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ORIGINAL ARTICLE

Problem modification as a tool for detecting cognitive flexibility in school children

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Abstract This paper presents the results of an experiment in which fourth to sixth graders with above-average mathematical abilities modified a given problem. The experiment found evidence of links between problem posing and cognitive flexibility. Emerging from organizational theory, cognitive flexibility is conceptualized through three primary constructs: cognitive variety, cognitive novelty, and changes in cognitive framing. Among these components, changes in cognitive framing could be effectively detected in problem-posing situations, giving a relevant indication of students' creative potential. The students' capacity to generate coherent and consistent problems in the context of problem modification may indicate the existence of a strategy of functional type for generalizations, which seems to be specific to mathematical creativity.

Keywords Problem posing · Cognitive flexibility · Creativity · Change in cognitive framing

Mathematical Subject Classification 97C30

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1 Introduction

In a previous study focused on students' behaviors in problem posing, Singer, Pelczer, and Voica found that the more the student advances in the abstract dimension of the problem and its context, the more mathematically relevant are his/her newly obtained versions (Singer et al. 2011). The present study draws on this idea. More specifically, a group of high achievers was asked to modify a given problem and their proposals were classified based on a set of criteria; this classification led to a description of creative behaviors in problem-posing activities. According to Silver (1994), problem posing refers to both the generation of new problems and the re-formulation or modification of given problems.

Beyond individual differences, high achievers in mathematics display, as a group, a set of common cognitive characteristics, which are challenged in competitions based on problem solving. Are, however, the winners of mathematical competitions indubitably creative? The question is meaningful because these students are usually highly trained and tend to approach problems algorithmically by applying already-known techniques. In addition, success in mathematics competitions supposes rapid reactions, which might be a disadvantage for some students, even highly creative ones (Kenderov 2006).

Consequently, a natural question is: how creative are the students that participate and win mathematics competitions? To study their creativity, we used problem-posing sessions: we postulate that this is a simple context in which the students are in a situation where they can generate (more or less) structured knowledge. In this context, we were interested in the nature of the changes the students proposed on a given problem, in order to understand if and how they might be creative in this type of task. Preliminary research in this direction (Singer and Voica 2011; Singer et al. 2011; Singer 2012; Singer and Voica 2012; Voica and Singer 2012) led us to questioning possible modalities to analyze mathematical creativity through problem posing.

2 Theoretical background

The topic of mathematical creativity has received much attention in the last decades from researchers who focused on defining and characterizing it (e.g., Ervynck 1991; Freiman and Sriraman 2007; Sriraman 2004), or on establishing possible measurement criteria (e.g., Haylock 1997b).

The literature on the topic generally accepts that there are correlations between problem solving and creativity (e.g., Maier 1970; Runco 1994). Silver (1997) argued that inquiry-oriented mathematics instruction which includes problem-solving *and* problem-posing tasks and activities can assist students to develop more creative approaches to mathematics.

The link between problem posing and creativity is still under discussion. Some authors (as, for example, Haylock 1997a; or Yuan and Sriraman 2011) are prudent in considering that there are correlations between creativity and mathematical problem-posing abilities. Other authors (as, for example, Brown and Walter 1993; or Jay and Perkins 1997) sustain that problem posing has the potential to stimulate creativity, possibly even more than problem solving. Accordingly, Csikszentmihalyi (1994) wrote that: "Many creative individuals have pointed out in their work that the formulation of a problem is more important than its solution and that real advances in science and in art tend to come when new questions are asked or old problems are viewed from a new angle…yet when measuring thinking processes, psychologists usually rely on problem solution, rather than problem formulation, as an index of creativity (…) They thus fail to deal with one of the most interesting characteristics of the creative process, namely, the ability to define the nature of the problem" (p. 138).

The nature of the problem is an important issue in a problem-posing context. However, this was not systematically taken into account. Starting from the well-known work of Torrance (1974), recent literature usually explores mathematical creativity through the following parameters: originality, fluency, and flexibility (e.g., Leikin 2009; Kontorovich et al. 2011). To study possible correlations between mathematical creativity of students and their ability to pose problems, we searched for a framework that could better describe these aspects in relation to contemporary trends of the knowledge society.

The students enrolled in our study are winners of mathematics competitions. This quality suggests that they are good problem solvers. However, in many cases, the participants in competitions rely on sustained training and the application of algorithms learned for categories of problems (possibly up to automation). From the perspective of contemporary society, we would be interested in those capabilities that enable students to manage their own learning and to assume identifying and solving problems arising in unpredictable contexts. For this reason, we consider that an analysis focused on key elements of organizational theory can better target the study of creativity.

Within the organizational theory framework, the relationship between problem posing and mathematical creativity can be discussed in terms of cognitive flexibility. Cognitive flexibility is seen as a person's ability to adjust his or her working strategies as task demands are modified (Krems 1995). Cognitive flexibility can be conceptualized as consisting of three primary constructs: cognitive variety, cognitive novelty, and change in cognitive framing (Furr 2009; Spiro et al. 1992).

