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ORIGINAL ARTICLE

Cultivating inquiry about space in a middle school mathematics classroom

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Abstract During 46 lessons in Euclidean geometry, sixth-grade students (ages 11, 12) were initiated in the mathematical practice of inquiry. Teachers supported inquiry by soliciting student questions and orienting students to related mathematical habits-of-mind such as generalizing, developing relations, and seeking invariants in light of change, to sustain investigations of their questions. When earlier and later phases of instruction were compared, student questions reflected an increasing disposition to seek generalization and to explore mathematical relations, forms of thinking valued by the discipline. Less prevalent were questions directed toward search for invariants in light of change. But when they were posed, questions about change tended to be oriented toward generalizing and establishing relations among mathematical objects and properties. As instruction proceeded, students developed an aesthetic that emphasized the value of questions oriented toward the collective pursuit of knowledge. Post-instructional interviews revealed that students experienced the forms of inquiry and investigation cultivated in the classroom as self-expressive.

Keywords Mathematical inquiry · Interest and disposition · Mathematical habits-of-mind · Spatial mathematics · Mathematical practices

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1 Introduction

Mathematical inquiry is a cornerstone of mathematical practice (Lakatos, 1976), but it is unclear how best to initiate students into this activity. In mathematics classrooms where students participate in discourse to support learning (Chapin, O'Connor, & Anderson, 2003), student questions are often encouraged to clarify the mathematical reasoning of participants, and to signal shifts in authority from teacher to student (Hufferd-Ackles, Fuson, & Sherin, 2004). Although these are valuable roles for inquiry, we are concerned here with a different sense of inquiry, one in which students pose questions that instigate mathematical investigation. To date, this form of student inquiry has received comparatively little attention. The few relevant studies have focused on small groups of students, not intact classrooms (e.g., Borasi, 1992). To explore the possibilities of sustained inquiry, we designed instruction to support middle school students (ages 11, 12) to initiate questions, formulate conjectures in light of their questions, conduct investigations that informed their questions and conjectures, draw conclusions from these investigations, and pose new questions. Our goal was to sustain a cycle of inquiry in which students would be agents of mathematical learning. The objects of inquiry were familiar topics in their schools' geometry curriculum, a sixth-grade unit of Connected Mathematics Project (CMP). Rather than adhering to the curriculum's scope and sequence, we let student inquiry, guided by instructors with a view of the mathematical horizon (Ball, 1993), determine the course of learning. We conjectured that a focus on student inquiry and investigation would incubate the development of interest in mathematical activity. These expectations were consistent with findings from studies in other disciplines of the long-term cultivation of interest. These findings suggest that opportunities to pose questions and conduct investigations related to these questions tend to generate and sustain interest (Azevedo, 2006; Barron, 2006; Hidi & Renninger, 2006; Renninger, 2009).

We emphasized geometry as the site for inquiry because spatial reasoning holds great promise for supporting the development of mathematical practices—habits-of-mind—for generating and revising mathematical knowledge. Goldenberg, Cuoco and Mark (1998) suggest: "Geometry, broadly conceived, can help students connect with mathematics, and geometry can be an ideal vehicle for building what we call a 'habits-of-mind perspective'" (p. 3). Harel (2008) proposes a similar distinction between ways of understanding and ways of knowing. He explains that a way of knowing emerges as one considers commonalities among the actions one takes to solve a particular problem or to develop a particular proof. Mathematical understandings ideally are intertwined with more general ways of knowing.

Clearly, posing questions that can be investigated is an important habit-of-mind, but to be productively sustained, questions must be coordinated with other habits-of-mind that support investigation, such as generalizing, reasoning with relationships, and investigating invariants (Driscoll, DiMatteo, Nikula, & Egan, 2007). Generalizing refers to a disposition to secure knowledge by seeking the broader structures or patterns that govern the generation of any particular case or set of cases. The distinction in geometry between a drawing, which exemplifies an instance, and a figure, which exemplifies a class with associated properties, suggests a pathway for seeking broader classes or patterns (Goldeberg & Cuoco, 1998). Reasoning with relationships refers to a disposition to search for the relationships among properties that undergird concepts. Geometry is replete with visual concepts such as planar figures and relations among them, such as congruency, that provide opportunities to construct these relationships (van Hiele, 1986). Investigating invariants refers to the propensity to search for constancy in light of change. This tendency is more often associated with the dragging operation of dynamic geometry tools (e.g., de Villiers, 1998; Hadas, Hershkowitz, & Schwarz, 2000), but as we later explain, change and invariance can be visualized with more conventional technologies, as well. We conjectured that inquiry would ground the need to develop these related habits-of-mind, and that the affordances of geometry's visually guided thinking, in turn, would nourish their development. We anticipated that during the course of instruction, student inquiry would increasingly reflect the operation of these three habits-of-mind.

We structured the investigation around several of the following questions. First, to what extent did students participate in generating questions? Was this practice widespread or confined to a few, elite students? Second, did student questions reflect an orientation toward habits-of-mind? Did this orientation develop over time? Third, what aesthetic (Sinclair, 2004) did students develop about mathematical questions and conjectures? What made a question or conjecture "good?" Fourth, were student reflections on their experiences consistent with mathematical agency and disposition to engage in mathematical thinking?

