ZDM Mathematics Education (2013) 45:253–265
DOI 10.1007/s11858-012-0473-3

ORIGINAL ARTICLE

Illumination: an affective experience?

Peter Liljedahl

Accepted: 28 October 2012 / Published online: 11 November 2012

Abstract What is the nature of illumination in mathematics? That is, what is it that sets illumination apart from other mathematical experiences? In this article the answer to this question is pursued through a qualitative study that seeks to compare and contrast the AHA! experiences of preservice teachers with those of prominent research mathematicians. Using a methodology of analytic induction in conjunction with historical and contemporary theories of discovery, creativity, and invention along with theories of affect the anecdotal reflections of participants from these two populations are analysed. Results indicate that, although manifested differently in the two populations, what sets illumination apart from other mathematical experiences are the affective aspects of the experience.

1 Introduction

> Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it's dark, completely dark. One stumbles around bumping into the furniture, and gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it's all illuminated. (Andrew Wiles, from Nova (1993))

Suddenly, it's all illuminated. In the time it takes to turn on a light the answer appears and all that came before it makes sense. A problem has just been solved, or a new piece of mathematics has been found, and it has happened in a flash of insight—in a flash of *illumination*. Literature is rich with examples of these instances of illumination—from Amadeus Mozart's seemingly effortless compositions (Hadamard, 1945) to Samuel Taylor Coleridge's dream of Kubla Kahn (Ghiselin, 1952), from Leonardo da Vinci's ideas on flight (Perkins, 2000) to Albert Einstein's vision of riding a beam of light (Ghiselin, 1952)—all of which exemplify the role of this elusive mental process in the advancement of human endeavours. In science, as in mathematics, significant advancement is often associated with these flashes of insight, bringing forth new understandings and new theories in the blink of an eye. But what is the nature of this phenomenon?

Simply put, illumination is the phenomenon of "sudden clarification" (Pólya, 1965, p. 54) arriving in a "flash of insight" (Davis & Hersch, 1980, p. 283) and accompanied by feelings of certainty (Burton, 1999; Fischbein, 1987). In sum, it is the experience of having an idea come to mind with "characteristics of brevity, suddenness, and immediate certainty" (Poincaré, 1952, p. 54). However, illumination is more than just this moment of insight. It is this moment of insight on the heels of lengthy, and seemingly fruitless, intentional effort (Hadamard, 1945).

In this article I explore this phenomenon more closely through the anecdotal reflections of preservice teachers and research mathematicians alike. In particular, I look at what it is that sets the phenomenon of illumination apart from more ordinary mathematical experiences. But first, I situate illumination within the broader context of mathematical discovery, creativity, and invention.

2 History of mathematical discovery, creativity, and invention

In 1902, the first half of what eventually came to be a 30 question survey was published in the pages of

P. Liljedahl (✉)
Simon Fraser University, Burnaby, BC, Canada
e-mail: liljedahl@sfu.ca

L'Enseignement Mathématique, the journal of the French Mathematical Society. Édouard Claparède and Théodore Flournoy, two Swiss psychologists, who were deeply interested in the topics of mathematical discovery, creativity and invention, authored the survey. Their hope was that a widespread appeal to mathematicians at large would incite enough responses for them to begin to formulate some theories about this topic. The first half of the survey centered on the reasons for becoming a mathematician (family history, educational influences, social environment, etc.), attitudes about everyday life, and hobbies. This was eventually followed up, in 1904, by the publication of the second half of the survey pertaining, in particular, to mental images during periods of creative work. The responses were sorted according to nationality and published in 1908.

During this same period Henri Poincaré (1854–1912), one of the most noteworthy mathematicians of the time, had already laid much of the groundwork for his own pursuit of this same topic and in 1908 gave a presentation to the French Psychological Society in Paris entitled *L'Invention mathématique*—often mistranslated to Mathematical Creativity (c.f. Poincaré 1952). At the time of the presentation Poincaré stated that he was aware of Claparède and Flournoy's work, as well as their results, but expressed that they would only confirm his own findings. This presentation, as well as the essay it spawned, stands to this day as one of the most insightful and reflective instances of illumination as well as one of the most thorough treatments of the topic of mathematical discovery, creativity, and invention.

> Just at this time, I left Caen, where I was living, to go on a geological excursion under the auspices of the School of Mines. The incident of the travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuschian functions were identical with those of non-Euclidean geometry. I did not verify the idea; I should not have had the time, as, upon taking my seat in the omnibus, I went on with the conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake, I verified the results at my leisure. (Poincaré, 1952)

So powerful was his presentation, and so deep were his insights into his acts of invention and discovery that it could be said that he not so much described the characteristics of mathematical creativity, as defined them. From that point forth mathematical creativity, or even creativity in general, has not been discussed seriously without mention of Poincaré's name.

Inspired by this presentation, Jacques Hadamard (1865–1963), a contemporary of Poincaré's, began his own empirical investigation into this fascinating phenomenon. Hadamard had been critical of Claparède and Flournoy's work in that they had not adequately treated the topic on two fronts. As exhaustive as the survey appeared to be, Hadamard felt that it failed to ask some key questions—the most important of which was with regard to the reason for failures in the creation of mathematics. This seemingly innocuous oversight, however, led directly to his second and "most important criticism" (Hadamard 1945). He felt that only "first-rate men would dare to speak of" (p. 10) such failures. So, inspired by Poincaré's treatment of the subject Hadamard retooled the survey and gave it to friends of his for consideration—mathematicians such as Henri Poincaré and Albert Einstein, whose prominence were beyond reproach. Ironically, the new survey did not contain any questions that explicitly dealt with failure.

In 1943 Hadamard gave a series of lectures on mathematical invention at the École Libre des Hautes Études in New York City. These talks were subsequently published as The Psychology of Invention in the Mathematical Field (Hadamard, 1945).