Cognitive variety refers to the diversity of mental templates for problem solving that exist in an organization (Eisenhardt et al. 2010), or to the diversity of cognitive pathways or perspectives (Furr 2009). Cognitive novelty refers to the concepts pertaining to the subject of study and the students' overall mastery of content (Orion and Hofstein 1994), or to the addition of external perspectives (Furr 2009). Previous experiences, particularly successful experiences, may lead to the phenomenon called cognitive framing: it is manifested by the persistence of trying to solve a new problem through the use of a certain strategy, previously practiced (Goncalo et al. 2010). In certain situations, it denotes an algorithmic fixation (in the terminology of Haylock 1997a).

For the reasons given above, cognitive flexibility might be a good candidate to help in identifying specific features of the creative abilities of high achievers in mathematics. In this study, we explore behaviors that high achievers express in problem-modification situations, relevant to describing students' creativity.

We further explain how the above cognitive parameters could be used in relation to problem posing. In a problem-posing context, we consider that a student proves cognitive flexibility when she or he poses different new problems starting from a given input (i.e., cognitive variety), generates new proposals that are far from the starting item (i.e., cognitive novelty), and is able to change his/her mental frame in solving problems or identifying/discovering new ones (i.e., change in cognitive framing, or even reframing).

Therefore, the more specific questions addressed by this study are: To what extent do high achievers in mathematics manifest cognitive flexibility in a problem-posing activity? What are the characteristics of cognitive flexibility for these children? The investigation of these questions will help us to see how a student's capacity to generate coherent and consistent problems in a problem-modification context might relate to mathematical creativity.

3 Method

3.1 Participants

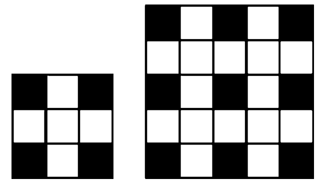
The participants in the study are 42 students from grades 4 to 6 who voluntarily answered a call for problems, from a total of 280 students who participated in a summer camp. The students in the camp were selected from 62,844 students via a two-round national competition. In each round of this competition, students had to answer 30 multiple choice problems with various degrees of difficulty, taking into account that wrong answers were costing penalties. About 10 % of all students in grades 4–6 in the country (cohorts of a total of approximately 600,000 students) participated in that competition and the winners (i.e., the students who achieved the highest scores in their classes in the second selective round) were invited to the camp. Therefore, compared with other students of their age, the participants in the camp possess mathematical skills above the average of the school population of that age.

3.2 Task description

Among many activities during the summer camp, children were asked to modify a given problem; we used therefore a *structured* problem-posing situation (in the terminology of Stoyanova and Ellerton 1996). To ensure that the students based their proposals on the global understanding of the initially given problem, and not just on a few formal changes of its text, we asked them to include the solution of the problem and of their proposals as well. Students were asked to pose as many problems as they wanted, starting from the given problem, to explain how they modified the initial problem, and to submit their proposals 2 days later. As a result, we received responses from 42 students, which are the sample of this research. Subsequently, we have interviewed 18 respondents. The choice of the respondents was made according to the nature of the problems they posed, their comments and solutions.

The following problem was initially given to students as a reference point. We will identify it as *the starting problem*:

> A squared kitchen floor is to be covered with black and white tiles. Tiles should be placed on the floor so that in each corner is a black tile, there are only white tiles around each black tile, and the number of black tiles has to be the biggest possible (in the picture there are two examples of such coverage). How many white tiles are needed if 25 black tiles are used?



We considered this starting problem to be of a medium level of difficulty because in the contest where it was provided to students (as a multiple choice question), about 31 % of the 31,998 participants in grades 5 and 6 indicated the correct answer.

From a mathematical view, this problem is based on three mathematical concepts: area, combinatorics, and recurrence relations. More specifically, for this problem the concept of area is connected to the idea of surface coverage and lies in the need of a compact surface. The problem question, which does not mean to count piece by piece, but to identify an effective method of counting, links this problem to combinatorics. Solving the problem involves identifying the fourth rank term of a recursive sequence whose first two terms are given both iconically and verbally in the problem.

3.3 Technical definitions

In general, when a student modifies a given problem, she/he changes some of the elements of that problem. The analysis of these changes will be made using the framework of Singer and Voica (2012). According to this framework, the text of a problem contains, in general: a background theme, parameters, (numerical) data, one or more operating schemes (or, simply, operators), constraints over the data and the operating schemes, and constraints that involve at least one unknown value of the parameter(s).

We describe these elements below for the starting problem:

- The *background theme* represents the general context in which the problem happens or is described; it simply means "what the problem is about". In our problem, the background theme refers to tiles placed on a kitchen floor.
- The background theme of a problem is characterized by one or more *parameters*; here, these parameters are: the pattern size, the variation of the total number of used tiles, and the variation of the number of black, respectively white, tiles.