2 Crafting a culture of inquiry

An epistemic enterprise like this one is inherently open-ended. Accordingly, we could not plot in advance anticipated trajectories of learning, as is commonly advocated for design research (Simon, 1995). Instead, we planned instruction from week to week to respond to the emerging history of inquiry and investigation. We were guided by our anticipations of the emerging mathematical horizon. For example, although questions about the area of polygons emerged comparatively early, we delayed their investigation until students had firmer grasp on the conceptual tools that we knew they would need to conduct their investigations. Designing instruction was guided by several heuristics that collectively aimed to root mathematical experience in everyday, even mundane, encounters with space (e.g., walking) and to elaborate and transform these experiences into mathematical systems by cultivating habits-of-mind, as we describe further in this section.

We viewed students' everyday experiences and conceptions of space, especially bodily motion, and their everyday forms of argument, chiefly propensities to categorize and classify, as rich resources for incubating mathematical activity. For example, we anchored students' learning about polygons to paths that they walked (Abelson & diSessa, 1980; Lehrer, Randle, & Sancilio, 1989) and related familiar properties of polygons, such as sides, to experiences of unchanging direction while walking (Henderson & Taimina, 2005). Our emphasis on bodily experience, or body syntonic approaches (Papert, 1980), reflected a conviction that mathematics often re-expresses, refines and extends commonplace experiences in the world (Hwang & Roth, 2011; Lakoff & Nunez, 2000). More specifically, we sought to map embodied experiences such as *no change in direction while walking,* or the *changing direction induced by body turns*, to corresponding fundamental properties of space, such as straight and angle. We often privileged questions related to classification, because instead of mere naming, classification was a forum for constructing and communicating definitions of mathematical properties, such *straight* sides, that could be readily related to embodied experiences (Lehrer, Jacobson, Kemeny, & Strom, 1999).

We encouraged and provided ways for students to inscribe space in multiple modes and at multiple scales (Latour, 1990; Roth & McGinn, 1998). For example, representations of polygons were generated by affixing tape along walked paths. These paths were re-generated on paper employing circular protractors, or alternatively, paper figures were re-described bodily. These acts of representational re-description were intended to foster closer examination of necessary and sufficient properties (e.g., What about an experience of walking a path needed to be represented on paper?) and meta-representational competencies (Which representations best communicated the intentions of their authors?, diSessa, 2004). For example, Fig. 1 displays a student-invented representation of the magnitude and direction of an angle considered as a rotation. The student explained that he had labeled the starting and ending points of rotation, but double-labeled the end point as both "end" and as 0, so that one could treat the end as zero if one wanted to turn in the other direction. By comparing and contrasting solutions to representational problems like these, students developed representational competencies.

We devised common forms of participant structures, such as small group tasks, that encouraged variability both in questions generated and in approaches to investigating questions. We complemented these smaller working groups with a whole-class dialogue format in which these variations were juxtaposed and related by encouraging students to build on another student's contributions or to suggest alternatives to them (Stein, Engle, Smith, & Hughes, 2008).

Critical to the conduct of the instructional design, we supported the development of questions and related habits-of-mind discursively. One form of conversational support was *labeling* forms of mathematical activity. For example, when a student, Vern, suggested that irregular polygons

Fig. 1 Representing degree and direction of rotation

"could all be the same (shape), but they could have different sides, lengths," the teacher (typically, one of us) noted, "Ahh so that's a *conjecture* you have." He wrote Vern's statement on the board, labeling it the "Vern conjecture," thereby attributing authorship to Vern and highlighting the contribution as important. The attribution of student *authorship* constituted a second form of support that was consistently practiced during instruction.

Questioning was cultivated during many episodes of classroom talk. For example, the teacher asked about a drawn figure: "What might we want to ask about it? What else might we want to know about this or anything related to it?" He later added that students might ask about "something that could take advantage of the information you can see here, but does something with it." The teacher also highlighted questions as worthwhile to investigate (e.g., "We haven't figured out the answer to this question, Mona's question, which *a lot of people think is worth investigating*.") Note that the teacher aligned his highlighting with a communal opinion, distinguishing it from his personal preference. At other times, the teacher more implicitly communicated about which questions were worthwhile. For instance, when students posed questions about a square, such as "how many vertexes?", the teacher immediately asked students to answer the question. Labeling student questions served two purposes. It sent the message that "investigate-able" was a quality of good questions and also indicated which questions fell into this category.

There were frequent opportunities to support *habits-of-mind*, such as generalization, often by asking students to justify the grounds of a claim beyond a single instance (e.g., Ellis, 2011). The instructor highlighted these habits-of-mind by meta-discursive talk about how mathematicians might proceed during the course of solving a particular problem or approaching a particular question. For example, when students were investigating questions about squares, rhombi, and other quadrilaterals, the teacher held up a tool made of four paper strips attached by brad fasteners that allowed for investigation of dynamic motion. He noted: "Now something that often is a good thing to do is investigate what happens when you *change* something. What stays the same and what changes?" Here, as in many other instances, discursive support was accompanied by tools or inscriptions or embodied activity, so that discourse was commonly multimodal. As instruction progressed, forms of discursive support initially raised by the teacher were increasingly appropriated by students. For example, during one class, students were talking about rhombi and their properties, and the conversation veered to the previously unexplored terrain of "fitting" (dissecting the rhombus) triangles. A student proposed a generalization: "Since he said that *I have a conjecture*…Every polygon

