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ORIGINAL ARTICLE

Learning to notice students' mathematical thinking through on-line discussions

Ceneida Fernández · Salvador Llinares · Julia Valls

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Abstract The goal of this research is to characterize prospective mathematics teachers' development of professional noticing of students' mathematical thinking in on-line contexts. Specifically, we are interested in how the participation in on-line discussions, when prospective teachers solve specific tasks, supports the development of professional noticing of students' mathematical thinking. Findings show that an aspect in which the on-line discussions, as an example of asynchronous collaborative communication interfaces, support this development is related to the role of writing; participating in an on-line discussion plays a significant role since the final written text is functional as regards the activity of interpreting students' mathematical thinking collaboratively.

Keywords On-line learning environments · Dialogic argumentation · Professional noticing · Students' mathematical thinking

1 Introduction

Previous research has shown different approaches used by mathematics teacher educators to help prospective teachers learn the knowledge needed to teach, and to develop useful skills in relation to noticing relevant aspects in mathematics teaching (Llinares and Krainer 2006). These studies suggest that the characteristics of the instructional tasks and the contexts of the teaching programs exert strong influences on what prospective teachers learn. For example, Masingila and Doerr (2002) recognized that the task that prospective teachers had to undertake in a multimedia case, tracing an issue through the case and their own teaching practice, influenced the new understanding gained. Similarly, Lin (2005) focused on identifying instructional scaffolding that prospective teachers could use to conceptualize mathematics teaching, pointing out that having prospective teachers watch video-cases and promoting discussions and writing to stimulate reflections helped them to refocus and deepen their awareness of students' learning. Llinares and Valls (2009) indicated that providing structured guidance—the task and specific questions in an on-line discussion—enabled prospective teachers to reflect on and integrate multiple aspects of teaching and also begin to develop a more complex view of teaching. All of these findings underline the important role played by the structure of learning environments, the type of task and the interaction with others in supporting prospective teachers' learning.

Nowadays communication and information technologies provide new conditions and tools to help prospective teachers interact with others in order to learn the knowledge needed to teach and to develop the skills to learn from practice (Borba and Villarreal 2005; Llinares and Olivero 2008).

In this context, on-line discussions encourage a collaborative process in which meanings can be negotiated and knowledge-building can be supported since they allow prospective teachers to spend more time constructing the arguments. So, it seems that extending part of the classroom discussion to an on-line discussion can be seen as an additional opportunity for prospective teachers to learn the

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C. Fernández (✉) · S. Llinares · J. Valls
Departamento de Innovación y Formación Didáctica, Universidad de Alicante, Alicante, Spain
e-mail: ceneida.fernandez@ua.es

knowledge needed to teach and to develop the skills needed to notice teaching–learning situations. Furthermore, on-line discussions might provide the focus for making a progressive discourse and simultaneously embodying the progress made while prospective teachers are working on a shared activity (Llinares and Valls 2010). Writing as a way of participating in on-line discussions might allow prospective mathematics teachers to become acquainted with and understand the topic they are writing about. In this way, participation in on-line discussions on a specific issue might encourage prospective teachers, for example, to move beyond descriptions of students' mathematical thinking to begin to endow them with meaning.

Recently, on-line learning environments integrating collaborative communication interfaces have been developed to encourage students to engage in dialogic argumentation. Dialogic argumentation occurs when different perspectives are being examined and the purpose is to reach agreement on acceptable claims or courses of actions. Clark and his colleagues (Clark et al. 2007, Clark and Sampson 2008) have identified a range of specific instructional features to promote these productive interactions among individuals or groups attempting to convince one another of the acceptability and validity of alternative ideas. Verillon and Rabardel (1995) have shown that such interfaces are powerful mechanisms for increasing prospective teachers' instrumentalization of ideas (instrumental activity). In the case of mathematics education, analysis of prospective teachers' on-line argumentations has shown evidence of this process, indicating different uses of theoretical knowledge in solving hypothetical professional problems (Llinares and Valls 2010; Prieto and Valls 2009). Another instructional feature of these on-line learning environments is the possibility of sharing text documents in order to enable prospective teachers to compare and refine their ideas through a process of dialogic argumentation (Roig, Llinares and Penalva 2010). These features have generated specific issues about how they can support the development of the professional noticing skill in specific mathematics domains.

1.1 Professional noticing of students' mathematical thinking

One of the professional skills that prospective mathematics teachers should develop as a component of becoming a mathematics teacher is professional noticing (Mason 2002; Sherin et al. 2010). Noticing what is happening in the mathematics classroom and endowing it with meaning from the perspective of mathematics learning is critical in enabling prospective teachers to conceptualize a contemporary view of mathematics teaching. Although this skill has been conceptualized from different perspectives in recent years, the connexion between them is in making sense of how individuals process complex situations (Santagata et al. 2007; Star and Strickland 2008; van Es and Sherin 2002). Mason (2002) considered noticing to be a fundamental element of expertise in teaching, characterized by: (a) keeping and using a record, (b) developing sensitivities, (c) recognizing choices, (d) preparing to notice at the right moment, and (e) validating with others. On the other hand, van Es and Sherin (2002) considered that noticing includes (a) identifying noteworthy aspects of a classroom situation, (b) using knowledge about the context to reason about classroom interactions, and (c) making connections between specific classroom events and broader principles of teaching and learning. In their studies, these authors have found that teachers can improve their noticing by changing what they notice and how they reason.