Hadamard's treatment of the subject of invention at the crossroads of mathematics and psychology is an extensive exploration and extended argument for the existence of unconscious mental processes. To summarize, Hadamard took the ideas that Poincaré had posed and, borrowing a conceptual framework for the characterization of the creative process from Wallas (1926), turned them into a stage theory. This theory still stands as the most viable and reasonable description of the process of mathematical invention.

For Hadamard, the phenomenon of mathematical invention consists of four separate stages stretched out over time. These stages are initiation, incubation, illumination, and verification (Hadamard, 1945).[1] The first of these stages, the initiation phase, consists of deliberate and conscious work. This would constitute a person's voluntary, and seemingly fruitless, engagement with a problem and be characterized by an attempt to solve the problem by trolling through a repertoire of past experiences (Bruner, 1964). This is an important part of the inventive process because it creates the tension of unresolved effort that sets up the conditions necessary for the ensuing emotional release at the moment of illumination (Davis & Hersch, 1980; Feynman, 1999; Hadamard, 1945; Poincaré, 1952; Rota, 1997). Following the initiation stage the solver, unable to come to a solution, stops working on the problem at a conscious level (Dewey, 1933) and begins to work on it at an unconscious level (Hadamard, 1945; Poincaré, 1952).

[1] For alternate models of creativity see Sriraman (2004).

This is referred to as the incubation stage of the inventive process and it is inextricably linked to the conscious and intentional effort that precedes it. After the period of incubation a rapid coming to mind of a solution, referred to as illumination, may occur. This is accompanied by a feeling of certainty (Poincaré, 1952) and positive emotions (Barnes, 2000; Burton 1999; Rota, 1997). Illumination is the manifestation of a bridging that occurs between the unconscious mind and the conscious mind (Poincaré, 1952), a coming to (conscious) mind of an idea or solution. Colloquially it is often referred to as the AHA! experience. What brings the idea forward to consciousness is unclear, however. There are theories of the aesthetic qualities of the idea (Sinclair, 2002; Poincaré, 1952), effective surprise/shock of recognition (Bruner, 1964), fluency of processing (Whittlesea and Williams, 2001), or breaking functional fixedness (Ashcraft, 1989). Regardless of the impetus, the correctness of this emergent idea is then evaluated during the fourth and final stage—verification.

So, although instances of creativity, discovery, and invention are often seen as being punctuated by the phenomenon of illumination, illumination is but one part of the process. Having said that however, illumination is THE aspect of the process that sets creativity, discovery, and invention apart from the more ordinary, and more common, processes of solving a problem—it is the marker that something remarkable has taken place.

3 Contemporary research on creativity

Creativity is a term that can be used both loosely and precisely. That is, while there exists a common usage of the term there also exists a tradition of academic discourse on the subject. A common usage of 'creative' refers to a process or a person whose products are original, novel, unusual, or even abnormal (Csikszentmihalyi 1996). In such a usage, creativity is assessed on the basis of the external and observable products of the process, the process by which the product comes to be, or on the character traits of the person doing the 'creating'. Each of these usages—product, process, person—has come to form an independent academic discourse on creativity (Liljedahl & Allen, in press).

The academic discourse that concerns itself with the products of creativity stipulates that first, and foremost, a product must be forthcoming—it cannot be a creative process if nothing is created. Secondly, this discourse demands that the product be assessed against other products within its field, by the members of that field, to determine if it is original AND useful (Bailin, 1994). If it is, then the product is deemed to be creative. Note that such a use of assessment of end product pays very little attention to the actual process that brings this product forth.

The second discourse to be discussed concerns the creative process. The literature pertaining to this can be separated into two categories, a prescriptive discussion of the creativity process and a descriptive discussion of the creativity process. Although both of these discussions have their roots in stage theories of creativity they make use of these stages in very different ways. The prescriptive discussion of the creative process is primarily focused on the initiation phase and is best summarized as a cause-and-effect discussion of creativity, where the thinking processes during the initiation stage are the cause and the creative outcomes are the effect (Ghiselin 1952). Some of the literature claims that the seeds of creativity lie in being able to think about a problem or situation analogically. Other literature claims that utilizing specific thinking tools such as imagination, empathy, and embodiment will lead to creative products. In all of these cases, the underlying theory is that the eventual presentation of a creative idea will be precipitated by the conscious and deliberate efforts during the initiation stage.

On the other hand, the literature pertaining to a descriptive discussion of the creative process is inclusive of all four stages (Kneller 1965; Koestler 1964). For example, Csikszentmihalyi (1996), in his work on 'flow' attends to each of the stages, with much attention paid to the fluid area between conscious and unconscious work, or initiation and incubation. His claim is that the creative process is intimately connected to the enjoyment that exists during times of sincere and consuming engagement with a situation, the conditions of which he describes in great detail.

The third, and final, discourse on creativity pertains to the person. This discourse is dominated by two distinct characteristics, habit and genius. Habit has to do with the personal habits as well as the habits of mind of people that have been deemed to be creative. However, creative people are most easily identified through their reputation for genius. Consequently, this discourse is often dominated by the analyses of the habits of geniuses as is seen in the work of Ghiselin (1952), Koestler (1964), and Kneller (1965) who draw on historical personalities such as Albert Einstein, Henri Poincaré, Vincent Van Gogh, D. H. Lawrence, Samuel Taylor Coleridge, Igor Stravinsky, and Wolfgang Amadeus Mozart to name a few. The result of this sort of treatment is that creative acts are viewed as rare mental feats, which are produced by extraordinary individuals who use extraordinary thought processes.

I sum up the different discourses on creativity as a tension between *absolutist* and *relativist* perspectives on creativity (Liljedahl & Sriraman, 2006; Liljedahl & Allen, in press). An absolutist perspective assumes that creative processes are the domain of genius and are present only as precursors to the creation of remarkably useful and universally novel products. The relativist perspective, on the other hand, allows for every individual to have moments of creativity that may, or may not, result in the creation of a product that may, or may not, be either useful or novel. From either perspective, however, illumination is at the heart of the creative experience—either punctuating the experience or bringing forth the creative products.