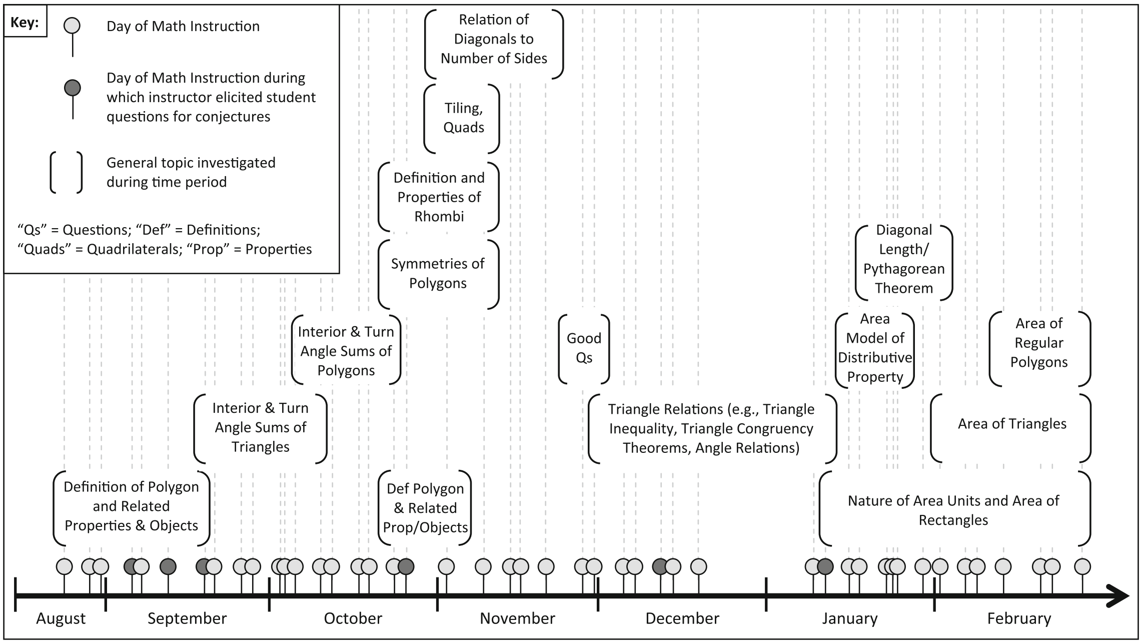

Fig. 2 A view of the trajectory of instruction

has to have at least one polygon inside of it…you could fit a square inside of a rectangle. You could fit a triangle inside of a square." Another student, without prompt from the teacher, responded, "*I have a question* for you Kira. Oh, I have a question. What can you fit in a triangle?"

To maintain a common ground (Staples, 2007) for classroom activity, we inscribed questions and conjectures on a list posted on the wall of the classroom. Whenever a student stated a new question or conjecture, we added it to the list and often noted the author. The list highlighted questions and conjectures as important and also served as a model of recordkeeping for students, who were expected to document questions and conjectures in their notebooks. For instance, writing two conjectures on the board, the teacher remarked: "I hope this is in everybody's notebook. There are now two conjectures along with our questions." These forms of recordkeeping made it possible to return to the questions and conjectures at later points in time. Sometimes the teacher related an investigation to a question on the list and on multiple occasions, the class looked at the list and discussed what had and had not been resolved. Later, the students spent a couple of days investigating a question or conjecture of their choice from the list.

The period of instruction in which inquiry about space was the dominant form of mathematical activity spanned approximately 6 months (46 lessons) during which students posed questions and conducted investigations about: (a) definitions and relations among polygons, including angle measures and sums and relations between diagonals and the number of sides; (b) relations within and among triangles, including triangle inequality and congruency theorems; (c) dissection of polygons via triangles; (d) isometries, symmetries and their implications for classifications previously developed for polygons, especially quadrilaterals; (e) area measure and its use as a model for the distributive property of multiplication over addition; and (f) the Pythagorean theorem. Figure 2 displays the overall trajectory of instruction, although there were many instances of overlapping topics, revisits of topics and briefer side conversations. The brackets show the topics that students investigated, as well as the time the investigations spanned. We did not include procedural topics, such as when students were learning how to use a tool (e.g., a protractor), or topics that lasted less than one class period. The circles at the bottom of the timeline represent days of math instruction. Six circles are shaded a dark gray to represent days when the teacher elicited questions or conjectures about a particular object or topic. The first three of the six such days occurred during the first month.

3 Method

3.1 Participants

Participants ($n = 18$, 11–12 years old, 10 male) attended an urban school that primarily served underrepresented youth in the southeastern United States. Most students (75 %) qualified for free or reduced lunch. Students in the target classroom were ethnically diverse (8 African American, 3 Caucasian, 1 Hispanic, 1 African, 1 Native American, 2 Kurdish, 1 Vietnamese, 1 Mixed).

