ZDM Mathematics Education (2010) 42:245–261
DOI 10.1007/s11858-009-0223-3

ORIGINAL ARTICLE

A five-phase model for mathematical problem solving: Identifying synergies in pre-service-teachers' metacognitive and cognitive actions

Asmamaw Yimer · Nerida F. Ellerton

Accepted: 28 October 2009 / Published online: 23 December 2009

Abstract Based on empirical data from a study of pre-service teachers engaged in non-routine mathematics problem solving, a five-phase model is proposed to describe the range of cognitive and metacognitive approaches used. The five phases are engagement, transformation-formulation, implementation, evaluation and internalization, with each phase being described in terms of sub-categories. The model caters for a variety of pathways that can be adopted during any problem-solving process by recognizing that the path between these five phases is neither linear nor unidirectional.

Keywords Mathematical problem solving · Metacognition · Cognition · Reflection · Pre-service teachers

1 Introduction

In prioritizing goals in the problem-solving process, Schoenfeld (2006) noted that "the quality of the decision-making … is very much a function of the individual's metacognitive skill" (p. 49). He also noted that the process he was describing was recursive "in the sense that goal prioritization and knowledge selection occur at multiple levels" (p. 49). These comments resonate strongly with the research described in this study, and with the five-phase model originally developed by Yimer (2004) in his doctoral dissertation to describe the cognitive and metacognitive processes used by pre-service teachers engaged in problem solving.

Major emphases in research on problem solving in the 1980s and 1990s included problem difficulty, distinctions between good and poor problem solvers, and problem-solving instruction, with studies of the role of metacognition becoming a major focus (Lester, 1994). In particular, researchers investigated the effect of metacognitive strategies on students' problem-solving performances and attitudes through control–treatment situations (Chicola, 1992; Marge, 2001; Willburne, 1997). Few studies, however, have examined the metacognitive processes in which students engage *during* problem solving. It was anticipated that the gathering of data on metacognitive processes while problem solving was in progress would open new perspectives on these processes, and that this would facilitate the development of clearer interpretations of the phases involved. This paper reports on a study which analyzed the metacognitive processes employed by a sample of pre-service teachers as they engaged in mathematical problem solving. Through analyses of these data, we further develop and interpret our five-phase model, which is summarized in Yimer and Ellerton (2006).

2 Literature review

2.1 Defining the role of metacognition in problem solving

Problem solvers are required to analyze a problem, understand it, evaluate the given information for its adequacy, organize knowledge and facts and devise a plan,

A. Yimer
Chicago State University, Chicago, IL, USA
e-mail: ayimer@csu.edu

N. F. Ellerton (✉)
Illinois State University, Normal, IL, USA
e-mail: ellerton@ilstu.edu

evaluate the feasibility of the plan before executing it, and evaluate the results for reasonableness (Polya, 1957; Schoenfeld, 1992, 2006, 2007). These behaviors are not limited just to the cognition involved in thinking and organizing information (Simon, 1973). Rather, regulation and monitoring of understanding, planning and evaluating the results are closely linked to thinking about one's thought processes, that is, to metacognitive behaviors (Marcus, 2007; Martinez, 2006; Veenman, Van Hout-Wolters & Afflerbach, 2006). Content knowledge is recognized as a necessary, but not sufficient, attribute for solving mathematical problems (Garofalo & Lester, 1985; Geiger & Galbraith, 1998; Schoenfeld, 1987; Silver, 1987). Carlson (2000) found that students' inefficient control decisions were major obstacles during their attempts to find solutions. Cognition without metacognitive thoughts rendered problem-solving endeavors incomplete (Artzt & Armour-Thomas, 1992; Berardi-Coleta, Dominowski, Buyer & Rellinger, 1995; Schoenfeld, 1985a, b). Artzt and Armour-Thomas (1997) went so far as to suggest that the main source of difficulties experienced by students in problem solving could be attributed to their failure to initiate active monitoring and subsequent regulation of their own cognitive processes.

Stillman and Galbraith (1998) noted that providing opportunities for metacognitive decisions to be made does not insure that they will be made. They also found that if such decisions were made, then there was no guarantee that these decisions would be appropriate. They concluded that a rich store of knowledge of metacognitive strategies as well as the appropriate application of this knowledge, developed over an extended period of use, were prerequisites to productive decision making.

Kuhn (2000) argued that metacognition did not appear abruptly, but rather emerged and developed under an individual's conscious control. In the USA, the National Council of Teachers of Mathematics (NCTM, 2000) included monitoring and reflecting on the process of mathematical problem solving as objectives in its problem solving standards. Articles that focus on giving students the opportunity to reflect on and evaluate their own thinking (e.g., Robert & Tayeh, 2007) continue to appear in journals, the target audiences of which are teachers. The implications of teaching metacognitive skills have also been explored by Lin, Schwartz and Hatano (2005).

Metacognitive processes can shape cognitive activities in which the individual is engaged (Garofalo, 1989; Mildren, Ellerton & Stephens, 1990) and can improve problem-solving performance (Artzt & Armour-Thomas, 1992; Goos & Galbraith, 1996). Metacognition is also believed to improve students' confidence when they tackle authentic tasks (Kramarski, Mevarech & Arami, 2002) and to help overcome obstacles encountered by students engaged in mathematical problem solving (Pugalee, 2001; Stillman & Galbraith, 1998).

2.2 Focus of research on the role of metacognition

The role of metacognition in students' solving of mathematical problems has received increasing attention in the research literature. In particular, research on the role of metacognition in mathematical problem-solving has been concerned with studying how problem solvers know, regulate and monitor their own thought processes (Lester, 1994). Other researchers have focused on how students identify and define the problem, mentally represent the problem, plan how to proceed in solving the problem and evaluate what they know about their own performance (Davidson, Deuser & Sternberg, 1994).

Quantitative and qualitative research on metacognitive aspects of mathematical problem solving has been reported. Crawford (1998), for example, used a quantitative pre-test–post-test design to study students' reflective thinking. Similar pre-test–post-test designs showed that students in treatment groups, who were encouraged to use metacognition, reflected more effectively and were more successful in their problem-solving attempts than those who did not (Chicola, 1992; Marge, 2001; Willburne, 1997).

Qualitative aspects of metacognition in mathematical problem solving have been the focus of studies with middle-school and high-school students (Artzt & Armour-Thomas, 1992; Goos, 2002; Goos & Galbraith, 1996; Goos, Galbraith & Renshaw, 2002; Pugalee, 2001; Stillman & Galbraith, 1998). A consensus seems to have developed that problem difficulty is not so much a function of task variables as it is of characteristics of the problem solvers. Geiger and Galbraith (1998) claimed that "it is the relationship between the learner and a problem that is of significance, not the perceived level of the problem as viewed within some hierarchy of abstraction" (p. 535). A study of the relationship between the problem and the problem solver has generated research in which learners are labeled as good or poor, or as expert or novice, problem solvers (Carlson, 2000; DeFranco & Hilton, 1999; Sriraman, 2003). During the problem-solving process, "good" problem solvers were observed evaluating the efficiency and effectiveness of their selected approach. They also showed a strong tendency to relate the given problem to other similar problems, employing self-reflection through a display of internal discussions (Carlson, 2000). Effective implementation and monitoring of control skills was found to be a key discriminant of success in problem solving (Geiger & Galbraith, 1998).

2.3 Investigating affect and metacognition

Affect is regarded by some as critical to the structure of competencies, accounting for success or failure in mathematical problem solving (see, e.g., DeBellis, 1998; Goldin, 1988; McLeod, Craviotte & Ortega, 1990). Students' perceptions of mathematics and of themselves as learners of mathematics can affect their ability to solve mathematical problems (Artzt & Armour-Thomas,1997; Schoenfeld, 1985b, 2007; Silver, 1987). This is particularly true if students believe that mathematical problems should be completed in a few minutes, that there is one correct answer to mathematical problems, or that only mathematical geniuses can solve mathematics problems. If students hold such beliefs, they are likely to spend only a limited time in finding single solutions to mathematical problems and may give up, because they have little faith in their ability to solve challenging problems. According to Krathwohl, Bloom and Masia (1964), beliefs are classified at the lowest of three levels of values, in which a person simply accepts a value. At the highest level, a value is interpreted as a conviction or a commitment. Thus, investigating the extent to which students value problem solving and the extent to which they value themselves as problem solvers are important aspects of metacognitive research.

