idea of epistemological obstacle in mathematics education (Brousseau 1997). In TDS, the central object is the notion of situation, which allows a first connection with Dewey. It is defined as the system of interaction between a student or a group of students, a teacher and some mathematical knowledge. As in Dewey, learning is seen as a process where adaptation plays an essential role, but in TDS adaptation and acculturation are necessarily combined. The distinction made between *a-didactic* and didactic situations, the notion of *devolution* and *institutionalization*, help conceptualize these processes of adaptation and acculturation and their interaction. If we compare with the teaching-through-problem-solving approach, despite evident similarities, significant differences must be mentioned. They concern first the criteria attached to the selection of problems. In Cai's description, emphasis is put on open-ended problems engaging important ideas regarding the concept(s) at stake, allowing for multiple correct answers and multiple approaches. In TDS, characteristics for problems aim at optimizing the adaptive dimension of learning and students' autonomy. The mathematical knowledge aimed at should appear as the optimal solution to the problem posed, and, through interaction with an appropriate milieu, students are expected to progressively build it collectively, rejecting or adapting their initial strategies if necessary. The milieu of the situation must offer means for envisaging and performing different actions and receiving feedback, for testing students' strategies and ideas, and supporting the necessary adaptations.

The learning potential of the situation does not rely thus on the same levers: sharing and discussion of a diversity of solutions in one case; optimization in the other case. Moreover, in TDS, emphasis is put on the existence of obstacles which must be explicitly addressed through appropriate situations. The well-known situation of puzzle enlargement is a situation helping students reject the familiar additive model which, at some point, becomes an obstacle to the crucial idea of a linear model or direct proportionality. Globally, this vision is coherent with Dewey's emphasis on the role of the learners' experiences in the learning process but it combines it with a strong epistemological sensitivity, captured by the idea of *fundamental situation*: a situation which makes clear the *raison d'être* of the mathematical knowledge aimed at.

Another important point in TDS regarding IBME is the distinction made between different dialectics: of action, formulation and validation. This distinction reflects important specificities of mathematical knowledge: something that helps us act on our environment but whose power is highly dependent on the specific languages it creates, and where validation obeys specific forms leading to apodictic statements. For having students experience these epistemological characteristics of mathematical knowledge, situations of action and attached pragmatic validation are not enough. Hence, the importance given in TDS to situations

> The example in the appendix, *Taxicab geometry,* illustrates these issues, showing how a system of interrelated problems can form a landscape of investigation for the students' work (Alrø and Skovsmose 2002, pp. 46–67).

of communication where a-didactic interactions with the milieu foster the elaboration of appropriate operational languages, and of situations of validation where interaction focuses on the validity of mathematical assertions. Beyond this attention paid to the epistemology of mathematics knowledge, in TDS emphasis is also put on the progressive organization of knowledge. The long-term didactic engineering on the extension of the field of numbers from integers to rational and decimal numbers developed by Brousseau is emblematic of it. Considering IBME, we also would like to insist on another affordance of the theory: implementing IBME supposes an appropriate didactical contract and the difficulties attached to the progressive negotiation of such a contract, compared with the usual contract, should not be underestimated.

Considering the views of IBE in the two first sections, we would like to stress that the conditions put on the problems implies that they are mainly under the control of the teacher or of the educational researcher and not of the student. In TDS, the notion *devolution* is thus attached to the necessity of making students consider these problems as their own and accept the responsibility of solving them. Moreover, regarding TDS, there is no doubt that the development of inquiry habits of mind, even if it should normally result from the way students' interaction with knowledge is organized and managed, is not a primary goal. The fact that metacognitive concerns only appear in terms of metacognitive shift is among the paradoxes of the didactic contract in the theory and makes clear the distance from the problem-solving tradition in that respect.

become part of students' reality. The term "Realistic" in RME can also be understood in opposition to the "unrealistic" position that abstract mathematics can be taught directly to the general population. If we want all students to learn mathematics in such a way that it could be useful for them in dealing with real-life problems and as a basis for further studies, the realistic approach is to help them construct the meaning of the abstract concepts and methods gradually through mathematization of meaningful real-life situations. Freudenthal would argue that, in general, it is not fruitful to teach the abstract mathematics directly to students. Accordingly, he was very harsh in his critique of the new maths movement.