3.2 Procedure

One of us (RL) served as the primary classroom instructor for mathematics during the school year (85 % of lessons) with occasional instruction from MK and the classroom teacher, Ms. Moskal (11 % jointly, 4 % by MM). Mathematics class was conducted twice each week, 1.5 h per session. Students also responded to occasional assessments for an additional 45 min on those weeks. Each class was videotaped and digitally rendered for further analysis. A small group of students was recorded with a second, wall-mounted camera throughout most of the year. Field notes were taken of whole group interactions during each class to contextualize the video recordings and inform the design of the next lesson(s). The choice of mathematical topics was informed by the school's grade-level standards for mathematics and the curriculum used by the classroom teacher. However, the conduct of any particular class was informed by students' questions and investigations, our interpretations of the mathematical implications of these questions and investigations, and by our judgments about productive "next steps" in light of students' reasoning. These judgments were further informed both by classroom interaction and by the results of periodic assessments. Students wrote summaries of their understandings and experiences at the end of every lesson, and the student journals were an additional source of data during the conduct of the study. Because we were committed to students' generation of mathematics, we authored only one definition, that of an angle as a rotation with a measure determined by the amount of rotation. One complete rotation was defined as 1 turn and was equivalent to 360°. From this humble outcrop, students developed other definitions, conjectures, theorems and proofs. When the space of questions grew unwieldy, the class negotiated which they judged most worthy of pursuit, or the instructor privileged some as particularly likely to yield richer understandings. We also posed occasional questions when we felt they would extend or enrich students' current activity.

At the end of the school year, each student reflected about their mathematical experiences during individual flexible interviews of approximately 45 min to 1-h duration. Seventeen interviews were conducted, as one student left the school during the semester. Questions included: (a) What did you like best about math this year? And, what did you like least? (b) Are there good questions and bad questions in mathematics? (Probe for examples); (c) What's it like doing math in your class on a typical day? (d) How is doing mathematics this year the same or different as doing mathematics last year (or before entering the teacher's class)? and (f) Which three of your classmates most helped you learn this year? Who helped the most? How did ___ help you? Students also responded to questions about how they reasoned about particular problems and concepts, but these are not addressed here. To describe student reflections, we first broadly read and interpreted each student's responses across the questions for the entire sample, and then re-considered each student's responses in light of the emerging themes. We were also guided by previous efforts in the field to consider students' dispositions toward mathematics (e.g., Boaler, 2002; Gresalfi, 2009).

3.3 Identifying student questions

We first viewed video of every classroom lesson during the course of instruction and identified all unique student questions that could be resolved by means of some form of investigation or proof. The following criteria were used to identify questions and to differentiate them from other types of statements. First, utterances that ended with rising intonation and included "what," "how" and related words were marked as questions. Second, the question had to refer either to mathematical objects, properties or relations, or to an epistemic, such as: "How could you know…without testing every one?" We counted as unique only one of a series of restatements of a question, and we did not include questions posed for evident purposes of communicative clarity, such as: "What do you mean by…?" We also excluded procedural questions, such as: "How do you use these (e.g., protractors)?" However, questions about a mathematical method to accomplish a goal were included. Finally, we excluded teacher revoicing (O'Connor & Michaels, 1996) of a student statement as a question. Once a student question was identified, it was transcribed verbatim and accompanied by a brief description of context to help situate it and to permit a more accurate interpretation of the student's intent. Describing the context included recording what the class was doing immediately before the question was asked, and, if necessary for interpretation, what was said immediately before the question was asked.

3.4 Coding student questions

To indicate the degree to which student-generated questions were oriented toward mathematical habits-of-mind, each question was coded along four dimensions, as follows.

The first dimension was the extent to which a question was oriented toward *Generalizing*. We distinguished among four levels of generalization. Questions at the lowest level typically focused on specific instances. Some were mathematically oriented, such as "Do(es) it (an instance) have any angles?" while others had no disciplinary intent, such as: "Did Mr. Einstein make one (an octagon)?" At the second level, questions were directed toward a particular instance but implicitly addressed a generalization. For example, during an investigation of the implications of the triangle inequality, a student asked: "Would 4, 3, 7 (lengths of sides) work?" Although it addressed an instance, the question suggests a more general orientation toward the boundary condition of the inequality. At the third level, questions explicitly referred to a generalization, but did not specify the aspect of the generalization of most interest. For example, "What is a rhombus?" or "Is there any degree above 360?" At the highest level, the question was clearly focused on general relations between properties or on relations between properties and classes, as illustrated by: "Don't you think that the number of diagonals that you make in a polygon will be the same as the number of sides?" or by: "How come not all regular polygons tile?"

The second dimension, *Reasoning with Relationships*, was the extent to which a question was oriented toward establishing relations, such as properties and classes, among mathematical elements. There were three levels of relational orientation. The lowest indicated no explicit relational reference. The second made an explicit reference to a pair, such as, "Why is the inner angle and the outer angle different?" or, "But is a circle a polygon?" or, "So would flipping be the same thing as turning…90°?" Questions at the third level explicitly referred to multiple relations, such as "Um, I was like on Tuesday when you guys were talking about how side, angle side and angle, side, angle, um. Would it still work even if you still switched them up?…Like when you say angle, side, side, could you put side, side, angle?" This question inquires whether the order of the relation among sides and angles matters for establishing the congruence of two triangles.

The third dimension, *Investigating Invariants*, was the extent to which a question was oriented toward finding invariance in light of change. At the lowest level, the question did not refer to change. At the second level, a question referred to change but did not specify a focal property of change, concentrating instead on global appearances, such as speculating about what would happen if one shape were pulled or stretched into another. The third level did specify a focal property of change, such as: "If you change the angles, will it still be a triangle?" At the fourth level, some questions were directed toward limits of change, as in extreme cases or boundary conditions. For example, during a discussion of the effects of an increase in the number of sides of a polygon: "Does it ever turn into a circle or does it look like a circle?" Other questions were directed toward the effects of variations in methods: "But wouldn't it come out the same if you tried it the center way?" This question referred to a variation on another student's method that employed diagonals to partition a hexagon into triangles, one where the triangles all met at the center.