4 Researching illumination

In this article I present the results of research done on the nature of illumination as experienced by both preservice elementary school teachers and research mathematicians alike. An absolutist would argue that this would not be possible as preservice teachers would be unlikely to have truly creative experiences. However, working from a relativistic perspective I see the phenomenon of mathematical illumination as being something that both preservice school teachers and research mathematicians alike have experienced, albeit in different contexts.

> Between the work of a student who tries to solve a problem in geometry or algebra and a work of invention, one can say there is only a difference of degree. (Hadamard 1945)

My central question in this research is to uncover what it is that sets illumination apart from other mathematical experiences. To achieve this, I am looking to compare the differentiating quality of illumination for the two populations mentioned—preservice teachers and research mathematicians.

But, how does one collect meaningful data on a phenomenon as rare and as fleeting as illumination?

> And yet, the task is inherently difficult. The absence of sufficient knowledge on this topic is not a matter of a mere negligence on the part of researchers. There are at least two reasons why collecting direct observational data on AHA! seems like an impossible mission. First, being a private phenomenon, it is directly accessible only to the experiencing subject. Second, being defined as an experience that happens suddenly and "without warning," it cannot be captured just when the observer has time and means to observe. These two difficulties, however, did not stop either Gestalt psychologists or the French mathematician Hadamard from tackling the issue. In both cases, the principal method of study was the subjects' self-report on their problem-solving processes, provided a posteriori. (Sfard, 2004)

And so it is in this study. Building on the utility of anecdotal reflections I initiated two distinct, but related studies. In what follows I present each of these two studies along with the pertinent details of their methodologies, results, and discussions.

5 Preservice teachers

The participants for this first study were undergraduate students at Simon Fraser University (SFU) enrolled in a *Mathematics for Teachers* (MfT) course. This course, like other MfT courses around the world, had been designed with the intention of providing its enrolees with a foundational understanding of elementary school mathematics. There is a focus on conceptual understanding of topics as opposed to an ability to replicate procedural algorithms. There is an attempt to look at specific strands of mathematics such as geometry and number theory in their entirety as opposed to the piecewise and fragmented way in which mathematics is often experienced in a spiralled curriculum. There is also an attempt to integrate an underlying appreciation for mathematical thinking and reasoning across all strands of the course.

In general, students enrolled in MfT at SFU are best described as *resistant*. Many of the students would describe themselves as either being math-phobic, math-incapable, or a combination of the two. They usually have negative beliefs about their abilities to do mathematics, poor attitudes about the subject, and dread the thought of having to take a mathematics course. Such being the case, there is a great effort made to alleviate some of these anxieties through the pedagogical approaches to the content, a heavy reliance on group work, and access to an open tutorial lab.

The data for this first study come from one of the options for an end-of-term project, wherein the students were asked to write about an experience with illumination in the context of mathematics (see Fig. 1). The project was worth 10 % of their final mark and they were given 4 weeks to work on it.

Rather than provide a definition of illumination participants were instead presented with an anecdotal account using language that was descriptive as opposed to definitive. This anecdote was identified as an AHA!, a label that has very clear colloquial meaning. This was in part due to the fact that I did not wish to corrupt their responses regarding the nature of illumination with ideas that would be included in such a definition.

Of the 112 students enrolled in the course, 76 students chose to write about their AHA! experience. Of these, three students clearly misunderstood the self-defining nature of the anecdote provided and, although assessed for the purposes of the MfT course, their responses were not analysed

Fig. 1 AHA! project

> *I had been working on the problem for a long time without any progress. Then suddenly I knew the solution, I understood, everything made sense. It seemed like it just CLICKED!*
>
> The above anecdote is a testament of what is referred to as an AHA! experience. Have you ever experienced one? The purpose of this assignment is to have you reflect upon such an AHA! experience and to explore exactly what you learned in that instance and what you think contributed to the moment. You will hand in:
>
> 1. A detailed explanation of the specific mathematical topic that you were studying and the difficulty you were having with it (including any incorrect or incomplete understandings that you had of the topic before the AHA!).
> 2. The story of the AHA! experience as you remember it, paying particular close attention to what you were doing before it happened, when it happened, and how it made you feel when it happened.
> 3. A detailed explanation of your new understanding of the mathematical topic.
> 4. A conclusion as to how, upon reflection, the AHA! experience contributes to mathematical learning in general, and for you in particular.
> 5. Anything else that you feel would contribute to the reader gaining insight into the moment as you experienced it.
>
> Your final product will be evaluated for completeness and clarity.

for the research presented here. The remaining set of 73 responses formed the data set.

The 73 accounts can be grouped into two categories: *teaching* and *discovering*. The first of these, teaching, pertains to illumination experienced in the passive reception of mathematical content. In total, there were 14 such accounts. Each of these told of an event in the lecture hall where something the instructor said or demonstrated caused them to understand a previously not understood piece of mathematical content. A much more common experience, however, was of the discovery type and accounted for the remaining 59 responses. These were descriptions of illumination that had occurred in the context of trying to work something out for themselves, either in solving a problem or working towards understanding some particular mathematics content.

These data were analysed using the principles of analytic induction (Patton, 2002). "[A]nalytic induction, in contrast to grounded theory, begins with an analyst's deduced propositions or theory-derived hypotheses and is a procedure for verifying theories and propositions based on qualitative data" (Taylor and Bogdan, 1984, p. 127 cited in Patton, 2002, p. 454).