A particular focus is the noticing of students' mathematical thinking. Jacobs et al. (2010) conceptualize this as a set of three interrelated skills:

- attending to students' strategies: the extent to which teachers attend to the mathematical details in students' strategies;
- interpreting students' mathematical understanding: the extent to which the teachers' reasoning is consistent with both the details of the specific students' strategies and the research on students' mathematical development; and
- deciding how to respond on the basis of students' understanding: the extent to which teachers use what they have learned about the students' understanding from the specific situation and whether their reasoning is consistent with research into students' mathematical development.

In the context of analyzing prospective teachers' professional-noticing skill of students' mathematical thinking, two relevant issues appear. Firstly, its characterization in specific mathematics domains; and secondly, how this skill is being developed (Levin et al. 2009). In this study, we focus on how prospective mathematics teachers' professional noticing of students' mathematical thinking could be developed, taking into account new learning environments based on on-line approaches. In this sense, asynchronous modes of communication might allow prospective teachers to spend more time constructing their arguments and also deliver a higher degree of joint elaboration and construction of interpretations. From these perspectives, the on-line debates, as examples of asynchronous collaborative communication interfaces, are a good context in which the skills of identifying, interpreting and making a decision can be developed. This is so since preservice mathematics teachers have to keep and use a record in order to communicate their interpretations to others, attempting to convince them of the acceptability and validity of alternative ideas.

1.2 A specific domain: the transition from students' additive to multiplicative thinking

The specific domain of students' mathematical thinking considered in this paper is the transition from additive to multiplicative thinking. For several decades, research has focused on the development of multiplicative reasoning, and, more particularly, on the transition from additive to multiplicative thinking. A characteristic of this transition is the difficulty that students of different ages have in differentiating multiplicative from additive situations. This difficulty is manifested in students who over-use incorrect additive methods in multiplicative situations (Hart 1984), and who over-use incorrect multiplicative methods in additive situations (Fernández et al. 2011b; Van Dooren et al. 2008). For example, the following additive situation can be modelled by the function $f(x) = x + b, b \neq 0$ since quantities are linked additively: "*Ann and Peter are running around a track. They run equally fast but Ann started earlier. When Ann has run 3 laps, Peter has run 6 laps. If Ann has run 5 laps, how many laps has Peter run?*" However, students used incorrectly multiplicative methods (proportional strategies) to solve it: "*Peter runs 2 times more laps than Ann ($6 = 3 \times 2$). If Ann has run 5 laps, then Peter has run $5 \times 2 = 10$ laps.*"

On the other hand, the problem "*Rachel and John are planting flowers. They started together but John plants faster. When Rachel has planted 4 flowers, John has planted 12 flowers. If Rachel has planted 20 flowers, how many flowers has John planted?*" is a proportional situation (a particular case of the multiplicative situations) since quantities are linked multiplicatively and it can be modeled by the function $f(x) = ax, a \neq 0$. However, students tend to incorrectly use additive methods to solve it, such as: *"As the difference between the flowers planted by Rachel and John is $4 + 8 = 12$, if Rachel has planted 20 flowers, then John has planted $20 + 8 = 28$ flowers."*

Recently researchers have provided relevant information about the transition from students' additive to multiplicative thinking in the context of proportional reasoning that is useful for teaching mathematics. This knowledge for teaching is the relation between students' use of additive methods (incorrectly) for proportional problems and students' use of proportional methods (incorrectly) for additive situations. Fernández and Llinares (2012) with primary and secondary school students and Van Dooren et al. (2010) with primary school students obtained profiles that describe students' behavior in the transition from additive to multiplicative thinking. Both studies showed that the number of students who solve proportional and additive problems using proportional methods increases through grades and the number of students who solve proportional and additive problems additively decreases through grades. Therefore, older students performed better on proportional problems but they were not able to discriminate proportional from additive situations (solving additive problems with proportional methods). Furthermore, there were a large number of students who were influenced by superficial task variables such as the type of ratio or the multiplicative relationship between quantities (integer or non-integer).

Therefore, to develop prospective teachers' professional noticing of students' mathematical thinking in this context, it is necessary that they identify the important aspects of proportional and non-proportional situations and reason about this in relation to the strategies used by students. Additionally, as teacher educators, we need to know how prospective teachers interpret students' understanding (taking into account students' difficulties in discriminating between both types of problems), and how they respond on the basis of this understanding.

In this study, we are interested in how the participation in on-line discussions and the proposed task (analysis of students' written work when solving proportional and non-proportional problems) could support the development of prospective mathematics teachers' noticing of students' mathematical thinking.

2 Method

2.1 Participants and context

Seven prospective secondary school mathematics teachers (PTs) enrolled on a post-graduate program participated in this experience. The post-graduate program qualifies them to teach mathematics in secondary education. The experience was carried out in one of the subjects called "Mathematical Learning in Secondary Education". One of the aims of this subject is for prospective teachers to learn to identify and interpret characteristics of secondary school students' mathematical thinking in order to make adequate teaching decisions. A topic in this subject is the development of proportional reasoning in the context of the relation between the additive and multiplicative thinking of secondary school students (12–16 years old).