- *The data* are (numerical or literal) values associated to the parameters; in our case, 25—representing the number of black tiles of the pattern.
- *The operating schemes* are actions suggested by the text of the problem; in our case, the operating scheme requires "coverage without gaps".
- *The constraints imposed over the data and the operators* are restrictions that state the relations of the background theme with the data and the operators. In our case, the constraints are given by the compliance of the described pattern, i.e., the placement of tiles on a square grid, where there are only white tiles around each black tile and in the corners are placed only black tiles, and the condition of maximality of the number of black tiles.
- *The constraints that imply at least one unknown value of a parameter* are those restrictions that state the relations among the data, the operating schemes, and the problem question; for us, it is about the relationship between the number of white tiles and the number of black tiles, according to the above construction (the number of white tiles of that pattern, if 25 black tiles are used).

To analyze the intrinsic quality of the problems, we postulate that a student's posed problem in a school context must be clearly formulated and solvable with the age-appropriate learning abilities. Therefore, we use two criteria, coherence and consistency, to classify students' proposals. Below, we briefly describe these criteria, to which we will return with examples in Sect. 4.2.

The coherence of a problem refers to its syntax; it refers to the rules and principles that govern the structure of a mathematical problem. More specifically:

- The text components (data, operations, constraints) are presented or can be identified in the text.
- The text components (data, operations, constraints) are recognizable as fulfilling their specific functions.
- The data are not redundant, or missing.

The consistency of a problem refers to its semantics; it supposes the existence of meaningful links among the elements of the problem. More specifically:

- The problem data are not contradictory.
- The text components (data, operations, constraints) are correlated.
- The elements of the text satisfy a certain assumed mathematical model.
- Information provided leads to at least one solution of the problem (or to the proof that there is no solution)

Within the problems obtained by modifying a given problem, consistency also requires that:

- At least one of the mathematical elements of the starting problem is identifiable.

3.4 Design and procedure

The qualitative analysis used in this study is based on analytic induction (Johnson 1998). Following the principles of analytic induction, the problems posed by students were carefully analyzed in order to determine some general categories. The students' posed problems have been analyzed in two stages: first, with the purpose to get a general view on students' approaches; and second, a more focused analysis based on the criteria that emerged at the first stage.

During the first stage, we looked at: the way of reasoning in solving the original problem; the accuracy and completeness of the statement of the posed problem; the students' explanations and comments on how they changed the starting problem; and the content and structure of their proposed solutions. Following this analysis, we identified types of problems, classified according to: the conceptual or procedural frame the students refer to when modifying the starting problem; coherence and consistency of the posed problems; and the mathematical concept most persistent in the modified problem.

In some cases, the proposal we received did not clearly communicate the student's view in posing that problem: more specifically, there were situations where the problem was incoherent, or the wording contained errors. In these cases, we related the problem statement with drawings, solutions, or patterns made by the student (if any), in order to acknowledge the student's intentions. In this way, we were able to include ambiguously formulated problems in a particular category.

After analyzing the students' proposals, we invited 18 students for interview. The main criterion for the selection of students for interview was the developmental potential of their proposals; thus, we chose students who proposed correct problems and solutions, but also students who made some interesting statements, even mathematically incorrect ones. For this second category of students, we wanted to see if errors were related to an insufficient explanation of a mental model, or were deep misunderstandings related to their mathematical skills and knowledge.

We structured the protocol interviews around questions such as: What did you change compared with the starting problem? Can the given data of your problem be changed? (And, if yes, can you devise new data that fit your problem?) Are there redundant/insufficient data in your proposal? Can you define a more general situation? Is there any interesting particular case? What happens if you change a small/large part of the problem?

We began the interview session by asking students to re-read their initial proposals. After this reflection time, the discussion started individually based on the interview protocol, but allowing children to express freely their own ideas. During each interview, a variety of supplementary questions were addressed, in order to depict students' ways of thinking. In addition, each student was asked to devise new versions of his/her own problem(s).

Each interview took from 10 to 30 min. Some students came back with new ideas after the interview sessions, during the next days, and the discussion continued in these cases. The interviews were video recorded and subsequently transcribed.

The initial classification of ambiguous or poorly formulated problems was based on assumptions regarding the students' wordings. We used interviews to question students concerning their proposals. In all these cases, we found concordance between students' intentions (which they did not translate into the proposed text with enough clarity) and our assumptions. Although not all the students were interviewed, we expanded (in the sense of analytic induction—see Johnson 1998) these findings to the entire sample, and thus we validated the classifications initially made.

4 Results

As said above, 42 students proposed changes to the starting problem, thus generating new problems. Some students suggested several versions in their initial proposals, and other students were explicitly asked to do so during the interviews. Consequently, a total of 66 problems have been developed by modifying the initial problem.

We analyzed the frequency of the changes for each component of the problem text (background theme, parameters, numerical data, operating schemes, and constraints). This analysis gave hints about the nature of the proposed modifications and the way in which these modifications changed the solving of the new problem compared with the starting one. We thus determined the coherence and the (mathematical) consistency of each posed problem.

In some cases, the problems obtained as a modification of the given one were predictable because they were on the surface, for example a few students only changed the colors of the tiles used in the starting problem. In many other cases, however, the changes were more consistent, showing various strategies that the students were able to use in problem modification.