McLeod (1989, 1992) argued that, if students reflect on their own cognitive processes, they can develop an awareness of their emotional reactions to problem solving, which will give them greater control over their cognitive processes. He foreshadowed the importance of further research into the intersecting cognitive and affective domains and called for the application of a wider variety of research approaches, including clinical interviews and observations. In particular, he believed that the understanding of linkages between cognitive and affective factors would be of relevance to teacher education and to the teaching and learning of mathematics (McLeod, 1992). This study takes up the challenges he foreshadowed.

2.4 Toward a model for the role of metacognition in problem-solving

Metacognition is now recognized as a quality that is important in mathematical problem solving. Researchers have developed models and frameworks that describe the cognitive and metacognitive actions which may occur during an individual's mathematical problem-solving attempts (Artzt & Armour-Thomas, 1992; Carlson, 2000; Carlson & Bloom, 2005; Garofalo & Lester, 1985; Geiger & Galbraith, 1998; Schoenfeld, 2006, 2007). Although several research studies have identified metacognitive behaviors exhibited by middle-school and high-school students, and by professional mathematicians, very little research has been carried out on the metacognitive behaviors of pre-service teachers who are engaged in mathematical problem solving. It is desirable that teacher educators are able to stimulate metacognitive behaviors in pre-service teachers so that the pre-service teachers, in turn, will be able to recognize and stimulate metacognitive behaviors of the next generation of young school students.

In what he described as a "rather speculative paper," Schoenfeld (2006) explored the reasons why individuals make the decisions they do when they are solving problems. After stating that his present model represents the culmination of more than 30 years of research, he said that he anticipated that many decades of research in this area lay ahead. He emphasized that the four categories of mathematical knowledge and behavior that he recognized over 20 years ago (Schoenfeld, 1985b), "resources (the knowledge base), heuristic (problem-solving) strategies, 'control' (monitoring and self-regulation, aspects of metacognition), and beliefs" (p. 62)—must *all* be examined if one is to understand an individual's success or failure in solving problems. Of particular relevance to our study were the following questions and comments:

> What was missing in this approach was a sense of how all these things fit together – a description of *mechanism*. How did the categories interact with each other? Why did people do what they did when they were in the midst of a problem solving attempt? (p. 62)

Schoenfeld went on to comment that describing people's decision making while they are actually solving problems "has the potential to be a theory of problem-solving-in-action" (p. 63). Indeed, the original five-phase model developed by Yimer (2004) and summarized by Yimer and Ellerton (2006) was based on "problem-solving-in-action."

3 The study

This study had two main purposes. The first was to identify and characterize the nature of metacognitive behaviors exhibited by a sample of pre-service teachers (referred to as "students" throughout this paper) as they engaged in mathematical problem solving. In addition to identifying metacognitive behaviors, any patterns in these metacognitive actions which might suggest a model to describe the metacognitive functioning of these students were noted. The second purpose was to identify and describe the ways in which these students valued problem solving and themselves as problem solvers, as well as how these values related to the metacognitive actions students employed during their engagement with mathematical problem-solving tasks.

3.1 Research questions

The following two questions guided this study.

1. What was the nature of students' individual metacognitive functioning and in what ways did this change during the course of study as each engaged in individual problem solving?
2. In what ways did the students value mathematical problem solving, in general, and themselves as problem solvers, in particular, and how did this relate to the metacognitive strategies they employed?

3.2 Research design and methodology

Seventeen pre-service teachers in a large university in the USA participated in this study. They were all studying a mathematics problem-solving course in which the problems were designed to draw attention to important aspects of number theory. The course was one of the final courses which students needed in order to gain an endorsement to teach mathematics in the middle school. Students taking this course had already passed a minimum of four other mathematics content courses, so could all be regarded as having achieved a firm foundation in mathematics. The class met for two 2-hour sessions for each of the 15 weeks of the semester. Cooperative group work and whole-class discussion were regular features of each class session.

Whole-class and small-group observations were conducted throughout the course in order to document the classroom environment, and to determine whether this environment facilitated the development and growth of metacognitive behaviors. Cobb and Yackel (1996) noted that for metacognitive behaviors to develop, a conducive environment that encouraged students to reflect on their own and others' thoughts needed to be established. All classes and small groups observed were audiotaped and transcribed.

During mathematical problem solving, students may exercise metacognitive behaviors at any stage (Pugalee, 2001). The design of the study, therefore, used a multiple-perspectives approach and included semi-structured task-based interviews, small-group and class observations, stimulated-recall interviews and students' written reflections.

This study investigated how the views of students about themselves as problem solvers related to the metacognitive actions they employed. Students were asked to talk aloud during both the task-based and stimulated-recall interviews. All task-based and stimulated-recall interviews were audiotaped and transcribed for later analysis.

Three task-based interviews with each of the 17 students were conducted at approximately equal time intervals during the semester so that any changes in students' metacognitive behaviors and problem-solving performances could be investigated as the semester progressed. The task-based interviews, each of which took 30–45 min, were not entirely think-loud sessions, but were guided by a semi-structured task-based interview protocol adapted from Newman's (1983) *Error Analysis Interview Protocol*. The task-based interview protocol began with the question "Please read the question out loud and explain to me, in your own words, what the question is asking you to do", and continued with questions like "How would you go about solving the problem?" Students were not prompted in any way with leading questions and were allowed to have lengthy periods of silence if they preferred it. The interviewer would ask questions such as "Could you tell me what you are thinking?" or "What have you been working on?" and so on. Each of the three task-based interviews involved the students in solving two of the six non-routine problems described in Sect. 3.3; all students completed the six problems in the same order. Data from the task-based interviews were used to address the first research question. Data from all sources described were used to address the second research question.

Stimulated-recall interviews were used to help identify individual metacognitive actions as students engaged in both small-group and whole-class discussion during class sessions. These interviews were conducted individually with students within a week of the class session. A transcript of the session was available, and audiotapes of relevant sections of the group or class discussions were played back to the student. Questions such as "Was there anything special you feel you contributed toward the group's discussion at that point?" and "What were you thinking at that moment (e.g. when… generated that idea)?" were asked.

As part of the requirements of their course, students were asked to produce written reflections on specific aspects of the course. Students were reminded that there were no right or wrong answers in reflections; rather they were encouraged to comment freely in response to the particular reflection. Five sets of written reflections were assigned and collected from all students. Reflection 1, for example, set out the following tasks for students in two parts:

1. In your own words, and from your perspective, what are some of the key features of mathematical problem solving? (no text-book definitions, please)
2. Reflect on one of the problems we have worked on in class. Comment on how you approached the problem, and whether others in your group thought about the problem in quite different (or similar) ways. In what ways did working on the problem in a group setting help (or hinder) the problem-solving process?

Reflection 5 contained the following three parts:

1. Reflect on what you have learned about yourself as a problem solver this semester.
2. As part of this reflection, comment on the problem-solving strategies you have developed that will be useful to you as a teacher.
3. To what extent are you now more aware of how you are monitoring your attempts to solve mathematical problems (compared with how you felt at the beginning of the course)?

Students were encouraged to treat reflections as a conversation with their instructor, and communication about concerns, difficulties and challenges felt by the students was encouraged.

3.3 Problems for task-based interviews

Six non-routine mathematics problems, listed in Sects. 3.3.1–3.3.6, were selected for individual task-based interviews. The first two problems were deliberately chosen to be relatively simple so that students were likely to start the interview session solving and talking about problems with which they felt at ease. Although the remaining problems were challenging to the students, they were chosen so that they were, nonetheless, within the students' reach.