A b-learning (blended learning) environment was designed for this topic, integrating face-to-face and on-line activities on a web platform (Fig. 1). In the face-to-face activities, prospective teachers work individually or collaboratively in order to solve the proposed different tasks during approximately 120 min once a week (for 8 weeks). Then, after the face-to-face activity, they shared their ideas in an on-line debate and synthesized them into a final joint

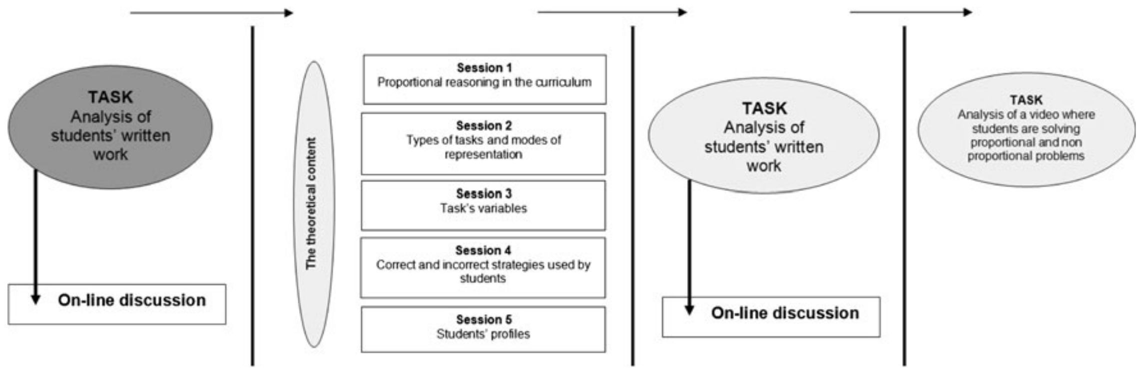


Fig. 1 Structure of the b-learning environment

report. Technical orientation sessions focused on the on-line discussion tool were offered before starting.

In each face-to-face meeting prospective teachers analyzed students' written work, read theoretical papers about the transition from additive to multiplicative thinking, and analyzed and discussed video-clips in which secondary school students solved problems with additive and multiplicative structures. They then continued the discussion in an on-line discussion. The general goal of the on-line discussion was to promote argumentation as a mechanism to initiate the development of the professional noticing skill and to provide an asynchronous context in which the prospective teachers could validate their interpretations with others. A specific goal of the on-line discussion was to help the prospective teachers to learn to see relevant aspects of students' strategies and analyze them by inquiring into the details of the students' thinking, arguing about possible interpretations of their thinking and validating their interpretations with others.

This paper focuses on the arguments generated by the prospective teachers when they worked on the first on-line discussion, since our intention was to analyze the extent to which an on-line debate could be a good instructional instrument to support the initiation of the development of the professional noticing skill. Our intention was to determine how the participation in an initial on-line discussion, focused on analyzing students' mathematical thinking in the context of the transition from additive to multiplicative thinking, could generate some cognitive scaffolds about this specific mathematical topic. We assumed that the meanings which each prospective teacher might begin to construct resulted from the initial interactions in which he/she participated and that the change in the initial development of the professional noticing skill is linked to the process of validation of the interpretations with others in the on-line discussion would relate to the dynamic of interactions that involve the use of language (written text).

2.2 The task

The prospective teachers had to examine the answers of four students to four problems (Fig. 2): two proportional problems (modeled by the function $f(x) = ax$, $a \neq 0$) (see, for example, problems 1 and 3 solved by the student 3 in Fig. 2) and two non-proportional problems with an additive structure (modeled by the function $f(x) = x + b$, $b \neq 0$) (see, for example, problems 2 and 4 solved by the student 3 in Fig. 2). Additive and proportional situations differ in the type of relationship between quantities. For example, in John and Kate's problem (problem 2, student 3 in Fig. 2) the relationship between John and Kate's laps can be expressed through an addition: Kate's laps = John's laps + 40 (the difference between quantities remains constant). On the other hand, in Sophie and Sarah's problem (problem 1, student 3 in Fig. 2), the relationship between the number of meters that Sophie and Sarah walked can be expressed through a multiplication: Sarah walks 2.5 ($50 = 20 \times 2.5$) times as many meters as Sophie. The first problem is an additive situation while the second situation is a proportional one. These differences between proportional and additive situations are considered in the problems with the sentences "they started together" or "John started later/Rachel started earlier" and "John/Sarah is faster" or "they go equally fast".

Moreover, in proportional situations there are two different multiplicative relationships between quantities (called internal and external ratios). For instance, in Sophie and Sarah's problem (Sophie walks 20 m and Sarah walks 50 m) if we multiply the number of meters walked by Sophie by 3 and correspondingly we multiply the number of meters walked by Sarah by 3 then we obtain another rate pair 60:150. On the other hand, if we double the number of meters walked by Sophie, we must double the number of meters walked by Sarah. The factor which relates quantities between or across magnitudes (for example, meters walked by Sophie and meters walked by Sarah) is constant (external ratios are constant), while the relationship