4.1 Conceptual frames that emerged in modifying the starting problem

A brief analysis of the proposals shows that students are particularly innovative in changing the theme of the starting problem: they propose problems that refer to, for example, athletes in training formation, trees in the orchard, cages at the zoo, etc. We did not pay attention to these changes that we considered irrelevant to our analysis, i.e., we did not make a classification of how the details that do not have mathematical consequences are changed. We noted, however, that most of the posed problems (58 of 66) refer to the arrangement of objects in the configuration suggested by the starting problem (but not necessarily identical with that). This brought us to conclude that the starting problem operating scheme (covering a square grid with non-overlapping pieces) is the text element most persistent for students.

A careful analysis concerned the constraints the students introduced in the text. The operating scheme of the starting problem is materialized, according to children's options for certain constraints, in a few types of configurations that occurred with relatively high frequency within the set of the posed problems of our sample. The fact that students come to pose problems with similar configurations shows the existence of certain frames of thinking in this group. We use the term *cognitive frame* as a conceptual and/or procedural mental reference system on which an individual relies when acting and taking decisions.

The analysis described above resulted in the identification of five categories of problems, differentiated according to the dominant cognitive frame the students seemed to rely on when they modified the starting problem. In the following, we accompany the description of each category with one significant example, in order to better clarify specific features.

For some students, the alternation of black and white tiles in the given figures of the initial problem is strong enough to lead them to use *the most familiar pattern* of this type, namely the chessboard model: in this case, the students have changed the tiles alternation rule and the numerical data. For example, Teofil (grade 6) suggested the following:

> *Problem 1.* For a pavement one uses white and red stones. Knowing that a red stone is surrounded by four white (Figure from below [this was the indication made by Teofil]) and that there were 63 red stones used, find the number of white stones that have been used.

We see that Teofil posed a partially ambiguous problem (because for the red stones in the perimeter, the condition "is surrounded by four white stones" does not apply). However, Teofil's drawing clearly shows that he thought of a model of a chessboard table. Many students of our sample

Fig. 1 Successive drawings made by Alexandru to explain the solution of his posed problem. He first made the drawings on the first line, then those on the second line

resorted to a chessboard table in posing problems. We considered it as a cognitive frame and, consequently, these problems were put into the category of *chessboard type*.

Other students kept the operating scheme and the constraints of the starting problem but varied the data. For example, Emilia (grade 4) suggested the following problem:

> *Problem 2.* On the square wall of the kitchen there are green and blue tiles. They are arranged so that each green piece is surrounded by blue pieces, and the number of green tiles is the biggest possible. The green pieces are in the corners. How many tiles are blue, if 36 are green?

We can see that Emilia actually resumed the text of the starting problem: her proposal preserved the configuration from the initial problem, but changed the number 25 to the number 36. Emilia proved that she understood the role of each constraint, not only because she mentioned them in her wording, but also because she rearranged the text, keeping all the constraints but in different order and formulation. Many students of the sample carefully resorted to the configuration of the initial problem, as Emilia did. We considered this as evidence that the cognitive frame in which they built the new problem is the same as the one of the starting problem. Therefore, we grouped these problems in the category of *the starting problem type.*

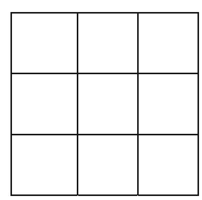
Other students went further from the starting problem model and proposed new configurations of a repetitive/iterative type (but different from the chessboard pattern). For example, Alin (grade 5) posed the following:

> *Problem 3.* Fairy's Flowers planted tulips and roses in the rectangular garden of her palace in a special way. Around each rose she planted tulips only (front, rear, side, oblique). Unfortunately, the garden is small so she only planted ten roses, and on the edges she planted tulips. Find out how many tulips are in the garden.

In this posed problem, Alin varies the data and the constraints of the starting problem. He uses a repetitive pattern to generate a configuration. Such iterative descriptions appear in a significant number of the analyzed papers. We classified these proposals into a *new recursion frame* category.

While the above-presented categories and configurations involve the handling of "objects" (ceramic tiles, animal cages, sportsmen, etc.) that merely differ by one parameter (e.g., by color or size), the following categories group problems that refer to much more varied "objects". Thus, some students retained from the starting problem only the operating scheme: filling a square grid, and changed all the other elements. For example, Paul (grade 4) posed the following:

> *Problem 4*. We have a 3×3 square as in the next figure. Knowing that on the first line there are odd numbers, on the second line there are even numbers, and on the third both odd and even numbers and that the sum of the first column is 3, the second is 14 and the third is 19, find the sum of all numbers and the numbers. Use numbers from 0 to 8.



The operating scheme of Paul's problem refers to filling in a grid of squares. The 9 "objects" handled by Paul (i.e., natural numbers from 0 to 8) are not interchangeable, as happens in the previous cases (where tiles were identical as shapes), but are all different and particular, and their layout on the grid takes into account their differences. In problems of this kind, students take some more degrees of freedom, which have the effect of changing/relaxing the constraints. However, there is a common pattern to these problems: they refer to a rectangular grid of squares, where placed "objects" are different. We included the problems displaying this layout in the *grid frame* category.

We also identified instances where the operating scheme of the starting problem is deeply changed—this happened in only three cases. For example, Cristiana (grade 6) suggested the following problem, which only keeps the idea of recursion from the starting problem:

> *Problem 5.* Maria must continue the sequence: 2, 12, 1112, 3112, 132112,… The teacher gave her some advice: "You must empty your mind of all other mathematical information." Can you help Maria solve the problem?