In RME, two forms of mathematization are distinguished: *horizontal and vertical mathematization* (Treffers 1987). Horizontal mathematization deals with the transformation of realistic situations and problems into mathematical terms and models; while vertical mathematization concerns the reflection and work on mathematizations themselves in which a new mathematical reality develops together with associated techniques and semiotic tools. The relationship between these two forms of mathematization is of a complementary nature, and the mathematization process which combines them can follow different routes. Another interesting distinction is that introduced by Streefland (1985) and then reworked by Gravemeijer (1999) and Gravemeijer and Doorman (1999) between *model-of* and *model-for* for approaching the different roles that a model may play in horizontal and vertical mathematization respectively. According to this distinction, initially a model (e.g. the linear model) is context-specific and attached to a particular situation; it is a model of this situation. But the repeated work with a given model gives it progressively a more generic character; it becomes a more abstract and general framework, a model for reasoning in a category of situations. In fact, Gravemeijer distinguishes four different levels of activities involved in the transition from the status of model-of to the status of model-for: situational, referential, general and formal.

In the RME educational perspective, a key principle is that of guided reinvention (Freudenthal 1991). A learning trajectory must be designed giving students the opportunity to develop their own mathematics and progressively transform informal and meaningful problem-solving strategies into more formalized methods in tune with those of the mathematical community. The teacher's guidance is essential for adequate design and piloting in the itinerary of such trajectories.

> The example *The rope triangle*, shows how inquiry in mathematics can be organized and guided towards a very specific learning objective within the framework of TDS. The example is used and analysed in the Danish part of PRIMAS.

### 4.3 Realistic mathematics education

The emergence of RME can also be traced back several decades to Freudenthal's *didactical phenomenology* (Freudenthal 1973). As evidenced by the acronym RME, the phenomena that invite the learning of mathematics need to be part of students' reality, but *reality* can be interpreted in different ways. Freudenthal (1991, p. 17) makes clear that reality denotes for him what is experienced as real in the meaning of shared and well-established understanding of a situation by a group of learners in a certain stage of a teaching process. Gravemeijer (1999, pp. 155–156) elaborates further on the notion and introduces *experientially real situations* to emphasize that the reality, which should be the point of departure for the mathematization and emergent modelling, can be constructed in the teaching process. These situations can be both real-life and mathematical situations, as mathematical objects progressively

RME is thus a problem-solving approach to teaching and learning which offers important constructs and experience for conceptualizing IBME. It shares with TDS a crucial attention paid to the necessities of conceptual development in mathematics, the importance attached to didactical design and to the role of the teacher, but these are encapsulated in a theoretical construction in which the core ideas are those of mathematization and model. Emphasis is put on the realistic nature of situations and phenomena under study more than on their fundamental character regarding the mathematical concepts. Even if it is acknowledged that mathematical objects and contexts progressively enter students' reality, this certainly also results in a different balance of internal and external contexts in the activities for students, and should make easier the connection with IBSE. Coherently with that perspective, the progression from local to more regional levels of knowledge is seen in terms of progression in the generality of models. Another difference of importance with TDS is that the vision of knowledge development looks more continuous, which certainly impacts learning trajectories and didactical design. Nevertheless Freudenthal's phenomenology has been combined with Bachelard's epistemology by different researchers, for instance Schneider (1991) in Belgium.

The example in the appendix *From the school theatre to the percentage bar* presents one of many concrete and well-analysed examples of RME-based activities in the research literature. However, this particular example illustrates well the RME perspectives for conceptualizing IBME. The example is from an elaborated learning–teaching trajectory on percentage by Van den Heuvel-Panhuizen (2003, pp. 18–29).

In the Mathe 2000 project[2] led by Eric Wittmann we find another slightly different research and developmental programme focusing on the design of learning environments. Here, more emphasis is put on the support of the students' progressive construction of important and very specific mathematical ideas and concepts. This programme, which has been quite influential in Germany at primary school level and in teacher education, refers explicitly to Dewey and Freudenthal as part of the philosophy behind the project. Wittmann (2001) presents and discusses the foundation of the project by placing it in a broader perspective, viewing mathematics education research as a design science.