The fourth dimension, *Scope*, was the extent to which a question created opportunities for exploring new mathematical terrain in light of the history of inquiry in the classroom. This dimension of questioning was not explicitly supported during instruction but instead emerged during the course of classroom activity. We made four distinctions along this dimension. At the lowest level, questions did not offer opportunities for exercise of any of the previous three habits-of-mind. For example, a student looked at a shape drawn on the board and asked: "How many vertexes?" Because this question was directed to a particular figure (no generalization), did not refer to other properties or figures (no reasoning about relations), and did not consider change and invariance, we ranked its scope at the lowest level. At the second level, questions clarified previously referenced properties or relations, or were intended to disrupt them. For example, "Is the circle the only non-polygon? What about an oval? Is that not a polygon?" and "Is it possible to have the interior sum (of a triangle) greater than 180°?" At the third level, the questions addressed a new context or relation, often marked by suppositional language. For example, during a discussion of angles as rotations measured in degrees, a student asked: "If, that if the circle's that way [gestures clockwise], wouldn't the negatives have to be that way [gestures counter-clockwise]?" At the highest level questions challenged the grounds of knowing. For example, when a student conjectured that if the triangle inequality held for the sum of the two shortest sides, it would be true for any combination, a second student asked: "How did she know they didn't work if she didn't test them out?"

Two of the authors served as the primary coders of 109 questions. A third coder coded all instances of disagreement without knowledge of any of the codes applied by the primary coders. Exact agreement between the two primary coders for the 109 questions was 66 % generalization, 74 % relations, 90 % for change, and 68 % for scope. Disagreements were most often at adjacent levels of the scheme (77 % of the time). Disagreements were nearly always resolved by agreement of one of the primary coders with the third coder. Rare instances of three-way disagreements were resolved by consensus.

3.5 Aesthetics of questions

We probed students' sense of the aesthetics of questions at the approximate mid-point of instruction, the 26th and 27th class sessions. During the first of these, students worked in small groups to decide: "What makes a good question or conjecture? Are all of these good questions and conjectures? Or, are which ones are *really good* and why? With your group, write down the top three qualities of a good

Table 1 Frequencies of questions by levels of generalizing

Weeks	No generalization	Implicit generalization	Explicit generalization	Explicit generalization and specifies aspect
1–4	9 (23 %)	6 (15 %)	10 (26 %)	14 (36 %)
5–25	7 (10 %)	11 (16 %)	15 (21 %)	37 (53 %)
Total	16 (15 %)	17 (16 %)	25 (23 %)	51 (47 %)

question or conjecture." After the small groups proposed criteria, the class judged each criterion and recorded those they agreed upon. During the second session, students made additional contributions to the criteria during a whole-class discussion.

4 Results

4.1 Questions and habits-of-mind

Most students posed multiple questions during the course of instruction about space, although two delayed until a new unit on decimals and fractions to pose their first questions. A third student left the class partway through the year before having posed a question. Tables 1, 2 and 3 display the frequencies of orientation toward the habits-of-mind evident in the 109 questions posed during the period of instruction, according to the coding scheme. The split of the data allowed us to compare the induction phase of the practice of inquiry (the first 6 classes in Fig. 2) and the remainder, more routine practice of inquiry during the remainder of instruction about space. By induction, we refer to the introduction of this new form of practice to students, one in which questioning was installed as a socio-mathematical norm in the classroom (Yackel & Cobb, 1996). Students' routine generation of questions and investigations was firmly established by the end of the first month of instruction.

The total frequencies for each habit-of-mind displayed in Tables 1, 2 and 3 suggest a strong orientation toward Generalizing and Reasoning with Relationships in students' inquiries. Only a comparative handful of the questions that students generated were oriented toward Investigating Invariants. However, when student questions were directed toward the higher levels of this habit-of-mind, exploring the effects of changes in properties or of the boundary conditions of extreme cases, the questions were also characterized by high levels of Reasoning with Relationships (100 %) and of Generalizing (94 %), in contrast to 46 and 65 %, respectively, for lower levels of Investigating Invariants. Hence, although they were less frequent, questions about Investigating Invariants had high potential for conducting fruitful mathematical work. The results suggest that, as anticipated, learning to pose fruitful questions for investigation was coordinated with developing dispositions to seek generalizations and explore relationships. The evidence is more equivocal about posing questions that situate investigation of change and invariance.

Table 2 Frequencies of questions by levels of reasoning with relationships

Weeks	No relations	One relation	Multiple relations
1–4	24 (62 %)	15 (38 %)	0 (0 %)
5–25	27 (39 %)	35 (50 %)	8 (11 %)
Total	51 (47 %)	50 (46 %)	8 (7 %)

Table 3 Frequencies of questions by levels of investigating invariants

Weeks	No change	Global morphing	Change properties	Variation in method or extreme case
1–4	35 (90 %)	1 (2 %)	0 (0 %)	3 (8 %)
5–25	56 (81 %)	0 (0 %)	8 (11 %)	6 (9 %)
Total	91 (83 %)	1 (1 %)	8 (7 %)	9 (8 %)

Table 4 Frequencies of questions by levels of scope

Weeks	No potential for investigation	Rethinking ideas	Elaboration in new context	Epistemic grounds
1–4	14 (36 %)	20 (51 %)	5 (13 %)	0 (0 %)
5–25	9 (13 %)	38 (54 %)	18 (26 %)	5 (7 %)
Total	21 (19 %)	58 (53 %)	23 (21 %)	5 (5 %)

Tables 1 and 2 further show that during induction, a slight majority of student questions were oriented toward generalization, but this tendency became much more pronounced as inquiry took root in the classroom. Similarly, during induction, the majority of student questions were not oriented toward Reasoning about Relationships but during the remainder of the instruction, the majority of student questions were oriented toward higher degrees of Reasoning with Relationships.