3.3.1 The age problem

John is 12 years older than Mary, but Mary is 15 years younger than Andrew. How old is Andrew compared with John? John has a brother Nick who is 2 years younger than John. Andrew has a sister, Julie, who is 6 years younger than Andrew. How old is Nick compared with Julie?

3.3.2 The banquet problem

Nathan and some of his friends are seated around a large circular banquet table. A tray containing 25 sandwiches is passed around the table with each person taking one sandwich as the tray reaches them. The tray is passed in this way until all 25 sandwiches have been taken. Nathan takes the first and the last sandwiches, but may also take some in between. How many people are seated around the table?

3.3.3 The locker problem

The new school has exactly n lockers and exactly n students. On the first day of school, the students meet outside the building and agree on the following plan. The first student will enter the school and open all of the lockers. The second student will then enter the school and close every locker with an even number (2, 4, 6, 8,…). The third student will then "reverse" every third locker. That is, if the locker is closed, he or she will open it; if the locker is open, he or she will close it. The fourth student will reverse every fourth locker. And so on until all n students have entered the building and reversed the proper lockers. Which lockers will finally remain open?

3.3.4 The egg vendor problem

An egg vendor delivering a shipment of eggs to a local store had an accident, and all of his eggs were broken. He could not remember how many eggs he had in the delivery. However, he did remember that when he tried to pack them into packages of two, he had one left over; when he tried to pack them into packages of three, he had one left over; when he tried to pack them into packages of four, he had one left over; when he tried to pack them into packages of five, he had one left over; and when he tried to pack them into packages of six, he had one left over. Nonetheless, when he packed them into packages of seven, he had none left over. What is the smallest number of eggs he could have had in the shipment?

3.3.5 The bridge problem

Mike was racing in a bike marathon. He heard the whistle of the Wabash Cannonball train approaching the bridge from behind him. He had carefully researched the path he would take and knew that the train traveled this stretch of track at 60 miles per hour. The marathon route involved the riders crossing a narrow railway bridge. Mike had counted the number of pillars in the bridge so that he could always estimate where he was on the bridge in case a train came. When he heard the whistle, he was $3/8$ of the way across the bridge. Being an amateur mathematician as well as a marathon biker, Mike calculated that he could just reach either end of the bridge at the same time as the train. How fast was Mike pedaling his bike?

3.3.6 The census problem

During a census, a man told the census taker that he had three children. When asked their ages, he replied "The product of their ages is 72. The sum of their ages is the same as my house number." The census taker ran to the door and looked at the house number. "I still cannot tell," she complained. The man replied, "Oh, that is right. I forgot to tell you that the oldest one likes chocolate pudding." The census taker promptly wrote down the ages of the three children. How old were they?

4 Results and discussion

To address the first research question, the results and analyses of the task-based interviews will be discussed. Contrasting examples of how two different students approached finding the solution to one of the six problems completed by the students for this study were chosen. This will be followed by a discussion of the results and analyses of stimulated-recall interviews, and of students' reflections, to address the second research question.

4.1 Analysis of metacognitive actions of individual students exhibited during individual, task-based interviews

Transcripts from individual, task-based interviews revealed different orientations and solution processes. Solution processes used by different students working on the same problem showed markedly different individual characteristics, degree of understanding, depth of analysis and control. Even the same student was found to exhibit different levels of sophistication and metacognitive behaviors across different problems. Nevertheless, certain patterns recurred in most individual's solution processes. The constant comparative method (Maykut & Morehouse, 1994) was used in unitizing and categorizing behaviors. This approach combines inductive category coding with a simultaneous comparison of all units of meaning identified. Thus, for example, cognitive/metacognitive behaviors identified in one student's problem-solving efforts on one problem were compared both with the same student's work on different problems, as well as with other students' cognitive/metacognitive behaviors identified in their work with the same problem and with different problems. The constant comparative method of data analysis was appropriate for this study since patterns in behaviors from several data sources had been coded and analyzed. Five groupings of cognitive/metacognitive behaviors were identified, and these groupings have been designated as the five phases: engagement, transformation-formulation, implementation, evaluation and internalization.

The five-phase cognitive/metacognitive model, which was formulated on the basis of the task-based interview data, is presented in Fig. 1. Each phase has been described

Fig. 1 The problem-solving model

COGNITIVE AND METACOGNITIVE CATEGORIES

1. **Engagement**: Initial confrontation and making sense of the problem.
   - A. Initial understanding (jotting down the main ideas, making a drawing)
   - B. Analysis of information (making sense of the information, identifying key ideas relevant information for solving the problem, relating it to a certain mathematical domain)
   - C. Reflecting on the problem (assessing familiarity or recalling similar problems solved before, assessing degree of difficulty, assessing the necessary store of knowledge one has in relation to the problem)
2. **Transformation-Formulation**: Transformation of initial engagements to exploratory and formal plans.
   - A. Exploring (using specific cases or numbers to visualize the situation in the problem)
   - B. Conjecturing (based on specific observations and previous experiences)
   - C. Reflecting on conjectures or explorations whether they are feasible or not.
   - D. Formulating a plan (devising a strategy either to test conjectures or devising global or local plans)
   - E. Reflecting on the feasibility of the plan vis-à-vis the key features of the problem
3. **Implementation**: A monitored acting on plans and explorations.
   - A. Exploring key features of plan (breaking down plan into manageable sub plans where necessary)
   - B. Assessing the plan with the conditions and requirements set by the problem
   - C. Performing the plan (taking action by either computing or analysing)
   - D. Reflecting on the appropriateness of actions
4. **Evaluation**: Passing judgments on the appropriateness of plans, actions, and solutions to the problem
   - A. Rereading the problem whether the result has answered the question in the problem or not
   - B. Assessing the plan for consistency with the key features as well as for possible errors in computation or analysis
   - C. Assessing for reasonableness of results
   - D. Making a decision to accept or reject a solution
5. **Internalization**: Reflecting on the degree of intimacy and other qualities of the solution process.
   - A. Reflecting on the entire solution process
   - B. Identifying critical features in the process
   - C. Evaluating the solution process for adaptability in other situations, different ways of solving it, and elegance
   - D. Reflecting on the mathematical rigor involved, one's confidence in handling the process, and degree of satisfaction

in terms of indicators or sub-categories, which were observed to take place in the solution processes of at least one individual.

Excerpts from two students' task-based interviews on the locker problem (Problem 3.3.3), and the corresponding behaviors and phases will be presented. The intent is to show similarities and differences in the observed behavior patterns of the two students rather than to compare the quality of their solutions. The two examples were chosen to demonstrate a range of different behaviors by two students as they worked individually through the same problem.

4.1.1 Karen's solution to the locker problem

Figure 2 shows numbered excerpts from Karen's solution to the locker problem, with corresponding behaviors and phases. Karen started her engagement with the problem by reading the problem (1) and restating it (2), thereby establishing her understanding of the problem. She noted that she had not come across similar problems (3) and drew a 16×16 grid to specialize the problem (4). Her drawing helped her to analyze the information thoroughly (5), and this helped her to make a new observation that prime numbers were closed (6). Her new observation about prime numbers sparked a monitoring move, which led her to look back and observe her drawing carefully (7).

Karen identified a pattern in her drawing, but instead of moving forward with her conjecture, she showed a regulatory behavior by considering more cases (8, 9). She did not extend her grid, but instead analyzed and identified which students reversed a certain locker by listing the factors of the locker number she considered. This analysis enabled her to state her conjecture (10, 11). She focused on justifying her conjecture by taking a specific case and closely observing the situation (12–14) until she identified a key mathematical idea in her analysis (15). She identified a new idea, that is, parity of numbers, but did not rush to state it. Rather, she went back and forth through the process, by which she had explicitly formulated her solution, with excitement (16–19). She reduced the entire problem into a problem of parity of numbers.

Karen's challenge was to justify whether all square numbers have an odd number of factors (20). She engaged in monitoring her work in the grid and her extended analysis (21, 22), as well as computing mentally without verbalizing anything (23). She then justified her solution with analysis supported by specific examples (24–26). She expressed her excitement at how difficult the problem was, and assessed her success and noted her satisfaction (27). She was motivated by what she had done and was curious whether the problem had elegant solutions (28).