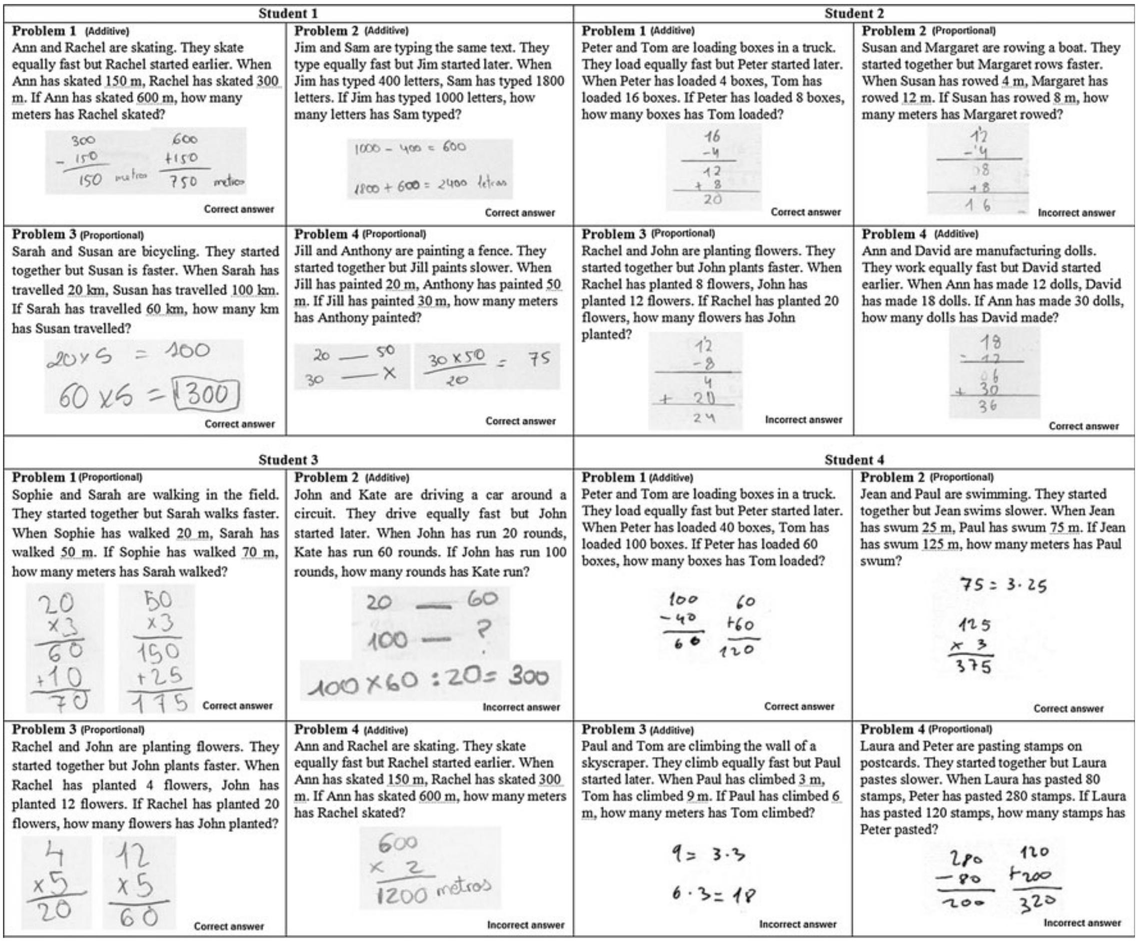


Fig. 2 The task solved by prospective teachers

between two quantities in a same magnitude (for example, meters that Sophie walked) is invariant (internal ratio).

The students' answers show different strategies used in proportional situations (the use of internal ratios, the use of external ratios, the building-up strategy, the unit rate and the rule of three as correct strategies). The additive strategy was used as a correct strategy in additive problems but as an incorrect strategy in proportional ones.

So, prospective teachers had to examine a total of 16 students' answers (four problems × four students) and respond to the following three issues related to the three skills of noticing of students' mathematical thinking (Jacobs et al. 2010):

- "Describe in detail what you think each student did in response to each problem" (related to prospective teachers' expertise in attending to students' strategies).
- "Indicate what you learn about students' understanding related to the comprehension of the different mathematics concepts implicated" (related to prospective teachers' expertise in interpreting students' understanding).
- "If you were a teacher of these students, what would you do next?" (related to prospective teachers' expertise in deciding how to respond on the basis of students' understanding).

To design the task, we selected students' answers taking into account previous research on proportional reasoning. We focus our attention on the research findings that describe different profiles of primary and secondary school students when they solve proportional and non-proportional problems (Fernández and Llinares 2012; Van Dooren et al. 2010). These students' profiles are:

- students who solve proportional and additive problems proportionally (student 3);
- students who solve proportional and additive problems additively (student 2);
- students who solve both types of problems correctly (student 1); and
- students who solve problems with integer ratios using proportional strategies (regardless of the type of problem) and problems with non-integer ratios using additive strategies (student 4).

This task has two special characteristics. Firstly, as the task consists of students' answers to different problems, the prospective teachers can identify students' strategies and relate them to the characteristics of the problems. Secondly, the task presents the answers of each student to four different problems. This task lets the prospective teachers consider the answers of one student to the four problems and therefore be able to interpret students' understanding (identifying students' profiles). The task was designed to give us information about the prospective teachers' different levels of noticing because there will be prospective teachers who just describe students' answers, others who might relate students' answers with the characteristics of the problem (to identify if the student strategy is correct or incorrect), and finally, others who might not only relate students' answers with the characteristics of the problem but might also relate all students' answers thus identifying the student profile.

2.3 The on-line debate

Once prospective teachers had worked individually on the task (face-to-face activity) they had to participate in an on-line discussion, sharing and justifying their answers to the task (Fig. 3) during a week.

Some technical considerations to manage the on-line discussion were introduced:

This on-line debate has been designed for the discussion of the following items linked to the task:

- *"Indicate what you learn about students' understanding related to the comprehension of the different mathematical concepts involved."*
- *"If you were a teacher of these students, what would you do next?"*

Some suggestions to take into account during your participation are:

- *Make sure the title of the message clearly reflects the content.*
- *Make sure that your message is written in the right place.*
- *Do not duplicate lines of debate (topics).*
- *Interact with other partners taking into account previous participations.*
- *Justify your participations.*
- *Take into account that the on-line discussion will be active during 7 days. So it is better to distribute your messages during that period.*

Remember that as a result of this on-line discussion you must hand in a report with the conclusions.