Table 4 displays the scope of student questions—the extent to which the question instigated reconsideration of concepts and methods previously investigated or extended established ideas and methods to new realms of inquiry. This table suggests that during the course of instruction, the scope of student questions increased and even included those that challenged the grounds of knowledge, part of a tendency to situate individual inquiry within a collective trajectory.

4.2 Aesthetics of questions

Table 5 displays the criteria for characteristics of good questions (and conjectures) that students generated. Some of them reflect the importance of personal agency, indicated by reference to personal authenticity—one does not pose a question or make a conjecture about something that one already knows—and by reference to individual curiosity, such as "eager to know." Other criteria suggest an orientation toward knowledge-building and communicative clarity. For example, "leads to other conjectures" and "helps you keep thinking about the topic" suggest a commitment to sustaining and elaborating knowledge. Similarly, one is obligated to pose a question that others can comprehend (e.g., clear, makes sense, specific focus), perhaps because without clear communicative intent, it is difficult, if not impossible, to build knowledge collectively. Students also stated that worthwhile conjectures follow from questions posed, so that conjectures are about potential answers to questions that have been posed.

4.3 The interplay of inquiry and investigation

We describe two points in time, chosen from the beginning and end of the 6-month period of instruction, to illustrate how questions arose and were taken up as investigations, and how, as they did so, the roles of the students and instructor shifted over time. The first excerpt, which focuses on students' investigations of the sum of angles for triangles, illustrates how the instructor initially supported students' posing of questions and helped them select or fashion questions they could actually manage, while still positioning students as responsible for authorship. The second excerpt describes students' entrée into what later became an investigation of the Pythagorean theorem. It illustrates how the students took on more responsibility, not only to pose questions, but also to immediately discuss and respond to those questions with general arguments.

4.3.1 Early entrée into inquiry: investigations of triangle angle sums

During the initial phase of instruction, students wrote directions for walking various polygons, an activity that required them to consider a shape's properties and the relations among those properties. They started the seventh day of instruction by using one student's directions to construct an equilateral triangle. During the discussion that followed, the class notated the triangle's interior and turn angle measures (see Fig. 3). RL then asked the students to pose questions about the figure: "What kinds of questions could we now ask, having made this thing? Cause in math, we don't make things unless we want to ask questions about them. So what kinds of questions could we ask about this thing that we've just made?…What else might we want to know? About this or anything related to it."

Students' initial questions ranged from, "If we hadn't started from the vertex and still turned 120, would it have still ended up the same way?" to, "Can there be a square above 90° angles?" to, "What's the area?" Along the way, RL commented about desirable features of questions (e.g., "Something that could take advantage of the information you can see here, but does something with it"). When students had no more questions to add, he encouraged them to ask about the properties of the triangle: "What if I said, 'think about the sums?'" One student asked about the sum of the sides ("How many, well, what would all the sides equal? What would be the sum of the sides?"), and RL

Table 5 Student-generated criteria for qualities of good questions/conjectures

1	Help you learn something that you do *not* already know
2	Something you are eager to know
3	Shows good curiosity and thinking
4	Has something you know about and something you want to find
5	Has questions for what we know
6	Topic of the discussion is inside the question. The question helps you keep thinking about the topic
7	Good conjecture follows from the question. It links to what you know
8	Questions are posed with good evidence. You can see why the question is relevant
9	Leads to other conjectures
10	Clear in wording and makes sense
11	Good detail, specific focus
12	Good math wording—good math vocabulary helps you know what the question (or conjecture) is about
13	Question helps you understand what people are saying and why

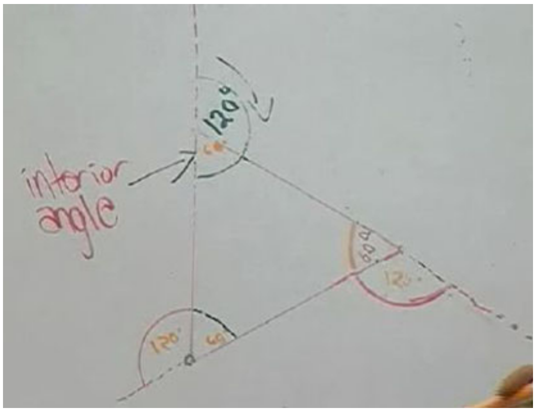

Fig. 3 Posing questions about a triangle

included this with their other questions. After a pause, RL pressed again, "Are there any other properties of the triangle that we could ask about the sum?" One student offered "angles," which, after probing by RL, was specified as "interior angles." RL revoiced this contribution as, "What is the sum of the interior angles?" Another student followed with, "What's the sum of the outer angles?" revoiced by RL as "turn angles." RL suggested that the students first try to answer the two questions about angles for the particular triangle they had been looking at.

Once they had accomplished this, RL prompted students to think more generally: "What I want to know is, do you think this (interior, exterior angle sums) is true for all triangles, or just for those that are equilateral?" He then further suggested that to start they might "draw a triangle that is not equilateral and find out what its turn angles are and what its interior angles are."