4.1.2 Ann's solution to the locker problem

Excerpts from Ann's solution to the locker problem and corresponding behaviors and phases are shown in Fig. 3. Ann read the problem and restated it (1, 2). She reread the problem and, as she did so, jotted down information and made a 20×20 grid. Observing the actions of the first four students, she concluded that she had reached a solution (3).

Ann was not comfortable, however, as she stated her solution and observed her grid. She realized that her solution was premature and wrong (4). She reread the problem and continued to try to find a pattern, make conjectures and test them The remaining numbered points in Fig. 3 can be read as continuing narrative describing Ann's actions as she worked through the problem. Finally, she expressed the difficulty level of the problem and her deviation from, and her return to, the right track in the process (15). Ann showed considerable metacognitive behaviors during her assessment and monitoring of her solution process. On the other hand, she was not observed reflecting on the reasonableness, consistency and elegance aspects of her solution.

The data from task-based interviews suggested that the paths that the problem solvers followed were mediated by rereading the problem. In other words, rereading served as a catalyst for metacognitive decisions to take place either in the form of choosing a path or other metacognitive actions within a specific cognitive phase in which problem solvers engaged. The engagement of problem solvers in controlling and regulating their actions in either selecting or abandoning a specific path corresponds to Casey's (1978) error analysis hierarchy. According to Casey, for example, a problem solver might decide to reread the problem to check that all relevant information had been taken into account. It should also be noted that these paths may be cyclic as students engage in a series of different paths between different phases.

4.2 Analysis of students' written reflections and affective aspects noted during the task-based and stimulated-recall interviews

In this section, affective elements exhibited by students during their written reflections and questions asked during task-based and stimulated-recall interviews will be presented and analyzed. In particular, the ways students viewed mathematical problem solving, the ways they viewed themselves as problem solvers, and the extent to which their views and the metacognitive behaviors they exhibited were related will be addressed.

Fig. 2 Excerpts and coding of Karen's solution to the locker problem

Excerpts	Behaviour	Phase
(1) Reads the problem (2) It is asking to determine which lockers will remain open by the end of the process (3) Wow! I haven't come across such problems before. (4) Let me see what it looks like with 16 student and 16 lockers in my chart	- Initial engagement - Restating the problem - Assessing familiarity - Exploring the problem with specific cases	Engagement
(5) I see that there are students who will not touch some lockers. For example, the 2nd student closes the 2nd locker, the 3rd student will not touch the 2nd locker and so on. (6) I see that there are only two students touching prime numbers. The 1st opens and the 2nd student closes. That means prime numbers are closed for ever. (7) Let me closely look at my chart (8) I see that lockers 1, 4, 9, 16 are open and the rest are closed in my chart. (9) Wait a minute! No! No! I better see what happens to 25 and 36… (10) I think square numbers will be open. That is my conjecture for the moment (11) I can show that every locker with a square number will be open by listing the students touching that locker. But my problem is I do not know about other lockers.	- Analysis of information - New observation - Monitoring the process - Identifying a pattern - Reflecting on conjectures - Considering more cases - Making conjectures - Formulating a plan - Reflecting on the plan	Transformation-Formulation
(12) I need to justify it. But I have to list the number of students reversing each locker. (13) Let me try the 24th locker. There are eight students with numbers 1, 2, 3, 4, 6, 8, 12, 24 touching it. (14) If I arrange these numbers in order and regroup them in pairs, I see that the 24th locker is closed. (15) I think these numbers are factors of 24. I listed them earlier. I should have known them. (16) Hold on! These are even. And the locker is closed. I am excited. (17) Let me pick the open locker I determined for the chart. I will take locker 16 and list its factors. (18) Yes! It has odd number of factors. (19) I think if the locker number has even number of factors, it will be closed and will be open if it has odd number of factors.	- Exploring the essence of the plan. - Assessing the plan with specific cases - Exploring the plan vis-à-vis the specific cases - Identifying key mathematical idea - Performing the plan - New observation - Excitement - Monitoring the performance on the plan - Get a solution - Formulating a solution explicitly	Implementation
(20) Do all square numbers have odd number of factors? I have no clue. (21) But a square number can be expressed as a product of a number by itself. So what? How does it help me? (22) Let me see what I did. The chart does not tell much except showing which locker is closed.	- Assessing plan for consistency - Assessing reasonableness of the solution - Justifying the solution - Reflecting on the solution	Evaluation
(23) (No verbal communication) (24) Yes (25) All square numbers have odd number of factors. E.g., 24: 1 x 24, 2 x 12, 3 x 8, 4 x 6 16: 1 x 16, 2 x 8, 4 x 4 36: 1 x 36, 2 x 18, 3 x 12, 4 x 9, 6 x 6 You do not count 4 x 4, 6 x 6 as two factors but one. (26) Lockers with square numbers are open. My conjecture was true.	- Monitoring the process -Mental computation - Justifying the solution	Evaluation
(27) Wow! That is wonderful. It was challenging. But I am really happy because I had no idea at first. I didn't think I will go this far. (28) Is there a short way of solving it? I will love to see that.	- Excitement - Assessing difficulty - Confidence in handling the solution - Inquiring for elegance	Internalization

Fig. 3 Excerpts and coding of Ann's solution to the locker problem

Excerpts		Behaviour	Phase
(1)	Read the problem	- Initial engagement - Restating the problem -Reread -Jotted down - Made a drawing	Engagement
(2)	I think the problem is clear. It is asking to determine which lockers will be open.		
(3)	Lockers 1, 4, 6, 8, 11, 12, 13, 16, 17, 18, 19, 20 are open.	- Stated a solution	Implementation
(4)	I do not think it is right. Some students between 5-20 may reverse these lockers	- Observed what she did - Reflected on the solution	Evaluation
(5)	I have a question. Does the 3rd student start from the 3rd locker or he starts from the 1st locker?	- Reread the problem - Assessing understanding	Engagement
(6)	I think 1, 4, 9, 16 are open so far. These are square numbers. Well square numbers will be open. This is my conjecture.	- Identified a pattern - Made a conjecture	Transformation-Formulation
(7)	Let me see 25. The 1st, 5th, and 25th students reverse this locker. Only 3! These are factors of 25	- Identified new idea	
(8)	Wow! It is easy now compared to that confusing chart I made. A locker will be open or closed depending on the number of factors it has.	- Excitement - Analyzing the information with respect to the new idea	Implementation
(9)	I see something else too. If the factors are even the locker is closed and open if they are odd.	- Making a new conjecture	
(10)	If you pair the factors, you can label open and closed. In case of even factors, the locker is completely paired and is closed. In case of odd factors, there is one unpaired factor and that makes the locker open	- Analysis of the information vis-à-vis the new idea - Justifying the new conjecture	
(11)	I think that is it. The problem is solved. Locker numbers with even numbers of factors are closed and those with odd number of factors are open	- Stated solution	Evaluation
(12)	Oh! I do not think I have answered the question yet.	- Reread the problem - Observed what she did - Monitoring the solution - Reflecting on the solution	Evaluation
(13)	I will stick to my first conjecture that square numbers are open. But do square numbers have odd factors?	- Assessing for consistency of her conjectures - Assessing for consistency of solution with the conditions of the plans	
(14)	Isn't it true that square numbers are expressed as the product of a number by itself? Well, we cannot count it as two factors. That is it.	- Identified a key mathematical idea. - Justified her solution	
(15)	It was a hard problem until I get into the right track. It took me a long way to see what I know is true	- Assessing difficulty - Reflecting on the process	Internalization

4.2.1 Students' views of mathematical problem solving

In their first written reflection, students were asked to identify and reflect on key features of problem solving, and to describe how they viewed these features in relation to their understanding of problem solving. The instructor explained that there were no "right" or "wrong" answers to reflections, and that reflections were one way students could share with her how they felt on certain aspects of the problem-solving processes. The key features of problem solving identified by students in their first reflection are summarized in Table 1.

