ZDM Mathematics Education (2012) 44:581–585
DOI 10.1007/s11858-012-0449-3

ORIGINAL ARTICLE

Revisiting the didactic triangle: from the particular to the general

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Accepted: 24 July 2012 / Published online: 7 August 2012

Abstract The basic notion of a didactic triangle is explained with historical annotations on its origins and subsequent theorization in the literature. Instances of its application to classroom environments to demonstrate its representational capabilities are presented. Generalizations of the triangle are proposed that integrate the role of technology, the researcher in mathematics teaching developmental research, and mediating complexes in the student–teacher–content interfaces. Further, the use of the didactic triangle as a heuristic device is also discussed.

Keywords Classroom developmental research · Didactic triangle · Mathematics classrooms · Mediating objects · Theorizing classrooms · Mathematics teaching development · Researching mathematics classrooms

1 Introduction

The didactic triangle in which student, teacher, and content form the vertices (or nodes) of a triangle is the classical *trivium* used to conceptualize teaching and learning in mathematics classrooms. Even though this representation may seem canonical to an extent and "simplify" the complexity of what occurs within the classroom during a mathematics lesson, it serves as a starting point to theorize the dynamics of teaching–learning, as well as situating and contextualizing each element in relation to the others.

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A question that can be posed regardless of how one conceptualizes a classroom is, what constitutes development when applied to the teaching of mathematics? Responses to this question might focus on the nature of the tasks and activities in which teachers engage their students. The introduction of 'inquiry' tasks, problem solving activities, and open, rather than closed tasks can all be taken as evidence of teaching development, indeed there has been given considerable attention, over the past two decades and recently, to the nature of tasks in the literature of mathematics teaching development (as noted by Berg, Fuglestad, Goodchild, and Sriraman, 2012). In classrooms tasks are a 'mediating artifact' used by the teacher with the intention of leading (enabling or facilitating) students to develop new understanding or knowing, that is tasks are used in a wider context of teaching. Over a decade ago Stigler and Hiebert (1999) reported on a meeting in which 'distinguished researchers and educators from Germany, Japan, and the United States' (p. 25) were invited to review and discuss the classroom recordings made for the TIMSS video study. One participant shared his reflections after viewing video recordings made in Japanese, German and US mathematics classes as follows:

> In the Japanese lessons, there is the mathematics on one hand, and the students on the other. The students engage with the mathematics, and the teacher mediates the relationship between the two. In Germany, there is the mathematics as well, but the teacher owns the mathematics and parcels it out to students as he sees fit, giving facts and explanations at just the right time. In the U.S. lessons, there are the students and there is the teacher. I have trouble finding the mathematics; I just see interactions between students and teachers. (Stigler and Hiebert 1999, pp. 25–26)

The observations made over a decade ago are still relevant today. In numerous studies that analyze episodes of mathematics teaching in which the teachers claimed to be using inquiry, and open tasks, sometimes the focus of the lesson radically shifts from a learner centered and discourse oriented pedagogy onto a very traditional focus on the mathematics when constraints such as the duration of the lesson become more important for the teacher as opposed to letting the lesson develop regardless of such constraints (Törner, Rolka, Rösker, and Sriraman, 2010). A question that drives developmental research in mathematics education then is: *How might teachers be empowered to become aware of and work on relationships between themselves (the teacher), their students and the mathematics?*

The question above is rooted-in-a-conception of teaching-and-learning represented by the 'didactic triangle' (mathematics, student, and teacher), which lies at the heart of the concerns addressed in mathematics teaching developmental research. It is implicated for example in the 'teaching triad' (management of learning, sensitivity to students, mathematical challenge) proposed by Jaworski (1994), and in Brousseau's (1997) theory of didactical situations in which the teacher sets out to create a milieu in which the students engage with the mathematics in an *adidactical* situation. Research and development activity that has focused on problem solving, inquiry and investigation, use of digital technologies in mathematics teaching, and teachers' engagement with students in classes is fundamentally concerned with students' engagement with mathematics, and the mathematical challenge they experience. Researchers taking these issues as the focus for their inquiries address the fundamental relationships represented within this didactic triangle.

This issue of ZDM (no. 5, 2012) brings together leading researchers and thinkers in the field of mathematics education to address, from the perspective of their own research the relationships between mathematics, student and teacher:

- How does/can the introduction of inquiry or investigational tasks impact upon the relationships within the didactic triangle?
- How does/can the development of a problem oriented approach to mathematics teaching and learning affect the relationships within the didactic triangle?
- How does/can the introduction of digital technologies to teaching and learning mathematics affect the relationships within the didactic triangle? Does the technology introduce another 'vertex' such that it is necessary to refer to a didactic quadrilateral?
- How do/can teachers transform the relationships between mathematics, students and themselves?
- How can those working in teaching developmental projects influence teaching so that teaching, and the didactical relationships accommodate new artifacts (inquiry tasks, problems, ICT, etc.?).
- How can the triangle be extended or generalized to incorporate developments in technology and the role played by researchers in teaching development?