The on-line discussion was integrated in this instructional design in order to allow prospective teachers to negotiate the meanings required for solving the task by dialogic argumentation. Negotiation of meanings is understood as a way of validating the interpretations, affirming, refusing or questioning the presented ideas to reach mutual understanding, and attempting to convince others of the acceptability and validity of alternative ideas about the students' mathematical thinking (Derry et al. 2000; Clark and Sampson 2008). Here, we assume that written texts by the prospective teachers in the on-line debate can be considered tools for the development of a higher level of noticing (intellectual artifacts) (Resnick et al. 1997). Therefore, we expected that participation in the on-line discussion could be perceived by the prospective teachers as a useful task in relation to the activity of interpreting students' mathematical thinking.

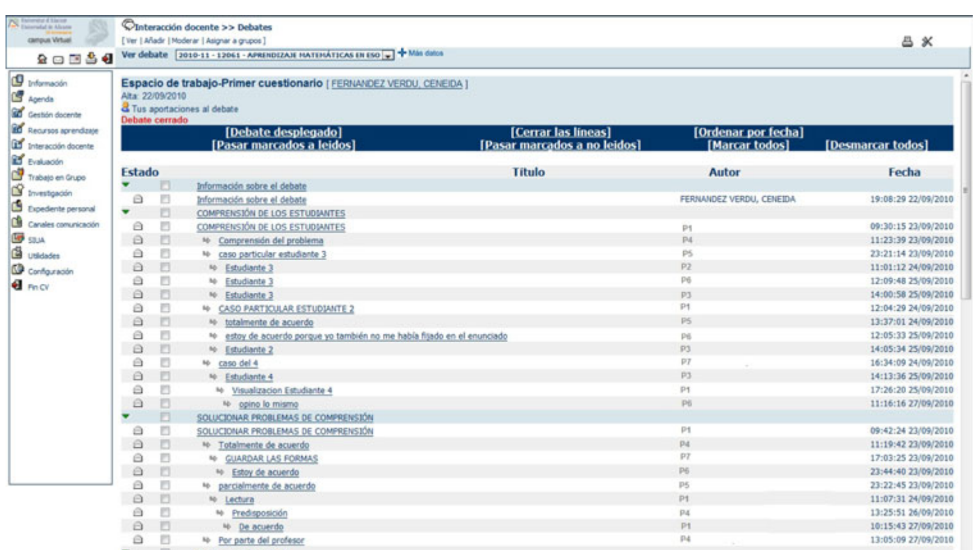


Fig. 3 Multiple prospective teachers' participations illustrating the interaction in the on-line discussion

2.4 Analysis

We carried out the analysis of the prospective teachers' answers to the task and the participations in the on-line discussion in two phases (Derry et al. 2000). In the first phase we analyzed the conceptual quality of the prospective teachers' answers to the task and of the participations in the on-line discussion. We considered these participations, to the extent that they allowed preservice teachers to attempt to convince others of the acceptability and validity of their ideas, to be examples of dialogic argumentation. To do this, taking into account the mathematical elements of proportional and additive situations (Table 1) and the strategies used by students (Table 2), we analyzed whether the prospective teachers had identified the strategies and integrated the mathematical elements in the written text they produced (relating the characteristics of the problem and the strategy) when they answered the task individually and when they participated in the on-line discussion. So, we determined whether the prospective teachers' answers indicated attention to these mathematical details.

We also considered the extent to which the prospective teachers identified the different students' profiles (whether the prospective teachers took into account globally all the students' answers). For example, whether the prospective teachers identified that the additive strategy was used correctly in the additive problems but also incorrectly in the proportional problems, or whether the proportional strategy was used correctly in the proportional problems but also incorrectly in the additive problems. In this way, we analyzed how the prospective teachers interpreted the students' understanding. Finally, we analyzed whether the prospective teachers were able to include considerations of students' understanding in their decisions as to how to respond (teaching actions, third task question).

We assigned each initial answer to the task or participation in the on-line discussion to a level of noticing. In order to make this assignment, we used a framework based on the four levels for characterizing the development of noticing in the specific domain of the transition from students' additive to multiplicative thinking (Fernández et al. 2011a). These four levels are:

Table 2 Strategies used by students

Proportional situations	Additive situations
Internal ratios	Additive strategy
External ratios	
Building-up strategy	
Unit rate	
Algorithm of "Rule of three" (cross-product of two ratios considered as equivalent fractions)	

- *Level 1* The prospective teachers do not discriminate proportional from additive situations. These prospective teachers only describe students' answers without relating the characteristics of the problem with the students' answer.
- *Level 2* The prospective teachers discriminate proportional from additive problems relating students' answers with the characteristics of the problems, but they do not justify their answers attending to the mathematical elements of each situation.
- *Level 3* The prospective teachers discriminate proportional from additive problems relating students' answers with the characteristics of the problems and they justify their answers attending to the mathematical elements of each situation. However, they do not identify students' profiles.
- *Level 4* The prospective teachers discriminate proportional from additive problems justifying through the mathematical elements and identify the students' profiles.

In the second phase of the analysis, we identified shifts in the level of the prospective teachers' noticing of the students' mathematical thinking. Furthermore, we considered how participation in the on-line discussion supported these shifts, analyzing the way in which the prospective teachers interacted with others: providing new information, clarifying previous participations and agreeing or disagreeing with previous participations.