In addition, some students said that they viewed problem solving as a process which went beyond determining a solution. For example, in Reflection 1, Liz stated: "Although 'figuring out' or finding the solution to a problem is important, as with many fields of study, the process can sometimes be more valuable." Lucy referred to

Table 1 Key features of problem solving identified from students' reflections

Key feature	Frequency
Understanding the problem (restating, jotting down the information, drawing a picture, understanding mathematical concepts)	13
Making connections (relating it to other problems)	6
Planning (identifying formulas, specializing, generalizing, breaking it into steps, looking for a pattern)	10
Solving (acting on it) (using multiple methods, communicating, practicing, recording work)	7
Checking (checking for errors, checking for reasonableness, working for elegance)	3
Reflecting (critical thinking, thinking out of the box)	4
Affective elements (confidence, patience)	2

the relevance of the process: "The solution obviously concludes the problems, but for some problems, the use of the processes is more beneficial in the long run."

Some students seemed to be aware that problem solving is a challenge that has to be faced, and that mathematical problem solving involves many decisions in terms of conceptual background, relevance of a strategy and other control mechanisms. For example, Ashley, in a stimulated-recall interview session described the challenging nature of mathematical problem solving in this way: "Problem solving is not to be preferred for convenience because it is not at all convenient. It is indeed challenging. But I get the greatest satisfaction when I figure out the solution of a challenging problem instead of a routine computation." Ann, in a task-based interview session, stated

> In life, I solve problems. I am very good at analyzing real-life problems. … Although, like life problems, you still analyze situations, you do not see conditions clearly in mathematics. They are hidden in the knowledge you have learned early. That makes me really uncomfortable in mathematics.

Liz, in a stimulated recall interview session, compared her school experience and her current situation in terms of her conceptions of problem solving:

> There is a shift in thinking. This shift is actually from recalling formulas and applying formulas to exploring the problem, making sense of it, and devising a solution strategy. We were not either expected or able to do such things at all in our school systems. Here the problems we have been solving are not necessarily difficult. In fact, our high school computations in calculus or trigonometry may be more complicated. But all we do is to plug them in a formula and get the right answer. Here it is different. All answers are acceptable as long as they make sense. What makes one strategy different from another is whether it is more elegant than the others or not.

Students' responses to the question "How do you value mathematical problem solving?" during the stimulated-recall interviews and task-based interviews tended to reflect their view of themselves as problem solvers. This shift of associating one's view of problem solving to one's own perception of self as a problem solver is consistent with Geiger and Galbraith's (1998) description of problem solving as the relationship between the task (the problem) and the problem solver.

4.2.2 How do students view themselves as problem solvers?

During task-based interviews and stimulated-recall interviews, and in their written reflections, students were asked how they viewed themselves as problem solvers and/or how they viewed problem solving in general. Responses revealed students' overall awareness of problem solving processes, in general, and of metacognition, in particular. At the end of task-based interview sessions, students compared where they were before and where they were currently in terms of particular characteristics about themselves, the tasks and the strategies they had used.

4.2.3 Person, task and strategy variables

Three variables, personal, task and strategy variables, have been used to characterize metacognitive knowledge (Flavell, 1976, 1987), and one's awareness of cognitive processes (Garofalo & Lester, 1985). Students reported that although they felt they had improved, they still identified weaknesses on their own part. Most students seemed to think that they needed to make improvements in their metacognitive behaviors if they wanted to improve their problem-solving performances.

For example, with regard to person variables, Liz stated in a task-based interview: "I was rigid and usually do not try for alternative solutions … I am improving and getting to be flexible." Some students reported that they were adjusting themselves to "the new culture." For example, Erin, in a task-based interview, stated: "I have developed interest and patience to see problems differently when one strategy fails. This is … new to me." Lucy, during one of her task-based interviews, said: "I have developed confidence and patience in handling problems. I am being flexible and confident in explaining and justifying what I am doing."

Some students' reflections included statements that aligned metacognitive behaviors with task variables. For example, "I found that anything I can relate to my own life, I can solve easier … I become more involved in the problem and look harder for a solution" (Mary in Reflection 5). Other comments included "hating rate problems" (Karen), "liking logical problems" (Katie, Liz), and "liking visual problems" (Lynn, Ann).

Statements that would be aligned with the strategy variable include "I am appreciating multiple strategies" (Beth), "I am good at identifying patterns" (Jackie, Katie), "I skim first and reread and analyze information" (John, Dianne, Liz, Ashley, Trish), and "I like checking and reflecting" (Lynn, Quinn, Lucy, Cindy). Figure 4 summarizes examples of person, task and strategy variables.

In the classification of behaviors as person, task or strategy variables, some overlap was found. Such overlap is consistent with Lester's (1985) findings in that the strategy variable tended to blend with the person and task variables.

Students' views about problem solving and themselves as problem solvers were revealed through their reflections. For example, "When I am solving problems now, I notice that when I start to work out a process I think where the process will ultimately bring me … and if the process will help me in finding a solution." (Lucy in Reflection 5). John, in Reflection 5, addressed how his previous school experiences had affected his performance:

> I knew that I had to do a few equations to get the exact distance the fly traveled. The equations were getting more difficult … but, what I like most about this problem was its simplicity. I think that too often students fall into a traditional way of solving problems. That is what happened to me that day.

Mary reflected on the nature of tasks and their relevance to the solver as being important when she stated in Reflection 5:

> I feel that a lot of times math gets a bad rap from middle school students because they may find it boring or difficult. But if they are given logical problems and ones that relate to their everyday life, they may be more inclined to want to work on the problems and make those connections.

Mary's statement went beyond a mere view of problem solving and addressed the impact of tasks on the attitudes and beliefs of students.

Fig. 4 Examples of person, task, and strategy variables

Person variable	Task variable	Strategy variable
- I do not easily give up - I get frustrated if I do not understand the problem - I am developing patience and confidence - I am very slow - I am being flexible - I started feeling autonomous - I am developing interest in problem solving - I started being reflective	- I started liking word problems - I like tasks that are related to my experience - I like visual problems - I like problems that have a definite answer - I like logical problems but also like algebra - Number theory and differential equations are my favourite subjects - I hate rate problems and have never been good at them - I like problems if I can solve them - I hate geometry	- I skim first and reread it thoroughly - I analyze the information before I start using a formula - I look back and check my calculations and if my answer makes sense - I still have problems in checking - I am improving in checking my work - I am good at identifying patterns - I like to reflect if I get time - I like multiple strategies to a problem - I like making a drawing - I like visualizing a problem by drawing or mentally - I spend much time in understanding - I focus on the last sentence - I like breaking down the problem into several steps

Some students' reflections showed awareness of their own improvement when they evaluated prior school experiences. Cindy, for example, reflected on how inappropriate some of the mathematics teaching she had experienced had been: "It was like a light bulb went off in my head when we talked about how the teacher does all of the work and when doing so it takes away from how the students should learn." (Reflection 5) During a task-based interview, Beth expressed her view about problem solving and her growth as a problem solver:

> I used to give up when I do not get a solution right away. But now I am aware that there are multiple ways of solving a problem and I will try it over again. Even if I may not be able to solve a problem, I can at least have a good feel whether I am in the right track or not or whether I have enough knowledge to solve it or not.

Recognizing one's limitations during problem-solving attempts can be an important part of monitoring one's actions.

In the fourth reflection, students were asked to reflect on how they monitored their problem solving efforts and to report on whether they felt they had improved in that regard. Liz, for example, used a maze metaphor to describe the relevance of setting subgoals during problem solving and the role of monitoring the process as a whole:

> Mathematical problem solving always involves a series of steps, or small solutions that eventually lead to the main solution of the entire problem. Finding these solutions is like traveling through a maze. With each new step a decision must be made on how to find the way to the following step … You could turn several corners (accomplish several steps successfully) and still find yourself at a deadend. In those cases the problem solver can either back-track going back one step at a time until he or she finds where another "turn" can be taken, or begin again.