In *Problem 5*, Cristiana retained from the original problem nothing other than a mathematical concept (the recurrence relation), which she further used to pose a new problem. She even addressed a nice advice to possible solvers to make them focusing on the recursion idea.

In conclusion, we classified the problems of the sample into five categories, according to the dominant cognitive frame that emerged from the posed problems and their solutions. These categories are:

1. problems using the starting problem configuration ("starting pattern frame");
2. problems using the chessboard configuration ("chessboard frame");
3. problems that used a new, different distribution pattern ("new recursion frame");
4. problems based on filling a grid with different objects ("grid frame");
5. problems that do not appeal to a configuration ("other").

Table 1 contains statistical data referring to these five categories.

The relatively small number of categories into which we classified the posed problems shows that the modifications of the starting problem induce certain thinking patterns (at least in our sample). Thus, some students remained trapped in the procedure requested by solving the starting problem, while other students tried to move further from the initial pattern. The data of Table 1 show that most of the posed problems (about 77 %) are of a recursive type, being placed in the first three categories. We can also see that about half of the problems (47 %), i.e., the problems of the first two categories, remain in a familiar frame while the rest of them are based on new frames.

Table 1 Posed-problem distribution by category

Category	Starting pattern frame	Chessboard frame	New recursion frame	Grid frame	Other
Number of posed problems	21 (32 %)	10 (15 %)	20 (30 %)	12 (18 %)	3 (5 %)

The percentages refer to the total number of the posed problems

4.2 Coherence and consistency of the posed problems

The emergent categories identified in the previous section are as yet insufficient to infer that these students developed a consistent cognitive frame of the problem under discussion. So far we have tried to identify conceptual categories to include the students' problems; we now further focus the analysis on the intrinsic quality of the posed problems, which we express in terms of coherence and consistency.

In the following paragraphs, we use one of the posed problems (Problem 4 from above, posed by Paul) to clarify the use of this taxonomy. Briefly, Paul's problem requires the distribution of (natural) numbers in the given grid, subject to certain restrictions. The connection to the starting problem is made through the operating scheme, that is, covering a square grid (which here means placing a symbol in each square of the grid). Based on Paul's solution to his problem we supposed that he intended to ask for identifying a distribution that verifies the constraints of the wording—a supposition that was confirmed during the interview. We classified Paul's problem as consistent since:

- The data are not contradictory. For example, by adding the sums of the three columns, one really gets the sum of the numbers to be distributed.
- The data correspond to a valid pattern: nine numbers must be distributed in a grid with nine boxes.
- In addition, the problem has solution (which is not unique!), and one of the concepts associated with the starting problem—namely combinatorics—is used in solving the problem.

Therefore, Paul's problem is mathematically consistent because it satisfies all the above conditions. However, we classified it as lacking coherence, for the following reasons:

- One of the data/constraints is missing: it is unclear if the (natural) numbers from 0 to 8 are used once each, or numbers can be repeated.
- The questions are ambiguously formulated. The first question ("find the sum of numbers") can be answered without actually solving the problem—therefore this is a kind of redundancy, and the second question ("find the numbers") is unclear—the numbers are the ones from 0 to 8.

For these reasons, we classified Paul's problem as incoherent, but consistent. In an analogous manner, we classified all students' questions on the basis of coherent/consistent criteria. The statistical data corresponding to this classification are presented in Table 2.

Without claiming that this is a statistically valid study, some comments can be deduced from this table in association with the qualitative analysis. We will do this in the next paragraphs, while a more detailed analysis follows in Sect. 5.

Most coherent and/or consistent problems were in the category of the chessboard pattern (50 %), followed by the starting pattern frame (48 %). The chessboard is assimilated by the students with a familiar pattern. For example, Paul (grade 4) gave us the explanation below during the interview about one of his problems on the chessboard model:

> *Interviewer*: But why did you pose a problem with a chessboard, if there was [in the starting model] a problem with a kitchen and…?
> *Paul*: Because it seemed to me that the drawing is like a chessboard. I changed a bit, to look like, and then I realized my problem.
> *I*: But is this drawing (i.e., chessboard type) similar to the given one? What makes them similar?
> *P*: There were arranged squares on the columns…

Overall, a significant percentage of problems (23 %) were mathematically consistent, but incoherent. This shows that, although these high-achiever students effectively handle mathematical concepts and problem-solving techniques, they fail to transpose those in clear and concise wordings.

4.3 Persistence of math concepts in the posed problems

As noted above, our starting problem is based on three mathematical topics: area, recurrence relation, and combinatorics. In this section, we consider to what extent these three topics have been retained in the newly issued problems.

The concept of area is found in the compact coverage of a surface (without overlapping or gaps). This concept is marginal in the starting problem, because essential there are the description of the recursive pattern and the combinatorial elements used in counting. In our sample, only two of the students refer to the concept of area in their proposals.