## 4.4 Modelling perspectives

In RME, modelling, and especially mathematization, plays an essential role as a vehicle for the conceptual knowledge aimed at with no clear distinction being made between mathematization of extra-mathematical situations and mathematization within mathematics. The modelling perspective is increasingly influential in mathematics education research (Blum et al. 2007; Kaiser et al. 2011). The focus is on applications of mathematics in extra-mathematical situations and the fundamental claim is that such applications always implicitly or explicitly involve some form of mathematical model, which establishes the connections between a mathematical system and the extra-mathematical situation. Modelling is thus denoting the often cyclic process through which a mathematical model is developed or can be understood as a model of some particular situation. Although linked to other mathematical competences—such as PS competence—and mathematical knowledge, modelling constitutes a competence in its own right, which needs to be developed through appropriate modelling activities in the curriculum (Niss et al. 2007, pp. 3–8).

In relation to IBME, the concept of modelling offers a systematic way of understanding and working with the relationship between mathematics and problem situations or phenomena in other disciplines and in extra-mathematical contexts in general. From a learning perspective, modelling can thus be a bridge between the mathematical concepts and ideas and real-life experiences. Through modelling activities the learner can make sense of the concepts as well as gain new insights into the problem situations modelled. Although a modelling process can be conceptualized in different ways for different purposes, at least six sub-processes are normally involved (Fig. 3).

Analytically, the modelling process can be described as a cycling process where reflections along the process can lead to changes in previous sub-processes and thereby initiate new loops in the modelling cycle. The similarity with the diagram shown in Fig. 2 is striking. We see a trans- disciplinary structure of a dynamical inquiry process behind both processes. Working with modelling in mathematics and in other subjects can thereby lead to valuable understanding of inquiry as a more general process with different particular realizations in different disciplines and contexts. The notions of reflection and critique are well developed in the modelling perspective and, as addressed in Blomhøj and Kjeldsen (2011), can be related to the relevance and consequences of an actual application of a mathematical model in a given societal context; to the analysis of the modelling process behind an existing

[2] See http://www.mathematik.uni-dortmund.de/ieem/mathe2000/engl.html.

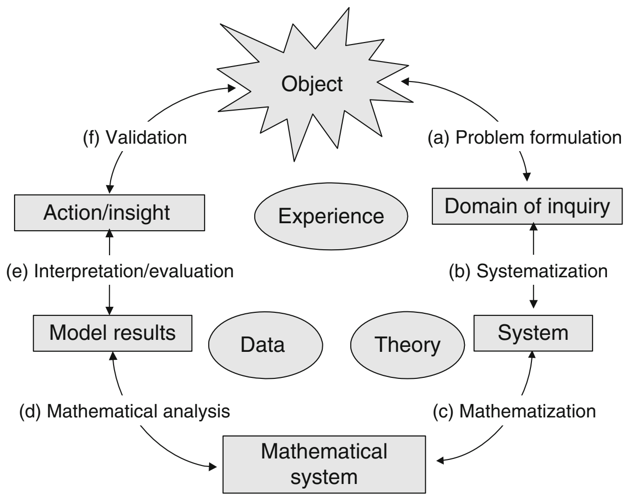


**Fig. 3** A mathematical modelling process with six sub-processes and indication of knowledge base consisting of elements of different epistemological nature (Blomhøj 2004, p. 148)

model; and to each of the sub-processes involved in an actual modelling process.

The modelling perspective on mathematics teaching resonates with Dewey's quest for taking the point of departure in real-life problems and for building on the students' experiences, and also with his thoughts about education for democracy, since mathematical models are playing important roles in the formation of modern societies (Skovsmose 1994, pp. 42–56). There is a need for competences not only for carrying out modelling in the support of the technological development but also for competences for understanding and critique of modelling and societal applications of models.

> and advise him in this pursuit.'[3] (Chevallard and Matheron 2002, p. 148)

This vision inspired the conceptualization of a didactic schema called *Herbartian schema.* From here Chevallard developed the notions *an activity of study and research* organized around the production of an answer to a given question and its extension *a programme of study and research* (PSR) initiated by a question with strong generative power. Chevallard (2011) distinguishes between an *open* and a *finalized* PSR. In the case of a finalized PSR, the initial question is selected in order to make students meet specified *mathematical praxeologies*[4] targeted by the educational institution. For an open PSR, this is no longer the case: the answer to the initial question is not necessarily known in advance, nor is the range of praxeologies that the production of this answer will engage and lead to studying. Moreover, there is no a priori reason that what is needed should be limited to mathematical praxeologies. We enter with open PSR into what Chevallard calls the paradigm of 'questioning the world' that he opposes to the dominant paradigm of 'visiting works' (Chevallard, 2013). In such a paradigm, inquiry is a central idea, as shown by the definition of open PSR in terms of open and codisciplinary inquiry:

> For that reason, I call *open and codisciplinary inquiry* such an inquiry: one creates conditions C—those especially required by the situation of inquiry on question Q—and the praxeological needs that these conditions will make appear will be observed and dealt with. (Chevallard 2011, p. 99, our translation)

| |
|---|
| The example *The asthma drug project* is a quasi-authentic problem for modelling which can be worked with from lower secondary level to teacher education and university level. The project can lead to students' reflection on both the modelling process and on the model's validity in a context of application. |

### 4.5 Anthropological theory of didactics

ATD is also a theoretical frame whose design perspective seems especially adapted to IBME. As recalled by Matheron (2010), the perturbation created by the introduction of interdisciplinary projects in French high schools in the 2000 curricular reform played an important role in the development of this design perspective that, interestingly, Chevallard connected with Herbart's views, as shown by the quotation below:

> The situation seems to echo that which, long ago, Herbart (1776–1841) argued for: 'The university professor no longer teaches (*Lehrender*), the student is no longer taught (*Lernender*); instead, he pursues personal research, the professor's task being to guide

Beyond the inversion of priorities between questions and praxeologies at stake in open PSR, another important feature of this approach which differentiates it from the TDS, even if it emerged in the same educational culture and shares the same epistemological values, is the role given in the inquiry process to answers existing already in the culture. Along the inquiry process indeed, the milieu moves as a dynamic entity. In particular, existing answers

[3] The authors refer the reader to Cauvin (1970). The passage quoted by Cauvin belongs in fact to the *Königberger Schulplan* (Konigsberg's study plan) written in 1809 by Wilhelm von Humboldt.

[4] In ATD, practices are modelled in terms of praxeologies combining a practical block made of a type of task and a technique for solving the tasks of this type, and a theoretical block made of a technology, that is to say a discourse for explaining and justifying the technique, and a theory on which the technology itself is based.

to the questions at stake produced by the culture, which can be found in different sources, and more and more through the use of the internet, can enter the scene at the initiative of the students or the teacher. As cultural answers they have in ATD the status of *media* and they must not be just taken as granted. In order to contribute to the milieu and to the elaboration of the answer ending the inquiry process, they must be questioned, critically analysed and reworked, what is theorized in terms of *media-milieu dialectics*.

Thus, ATD provides a coherent framework that seems able to support the conceptualization of different forms of inquiry processes, more or less open, and to support the local or regional ambitions of these. The structuring of praxeologies in pointwise, local, regional and global praxeologies is also an important tool for that purpose. More than other frameworks, it has also elaborated constructs for approaching the fact that inquiry processes, even when carried out in schools, do not develop in a closed and limited environment; furthermore, it claims that existing cultural resources and answers are generally accessible, and that learning to productively and critically use them is today an essential dimension of IBE. Regarding existing realizations, despite the existence of groups such as the AMPERE group in France, and of several doctoral theses exploring the potential offered by this approach, there is no doubt that the experience gathered is much more limited than that available for the approaches previously described. This is especially the case for open programmes of study and research.

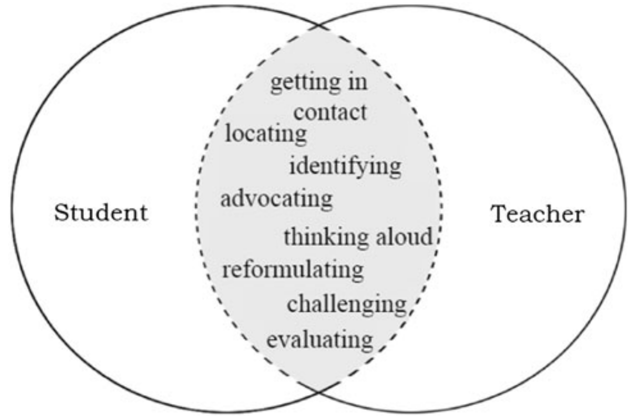


**Fig. 4** The inquiry-cooperation model (Alrø and Skovsmose 2002, p. 63)

fostering and sharing *reflections* and for encouraging and valuing *critique*. These elements are captured in the *inquiry-cooperation model* summarized in Fig. 4.