Table 1 Mathematical elements of the situations

Proportional situation $f(x) = ax$, $a \neq 0$	Additive situation $f(x) = x + b$, $b \neq 0$
The function passes through origin (related to the sentence *"they started together"*)	The function does not pass though origin (related to the sentence *"they started later or earlier"*)
The value of the slope changes (related to the sentence *"someone goes faster or slower"*)	The value of the slope remains constant (related to the sentence *"they go equally fast"*)
External ratios are constant and internal ratios are invariant	The difference between quantities remains constant

3 Results

In the first part of this section we describe what the prospective teachers noticed about the students' mathematical thinking in their individual answers to the task. Then, in the second part, we identify the shifts in the level of prospective teachers' noticing of students' additive and multiplicative thinking and how interactions supported this change in the on-line discussion.

3.1 What the prospective teachers noticed about the students' mathematical thinking

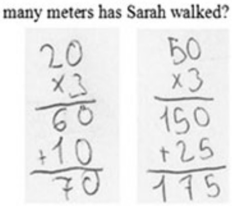
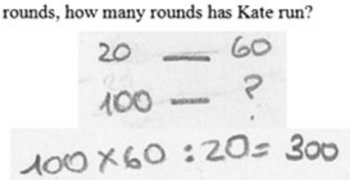
Taking into account the professional noticing of students' mathematical thinking levels of expertise (Fernández et al. 2011a), initially four out of the seven prospective teachers (PT2, PT4, PT5 and PT6) were classified in level 1 (prospective teachers who do not discriminate proportional from additive situations). One prospective teacher (PT7) was classified in level 2 (he discriminated both types of situations but did not give justifications using the mathematical elements). Finally, two prospective teachers (PT1 and PT3) were classified in level 3 (they differentiated both problems, they gave justifications based on the mathematical elements but they did not identify students' profiles).

So, initially, most of the prospective teachers described the operations made by students but without relating the students' strategy with the characteristics of the problem in order to interpret the students' understanding. For example, prospective teacher PT6 only described the operations that student 3 made to solve the problem 1 (Fig. 4, Sophie and Sarah) and problem 2 (Fig. 4, John and Kate), but he was not able to relate the characteristics of the situation with the strategy used by the student. PT6 indicated that the student solved the country walk situation (problem 1) using a building-up strategy ("*he/she tries to go from 20 to 70 using multiplications and additions*"), and "*translating*" these relationships to the second measure space (meters that Sarah walks).

However, in relation to the student's answer in problem 2 (John and Kate), prospective teacher PT6 indicated that the student used a proportion but did not recognize that it is an additive situation and that the procedure used by the student is wrong (based on multiplicative relationships instead of additive relationships). This type of answer does not show understanding of the differences between the two situations and so the prospective teacher was classified in level 1.

Therefore, in some cases, some prospective teachers had difficulties in relating the types of problems and the characteristics of students' strategies in order to interpret individual students' mathematical thinking.

Fig. 4 What prospective teacher P6 noticed in the students' answers to additive and proportional situations

Student 3	
Problem 1 Sophie and Sarah are walking in the field. They started together but Sarah walks faster. When Sophie has walked 20 m, Sarah has walked 50 m. If Sophie has walked 70 m, how many meters has Sarah walked? 	*The student tries to solve the problem using proportions. He/she tries to go from 20 to 70 using multiplications and additions. The student knows that has to go from 20 to 70 so he/she multiplies by 3 and then adds 10. So we have to do the same operations with 50. We obtain 175. Therefore, I think that this student does not know proportions but he/she solved the problem correctly*
Problem 2 John and Kate are driving a car around a circuit. They drive equally fast but John started later. When John has run 20 rounds, Kate has run 60 rounds. If John has run 100 rounds, how many rounds has Kate run? 	*This student used the method of proportions. Although he/she did not write 20:100 = 60: x, he/she wrote 100×60/20*

On the other hand, the lack of noticing influenced the prospective teachers' decisions as regards teaching actions. In this way, when the prospective teachers did not differentiate additive and proportional situations, they indicated general teaching actions such as asking the students for more explanations about their answers or explaining the use of procedural approaches to solve problems.

3.2 How interactions supported the development of the prospective teachers' noticing of students' mathematical thinking in the on-line discussion

The interaction motivated by the interpretation of the students' answers collaboratively made the prospective teachers start to attend and interpret jointly students' answers. In this way, interactions in the on-line discussion supported the change in the prospective teachers' level of noticing (Table 3). Some evidence of these changes is that the prospective teachers were able to focus on the characteristics of the problems and to identify some students' profiles.

We describe some examples of how the prospective teachers shifted their focus on what was noticed and how they kept and used the records in order to validate them with others. For example, prospective teacher PT1 related the characteristics of the situations with the strategy used by the student discriminating proportional from additive situations when solving the task individually. He underlined the importance of the sentences "they load equally fast but Peter started later" and "they started together but Jean swims slower"; but prospective teacher PT4 did not discriminate the situations (he/she only described students' strategies, as also did prospective teacher PT6). The interaction between PT1 and PT4 in the on-line discussion (where PT4 agreed with PT1 and added more information) let prospective teacher PT4 start to discriminate both types of problems, identifying the mathematical elements of the situations.

Table 3 Prospective teachers' level of expertise before and after on-line discussion

	Level of noticing before on-line discussion	Level of noticing after on-line discussion
Level 1	PT2, PT4, PT5, PT6	PT2
Level 2	PT7	
Level 3	PT1, PT3[a]	PT4, PT5, PT6, PT7[b]
Level 4		PT1, PT3[c]

[a] Both prospective teachers identify 2 out of 4 students' profiles

[b] Some of them identify 1 or 2 out of 4 students' profiles

[c] Both prospective teachers identify 3 out of 4 students' profiles

- Students' understanding (PT1—09:30:15 23/09/2010)

  Students use elemental operations (such as addition, subtraction …) correctly. However, they do not usually read the problem well and interpret in the same way the sentences "starting later" and "being slower".