Ellerton (2003) developed a modified version of the Garofalo and Lester's (1985) cognitive–metacognitive framework for analyzing students' reflections. Her framework comprised report, discussion, monitoring and self-evaluation. As students developed problem-solving skills and came to understand the role of reflection in their problem-solving experiences, they began to show evidence of higher metacognitive behaviors (monitoring and self-evaluation). In Reflection 4, for example, Trish noted that:

> By use of my metacognition, I remind myself to keep checking my solution with the problem to make sure I am doing it correctly. There is nothing worse than solving a very in-depth problem, and then realizing that you used the wrong information to solve it, making your solution incorrect … I also take a problem step-by-step when I am trying to solve it. There is nothing worse than rushing into it and then skipping over important information.

In Reflection 5, Trish built on her previous analysis as she evaluated herself in terms of her awareness of controlling her problem-solving process. She stated:

> When solving problems, I am aware of what I am doing to solve it. I am now able to know internally which attempts have worked and which have not while specializing, and why … Internal monitoring is a great discovery that I have made because it gives me a guide to problem solving and internal guide is what I have developed.

4.2.4 Relationships between students' views and corresponding metacognitive behaviors

Having access to a range of metacognitive behaviors allows an individual to reflect on the problem-solving process and to monitor and regulate performance. However, such access does not guarantee successful problem-solving attempts. Notwithstanding these notes of caution, the use and development of metacognitive behaviors can help problem solvers utilize resources efficiently, anticipate and tackle difficulties, think of multiple ways of attacking a problem and avoid arbitrary and fruitless approaches (Schoenfeld, 1985b).

Although metacognitive behaviors emerge and develop under favorable conditions (Kuhn, 2000), the results in this study demonstrate that the situation is more complex than is frequently assumed. Students, for example, did not exhibit consistent metacognitive behaviors across problems. Jackie, for example, stated that mathematics was challenging and that she liked the challenge. This was the reason she had majored in mathematics. However, Jackie was observed taking up the challenge in some problems, but *not* appreciating the challenges in other problems.

Trish was reflective in most cases and presented her views about problem solving and herself as a problem solver in thoughtful ways. Although she was active in group discussions, she was observed being rigid and unreasonable in the locker problem.

John's and Lynn's behaviors also demonstrated the complexity and inconsistency of emerging metacognitive actions. John, who had a strong mathematical background, commented in his reflections that he was patient and would persevere with problems in the hope of finding an elegant solution. But with the egg vendor problem, he was unwilling to be flexible and failed to solve the problem.

In contrast, he found an elegant solution for the bridge problem. Lynn, in her background survey, reported that she had decided to major in mathematics because she liked to understand and work on all problems, but in particular, on those on which most people got stuck. Later, in her final reflection, she stated:

> If I had trouble or was stuck on a task, I was not as aware of options to go through to "unstuck" myself. I am now more aware of processes to go through to solve a potentially difficult problem … Throughout the semester I have refined my ability to use or decide which strategy would be best for a particular situation.

However, despite her view about problem solving and herself as improving and "being aware", she did not justify why the lockers with square numbers would be open. In the bridge problem, she identified a global plan to use $d = rt$ and set up equations. She suddenly decided, though, that this was a problem that may not have a solution and gave up.

Students reported that they had grown during the semester in terms of their problem-solving behaviors. They stated that they had moved from lacking patience to being patient, from not staying on a task to being aware that problem solving takes much more time, and from giving up when stuck to trying out multiple strategies. Students also reported that they had grown with respect to reflecting and monitoring their problem-solving approaches. For example, John described how he had improved in developing his thought processes:

> I had a mathematics content exam this week. I was preparing myself for this exam. It was on applications of derivative to determine maxima and minima. I used to memorize all possible tests and conditions and try to decide which of the facts fit the problem. This last weekend, I thought differently. Instead of memorizing these facts, I tried to make sense out of them, how they are derived and why they make sense in maximizing and minimizing functions. It so happened that I see the secrets clearly and I was very much satisfied. I think this new behaviour must be the effect of the strategies I have acquired in this course. The procedure I used to memorize comes naturally without being memorized.

John's experiences in developing metacognitive strategies had been successfully applied to his studies outside the problem-solving course.

Some students were observed to analyze information correctly in such a way that their analysis led them to a solution plan and a solution. These same students, however, sometimes engaged in random guesses. For example, many of the students analyzed the information provided for the egg vendor problem and realized that they were looking for an odd multiple of 7. These same students were observed listing and checking *all* multiples of 7. Others, in the same problem, in addition to being aware that they were looking for odd multiples of 7, also observed that since 5 was involved, the number they were seeking should end in either 1 or 6. Instead of combining these ideas and devising a way to execute their plan, they engaged in listing and checking multiples of 7 that end in 6 that were supposed to be *excluded*. Goos (2002) named this as metacognitive mirage when students abandon a good idea without thoroughly analyzing it and without finding a good reason to abandon it. It should be recognized that many of the metacognitive approaches being adopted by students were *new* skills. A level of immaturity in the application of such emerging skills was almost inevitable.

Although students' reflective skills had improved, and although they were less easily frustrated when faced with unfamiliar problems, the metacognitive behaviors they employed while solving problems were not always consistent. A student who was metacognitively rich in one problem showed poor metacognitive skills at other times. Regardless of the difficulties in matching students' views with their metacognitive behaviors, however, during the semester, students' beliefs and views about problem solving became more open and their awareness and regulation of their problem-solving approaches more acute.

5 Conclusions

One of the goals of this study was to identify and characterize metacognitive behaviors that emerged during students' engagement in mathematical problem solving. Garofalo and Lester (1985) and Artzt and Armour-Thomas (1992) have presented extensive discussion and examples that characterize metacognitive behaviors as distinct from cognitive behaviors. Schoenfeld (1985b), although focusing on one aspect of metacognition—control—also addressed what metacognitive behaviors might look like. Geiger and Galbraith (1998), in an attempt to develop a diagnostic framework to evaluate students' approaches to solving problems, put forward a script analysis framework that encompassed engagement, executive behaviors, resources and beliefs.

Although these frameworks are all variations of Polya's model, they differ in technical ways depending on the authors' emphases. For example, rereading is believed to be a metacognitive behavior (Garofalo & Lester, 1985), but it is regarded as cognitive by Artzt and Armour-Thomas (1992). During the task-based interviews as well as small-group discussions in the current study, students were observed rereading the problem for different purposes. They were observed rereading to understand and analyze the problem, rereading to assess consistency of a plan with the conditions of the problem, and rereading to assess reasonableness of results obtained.

In this study, no attempt was made to label particular phases or sub-categories as either cognitive or metacognitive. The cognitive/metacognitive model proposed in Fig. 1 allows for this blurred distinction between cognitive and metacognitive behaviors by having separate phases and sub-categories that *together* can be used to describe each student's behavior while solving a mathematical problem. This approach allows for the possibility that, for some students, particular behaviors may be predominantly cognitive, while for others (or for the same student working through a different problem), the same behaviors may be predominantly metacognitive. Pathways between the different phases and sub-categories are frequently seamless, as a student moves from the first reading of a problem to its solution. That is not to say that the pathways are necessarily smooth, but rather that the students are able to make mental transitions between actions and reflection and back to actions again without apparent effort. The symbiotic relationship between metacognitive and cognitive behaviors proposed in this study is an attempt to model that seamlessness.