A cursory look at the data revealed that the participants were attending, in particular, to the last part of question 2 of the assignment—"and how it made you feel when it happened". The length of submissions for this assignment varied from four pages to 15 pages with the majority being in the range from six to eight pages. With one small exception the participants dispensed with addressing all of questions 1, 3, 4, 5, and the first part of question 2 (above) of the assignment in two pages or less. The remaining pages were filled with their attention to how they felt.

As such, I began a process of extracting affective themes from the data using a constant comparison method similar to ones used in grounded theory (Glaser & Strauss, 1967). The difference being, however, that whereas grounded theory begins with the data, analytic induction begins with theory.

The theories I began with come out of the research literature on affect. In particular, I drew on the literature from McLeod (1992), who describes affect as being comprised of beliefs, attitudes, and emotions, and DeBellis & Goldin (2006), who include among these a fourth aspect—values. Beliefs can be defined as subjective knowledge—what someone holds to be true. For mathematics students it pertains to what they hold to be true about mathematics both with respect to the teaching and learning of mathematics. A qualitatively different form of belief is with regards to what a person holds to be true about their ability to do mathematics, often referred to as efficacy, or self-efficacy. Self-efficacy, like the aforementioned mathematical beliefs, is a product of an individual's experiences with mathematics.

Attitudes can be defined as "a disposition to respond favourably or unfavourably to an object, person, institution, or event" (Ajzen, 1988, p. 4). Attitudes can be thought of the kinds of emotional responses that students have in particular contexts (DeBellis & Goldin, 2006).

Emotions, unlike beliefs and attitudes, are much more unstable (O'pt Eynde, De Corte, & Verschaffel, 2001).

They are rooted more in the immediacy of a situation or a task and as a result are often fleeting. Students with generally negative beliefs and attitudes can experience moments of positive emotions about a task at hand or, conversely, students with generally positive outlooks can experience negative emotions.

Values include morals and ethic and refer to deep personal truths that help guide "long-term choices and shorter-term priorities" (DeBellis & Goldin, 2006, p. 135).

5.1 Results

In what follows I present the affective themes present within the data. For each theme I use excerpts that exemplify the theme while at the same time being representative of all of the students' responses pertaining to that theme.

5.1.1 *Emotions*

All but one of the participants in this study mentioned something about the positive emotion they felt after their AHA! experience. Although their comments varied in length and details with regard to these emotions, each of them stated in one way or another that it felt 'great'. In what follows I provide a partial list of some of these comments.

John: It felt great.
Ruth: I was so relieved; I could barely contain my happiness.
Jenny: This was the best feeling.
Christina: I never knew I could feel so good while doing math.
Keri: Wow!
Stacy: The joy I felt was like none other.
Natalie: It made me feel like I could do anything.

It is clear that illumination produced a positive affective response in the students. However, as will be shown in the next two sections, the AHA! produced more than simply a 'good' feeling. It contributed to a positive change in the beliefs and attitudes of many of the students.

5.1.2 *Change in beliefs*

Of the 73 students whose projects were analysed 61 of them discussed their beliefs. Moreover, each of these 61 students did so in the context of a change in their beliefs through their AHA! experience.

Susan describes how the experience has changed her beliefs on both her ability to solve problems and the process she uses to produce a solution.

Susan: The AHA! experience is inspiring. It makes students believe that they solved that question through reasoning and deep thought, and inspires him or her to seek more of these moments to obtain a sort of confidence and further knowledge.

This was a common theme, often manifesting itself in discussions of newfound confidence as expressed by Steve and Andrea.

Steve: Initially this course made me very unsure of myself but now I am confident when working out problems among my homework group. Previously, I naturally deferred to them, but after this AHA! experience I got confidence in my answers.

Andrea: In reflecting upon this AHA! experience I feel a sense of pride that I accomplished this mathematical idea by myself. I am relieved to know that I do not have to depend on others to help me along. This moment also gave me a self-confidence boost in the sense that I may have something to contribute to others, for example my group members.

James reflects on how the absence of these experiences may have contributed to his belief that he was not good at mathematics.

James: For myself, I wish that I'd had more of these moments in my earlier years of high school then I would maybe not have so readily decided that I was not good at math.

The belief of what 'it takes' to be good at math is altered for Lena as she expresses that she now sees that it is not an issue of intelligence.

Lena: Knowing that I could stare at a problem and in time I would understand, gave me more confidence that I could be successful in math. It really is not an intelligence issue.

Karen sits on the border between beliefs in her ability to do mathematics and her belief in what it takes to do mathematics.

Karen: I used to think that if you couldn't get it right away you didn't know how to do it. This is the longest I've ever worked on a problem. I had just about given up when it just came to me. I now know that sometimes it just takes time.

What is interesting is the variety of beliefs that were affected by experiencing illumination in the context of mathematics. Although most of them centre on their own conceptions of their abilities to do mathematics some students expressed how their beliefs about mathematics have changed, as seen in Paula's statement.

Paula: I used to think that math was all about the right answer, but now I am more aware of the value of the process.

5.1.3 Change in attitudes

Because attitudes are the manifestations of beliefs it was sometimes difficult to distinguish between the two. In fact, almost every expression of a change in attitude had a discernible change in beliefs associated with it and has been counted in the 61 responses discussed above. Charlotte and Stephen express a change in optimism and expectations, respectively.

Charlotte: I have a better attitude now; I'm more optimistic. This is helpful in learning as complete thought processes can be impeded by a dejected attitude.

Stephen: Also, I enjoy math now. I feel like this success stimulated more success. Now I have raised my expectations in math.

Carla has come to terms with her lack of knowledge of mathematics and found within it a new attitude for success.

Carla: I've decided that I really don't know a lot of math. But who cares? I know enough. And I know how to think enough to find the answers. And I know how to ask for help. And I don't care so much about the end result.

However, a few students clearly demonstrate a change in attitude without expressing an obvious change in beliefs. This is best demonstrated in Kristie's comment.