  – Problem understanding (PT4—11:23:29 23/09/2010)
    I agree with you. Students do not differentiate between "doing equally fast an action but starting at different times" and "starting at the same time but doing an action faster". We have to find out if students did not read the problem well or they had difficulties in understanding the concept of proportionality (the difference between proportional and non-proportional problems).

So, the interaction between PT4 and PT1 shows how PT4 used the characteristics of both situations in relation to the students' strategies to agree with his colleague, indicating a higher level of noticing.

However, the identification of students' profiles was more difficult. Only prospective teachers who had identified initially the characteristics of the problems and related them to students' strategies were able to start to identify the profiles. The interaction between prospective teachers PT7, PT3, PT1 and PT6 is an example of how interaction let these prospective teachers identify students' profiles.

This interaction started with the participation of PT7, who finished with a question. This prospective teacher identified that student 4 had solved one of the two proportional problems correctly but solved the other problem incorrectly and the same happened with the additive problems (Fig. 2). The participation of prospective teacher PT3 was not relevant (she only agreed with PT7 without providing any justification or any answer to the question formulated by PT7). However, PT1 answered the question formulated by PT7, focusing on the multiplicative relationships between quantities (we have to underline that this prospective teacher had not identified this profile when he solved the task individually). In this way, PT1 indicated that the student solved the two problems proportionally with an integer relationship between quantities (triple) but when the relationship was non-integer the student solved the problem additively. That is to say, the prospective teacher noticed that the presence of a non-integer relationship changed the way in which the student thought about the problem. Finally PT6 agreed with PT1 and clarified the participation of PT1.

- Student 4 (PT7—16:34:09 24/09/2010)

  This is a strange case because there are two proportional problems but one is solved correctly and *the other incorrectly. And there are two problems in which they do not start at the same time and again, one is solved correctly and the other incorrectly. How can we explain this? The student could not understand the problem or he/she could have had some difficulties. We have to ask students to explain their answers.*

– Student 4 (PT3—14:13:36 25/09/2010)

It is true that this is a strange case. As you said, if we ask for more explanations, students could understand when he/she can use the strategy. For example, when he/she wrote 100 – 40 = 60, she should have written "60 boxes loaded by Tom when Peter started to load".

- Answer of student 4 (PT1—17:26:20 25/09/2010)

As regards this student, we could say that he/she did not discriminate proportional from additive problems. However, two problems were solved by the same strategy because the multiplicative relationship between quantities is integer ("the triple", the multiplicative relationship between 25 and 75 and the multiplicative relationship between 3 and 9). The other two problems have a non-integer multiplicative relationship between quantities and they are solved looking for a difference and using it. So when students had difficulties in looking for the relationships between quantities, they used a constant difference instead of a multiplicative relationship.

– I agree with you (PT6—11:16:16 27/09/2010)

If there is an integer multiplicative relationship between quantities, the student obtains a correct result. However, if there is a non-integer multiplicative relationship, the student uses another method.

So this interaction shows how the participation of PT7 and the question formulated let prospective teacher PT1 think about this issue and recognize one of the students' profiles that had not been identified before. On the other hand, PT6, who had not discriminated proportional from additive situations, agreed with PT7 and clarified his/her participation.

Another example that shows the shift of the prospective teachers' level of noticing is the interaction between PT4, PT7, PT6 and PT1. In a previous interaction with PT1, PT4 identified the characteristics of proportional and non-proportional situations (mathematical elements of the situations) and their relation with student's strategies, so this led him to start a new interaction proposing that student 1 had solved the four problems correctly, discriminating between both types of problems (this is evidence for identifying the profile of student 1). However, PT7 disagreed with PT4, arguing that one of the students' answers was wrong. Participations of PT6 (indicating disagreement with PT7) and PT1 (showing disagreement with PT7 and justifying his/her opinion) allow prospective teacher PT7 to recognize his/her misconception.

- Student 1 (PT4—10:33:41 24/09/2010)

Student 1 has solved the four problems correctly, discriminating between "if they go equally fast" or "if they do the action faster or slower".

– I don't agree with you (PT7—16:12:46 24/09/2010)

The answer of this student to problem 2 is not correct. The answer would be: As Jim started later, the difference between the letters typed by Sam and Jim is 1400, Sam will have typed 1400 letters more than Jim. So, if Jim has typed 1000 letters, Sam will have typed 2400 letters.

- I don't understand you (PT6—12:30:32 25/09/2010)

You have obtained the same result as the student.

- Correct (PT1—12:48:37 26/09/2010)

I agree with PT6. The answer is correct but instead of obtaining the difference between the letters typed by the two people, the student has obtained the difference between the letters typed by Sam (initially and after).

- Correct (PT7—18:51:41 28/09/2010)

I agree with all of you. I didn't realize that.

The content of the participations and the way in which interactions were developed indicated that the prospective teachers attended to specific features (characteristics of the proportional and additive situations and students' strategies). These interactions, which were motivated by the characteristics of the proposed task and by the necessity of validating their own interpretations with those of others, supported the changes in the level of noticing.

Further, the prospective teachers' teaching decisions also changed after the participation in the on-line discussion. All the prospective teachers stressed the necessity to differentiate between proportional and non-proportional problems. Thus, when they were able to identify the students' profiles, they proposed to focus on the type of ratio and on the use of qualitative problems instead of missing-value problems in order to make the students focus their attention on the multiplicative or additive relationship between quantities.