The five-phase model proposed in the current study, shown schematically in Fig. 5, has some of the characteristics of earlier models and frameworks. For example, the engagement phase corresponds to Garofalo and Lester's (1985) orientation category, to Artzt and Armour-Thomas's (1992) understanding and analyzing phase, and to Geiger and Galbraith's (1998) engagement and resources categories. The transformation-formulation phase corresponds to the planning and exploring phase of Artzt and Armour-Thomas (1992), in part to the organization phase of Garofalo and Lester (1985), the control phase of Schoenfeld (1985b) and the executive behaviors phase of Geiger and Galbraith (1998). The implementation phase corresponds to the implementing phase of Artzt and Armour-Thomas (1992), in part to the organization and execution phases of Garofalo and Lester (1985), the executive phase of Geiger and Galbraith (1998) and the control phase of Schoenfeld (1985b). The evaluation phase corresponds to the verification phase of Garofalo and Lester (1985), partly the control phase of Schoenfeld (1985b), the executive behavior phase of Geiger and Galbraith (1998) and the verifying phase of Artzt and Armour-Thomas (1992). Finally, the internalization phase includes some elements from the execution and verification phases of Garofalo and Lester (1985), the belief systems phase of Schoenfeld (1985b), the beliefs phase of Geiger and Galbraith (1998) and the exploring phase of Artzt and

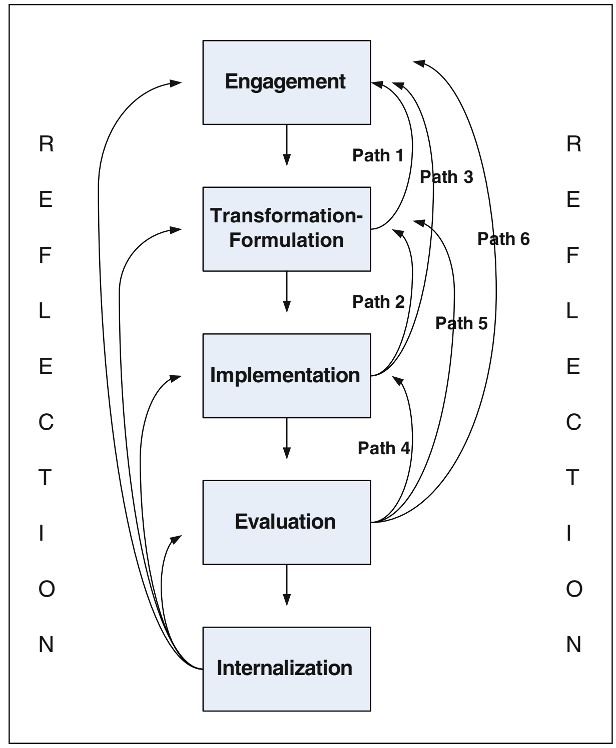

Fig. 5 Flowchart for the five-phase problem-solving model embedded in a context of reflection

Armour-Thomas (1992). It is, perhaps, not surprising that intervention programs aimed at enhancing metacognitive behaviors (see, for example, Mevarech and Kramarski, 1997; Mevarech and Fridkin, 2006) focus on characteristics closely aligned to those identified in the various models mentioned here.

The model described in Fig. 1, and shown schematically in Fig. 5, has several distinctive characteristics, which differentiate it from other models. First, and most importantly, reflection is an integral part of each category and of the entire model, and the model is portrayed in Fig. 5 as embedded in a context of reflection. Second, the last phase, internalization is not present in other models as a separate phase.

The internalization phase takes into account the degree of intimacy a problem solver has with the problem-solving process, in general, and the problem solver's search for elegance and extension, in particular. In this phase, problem solvers have the opportunity to reflect on the mathematical rigor of their solution, search for more elegant solutions and express their level of satisfaction with what they have achieved. They might also reflect on their confidence in handling similar problems. These reflections and inquiries take place only after the individual has solved the problem.

Evidence of internalization phases by problem solvers can therefore be regarded as reflecting the level of emerging metacognitive maturity. The engagement phase, for example, is not concerned with a mere reading and restatement of the problem. Rather, it requires problem solvers to analyze the problem thoroughly, relating it to a mathematical domain. As students analyze and reflect on the degree of familiarity and difficulty, they are forced to assess their conceptual background. The transformation-formulation phase is not just stating a plan. To begin with, a statement of plan does not take place automatically in non-routine problems. This phase then represents the gradual process that takes place when students coordinate the analysis they made at the engagement level when they tried out specific cases and made conjectures. Once a plan is stated, students are required to reflect on the feasibility of the plan before they act upon it.

The implementation phase does not simply involve embarking on a plan, but rather reflects regulatory moves in terms of exploring more of the key features of the plan with respect to the conditions of the problem. This phase, like the previous ones, requires students to make continuous assessments of their knowledge base.

In the evaluation phase, data from the previous phases and decision making are considered. Although the evaluation phase follows the implementation phase for the sake of presentation, it can take place after any phase and at any time. Sound evaluations are essential if appropriate decisions are to be made.

The internalization phase goes beyond reflecting on a particular problem, beyond reflecting on a plan, and beyond reflecting on implementation and on the plausibility of decisions made. Rather, internalization takes place when students' emerging metacognitive qualities help them assimilate the paths they took through the previous phases. Internalization can occur when students assess their confidence and levels of satisfaction with the problem-solving process.

Not every student showed evidence of each of the sub-components in every phase. A student may exhibit one sub-component in a phase, but may not exhibit other sub-components or phases. No one student was found to exhibit all of the phases and their corresponding sub-components for a single problem-solving task.

A key aspect of the model is that it can take account of various pathways between the phases. In other words, this is *not* a linear model, and a student may move back and forth through several phases without attaining the internalization phase (see Fig. 5). Some students can reach the internalization phase without passing through any of the other phases. This latter scenario, however, needs to be investigated further as it is possible either that these students had not made any of their thinking apparent during this problem-solving session, or that the problem context involved a known problem type for these particular students at that time. Some students may not exhibit some of the phases beyond the engagement phase and may never reach the internalization phase. This may be due to the nature of the problem, since some problems may be solved by analyzing the information without formulating a plan.

The five-phase model discussed in this paper represents a bringing together of the work of many researchers from many parts of the world, seeking to clarify what many of us do every day: solving non-routine mathematics problems. In particular, the fundamental work of Polya (1957), the development of Polya's ideas by Schoenfeld (1985a) coupled with the pioneering error analysis work of Newman (1977) and Casey (1978), and more recently the role of reflection in "giving students permission" to think about their approaches while they solve mathematics problems (Ellerton, 2003), have helped bring to convergence the five-phase model discussed here. This model reflects the synergies evident between metacognitive and cognitive actions, and attempts to capture the flexibilities and variations in approaches and pathways taken by different students as they work on the same problem, as well as the pathways taken by the same student to tackle different problems. Future research needs to continue to focus on "problem-solving-in-action," and how teachers' ways of working might facilitate students' cognitive and metacognitive development.

References

Artzt, A. F., & Armour-Thomas, E. (1992). Development of a cognitive–metacognitive framework for protocol analysis of mathematical problem solving in small groups. *Cognition and Instruction, 9*, 137–175.

Artzt, A. F., & Armour-Thomas, E. (1997). Mathematical problem-solving in small groups: Exploring the interplay of students' metacognitive behaviours, perceptions, and ability levels. *Journal of Mathematical Behavior, 16*, 63–74.

Berardi-Coleta, B., Dominowski, R. L., Buyer, L. S., & Rellinger, E. R. (1995). Metacognition and problem solving: A process-oriented approach. *Journal of Experimental Psychology, 21*, 205–223.

Carlson, M. P. (2000). A study of the mathematical behaviour of mathematicians: The role of metacognition and mathematical intimacy in solving problems. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics* Education (Vol. 2, pp. 137–144). Hiroshima, Japan: PME.

Carlson, M., & Bloom, I. (2005). The cyclic nature of problem-solving: An emergent multi-dimensional problem-solving framework. *Educational Studies in Mathematics, 58*, 45–75.

Casey, D. P. (1978). Failing students: A strategy of error analysis. In P. Costello (Ed.), *Aspects of motivation* (pp. 295–306). Melbourne: Mathematical Association of Victoria.

Chicola, N. A. (1992). *The effect of metacognitive strategy instruction on sixth-graders' mathematics problem-solving ability.* Unpublished doctoral dissertation, University of Colorado.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*, 175–190.

Crawford, P. (1998). *Fostering reflective thinking in first-semester calculus students*. Unpublished doctoral dissertation, Western Michigan University, Kalamazoo.

Davidson, J. E., Deuser, R., & Sternberg, R. J. (1994). The role of metacognition in problem solving. In J. Metcalf & A. P. Shimamura (Eds.), *Metacognition* (pp. 207–226). Boston, MA: The MIT Press.