Kristie: I must admit that math is challenging for me … after the AHA! experience you feel like learning more, because the joy of obtaining the answer is so exhilarating. It almost refreshes one's mind and makes them want to persist and discover more answers. It gave me the inspiration and the determination to do the best that I can do in the subject.

Kristie has most definitely changed her attitude about the pursuit of mathematics in that she is feeling inspired and determined to succeed in the course. What is not clear is whether or not this is as a result of a new belief that she can succeed.

5.1.4 Mathematical understanding

Although the data were coded with a theory of affect in mind, analytic induction allows for the emergence of themes not anticipated from a priori theories. One of the themes that emerged in this way was the theme around mathematical understanding achieved in the moment of illumination. I present here Kim's comments as an example.

> My AHA! experience came during the first midterm. The question that I was having difficulty with was the very first one. That was already a bad sign. The question was concerning patterns and involved stars arranged in a triangular fashion.
>
> P(1) = **
> *
>
> ***
> P(2) = **
> *
>
> ****
> ***
> P(3) = **
> *
>
> … …
>
> The question asked how many stars would be in P(12) and how many stars in P(121)? I was lost. I added up all the stars for each number and compared them to each other and tried to figure out the pattern there. I drew diagrams that made no sense. I tried to use formulas that I had memorized, placing n's and (n − 1)'s and (n + 1)'s anywhere, but that was not yielding anything correct. Everything I thought I remembered was completely disintegrating out of my brain and as the minutes ticked I began to panic. As I turned the page there was a similar question as the first. I was lost again. The exam is going very poorly. I ultimately decided to skim throughout the rest of the test and purely answer whatever I could and if I had moments to spare I would go back to the puzzling questions. I went back to question one and looked at what I had written down thus far. Then I erased it all. I started to draw more figures representative of the ones given and then it hits me! I realized that the number of stars in the first row of any P(n) was one larger than n and that they descended by 1 to 1. That is P(4) starts with 5, then 4, 3, 2, 1. I now remember the very first math class where we discussed patterns and Gauss and how to add all the numbers together up to 100. So, P(12) = 13 + 12 + 11 + … + 1. […] So, P(12) = 182/2 = 91.
> Thus, the answer is 91 stars for P(12). I could now answer the rest of the questions with ease. I began to relax …

In general, all of the participants had clear recall of the mathematics involved in their AHA!'s. Analysis of this mathematics and the mathematical understandings surrounding the phenomena revealed that prior to their AHA! experiences there existed, as it did in the mind of Kim, a piece of mathematics that they either did not understand or a problem that they could not solve—they were ‘stuck’. After their AHA! they now understood this mathematics or could solve the problem—they were ‘unstuck’. The AHA! was clearly in the middle of all this and intimately involved in the transition from being ‘stuck’ to being ‘unstuck’. However, at the level of mathematical ideas this transition from ‘stuck’ to ‘unstuck’ was minute. It was unremarkable and in many cases indistinguishable from the progression of ideas that surrounded the phenomenon. From an absolutist perspective it could be claimed, for a number of reasons that because the product of the experience was not significant that the experience was not creative. However, from a relativist stance it is clear that there was an experience of some significance, but that the importance was not played out at the level of mathematics.

5.2 Discussion

For the preservice teachers the AHA! experience was an affective experience. There were clear and powerful positive emotions associated with the phenomenon. For most participants, accompanying these positive emotions were positive changes in beliefs and attitudes. There was also clear evidence that for each participant the phenomenon involved a change in mathematical understanding. However, the shift in understanding was unremarkable in relation to the progression of mathematical ideas that surrounded the AHA! Nonetheless, the participants recalled the details of the experience both affectively and mathematically. DeBellis and Goldin (2006) would posit that the intensity of the emotions *encoded* the experience for the participants, building a bond between the affective aspects of the experience with the mathematical content involved.

6 Research mathematicians

For the second study a portion of Hadamard’s original questionnaire was resurrected and given to research mathematicians in order to elicit reflections on their own encounters with the phenomenon of mathematical creativity, invention, and discovery in general and illumination in particular. Hadamard’s original questionnaire contained 33 questions pertaining to everything from personal habits, to family history, to meteorological conditions during times of work (Hadamard, 1945). From this extensive and exhaustive list of questions the five that most directly related to the phenomena I was interested in were selected.

1. Would you say that your principal discoveries have been the result of deliberate endeavour in a definite direction, or have they arisen, so to speak, spontaneously? Have you a specific anecdote of a moment of insight/inspiration/illumination that would demonstrate this? [Hadamard #9]
2. How much of mathematical creation do you attribute to chance, insight, inspiration, or illumination? Have you come to rely on this in any way? [Hadamard #7]
3. Could you comment on the differences in the manner in which you work when you are trying to assimilate the results of others (learning mathematics) as compared to when you are indulging in personal research (creating mathematics)? [Hadamard #4]
4. Have your methods of learning and creating mathematics changed since you were a student? How so? [Hadamard #16].
5. Among your greatest works have you ever attempted to discern the origin of the ideas that lead you to your discoveries? Could you comment on the creative processes that lead you to your discoveries? [Hadamard #6]

These questions, along with a covering letter, were then sent to 150 prominent mathematicians in the form of an email. Hadamard set excellence in the field of mathematics as a criterion for participation in his study. In keeping with Hadamard’s standards, excellence was also chosen as the primary criterion for participation in this study. As such, recipients of the survey were selected based on their achievements in their field as recognized by their being honoured with prestigious prizes or membership in noteworthy societies. In particular, the 150 recipients were chosen from the published lists of the Fields Medalists, the Nevanlinna Prize winners, as well as the membership list of the American Society of Arts and Sciences. The 25 recipients, who responded to the survey, in whole or in part, have come to be referred to as the *participants* in this study. Of these 25 participants all but one agreed to allow their name to appear alongside their comments.