4 Discussion

This study contributes to the research based on how interactions in an on-line discussion support the development of prospective teachers' useful skills in noticing relevant aspects in mathematics teaching. We contribute to this field, characterizing prospective teachers' noticing of students' mathematical thinking in the transition from students' additive to multiplicative thinking. In this context, understanding the students' mathematical thinking about the relation between additive and multiplicative situations is an important topic that prospective teachers have to learn to notice. That is to say, they have to attend to noteworthy events, reason about such events, and make teaching decisions on the basis of the analysis of these observations (van Es 2010), and they can do this by keeping and using records and validating them with others (Mason, 2002).

Findings show that, initially, the prospective teachers had difficulties attending to and interpreting the students' mathematical thinking in the domain of multiplicative and additive thinking. Some of them described students' answers without including mathematically significant aspects about the structure of the situations or about students' strategies, and therefore they were not able to identify students' profiles. However, the participation in the on-line discussion of prospective teachers who had different levels of noticing led to some of them beginning to develop the noticing of students' mathematical thinking. In this way, when prospective teachers with a lower level of noticing interacted with others with a higher level of noticing, they changed their interpretations to reach mutual understanding. This process led prospective teachers with a lower level of noticing to develop new understanding of students' mathematical thinking. Below we discuss two aspects that help us to understand the development of noticing in on-line contexts: the dialogic argumentation generated within the discussion; and the role of sharing a written text with others in an on-line discussion.

4.1 Designing environments that encourage dialogic argumentation

Designing on-line argumentation environments and incorporating tools that enable prospective teachers to share text documents provides opportunities for the development of noticing. As our findings suggest, presenting and defending interpretations of students' mathematical thinking with other prospective teachers with a higher level of noticing led some prospective teachers to move to higher levels of noticing themselves. This shift to a higher level of noticing is linked to the role played by dialogic argumentation generated within asynchronous threaded online forums as a social and collaborative process to solve professional problems. A characteristic of the on-line discussion that can help to explain the change in the level of noticing is the progressive discourse built thanks to the integration of ideas about proportional and non-proportional situations and about the characteristics of students' proportional reasoning. The on-line debate and the characteristics of the task played a relevant role in this change.

The learning environment was addressed with a specific subject of discussion (the interpretation of students' mathematical thinking in the relationship between additive and multiplicative situations), with a clear goal of discussion (communicating and validating their interpretations with others in order to learn more about this topic), and with the rules that the prospective teachers had to use in order to participate in the on-line discussion (attempting to convince other prospective teachers about the acceptability and validity of alternative ideas). In this context, our results indicate that the specific goal and the structure of the learning environment influenced the change of the prospective teachers' level of noticing. This was evidenced by the conceptual quality of their contributions, in particular when we consider the impact on their final arguments of engaging in dialogic argumentation with a partner who held a different interpretation.

These features of the on-line learning environment have helped prospective teachers to self-regulate their dialogic argumentation by mirroring specific aspects of argumentation processes back to them (e.g. the interchanges described above). Furthermore, it has offered the opportunity to integrate specific characteristics to reify focal aspects of the argumentation such as the characteristics of proportional and non-proportional situations and their relation with students' strategies.

4.2 Writing in the on-line discussion as a tool for the development of noticing

An aspect in which such on-line discussions support the development of prospective teachers' noticing of students' mathematical thinking is related to the role of writing. We suggest that beginning to notice aspects of mathematical problems that influence students' learning is linked to the activity of writing. Firstly, the activity of writing plays a significant role since the final written text is functional with respect to the joint activity in which the prospective teachers are involved. In the example described in this paper, the text produced by prospective teachers helped some of them attend to the mathematical elements of proportional and non-proportional situations and link these elements with the characteristics of students' strategies. So, the text produced is functional in relation to the goal of the on-line discussion. Secondly, the text is related to a topic in which the prospective teachers are interested and about which they believe they should learn more. Thirdly, the participation in the on-line discussion has an explicit goal: validating the interpretations with others through writing a text in order to have the possibility of comparing and refining their ideas through a process of dialogic argumentation. These characteristics translate our attention to the nature and function of participations in the on-line discussion (the written text and the technical instructions to manage the on-line discussion) as a key aspect of the development of noticing.

Wells (2002) argues that it is in writing where the power of written language to create new meanings can be fully exploited. In this sense, creating a written text to convince others in an on-line discussion allows prospective teachers to both develop their skill of noticing the students' mathematical thinking and to deepen their individual understanding. The above argumentation examples from the on-line discussion can be considered to show that the written text enables prospective teachers to create points of focus around which the negotiation of meaning and reciprocal understanding was organized. The points of focus were created by noticing relevant aspects of the relation between the characteristics of proportional and non-proportional situations and the students' strategies. So, the above examples of interactions in the on-line discussion can be considered to be evidence of how participation in this learning environment supports the processes of reification of ideas about students' mathematical thinking and the shifts to a higher noticing level. However, the role played by writing in learning about mathematical knowledge for teaching continues to be an issue for further research.

These results indicate that professional noticing can be learned and that b-learning environments could help prospective teachers to develop this skill. The findings of this type of research would be more powerful if we could trace the trajectory of development across a full discussion. In order to do this it would be necessary to design learning environments with a longer duration and consider a larger sample of prospective teachers. In this context, an area for future work involves exploring the adaptability of the characteristics of the task to other aspects of professional noticing such as mathematical discourse in the classroom and other mathematical domains. Finally, more studies are needed to develop ways of translating the new knowledge about the impact of on-line learning environments on the development of professional noticing to teacher education programs.

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