DeBellis, V. A. (1998). Mathematical intimacy: Local affect in powerful problem solvers. In S. Berenson, et al. (Eds.), *Proceedings of the 20th annual meeting of PME-NA* (Vol. 2, pp. 435–440). NC: Raleigh.

DeFranco, T. C., & Hilton, P. (1999). Distinguishing features of mechanical and human problem solving. *Journal of Mathematical Behavior, 18*, 79–84.

Ellerton, N. F. (2003). Metacognitive benefits accruing from teacher education students' reflections on problem solving. In H. S. Dhindsa, L. S. Bee, P. Achleitner, & M. A. Clements (Eds.), *Studies in science, mathematics and technical education* (pp. 202–211). Gadong, Brunei Darussalam: Universiti Brunei Darussalam.

Flavell, J. H. (1976). Metacognitive aspects of problem solving. In L. B. Resnick (Ed.), *The nature of intelligence*. Hillsdale, NJ: Lawrence Erlbaum.

Flavell, J. H. (1987). Speculations about the nature and development of metacognition. In F. E. Weinert & R. H. Kluwe (Eds.), *Metacognition, motivation, and understanding* (pp. 21–29). Hillsdale, NJ: Lawrence Erlbaum.

Garofalo, J. (1989). Beliefs and their influence on mathematical performance. *Mathematics Teacher, 82*, 502–505.

Garofalo, J., & Lester, F. K. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education, 16*, 163–176.

Geiger, V., & Galbraith, P. (1998). Developing a diagnostic framework for evaluating student approaches to applied mathematics. *International Journal of Mathematics, Education, Science, and Technology, 29*, 533–559.

Goldin, G. A. (1988). Affective representation and mathematical problem solving. In M. J. Behr, C. B. Lacampagne, & M. M. Wheeler (Eds.), *Proceedings of the 10th annual meeting of PME-NA* (pp. 1–7). DeKalb, IL: University of Northern Illinois.

Goos, M. (2002). Understanding metacognitive failure. *Journal of Mathematical Behavior, 21*, 283–302.

Goos, M., & Galbraith, P. (1996). Do it this way! Metacognitive strategies in collaborative mathematical problem solving. *Educational Studies in Mathematics, 30*, 229–260.

Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics, 49*, 193–223.

Kramarski, B., Mevarech, Z. R., & Arami, M. (2002). The effects of metacognitive instruction on solving mathematical authentic tasks. *Educational Studies in Mathematics, 49*, 225–250.

Krathwohl, D. R., Bloom, B. S., & Masia, B. B. (1964). *Taxonomy of educational objectives: Handbook II: Affective domain*. New York: David McKay Company Inc.

Kuhn, D. (2000). Metacognitive development. *Current Directions in Psychological Science, 9*, 178–181.

Lester, F. K. (1985). Methodological considerations in research on mathematical problem solving instruction. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 41–69). Hillsdale, NJ: Lawrence Erlbaum.

Lester, F. K. (1994). Musings about mathematical problem-solving research: 1970–1994. *Journal for Research in Mathematics Education, 25*, 660–675.

Lin, X., Schwartz, D. L., & Hatano, G. (2005). Toward teachers' adaptive metacognition. *Educational Psychologist, 40*, 245–255.

Marcus, A. (2007). Coding strategic behavior in mathematical problem solving. In D. Kuechemann (Ed.), *Proceedings of the British Society for research into learning mathematics* (Vol. 27, pp. 54–59).

Marge, J. J. (2001). *The effect of metacognitive strategy scaffolding on student achievement in solving complex math word problems.* Unpublished doctoral dissertation, University of California, Riverside.

Martinez, M. E. (2006). What is metacognition? *Phi Delta Kappan*, pp. 696–699.

Maykut, P., & Morehouse, R. (1994). *Beginning qualitative research: A philosophic and practical guide*. London: The Falmer Press.

McLeod, D. B. (1989). The role of affect in mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 20–36). New York: Springer.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualisation. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–596). New York: Macmillan.

McLeod, D. B., Craviotto, C., & Ortega, M. (1990). Students' affective responses to non-routine mathematical problems: An empirical study. In G. Booker, P. Cobb, & T. N. de Menicuti (Eds.), *Proceedings of the fourteenth international conference on the psychology of mathematics education* (Vol. 1, pp. 159–166). Mexico: PME.

Mevarech, Z. R., & Fridkin, S. (2006). The effects of IMPROVE on mathematical knowledge, mathematical reasoning and metacognition. *Metacognition and Learning, 1*, 85–97.

Mevarech, Z. R., & Kramarski, B. (1997). IMPROVE: A multidimensional method for teaching mathematics in heterogeneous classrooms. *American Educational Research Journal, 34*, 365–395.

Mildren, J., Ellerton, N., & Stephens, M. (1990). Children's mathematical writing: A window into metacognition. In K. Clements (Ed.), *Whither mathematics* (pp. 356–363). Melbourne: Mathematics Association of Victoria.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Newman, M. A. (1977). An analysis of sixth-grade pupils' errors on written mathematical tasks. *Victorian Institute for Educational Research Bulletin, 39*, 31–43.

Newman, M. A. (1983). *Strategies for diagnosis and remediation*. Sydney, Australia: Harcourt, Brace Jovanovich.

Polya, G. (1957). *How to solve it*. Princeton, NJ: Lawrence Erlbaum.

Pugalee, D. K. (2001). Writing, mathematics, and metacognition: Looking for connections through students' work in mathematical problem solving. *School Science & Mathematics, 101*, 236–245.

Robert, S., & Tayeh, C. (2007). It's the thought that counts: Reflecting on problem solving. *Mathematics Teaching in the Middle School, 12*, 232–237.

Schoenfeld, A. H. (1985a). Making sense of "out loud" problem solving protocols. *The Journal of Mathematical Behavior, 4*, 171–191.

Schoenfeld, A. H. (1985b). *Mathematical problem solving*. New York: Academic Press.

Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In *Cognitive science and mathematics education* (pp. 189–215). Hillsdale, NJ: Lawrence Erlbaum.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grows (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York: Macmillan.

Schoenfeld, A. H. (2006). Problem solving from cradle to grave. *Annales de Didactique et de Sciences Cognitives, 11*, 41–73.

Schoenfeld, A. H. (2007). Problem solving, teaching, and more: Toward a theory of goal-oriented behavior. In: *Proceedings CIEAM* (Vol. 59, pp. 48–52). Debogoko-Hungary.

Silver, E. A. (1987). Foundations of cognitive theory and research for mathematics problem-solving instruction. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 33–60). Hillsdale, NJ: Lawrence Erlbaum.

Simon, H. A. (1973). The structure of ill-structured problems. *Artificial Intelligence, 4*, 181–202.

Sriraman, B. (2003). Mathematical giftedness, problem solving, and the ability to formulate generalizations: The problem solving experiences of four gifted students. *The Journal of Secondary Gifted Education, 14*, 151–165.

Stillman, G. A., & Galbraith, P. L. (1998). Applying mathematics with real-world connections: Metacognitive characteristics of secondary students. *Educational Studies in Mathematics, 36*, 157–195.

Veenman, M. V. J., Van Hout-Wolters, B. H. A. M., & Afflerbach, P. (2006). Metacognition and learning: Conceptual and methodological considerations. *Metacognition and Learning, 1*, 2–14.

Willburne, J. M. (1997). *The effect of teaching metacognition strategies to preservice elementary school teachers on their mathematical problem-solving achievement and attitude.* Unpublished doctoral dissertation, Temple University, PA.

Yimer, A. (2004). *Metacognitive and cognitive functioning of college students during mathematical problem solving*. Unpublished doctoral dissertation, Illinois State University, IL.

Yimer, A., & Ellerton, N. F. (2006). Cognitive and metacognitive aspects of mathematical problem solving: An emerging model. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures, and learning spaces* (pp. 575–582). Adelaide, Australia: Mathematics Education Research Group of Australasia.