After responses to the aforementioned five questions had been received participants were sent a further two questions, again in the form of an email. These additional questions were designed to specifically focus on the phenomenon of illumination—again referred to as the AHA! experience. These questions were:

6. How do you know that you have had an AHA! experience? That is, what qualities and elements about the experience set it apart from other experiences?
7. What qualities and elements of the AHA! experience serve to regulate the intensity of the experience? This is assuming that you have had more than one such experience and they have been of different intensities.

As in the first study these data were coded using a methodology of analytic induction (Patton, 2002). This time the coding was informed by the theories of affect, discovery, invention, and creativity. As before, I was looking for the distinguishing characteristics of illumination.

6.1 Results

Through the process of recursively coding for relevant attributes 15 themes eventually emerged out of the data. These were: (1) the role of sleep, (2) meta-cognition, (3) mathematical landscape, (4) gaps in the landscape, (5) failures and wrong ideas, (6) prolonged conscious work, (7) deriving and re-creating mathematics, (8) the sense of significance, (9) the contribution of chance, (10) the de-emphasis of details, (11) the role of talking, (12) the role of time, (13) simplicity, (14) the role of surrounding information, and (15) the context of illumination. All of these themes are present in the work of Hadamard (1945) and many in the work of Poincaré (1952). However, the analysis extend the results present in themes 8, 9, 10, and 11 beyond both Hadamard's (1945) and Poincaré's (1952) contributions to the field. For purposes of brevity and relevance, in what follows, I present partial results from four of these 15 themes. For a more thorough presentation of results and how they link to Hadamard (1945) see Liljedahl (2009).

6.1.1 The context of illumination

With respect to this article, the second part of question one is most relevant. I asked the participants to provide a specific anecdote that demonstrated the role of insight, illumination, or inspiration. One of the things that became immediately apparent in analysing the mathematicians' responses to this question is that, unlike with the preservice teachers, the particulars of mathematical ideas are almost wholly absent from the mathematicians stories. This is exemplified nicely in Dusa McDuff's response to question one.

> In my principal discoveries I have always been thinking hard trying to understand some particular problem. Often it is just a hard slog, I go round arguments time and again seeking for a hole in my reasoning, or for some way to formulate the problem/structures I see. Gradually some insights builds and I get to "know" how things function. But the main steps come in flashes of insight; something clicks into place and I see something clearly, not necessarily what I was expecting or looking for. This can occur while I am officially working. But it can also occur while I am doing something else, having a shower, doing the cooking. I remember that the first time I felt creative in math was when I was a student (undergrad) trying to find an example to illustrate some type of behaviour. I'd worked on it all the previous evening with no luck. The answer came in a flash, unexpectedly, while I was showering the next morning. I saw a picture of the solution, right there, waiting to be described. (Dusa McDuff)

Although the details of the mathematics are missing the essence of the experience remains. There is a vagueness about the mathematical ideas involved that is reminiscent of the way Poincaré (1952) talked about his experiences.

> I must apologize, for I am going to introduce some technical expressions, but they need not alarm the reader, for he has no need to understand them. I shall say, for instance, that I found the demonstration of such and such a theorem under such and such circumstances; the theorem will have a barbarous name that many will not know, but that is of no importance. What is interesting for the psychologist is not the theorem but the circumstances (p. 52).

And the circumstances were interesting. Almost all of the mathematicians surveyed highlighted in their anecdotes the context of their experience, and they do so in the absence of any mathematical details. It is almost as though the ideas themselves are ancillary to the experience of illumination.

> While at a meeting in Philadelphia, I woke up one morning with the right idea. (Dick Askey)

> And relevant ideas do pop up in your mind when you are taking a shower, and can pop up as well even when you are sleeping (many of these ideas turn out not to work very well) or even when you are driving. (Dan J. Kleitman)

> In regard to illumination, I would like to add that in my case the best instances have been at night when I am laying in bed, somewhere between consciousness and sleep. (Demetrios Christodoulou)

> I distinctly remember the moment early in our collaboration when I saw how to get past one of the major technical difficulties. This happened while walking across campus after teaching a class. (Wendell Fleming)

> It may have been in the shower that it just occurred to me that the work of some of the classical authors could be generalized in a certain way […] I can be talking to a colleague or my wife or eating breakfast and suddenly, like a voice from the blue, I get told what to do. (Jerry Marsden)

> The overt work is much the same as it always was. The covert work (in bed, on the subway, in dreams) is harder now. (Henry McKean)

> I'm convinced that I do my best work while asleep. The evidence for this is that I often wake up with the solution to a problem, or at least with a clear idea of how to proceed to solve it. (Charles Peskin)

6.1.2 The contribution of chance

The analysis of the data revealed that there are two types of chance, intrinsic chance and extrinsic chance. Intrinsic chance deals with the luck of coming up with an answer, of having the right combination of ideas join within your mind to produce a new insight. This is nicely articulated in Kleitman's comments.

> And relevant ideas do pop up in your mind when you are taking a shower, and can pop up as well even when you are sleeping (many of these ideas turn out not to work very well) or even when you are driving. Thus while you can turn the problem over in your mind in all ways you can think of, try to use all the methods you can recall or discover to attack it, there is really no standard approach that will solve it for you. At some stage, if you are lucky, the right combination occurs to you, and you are able to check it and use it to put an argument together. (Dan J. Kleitman)

Intrinsic chance was discussed by Hadamard (1945) as well as by a host of others under the heading of the "chance hypothesis". Extrinsic chance, on the other hand, deals with the luck associated with a chance reading of an article or a chance encounter, either of which contribute to the eventual resolution of the problem that one is working on.