While compatible with perspectives analysed above, this model adds to our understanding of the importance and structure of dialogical interactions between teacher and students in mathematics teaching, and provides a model which can be seen as a tool for supporting teachers in their interactions with the students. Moreover, in this research strong emphasis is put on reflection and critique as crucial elements in dialogical learning in mathematics. Here, we thus see a potential for linking the micro and local levels of analysis proper to dialogical approaches. For the more global vision of teaching practices for developing citizenship and strengthening democracy the dialogical interplay between teacher and students is crucial. Hence, we need more focus in research on the dialogical aspect of learning in mathematics in order to pursue Dewey's vision of education for democracy by means of IBME.

The example *Teaching mathematics from magnitudes in grade six from areas* is a PSR on areas which illustrates this approach. It belongs to a series of PSR developed at the IREM of Poitiers between 2009 and 2012 around different magnitudes. This project shows how the inquiry process can develop from a few questions with strong generating power and take in charge a substantial part of the sixth grade curriculum.

## 4.6 Dialogical and critical approaches

In Dewey's educational vision, particular attention is given to the interaction between teacher and students and between students themselves in the inquiry process. This is also the case for the theoretical approaches mentioned above. However, in none of these are the dialogical interactions at the core of the theory, as is the case in the work of Alrø and Skovsmose (2002). From analysing classroom interactions and from theoretical analyses, these researchers have identified elements of dialogue which are necessary for communicating and negotiating *intentions*, for

In the appendix the example *Terrible small numbers and salmonella in eggs* illustrates how inquiry in and with mathematics can be a link to the idea of education for democracy in a society which has to deal with different types of risk phenomena. The example also illustrates the role of dialogue in fostering reflection and critique in students' mathematical activities (Alrø and Skovsmose 2002, pp. 195–230).

## 4.7 Inquiry by teachers and inquiry in teaching

In order to be able to plan for and support IBL for students, the teachers need to experience and exercise inquiry in mathematics themselves. Moreover, they need to develop an inquiry stance towards their own teaching, individually, which is very difficult, in teams or in networks (Krainer 2008). Therefore, IBME raises a quest for developmental work and professional development to support teachers in experimenting with and developing their own inquiry-based practice of mathematics teaching. In her work on co-learning communities, Jaworski (2004, pp. 8–9) has investigated the connections between the processes of inquiry at the following three levels in the educational system:

- Inquiry in mathematics: Students in schools learning mathematics through exploration in tasks and problems in classrooms;
- Inquiry in teaching mathematics: Teachers using inquiry to explore the design and implementation of tasks, problems and activity in classrooms;
- Inquiry in research which results in developing the teaching of mathematics: Teachers and didacticians researching the processes of inquiry in mathematics and mathematics teaching.

We find these ideas particularly relevant for designing professional development activities promoting IBME. To some extent the interplay between these three levels is built into the modules for teachers' professional development implemented in PRIMAS (see http://www.primas-project.eu).

For us, these ideas raise a quest for a strong interplay between research and professional development activities in relation to IBME and a challenge to understand better how *communities of inquiry* among mathematics teachers at schools can be established and maintained.

## 5 Towards a conceptualization of IBME

The analyses developed above lead to a broad conceptualization of IBME as an educational perspective which aims to offer students the opportunity to experience how mathematical knowledge can meaningfully develop. Thus, IBME becomes a powerful means of action, through personal and collective attempts at answering significant questions, making these experiences not just anecdotic but inspiring and structuring for the entire educational enterprise. As for IBSE, inquiry-based practices in mathematics involve diverse forms of activities combined in inquiry processes: elaborating questions; problem solving; modelling and mathematizing; searching for resources and ideas; exploring; analysing documents and data; experimenting; conjecturing; testing, explaining, reasoning, arguing and proving; defining and structuring; connecting, representing and communicating. These actions contribute to the students' knowledge and competences, but also to the formation of habits of mind for inquiry. However, the terrain for inquiry in IBME is broader than that of IBSE. Today mathematics is used in many different domains of science and societies in general in ways which are forming our understanding of the world. These applications which cover nearly all fields of human activity are sources of significant questions for IBME at various levels of the educational system. Many daily-life phenomena can be described, investigated and understood with the help of mathematics in combination with science or common sense, and are therefore a rich source for IBME especially in primary and secondary teaching. Moreover, mathematical objects themselves (numbers, geometrical shapes, algebraic symbols, graphs and more and more sophisticated objects) are an essential source of mathematical inquiry as they have been from the origins of this science, and this can and should be included in mathematics from the start of schooling.

