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ORIGINAL ARTICLE

# Potential scenarios for Internet use in the mathematics classroom

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**Abstract** Research on the influence of multiple representations in mathematics education gained new momentum when personal computers and software started to become available in the mid-1980s. It became much easier for students who were not fond of algebraic representations to work with concepts such as function using graphs or tables. Research on how students use such software showed that they shaped the tools to their own needs, resulting in an intershaping relationship in which tools shape the way students know at the same time the students shape the tools and influence the design of the next generation of tools. This kind of research led to the theoretical perspective presented in this paper: knowledge is constructed by collectives of humans-with-media. In this paper, I will discuss how media have shaped the notions of problem and knowledge, and a parallel will be developed between the way that software has brought new possibilities to mathematics education and the changes that the Internet may bring to mathematics education. This paper is, therefore, a discussion about the future of mathematics education. Potential scenarios for the future of mathematics education, if the Internet becomes accepted in the classroom, will be discussed.

An earlier and significantly different version of this paper was presented as the Keynote Address at the Fields Symposium on Digital Mathematical Performance, June 2006, available at http://www.edu.uwo.ca/dmp, and was also published in the proceedings of the conference under the title “Humans with Media: a performance collective in the classroom?” (Borba, 2007).

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## 1 Introduction

It has been almost 30 years since the discussion on multiple representations gained the main stage in mathematics education. Many authors (e.g., Kaput, 1989) argued that the availability of multiple representations of a given mathematical object, such as function, was transforming the way students could learn mathematics. In a more general sense, authors such as Borba and Villarreal (2005) have stressed, to different degrees, how the use of different media changes mathematics itself. Kieran and Yerushalmy (2004), in an extensive review of the role of technological environments in algebra teaching and learning, show how different softwares have changed the nature of curriculum in the mathematics classroom. They show how the use of spreadsheets, e.g., changes the focus of part of the mathematics curriculum from the notion of variables to the notion of unknown, and how the use of software packages such as Visualmath can make the teaching of algebra more function oriented. Many authors included in their review, as well as another prepared by Ferrara, Pratt and Robutti (2006), illustrate how it is now common in some classrooms in countries from different parts of the world (e.g., Belgium, Brazil, France, Italy, Japan, and New Zealand) to use simulation and visualization in mathematics to establish conjectures and verify results. However, the literature cited does not offer insights into how widely spread computers are in classrooms that are neither the focus of research nor the implementation of such computer-based approaches. Computers have become important actors in some branches of mathematics as well, and centers in Canada and the United Kingdom have been devoted to experimentation in mathematics. The Centre for Experimental and Constructive Mathematics (http://www.cecm.sfu.ca), e.g., has been advocating an important role for experimentation in mathematical heuristics for more than 10 years.

As different educational software designs became available in the 1980s, different mathematical activities could be developed in the mathematics education research community. The classic examples are related to the teaching and learning of geometry (Laborde, 1998), functions, and calculus (Confrey & Smith, 1994). The availability of dynamic geometry software can transform the types of tasks that can be developed in the classroom (Marrades & Gutierrez, 2000; Arzarello et al., 2002; Arzarello & Edwards, 2005; Mariotti, 2002; Ferrara, Pratt & Robutti, 2006; Laborde et al., 2006). For instance, problems can be assigned in a way that encourages students to try out different constructions and verify whether there are invariants after dragging a figure. The coordination of multiple representations as a way of knowing different aspects of functions and other calculus topics was also transformed as different softwares became available. Confrey and Smith (1994) argued about the importance of the software design for learning. In the case they analyzed, they showed how the design of *Function Probe* (Confrey, 1991), which allows for a graph of a function to be dragged, offered students new possibilities for coordinating representations. Being able to drag a graph and see the change in a table of $x$–$y$ values of the original function was a way of bringing the transformation of functions to the forefront. The influence of humans in the design of software shapes the way other humans learn using that software.

More recently, Borba (2004) expanded on these ideas with the conjecture that mathematics also becomes transformed when one moves from a face-to-face context to online distance courses for teachers. For instance, there are online environments in which writing in a chat or a forum is the only way for participants to communicate among themselves. Analysis of an online course offered for mathematics teachers showed how writing shapes the mathematics discussed in this context. A comparison was made to show differences and similarities between solutions to a similar problem presented in a face-to-face classroom and an online course. The analysis was carried out based on the idea that when technology changes, the possibilities for mathematics are also altered. Many years of research with software and with the Internet have also led to developments in theoretical discussions about epistemology.