> But chance is a major aspect: what papers one happens to have read, what discussions one happens to have struck up, what ideas one's students are struck by (never mind the very basic chance process of insemination that produced this particular mathematician). (Connor)

> As my anecdote indicates, it is not just chance, but rather inspiration in the presence of lots of surrounding information. The surrounding information is really crucial, I believe. (Jerry Marsden)

> Do I experience feelings of illumination? Rarely, except in connection with chance, whose offerings I treasure. In my wandering life between concrete fields and problems, chance is continually important in two ways. A chance reading or encounter has often brought an awareness of existing mathematical tools that were new to me and allowed me to return to old problems I was previously obliged to leave aside. In other cases, a chance encounter suggested that old tools could have new uses that helped them expand. (Benoit B. Mandelbrot)

> Thus while you can turn the problem over in your mind in all ways you can think of, try to use all the methods you can recall or discover to attack it, there is really no standard approach that will solve it for you. At some stage, if you are lucky, the right combination occurs to you, and you are able to check it and use it to put an argument together. (Dan J. Kleitman)

The idea that mathematical discovery relies, at least in part, on the fleeting and unpredictable occurrences of chance encounters is starkly contradictory to the image projected by mathematics as a field reliant on logic and deductive reasoning. Extrinsic chance, in particular, is an element that has been largely ignored in the literature.

6.1.3 The role of significance

Several of the mathematicians mentioned significance as one of the characteristics that, both sets the AHA! experience apart from other mathematical experiences, and regulates the intensity of the experience. However, their usage of the term is problematic. Consider the following three excerpts:

> It is, in my experience, just like other AHA! experiences where you suddenly "see the light". It is perhaps a little more profound in that you see that this is "important". I find that as one gets older, you learn to recognize these events more easily. When younger, you often don't realize the significance of such an event at the time. (Jerry Marsden)

> What I think you mean by an AHA! experience comes at the moment when something mathematically significant falls into place. (Wendell Fleming)

> The depth of the experience depends on how profound the ultimate result is. (Michael Atiyah)

Each of these three comments uses significance (or important, or profound) in a post-evaluative sense. That is, they speak of how significant a new idea will turn out to be once verified. However, such a thing cannot be known in the instance of illumination. It is only through verification, a potentially lengthy process, that this can be truly ascertained. If age has an effect on this, as Marsden claims, it would only serve to shorten the process of verification, not eliminate it as would be necessary for the significance to truly be known in the instance of illumination. This is particularly pertinent to Fleming's comment that the AHA! occurs "when something mathematically significant falls into place".

However, such statements cannot be ignored. In order for the mathematicians to be associating the AHA! experience with the significance of the idea that reveals itself at the time of illumination then one of three things must be occurring. First, they are consciously suspending any evaluation of an experience as to whether or not it is an AHA! experience until after all the results have been checked. This is an absolutist stance with a focus on product and is likely what Atiyah is doing in ascertaining the depth of the experience. Secondly, their recollection of the experience is being influenced by the outcome of the eventual evaluation of the idea that was presented to them during their AHA! experience. In psychology this is referred to as memory reconstruction (Whittlesea, 1993). The third option, and the one that I find most likely, is that although the absolute significance of a find may ultimately be verified, at the time of illumination what they are, in fact, experiencing is a sense of significance. That is, at the moment of illumination the mathematicians are living the experience of the creative process. This possibility is displayed in Fleming's complete passage.

> What I think you mean by an AHA! experience comes at the moment when something mathematically significant falls into place. This is a moment of excitement and joy, but also apprehension until the new idea is checked out to verify that all the necessary details of the argument are indeed correct. (Wendell Fleming)

Although Fleming begins the passage with an absolute usage of significance, stating that the AHA! occurs when something "mathematically significant falls into place", he softens this stance in his second sentence where he acknowledges that he is filled with a feeling of apprehension until the results are verified. Fleming's statement clearly indicates that regardless of how he uses the term significance, he sees it as temporary and tentative. The same theme is also nicely demonstrated by Huber, where he makes a distinction between the AHA! experience and the EUREKA! experience.

> First I would have a promising, brilliant(?) idea (the AHA! event) which would induce me to drill. But the EUREKA event ("I found it!") at best would come hours or days later, if and when the oil would begin to gush forth. That the idea had been brilliant and not merely foolish would be clear only in retrospect, after attempts to verify and confirm it. And later on one tends to suppress and forget foolish ideas because they are embarrassing (but they are indispensable companions to the brilliant ones!). (Peter J. Huber)

6.1.4 Prolonged conscious work

All of the mathematicians mention that prolonged conscious work is an important part of the creative process. This is clearly in alignment with the four stages of the creativity and invention first posited by Wallas (1926) and made popular by Hadamard (1945). However, some of the mathematicians go further to point out that the more intense this period of conscious work is the more intense the AHA! is.

> The harder and more prolonged the prior work, and/or the more sudden and unexpected the insight, the more intense is the AHA! experience. (Connor)

> It depends how long one has worked, how many silly mistakes one made. (Henry McKean)

This prolonged conscious work creates the tension of unresolved effort that sets up the conditions necessary for the ensuing emotional release at the moment of illumination (Davis & Hersch, 1980; Feynman, 1999; Hadamard, 1945; Poincaré, 1952; Rota, 1997).

6.2 Discussion

In comparison to the preservice teachers the research mathematicians are not nearly as forthcoming with the mathematical details of their AHA! experiences. This is not to say that they do not recall them, for I suspect that these details will have been encoded (DeBellis and Goldin, 2006) within their experience in the same way that illumination encoded the details for the preservice teachers. It is more likely that the mathematicians chose to withhold the details because they are acutely aware that these details are beyond me as a researcher. This is not true of the details shared by the preservice teachers. What is interesting, nonetheless, is that the anecdotes of their experiences with illumination are not diminished by this omission. These experiences can stand on their own and away from the details of the mathematics. What remains are the stories of the context within which illumination occurred— suddenly and without any prior warning, perhaps due to luck, and often while doing something completely different—and of the intensity of these experiences as regulated by the intensity of the preceding conscious work.