The recurrence relation occurs, in the starting problem, both through the given drawings which are the first two terms of a series of geometric figures, and the verbal description of the pattern. Therefore, the idea of recursion is highlighted twice. For this reason, we expected that the recurrence relation would have a large frequency within the set of the posed problems. Although 77 % of the posed problems are based on recursion, we found that steps of a recursive development were explicitly included in the text of the proposals in very few cases.

One among the few exceptions is the problem posed by Alexandru (grade 4):

> *Problem 6.* For a puzzle, only red and purple pieces are used. Knowing that each purple piece is surrounded by only red pieces, that in each corner there is a purple piece and that there are 25 purple pieces, determine the number of red pieces.

At a first reading, Alexandru's problem seems very close to the starting problem: only the constraints regarding the square distribution and the maximality are missing. However, analyzing Alexandru's solution, in which drawings of various sizes successively appear (Fig. 1), we found that, in fact, he developed another mental pattern that he could not explain in words.

Alexandru's problem is fundamentally based on recursion. His drawings describe the recursive pattern he imagined: at each step, the previous drawing is completed by adding a "ring" outside. Recursive thinking is also suggested by another detail. The first two calculations, although poorly explained, lead to the correct result. Alexandru observed a certain regularity, i.e., that in cases where the sides are each $n = 3$ or 5 units, the number of "red" squares (these are the white small squares of his drawings) is computed using the formula $(n - 2) \times 4 + 4$, and then he extrapolated (erroneously!) this formula to the next cases. This shows again that he focused on the idea of recursion and tried to find "universal" formulas of computing.

Table 2 The distribution of the posed problems on the basis of the coherence/consistency criteria

Criteria	Starting pattern frame	Chessboard frame	New recursion frame	Grid frame	Other	Total
Inconsistent and incoherent	5 (24 %)	1 (10 %)	7 (35 %)	3 (25 %)	1 (33 %)	17 (26 %)
Coherent, but inconsistent	2 (9 %)	1 (10 %)	1 (5 %)	4 (33 %)	–	8 (12 %)
Incoherent, but consistent	4 (19 %)	3 (30 %)	4 (20 %)	3 (25 %)	1 (33 %)	15 (23 %)
Coherent and consistent	10 (48 %)	5 (50 %)	8 (40 %)	2 (17 %)	1 (33 %)	26 (39 %)

The percentages refer to the total number of problems from each category. The last column contains the number of problems of a certain type and their percentage of the total number of posed problems

A recurrence relation is also the essential mathematics element in the next problem, posed by Matei (grade 5):

Problem 7. We have the square next to here (note: *the figure drawn by Matei is a 10 × 10 grid, numbered horizontally A, B, C,…, and vertically 1, 2, 3,…*). We have to cover it with black in this order: A1, A2,…, A10, B1, B3,…, B9, B2, B4,…, B10, C2, C4,…, C10, C1, C3,…, C9, D1, D2, D3,…
(a) What is the 48th covered box?
(b) When will the G7 box be covered?

In this problem, we note that recursion appears explicitly as a way to build the wording. The problem keeps the background theme (grid squares) and the operating scheme (coverage with no gaps) of the starting problem. It is interesting that Matei uses the same word (coverage) to describe the operating scheme of the new problem, when in fact, more appropriate would be "numbered" or "scrolled".

Another key concept of the starting problem is combinatorics. In the starting problem, combinatorics appeared in connection with the counting of the pattern tiles. Some students posed problems that emphasized exactly this component of the starting problem. For example, Miruna (grade 5) suggested the following:

Problem 8. A dance show will take place in a rectangular room. The dancers' costumes will be of two colors: purple and pink. Choreography was made so that in every corner of the scene is one dancer dressed in purple and the purple dancers will be surrounded by only dancers dressed in pink. How many dancers are dressed in pink, if those dressed in purple are 48?

Miruna kept the distribution of the starting problem (ignoring the maximality constraint), gave up the constraint on the square shape of the model and changed numerical data. These changes—apparently small—required a different approach to solving the problem. More specifically, in solving her problem, Miruna decomposed 48 as a product of two factors, then she identified effective arrangements that correspond to each decomposition, and finally, she found the solution for each case.

5 Discussion

As we have seen above, some of the problems posed by the students of our sample were coherent and consistent, while others did not fulfill these qualities.

Analyzing the data from Table 2, we find that the percentage of coherent and consistent problems decreases as students move from the familiar frame of the starting problem. The percentage of consistent and coherent problems in the category chessboard type frame is even higher because, actually, this is the closest pattern to the problem, seen in a familiar environment. We believe that this situation is generated by operating within a familiar frame, where most constraints imposed over the data and the operators and the constraints on the parameters do not need to be explicitly presented in the wording. For example, resorting to the chessboard table, the maximality constraint on the number of black tiles is automatically fulfilled, and the constraint concerning the positioning of certain tiles in the corners is irrelevant, because it arises from the correlation with the chessboard size.

A frequent reason for lacking coherence is ignoring the maximality constraint (the constraint regarding the maximal number of black tiles, or its equivalent in the posed problems). Relatively many students did not explicitly put this constraint in their wordings, but they have used it in the drawings meant to help in solving the problem. This shows that in decoding the starting problem, students (of these ages) especially focused on the iconic code of processing: they understood the constraints of the starting model mostly with the help of graphical information (given in images), and less based on linguistic information (the wording itself).