At the beginning of the next class, one student, Diyari, shared a conclusion: "The interior angle sum is 180° and the turn angle sum is 360° and they're…And it works for all of them…it works for all triangles." Several students suggested that this was contrary to their experiences. The instructor highlighted these competing claims: "So Kate actually tried this and Kate says Diyari, your *conjecture* is false. Because, let's look at Kate's reasoning. Because she found one instance that it's not true of. Therefore, you can't say *for all* triangles." The students spent the next few classes investigating Diyari's conjecture and eventually explained why it was true.

These episodes illustrate how RL facilitated students' entrée into inquiry and investigation. He started by asking them to pose questions about a triangle that already had the interior and turn angle measures labeled, and thus highlighted. As students asked initial questions, he provided commentary about desirable features of questions, and focused their attention further by suggesting that students ask questions about "sums" of properties. This prompted a student to ask the question about the sum of the side length measures. When a student offered the property of "interior angles," the instructor revoiced this property as a question, and, in doing so, modeled asking a question. Although we did not include this question as an instance of student generation, it prompted a student question about outer angles. Because the students' questions about sums of sides and sums of "outer" angles were both about a particular case and did not explicitly describe relations or change, they were coded at the second lowest level for generalization and the lowest levels for relations, change and scope. However, the questions provided a point of entrée for the instructor's more general question about the sums of angles for all triangles. Moreover, by suggesting initial forms of investigation, the instructor provided students access for exploring the question themselves and grounded Diyari's contribution of a conjecture. "Diyari's conjecture" then became the focus of investigation for the next several class periods.

4.3.2 Student initiators of inquiry: seeding the pythagorean theorem

As they investigated area measurement, students drew a square inch and then a square foot. While constructing these, Ned noticed that the diagonal of the squares was not the same as the length of the side: "Dr. Rich, why won't this work right here? Why won't this make a foot right here?…I tried that with a square inch and it still didn't work." RL replied: "No kidding. So no matter what square you did, it didn't work? Huh. Is that a question we need to ask ourselves?" Ned enthusiastically agreed, and RL requested, "Phrase your question so that we can all understand it. Ms. Moskal will add it to our question list." While working on the wording of his question, Ned decided to "bring back up the 1–3" notation the students had used earlier to refer to diagonals (meaning the length connecting vertex 1 to vertex 3). This is noteworthy because Ned not only considered the notation relevant in this new context, but he also identified it as a way to help communicate his question to the class, as RL had requested. Several other students had similar questions. Ned represented their question (see Fig. 4): "OK, from 1 to 2 (gesturing a path from 1 vertex to another on the same side), 2 to 3 (same gesture as preceding), 3 to 4 (gestures) and 4 to 1 (gestures) are all one foot…So if those (gestures perimeter) on the outside are correct for one foot, why isn't 1 to 3 one foot too?"

RL then asked students what they considered the relationship to be—shorter than or longer than, or it depends on the square. Jomerd suggested that a diagonal of a square could not possibly be the same length as a side. He demonstrated with his notebook (the more general case) as a stand-in for the square. Holding his finger at the upper left

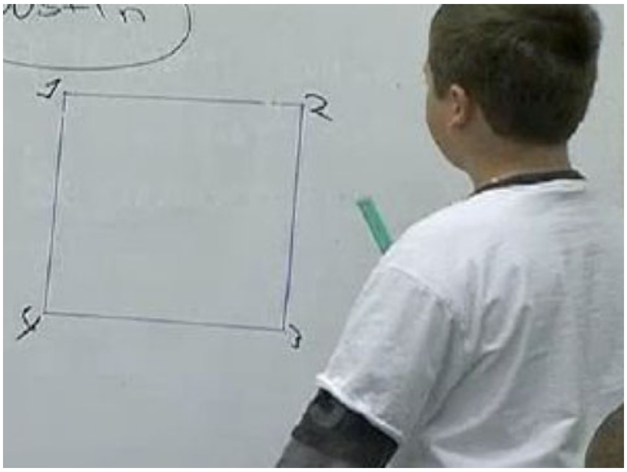

Fig. 4 Ned poses the question about diagonal length

corner, he asked the class to imagine sweeping a circle with this vertex as the center and the length of the adjacent side as the radius. The circle never intersected the opposite vertex, illustrating that the diagonal length was longer than the side length. Lavona offered an alternate argument. She noted that when they had investigated diagonals, she had noticed that the diagonals radiating from a vertex become longer and then shorter again as one sweeps through the shape. She suggested that the same phenomenon was happening with the square, implying that the diagonal must be longer than the sides. This student-initiated question and their arguments served to motivate investigation of whether or not there was *any* relation between the sides and diagonal of the square, seeding entrée to the Pythagorean theorem.

In this excerpt, multiple students presented their observation of the difference in length between the diagonal and the side of the square as a question, suggesting a disposition to ask questions. Unlike the questions about angle sums posed in the earlier excerpt, this student-generated question explicitly described more than one relation, and was thus coded at the highest level for relations, and presented greater potential for future investigations, coded at the third level for scope. Although Ned's question was about a particular case and hence classified at a lower level for generalization, students readily responded to it with arguments that justified the general case about why the length of the diagonal could not be equal to that of the sides. In this instance, students assumed authority to pose a relevant relational question while others readily responded with general approaches for reasoning about the question. Moreover, Ned, Jomerd and Lavona's contributions all built upon previous investigations, notations, and definitions that the class had collectively developed. These experiences served as resources for the students, and, because of their shared grounds, were accessible to others in the class. For instance, when examining Jomerd's circle argument, Cordell remembered aloud that they had constructed circles in a similar way several months earlier. As before, the instructor facilitated discussion about the topic and suggested productive pathways to furthering the mathematics.