2 Pedagogical traditions of mathematics teaching: past to present

In many Northern European countries the pedagogical tradition of teaching mathematics was influenced by Wilhelm von Humboldt's educational program which emphasized student (or child) centered Bildung and to achieve this aim, the content (namely mathematics) became the focal point of lessons (Sriraman and Törner, 2008). It is important to note that for Humboldt[1] the content (of both language and mathematics) was to be delivered to the students in a non-mechanical (or non-procedural) manner. Consequently "Stoffdidaktik" (content-based didactics) was the manner in which mathematics lessons were idealized as the appropriate means of delivery to students for nearly a century in those countries influenced by von Humboldt's program. The classical Stoffdidaktik tradition in Germany asserts the need to continually develop the *pedagogy of mathematics*. Over a much shorter period of time one has seen a shift from the focus on content, to the teacher, and with the renewal of the "constructivist" program in mathematics education, the focus shifted to the student, and classrooms were again idealized as 'learner-centered' repeating a 200-year-old cycle. However there is an important distinction to be made here, between the emphases on the *pedagogy of mathematics* versus the *pedagogy of teaching*. Today this dichotomy has been packaged in terms of teachers' *subject matter knowledge* versus *pedagogical content knowledge* and is amongst others subsumed under the general category of 'Mathematical Knowledge for Teaching' (Ball, Thames, & Phelps, 2008). Meaningful distinctions aside, if one views the classroom as a whole using the gestalt[2] conception, where the whole cannot be analyzed by focusing on the parts decomposed to simplify the system, then it would be helpful to have examples of research that demonstrate such

[1] For a more detailed treatment of Humboldt's visionary Allgemeinbildung, please refer to Kaiser (2002). Historical aspects of the development of Stoffdidaktik are discussed in detail in Sriraman and Törner (2008).

[2] The use of the lower case gestalt to signify viewing the classroom holistically is not to be confused with the Gestalt theory of Ernst Mach.

a possibility. Lampert's (1989, 1990, 1991) classrooms demonstrate a unique way in which the learner, the teacher and problems come together as a gestalt (or a whole) in which students are led to think about mathematical operations as a basis of understanding relationships in mathematics. Lampert's classroom lessons are designed to help students develop mathematical knowledge in the way mathematicians discover new knowledge. The teaching agenda in Lampert's lessons is unique because they intertwine content and discourse. One part of the agenda is related to the goal of students acquiring the technical skills and knowledge in mathematical content, and the other part is working toward the goal of students acquiring the skills and dispositions necessary to participate in disciplinary discourse (mathematical practice) (Lampert, 1990).

Today the classroom is regarded as a complex system in which the teacher, the learner and content are dynamically situated in relation to each other, with the understanding that perturbing a lesson, can affect its flow and outcome (Törner, Rolka, Rösken, and Sriraman, 2010). Schoenfeld (2012) argues for broadening the traditional framing of the didactic triangle that focuses on the nodes, to encompass the analysis of the interactions that occur between various nodes, i.e., view classroom activities from a more social/cultural perspective. He presents a conceptual framework, a sociocultural lens to examine mathematically productive classrooms. In a similar vein, Herbst and Chazan (2012) elaborate the notion of the didactic triangle by philosophizing on "rationality" as a basis under which interactions can be justified. Their focus is on the interactions between the person and the larger system under which their actions are regulated. Jaworski (2012) also discusses how the system (namely the classroom) gets perturbed with the introduction of a researcher/didactician within the setting and proposes an additional node to the triangular configuration to include the didactician as an integral part of a system in which teacher development occurs.

3 Extending the triangle

The advent of new technologies in the late twentieth century, which made algebra, geometry and calculus accessible via computer algebra systems and graphical technologies particularly dynamic geometry (Moreno and Sriraman, 2005), brought the role of the teacher once more to the forefront. If the traditional content that were taught required procedural thinking, then the new technology made the "content" more or less obsolete. This resulted in numerous modeling based curricula in the US, such as the Core Plus Mathematics Curriculum (CPMC), Systemic Initiative for Montana Mathematics and Science (SIMMS) in the US sponsored by the National Science Foundation that integrated the new technologies into a non-traditional curriculum. Other regions of the world have also witnessed the dawn and implementation of hand held or computer based technologies in the classroom (e.g., SimCalc in Brazil and Cyprus).