On the other hand, the data of Table 1 show that about half of the posed problems remain in the familiar frame of the starting problem or of the chessboard frame, while the problems using different patterns are more dispersed.

Many students are inventive in changes not related to the mathematical nature of the posed problems: most often, they change the background theme, numerical data, or irrelevant elements (such as the colors of the pieces). When students make changes that affect the mathematical nature of the problem, they often lose consistency, a fact observed both in the initially posed problems, and in interviews. In analyzing the problems posed by each student, we noticed that, roughly speaking, as novelty increases, the quality of the problems decreases. This is also visible in the statistical data. Thus, 53 % of the posed problems are based on structural changes of the original problem—which means greater novelty—but only 39 % of them are coherent and consistent. Moreover, only 11 (31 %) of the 35 problems with a frame structurally changed compared with the starting problem are coherent and consistent.

We interpret these data as evidence of the fact that some of the students in the sample exhibit cognitive novelty, but at a low, non-spectacular level.

Comparing the problems posed by the same student during the initial task or during the interviews, we found that, in most cases, once a type of change has been made, the student then proposes variations of the same nature. For example, if he/she first changed the theme, when asked to formulate a new problem, the student changes the theme again. Or, if first he/she posed a problem of the chessboard frame category, then he/she proposes a problem falling into the same category. We believe that, globally, these data show that for the students in the sample, cognitive variety is limited. This conclusion is not surprising: in the context of problem modification, where the problem-posing process is structured by the imposed model, presumably it induces greater rigidity than a free problem-posing situation.

When correlating cognitive novelty and cognitive variety with the product quality (coherent and consistent problem), we can see that changes in cognitive frame become important in assessing students' creative behavior. In the previous examples (Problem 7 and Problem 8), Matei and Miruna were able to "think outside the box" in an innovative way, obtaining coherent and consistent problems. Matei adds an element—coverage order (which is insignificant in the starting problem)—and thus emphasizes the recursive nature of the problem. With this addition, Matei proves cognitive novelty—the description of the coverage order fundamentally transforms the problem, compared with the starting one. Matei manages to reframe: he masters the frame of the initial problem, but, additionally, he is able to think beyond this frame, and to come up with a problem that uses a different pattern. Miruna also managed to overcome cognitive framing and to customize her solving approach to the new situation. Although the question posed by Miruna made minimal changes from the starting problem, those changes modified the solving strategy by moving the focus on some combinatorial elements. The starting problem has a unique solution, but her new problem abandons some constraints, and thus the answer may not be unique.

These two students proved to possess cognitive frames of their problems. Moreover, they were able to connect the found solution to that frame and to propose modifications within that frame. In addition, these modifications maintained the mathematical consistency of the problem.

We wondered whether this capacity to make changes in cognitive framing is a condition for obtaining qualitative mathematical problems. Problem 6, posed by Alexandru, reinforces this assumption. His problem has some incoherence but it is mathematically consistent. Recursion is not explicit in the wording of Alexandru's problem, but appears in the drawings he made for solving. This shows the presence of a cognitive frame that allows him to identify the pattern and to apply it in some cases. Alexandru developed his problem within a cognitive frame probably built in time, on similar problems. He suggested a rule of development and proposed a computing formula, valid for two particular cases. However, to find the correct answer to his posed problem (in the absence of algebraic reasoning), it would be necessary to continue the inductive process: this involves formulating hypotheses, testing them on some cases, reformulating these assumptions and continuing the validation/invalidation and reformulation process. In other words, it would require changes in cognitive framing. What happened to Alexandru is the same type of phenomenon observed in organizations, where the existence of a cognitive frame allows organizational development to a certain point; then, a change in cognitive framing, or even reframing, is needed for the organization to grow.

As we have seen above, Matei and Miruna proved that they are able to change their mental frames, and their posed problems are coherent, while Alexandru showed a blockage in cognitive framing and his posed problem is not coherent. A common trait of the above examples is the persistence of only one of the mathematical elements of the starting problem. In general, we found that the focus on a single element of the starting problem (even if not acknowledged by students) essentially contributes to the generation of consistent problems. Extrapolating, we can say that the creativity-detecting mechanism used in this paper is sensitive to variations in cognitive framing, a fact that argues for the validity of the proposed analysis.

Some students who appeared to be creative did not have, or have not yet built, a cognitive frame. They solved both the starting problem and their posed problems by making drawings that matched the wordings. Typically, these students do not think of the possibility of another pattern with the same restrictions: as a result, most times their posed problems were not consistent. Therefore, the existence of a cognitive frame seems to be a necessary (but not sufficient) condition to generate coherent and consistent mathematical problems. This assumption is sustained by the fact that the students who failed to pose a coherent and consistent problem do not have deep understanding of the starting problem, even if they correctly solved it. Conversely, the students who mastered a cognitive frame of the starting problem managed to change it keeping its mathematical consistency. For example, the explanation given by Emilia (grade 4) for her choice of number 36 in her posed problem (see Problem 2) shows that she has control over an important condition of the problem:

Interviewer: But why didn't you change to 26?
Emilia: Because… **I wanted** 36…
I: But if you were to do another problem, what could you put instead of 36?
E: 81.
I: Why?
E: In order to be divided evenly… that $9 \times 9 = 81$, and the wall must be squared.
I: I see…
E:… or 4×4… 6×6…

To our surprise, Emilia proved that she is aware that the number in the problem has to be a perfect square.