4.4 Student reflections on their mathematical experiences

We identified four themes that characterized all students' reflections about their mathematical experiences. One theme, agency, threaded throughout student responses, and was indicated by first-person statements about mathematical constructions ("I made" and "make up your own methods"), challenge ("I actually liked it because then you really got to challenge yourself and didn't have to like hold yourself back"), control ("We bring up our own subject, it's like our own lesson"), and perceived personal impact on collective learning ("I had more conjectures there than any other part of math…making conjectures you can actually change the history of math, like it can like you can go around and tell people that and it can actually come true and you can put like in the math notebook and you find all these different types of discoveries that nobody else has known before. And that is like really really cool."). All students expressed one or more of these components of agency and typically expressed several of them.

The second theme, positive attitude, was expressed by indications that the class was "fun" or "cool" even though some students suggested that at times they were frustrated by lack of progress when they attempted to generate a conjecture or conduct an investigation.

The third theme, practices that generated knowledge, indicated that students reflected about how practices contributed to the development of mathematical knowledge. For example, one student noted: "Then I started to investigate that (question). And then I learned about that. And then I made a conjecture, so that led me to a new idea—because of that question." Another said more baldly: "Sometimes curiosity doesn't always kill the cat! Curiosity, like if you ask questions, you find out more. That's how you learn most of the time, is to ask questions."

The fourth theme was peer collaboration. All but two were nominated by classmates as assisting learning. The number of times that individuals were nominated ranged from 1 to 7, with a mean of 3.1. Nominations spanned the spectrum of mathematical achievement, with both higher and more modestly achieving students represented in the group of nominees who exceeded the mean nomination level. Most of the forms of help that students described were about understanding better (e.g., "like if I don't understand like a method or something like, he'll like put it like he'll like help me explain…they just help me like explain it more like in their words or like in sixth grade words and not in like Dr. Rich's words"). Some students noted helpful classmates introduced them to new ideas they had not thought of before, asked good questions, or contested a claim. For instance, one student explained that Jomerd and Diyari helped them learn: "Cause, it's like, it was like really, it wouldn't be like he would actually come up and help me, it would be like we would get into it, like we would be like, no it's not this, not it's that, and so he's like, so he would actually lay the problem out and his solutions…(and Diyari) usually helps me a lot because he usually jumps in when me and Jomerd are going at it. He says, 'No, you're both wrong, it's this.'" Arguing, she explained, "gives us new ways of thinking about it."

5 Discussion

Practices of inquiry are foundational for mathematical endeavor. Conjectures, refutations, theorems, and proofs all emanate from questions. In this research, we explored how to enlist comparatively young students in posing and refining questions that could initiate and guide mathematical investigation. We contrast this form of inquiry with more commonplace roles for questions, such as clarification. Space was the realm of inquiry, because, in our view, spatial mathematics is readily grounded in forms of everyday embodied experience that provide resources for inquiry and investigation.

When inducted into inquiry practice, students readily participated. The sheer number of questions posed by students exceeded our expectations, and the majority of students generated several questions during the course of instruction. Even during the formative phase of the first month of instruction, many of the questions posed by students were oriented toward related habits-of-mind. The propensity to pose questions informed by habits-of-mind, especially Generalizing and Reasoning with Relationships, increased as instruction progressed. The scope of student questions also increased over time, with increasing number of questions informed by the aim of clarifying the results of previous investigations or of extending methods or concepts to new realms.

Students' increasing sensitivity to the history of inquiry suggests the operation of what Knorr Cetina (1999) termed an epistemic culture: an arrangement of social, cognitive and material mechanisms that supported disciplinary-distinct ways of knowing. Here the epistemic culture of the classroom produced a collective horizon in which the conduct of individual inquiry and investigation was situated and sustained. Other indicators of the operation of an epistemic culture included the development of an aesthetic of inquiry—that is, students especially valued questions about things that were genuinely unknown to them or that expressed personal curiosity. Students favored questions and conjectures that contributed to others previously posed, and they especially valued individual agency.

Students' reflections about the nature of their mathematical experiences communicated the growth of disciplinary dispositions (Lehrer, 2009). Disciplinary dispositions go beyond likes and interests to encompass anticipations that disciplinary practices contribute to personal agency and identity. As students authored questions and conjectures, and conducted investigations, they authored, albeit clearly supported by teachers, a common ground of knowledge. The common ground was established by mathematical practices of wide scope (habits-of-mind), but these practices produced claims and evidence about particular mathematical objects and relations. This research draws attention to an oft-neglected aspect of mathematics, that of asking questions that can be profitably investigated. As one student reflected, "asking good questions is hard." Nevertheless perhaps because the design of instruction fostered agency and self-expression, students persevered in this challenge. Their perseverance could be viewed as a fortunate coincidence of this particular setting and the unique contingencies that emerged during the course of instruction. Yet, if students are encouraged to generate questions and to follow the implications of these questions, the resulting pathways of particular forms of student learning may be more variable than those observed under other conditions of instruction. It remains to be seen if this variability can be harvested and put to productive use more routinely in everyday situations of teaching and learning.

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