Ruthven (2012) refers to the work of David Tall who introduced an additional vertex (or node) to the triangle to represent the special role of technology nearly 20 years ago. As a result a didactical *tetrahedron* now containing technology in the fray can be used to interpret "several levels from that of the material resources present in the classroom to that of the fundamental machinery of schooling itself." To illustrate this extension of the didactic triangle, Ruthven presents contrasting cases of the use of dynamic geometry in English classrooms. In contrast to Ruthven, the paper by Rezat and Straesser (2012) proposes the addition of artifacts that mediate teaching and learning as an extension of the didactic triangle to the *tetrahedron*, and theorize situating the tetrahedron in a heuristic model that is socio-didactical in nature. As noted above, Jaworski (2012) also proposes the introduction of an additional node in the context of teaching development, but in this case the node is not connected to each of the existing vertices of the didactic triangle, to form a tetrahedron as in the case of Ruthven or Rezat and Straesser. Jaworski's additional node represents didacticians who work, research and learn alongside teachers but not separately connected to each aspect of the classroom setting.

4 The didactic triangle in the Norwegian milieu

Three papers in this issue come from Norway based on developmental research projects in schools and kindergartens. These studies address the didactic relationships, in the sense of ways in which the didactic triangle serves as a device (or heuristic as Ruthven explains) for focusing attention in both developmental activity and the analysis of developmental events. The adaptability of the heuristic to different settings is considered by Erfjord, Hundeland, and Carlsen (2012) in their report on the inquiry stances adopted by kindergarten teachers when orchestrating mathematical activities.

Bjuland (2012) analyzes the semiotic resources used by an experienced sixth-grade teacher when her students encounter inscriptions within written (mathematical) texts. Even though the dynamics of the semiotic bundles seems to be the main focus of the paper, the relationships between the content, student and teacher form the backbone of the analysis.

The last paper in this collection (Berg, Fuglestad, Goodchild, and Sriraman, 2012) presents analyses of teachers' discussions within mathematics teaching by using "mediation" as a central construct. Two episodes with upper secondary school teachers preparing tasks for use in their classrooms are used to illustrate that the focus on tasks places an emphasis on the task as object and its resolution as goal; with mathematics playing the role of a mediating artifact. Subject content in the didactic triangle is thus displaced by the task and learning mathematics appears to be relegated to a subordinate position. These three papers also serve as "empirical" ways in which the didactic triangle may be particularized or applied to interactions between the three nodes.

5 Summary: revisiting entails reconceptualizing

The collection of papers in this issue offer possibilities of how the didactic triangle might be interpreted and used as "a heuristic that identifies what are the fundamental components of any didactic system" (Ruthven, 2012). The papers serve to demonstrate the limitations of the parsimonious representation of didactical components and relationships represented by the triangle, as Ruthven observes "the didactical triangle may offer an overly idealised model." Schoenfeld (2012) also provides a thoroughgoing account and illustration of the application of the didactic triangle and argues that the didactic triangle, "as typically construed in the English-speaking literature (contrasting particularly with the French speaking), is too narrow" because it does not represent (directly) classrooms as cultural systems and consider the cultural forces that shape them. This idea is also present in the paper by Herbst and Chazan (2012) who focus on the teacher as an agent in a complex system of interrelated agents, and consider the professional obligations of teachers. Herbst and Chazan observe that for teachers "individual choice is possible but not cost free." Similarly Jaworski (2012) explores the dimensions of the teacher as a person in terms of 'identity' and 'personhood', and in the context of teaching development proposes a three dimensional composite of didactic triangles to represent communities of teachers (and didacticians working alongside teachers in teaching development activity).

The extension of the didactic triangle is an theme that several of the papers in this issue consider, and as Ruthven (2012) notes this is not a recent idea, as in the development of a didactic tetrahedron to provide a fourth vertex to represent digital tools and other resources. Berg et al. (2012) use the didactic triangle in their mathematics teaching developmental research, which is framed within cultural historical theory. They experience the limitations of the didactic triangle as a heuristic (as outlined by Ruthven, 2012 and Schoenfeld, 2012), but they maintain the two dimensional model and propose replacing the vertex in the didactic triangle that represents the teacher with a new vertex representing a 'mediating complex' that stands for the intricate array of social, cultural and material mediators of mathematical meaning (that includes the teacher and teaching–learning resources). An alternative and substantial development is proposed by Rezat and Straesser (2012) they set the didactic tetrahedron on top of the extended activity system proposed by Yrjö Engeström (1987), to create a socio-didactical tetrahedron.

Erfjord et al.'s (2012) proposition that the usual focus of the didactic triangle on the subject (mathematics) might not be appropriate for representing didactical systems in kindergartens. They propose as an alternative 'pedagogical mathematical activities' to be more appropriate for the kindergarten setting. Thus, they provoke questions about whether the didactic triangle, or tetrahedron, might need to be redefined to consider different levels of teaching and learning mathematics.

The papers included in this issue relate to research activity in classrooms and teaching development settings, at kindergarten, elementary and secondary levels, as well as working with mathematics teachers in developmental settings. In many ways the 'simple' representation of didactical systems depicted in the didactic triangle is argued to be inadequate. However, all the papers confirm the central position of mathematics, learner and teacher in researching and theorising teaching–learning processes in mathematics classrooms.

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