(Obviously, in the fourth grade, the notion of perfect square is not used.) Although she does not propose a generalization, Emilia predicts the pattern that can generalize the problem. She manifests deep understanding of the constraints imposed on the data and the operators of the problem and, thus, understands the effects of changing these data or the constraints of the problem. This situation is not unique among the students of our sample. All the examples presented in Sect. 4.3, of posed problems that have a focus on only one of the three mathematical concepts of the starting problem, show the students' capacity to make controlled incremental changes.

These examples led us to conclude that, in problem posing, some students express a functional type strategy—they vary one single element of the starting problem, to control the quality of the newly obtained statements. Due to minimal variation compared with the starting problem, their problems do not show cognitive novelty in the usual sense of the definition.

We are going deeply to understand this phenomenon. Emilia, as well as other students who posed coherent and consistent problems, seemingly made only small changes. Does it just mean that the students who are at a more advanced stage of mathematical understanding are less creative? The answer we found is that mathematical creativity is of a special type, which requires abstraction and generalization (Singer 2012). More specifically, the students that show the presence of cognitive frames, and have the capability to make incremental changes (small, controlled) of the starting problem, show creative potential as they prove cognitive novelty, cognitive variety, and change in cognitive framing. We found that these students have a specific creative expression: cognitive novelty and cognitive variety are present but limited compared with other students, while change in cognitive framing is specialized in the direction of abstracting to get the problem generalization. This behavior strongly manifests for the most mathematically able students and it seems to correlate with the specific nature of the mathematical domain(s).

6 Conclusions

This paper presents the results of an experiment in which students in grades 4–6 (11–13 years old) with above-average mathematical abilities posed new problems by changing a given one. The students of our sample were among the winners of national mathematics competitions. The quality of being a winner of competitions does not a priori ensure that the student is mathematically creative because, in competitions, the reaction speed and the ability to use algorithms practiced before are far more important than creativity in problem solving. Consequently, a starting question was: What tools could be used to identify mathematical creativity in high-achiever students? Our research showed that the analysis of cognitive flexibility in the context of problem modification might be a good indicator of mathematical creativity. Therefore, the students of our sample were challenged to develop new approaches to a starting problem.

A first analysis of the students' posed problems led us to classify them into five categories, namely: problems based on the configuration of the starting problem ("starting pattern frame"); problems using the chessboard configuration ("chessboard frame"); problems that used new, different distribution patterns ("new recursion frame"); problems based on filling a grid with non-identical objects ("grid frame"); and problems that do not appeal to a certain configuration ("other"). These categories of problems differ not only through the nature of the configuration used, but also through their level of complexity, as more independent factors intertwined in the problem. At first, we were tempted to say that the students who posed problems of a growing complexity compared with the starting problem are more creative because at least they moved far away from the starting problem, thus showing novelty. A qualitative analysis indicates, however, that a significant percentage of those posed problems were mathematically inconsistent or incoherent. Therefore, to answer the question: "How does creativity manifest in these students?" we considered other features of the posed problems.

We tried to find out if there is evidence of links between the quality of the students' posed problems and their cognitive flexibility: the existence of such a link would be an argument for an answer to the main question of this study.

Cognitive flexibility consists of three primary constructs: cognitive variety, cognitive novelty, and change in cognitive framing. The study shows that, among these, cognitive framing seems to be more relevant for mathematical problem posing and problem solving. Cognitive framing was assessed through the capacity to generate a specific pattern of thinking for a specific problem. We were interested in the students' ability to change their cognitive frames, or even more, in their ability to reframe. In a context in which high-achiever students have to solve problems, we get information rather on cognitive framing than on changes in cognitive framing. Conversely, if we place the same children in a problem-modification context, then their tendency is to deepen the understanding of the mathematical problem, maybe because the cognitive frame that acts at some point is well defined. Therefore, changes in cognitive framing could be effectively detected in problem-modification situations, giving a relevant indication of students' creative potential in mathematics.

In general, the students capable of reframing can do this merely by varying a single mathematical dimension of the problem. (There was no exception to this remark in our sample.) In this way, the students can control the consequences of these changes and may propose coherent and consistent problems. Therefore, the student's capacity to generate coherent and consistent problems in the context of problem modification may indicate the very existence of a strategy of functional type for generalizations, which is specific to mathematical creativity. It might be possible that exactly this capacity becomes part of a tool for measuring students' mathematical competencies and for screening mathematical creativity.

A secondary finding of the paper is that mathematical creativity is different from creativity in general. However, a brief discussion of the limitations of this study has to be taken into account. It is possible that the above conclusions are valid only for students such as those in the group we worked with (high-achiever students, aged between 10 and 13 years). Further research could analyze the implications of this study for other ages and samples.

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