Conceptualization of IBME must thus take explicitly into consideration this specific nature of mathematical inquiry and the essential contribution of internal inquiry to the development and structuration of mathematics whose cumulative character as a domain of knowledge should not be underestimated. These characteristics of inquiry in mathematics also affect the forms that experimentation and validation take in IBME. Experimentation can contribute to mathematical inquiry at its different stages, as is made more and more evident thanks to the affordances of digital technologies, and it does so in a diversity of forms that the standard vision of experimentation in IBSE is unable to capture. Moreover, whenever modelling is involved, the inquiry process is submitted to different forms of rationality. Mathematical rationality (Hanna and de Villiers 2012) guides validation for the part of the inquiry process carried out inside the mathematical model, but within the modelling process it necessarily interacts with other forms of rationality which regulate the external world considered.

The elements just mentioned only partially contribute to the conceptualization of IBME from an epistemological perspective. From its origins, an essential aim of IBE has been to promote values of emancipation and democracy. This aim affects the questions which are judged significant, especially through the importance attached to connections to real-life concerns as well as the way these questions are dealt with in the inquiry process, especially the vision of relationships between the different actors potentially involved within and outside the school system.

Conceptualization of IBME must take into account this essential dimension of IBME, considering the inquiry process as a collaborative process, and a process not necessarily limited to the space of the classroom or even the school. Of course, emphasizing this collaborative dimension does not mean that we negate the different institutional positions and roles of the different actors involved in the process, but rather consider their actions as joint actions (Sensevy 2011).

As we have seen, different theoretical frameworks can support the conceptualization of IBME along these lines, and its implementation in practice. As could be anticipated considering the current state of the field, their respective affordances are both partial and overlapping, the resulting picture resembling more a kaleidoscope than a unified structure. Nevertheless, at this stage of our reflections we see more complementarities than discordances and oppositions. Moreover, this theoretical diversity helps us to understand some of the tensions which seem inherent in IBME while at the same time providing tools for dealing with them. These are tensions between the development of inquiry habits of mind and the progression of mathematical knowledge paying necessary attention to curricular progression, tension between internal and external sources of mathematical activities, and tension between scientific and real-life interests. This being said, there is no doubt that IBME is likely to take a diversity of forms, according to the institutional conditions and constraints where it develops. A curricular organization in which specific time is allocated for the development of projects, or where interdisciplinary connections exist, offers more possibilities for implementing ambitious forms of IBME, while rigid organizations and forms of evaluation limit strongly the possibilities of implementing any form of it. Relying on the analyses developed in previous sections, we introduce below a list of *ten concerns* that we think are potentially useful for situating specific versions of IBME. We point to the importance attached to:

- the 'authenticity' of questions and students' activity in terms of connection with students' real life and link with out-of-school questions and activities;
- the epistemological relevance of the questions from a mathematical perspective, and the cumulative dimension of mathematics;
- the progression of knowledge as expressed in the curriculum;
- extra-mathematical questions and the modelling dimension of the inquiry process;
- the experimental dimension of mathematics;
- the development of problem-solving abilities and inquiry habits of mind;
- the autonomy and responsibility given to students, from the formulation of questions to the production and validation of answers;
- the guiding role of the teacher and teacher–student(s) dialogic interactions;
- the collaborative dimension of the inquiry process;
- the critical and democratic dimensions of IBME.

Existing forms of IBME, which rely on different constructs and theoretical tools, developed in different institutional contexts and educational cultures, seem to obey different distributions of priorities of these ten concerns, which produce different variants of IBME. To what extent can their respective strengths be productively combined and capitalized on in order to make IBME something more than a new slogan? To what extent can they be used for envisaging trajectories allowing educational systems and teachers to move in that direction? These questions are still wide open, but we can hope that the papers of this special issue and on-going European projects will contribute to answering them.

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