These descriptions speak to the affective aspects of the AHA! experience, and only to the affective aspects. Even the reflections on the creative products have affective overtones. The sense of significance, for example, that the mathematicians experience is a an affective response to the creative process in general and illumination in particular. Only after they have the time to evaluate the merits of the idea that came in a flash if illumination can they assume the absolutist stance that allows them to speak about creative products.

7 Conclusions

My main conclusion is that the essence of illumination in mathematics is in the affective domain. That is, what sets the phenomenon of illumination apart from other mathematical experiences is the affective component of the experience, and ONLY the affective component.

This is not to say that illumination is not a cognitive experience, for clearly it is. After all, it is the arrival of an idea that, in part, defines the phenomenon. I do not deny this, and the results of this study do not refute this. What the results do show, however, is that, while the affective component of illumination is consequential to the differentiation of it from other mathematical experiences, the cognitive component is not. The most obvious evidence for this comes from the fact that nowhere in the data were the details of the idea central to the discussion of the phenomenon. In the case of the preservice teachers where details were presented, their contribution to the account of the experience was negligible. Their mathematical *leaps* were unremarkable in comparison to the other steps forward through thinking or learning that surround the moment of illumination. Ironically, this notion is most succinctly stated by a mathematician.

> No, I don't find it different from understanding other things in life! (Henry McKean)

McKean's view is that the understandings gained from illumination are no different than the understandings gained from other sources.

For the mathematicians, the descriptions of the mathematics and the ideas that were implicated in illumination were absent, but not missed. The descriptions of the AHA! experiences did not suffer for this absences.

So what defines illumination? What sets it apart from other mathematical experiences with solving problems or learning mathematics? Affect does. That is, what serves to make the AHA! experiences extraordinary is the affective response invoked by the experience of an untimely and unanticipated presentation of an idea or solution. For the preservice teachers this unexpected presentation of a solution filled them with positive emotions, precipitating changes in beliefs and attitudes, and encoding the details of the experience. Although not as overtly emotive, the experiences of the research mathematicians during these flashes of illumination were equally affective in nature.

References

Ajzen, I. (1988). *Attitudes, personality, and behaviour*. Milton Keynes: Open University Press.

Ashcraft, M. (1989). *Human memory and cognition*. Glenview: Scott, Foresman and Company.

Bailin, S. (1994). *Achieving extraordinary ends: An essay on creativity*. Norwood: Ablex Publishing Corporation.

Barnes, M. (2000). 'Magical' moments in mathematics: Insights into the process of coming to know. *For the Learning of Mathematics, 20*(1), 33–43.

Bruner, J. (1964). *Bruner on knowing*. Cambridge: Harvard University Press.

Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37*(2), 121–143.

Csikszentmihalyi, C. (1996). *Creativity: Flow and the psychology of discovery and invention*. New York: HarperCollins Publishers.

Davis, P., & Hersch, R. (1980). *The mathematical experience*. Boston: Birkhauser.

DeBellis, V., & Goldin, G. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics, 63*(2), 131–147.

Dewey, J. (1933). *How we think*. Boston: D.C. Heath and Company.

Feynman, R. (1999). *The pleasure of finding things out*. Cambridge: Perseus Publishing.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*. Dordrecht: Kluwer Academic Publishers Group.

Ghiselin, B. (1952). *The creative process: Reflections on invention in the arts and sciences*. Berkeley: University of California Press.

Glaser, B. & Strauss, A. (1967). *The discovery of grounded theory: Strategies for qualitative research.* Chicago: Aldine Publishing Co.

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. New York: Dover Publications.

Kneller, G. (1965). *The art and science of creativity*. New York: Holt, Reinhart, and Winstone, Inc.

Koestler, A. (1964). *The act of creation*. New York: The Macmillan Company.

Liljedahl, P. (2009). In the words of the creators. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Mathematical creativity and the education of gifted children* (pp. 51–70). Rotterdam: Sense Publishers.

Liljedahl, P., & Allen, D. (in press). Mathematical discovery. In E. G. Carayannis (Ed.), *Encyclopedia of creativity, invention, innovation, and entrepreneurship*. New York: Springer.

Liljedahl, P., & Sriraman, B. (2006). Musings on mathematical creativity. *For The Learning of Mathematics, 26*(1), 20–23.

McLeod, D. (1992). Research on the affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–596). New York: Macmillan.

Nova (1993). *The proof*. Aired on PBS on October 28, 1997. http://www.pbs.org/wgbh/nova/transcripts/2414proof.html. Accessed 11 December 2003.

O'pt Eynde, P., De Corte, E., & Verschaffel, L. (2001). Problem solving in the mathematics classroom: A socio-constructivist account of the role of students' emotions. *Proceedings of 25th Annual Conference for the Psychology of Mathematics Education, 4*, 25–32.

Patton, M. Q. (2002). *Qualitative research and evaluation methods*. Thousand Oaks: Sage.

Perkins, D. (2000). *Archimedes' bathtub: The art of breakthrough thinking*. New York: W.W. Norton & Company.

Poincaré, H. (1952). *Science and method*. New York: Dover Publications, Inc.

Pólya, G. (1965/1981). Mathematical discovery: On understanding, learning and teaching problem solving (vol. 2). New York: Wiley.

Rota, G. (1997). *Indiscrete thoughts*. Boston: Birkhauser.

Sfard, A. (2004). Personal Communication.

Sinclair, N. (2002). The kissing triangles: The aesthetics of mathematical discovery. *International Journal of Computers for Mathematics Learning, 7*(1), 45–63.

Wallas, G. (1926). *The art of thought*. New York: Harcourt Brace.

Whittlesea, B. (1993). Illusions of familiarity. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 19*(7), 1235–1253.

Whittlesea, B., & Williams, L. (2001). The discrepancy-attribution hypothesis: The heuristic basis of feelings of familiarity. *Journal for Experimental Psychology: Learning, Memory, and Cognition, 27*(1), 3–13.