The idea that computers influence knowledge and the way we know led to the notion that other technologies, such as writing or even orality, influence the way one knows, as well. From such a perspective, the technology that surrounds us plays an active role in the process of knowing. These ideas led to the notion that knowledge is constructed by collectives of humans-with-media. Knowledge is not only produced by humans, but also by different media, such as orality, writing, or the new modalities of language that emerge from computer technology:

> We believe that humans-with-media, humans-media or humans-with-technologies are metaphors that can lead to insights regarding how the production of knowledge itself takes place (…). This metaphor synthesizes a view of cognition and of the history of technology that makes it possible to analyze the participation of new information technology 'actors' in these thinking collectives[1] (Borba & Villarreal, 2005, p. 23).

In such a perspective, knowledge is constructed by collectives of humans-with-media, in contrast to views that hold that the epistemological subject is one single person or collectives of humans. For instance, some members of our research group, GPIMEM,[2] have applied this notion to study different kinds of interactions that take place in online courses that employ chat and no form of orality, and have found that this transforms the nature of the mathematics produced in such environments (Borba, Malheiros & Zulatto, 2007). We have also shown how the possibilities for doing mathematics change in an online course when the interface changes from a chat to a videoconference, for example. Our research suggests that different interfaces change the nature of humans-with-Internet collectives in ways that are similar to classical examples of how the presence of software in the classroom may change the nature of the teaching of geometry, functions, or calculus. In this article, a different possibility of the theoretical construct humans-with-media will be explored. I will look into the future and try to study what could but does not yet exist (Skovsmose & Borba, 2004), as opposed to studying what does exist. More specifically, a discussion will be made about how the mathematics classroom can be transformed if the Internet is used intensively in education.

In a thorough review about technological environments and algebra teaching and learning, Kieran and Yerushalmy (2004) assure us that

[1] Thinking collective is a term used by Lévy to emphasize that knowledge is produced by collectives composed of human and non-human actors.

[2] Computer Technology, other Media and Mathematics Education Research Group (Sao Paulo State University, Department of Mathematics, Rio Claro, SP, Brazil). http://www.rc.unesp.br/igce/pgem/gpimem.html.

> even though computers have been around since the late 1940s, it was only in the late 1960s that mathematicians and mathematics educators began to feel that computing could have significant effects on the content and emphases of school level and university level mathematics (Fey, 1984). Early visionaries soon saw the ways in which computing technology could be harnessed in order to more fully integrate the multiples representation of mathematical objects in mathematics teaching (p. 100).

Fifty years later, it can be said that multiple representations, in particular, and computers, in general, have entered the educational research scene and made inroads into some "pockets", albeit in an unequal fashion, in different parts of the world. While the use of geometry and function software is common in many classrooms in the world, there are also many schools without computers. A look at the book edited by Atweh et al. (2007) gives us a window into the inequity in mathematics education related not only to current access to technology, but also to other factors. It was important, however, that the early visionaries foresaw possible ways to use computers to change mathematics teaching and learning, because otherwise computers may never have come to be used in education at all. In this paper, I would like to envision what the mathematics classroom could look like if the Internet were to in fact enter the mathematics classroom; or more appropriately, if the Internet were to permeate the classroom via wire and wireless connections.

I will begin with a discussion regarding how different media have shaped the notions of problem and knowledge, followed by a discussion of how technology, thinking, and mathematics education are intertwined, with a particular focus on the relationship between the Internet and mathematics education. The relationship between modeling (an established trend in mathematics education) and use of the Internet will be addressed, as well as new possibilities for mathematics education that draw on the performance arts. Afterward, I will engage in an exercise of looking into the future to envision how modeling and digital mathematical performance may present alternatives to the way the notion of problem is currently addressed in classrooms without the Internet. A new notion of problem in which students are the proponents may be important in schools where the Internet is allowed in the classroom and responses to standard, traditional problems are easily found.

## 2 Intershaping relationship: technology and cognition

The relationship between cognition and technology has been under discussion for some time, but it was transformed substantially as computers became available for educational purposes. Many mathematics educators have discussed this issue since the mid-1980s and more intensively in the 1990s. Reviews of much of the literature on this theme can be found not only in the two reviews already presented but also in the special issue of *Educational Studies in Mathematics* in honor of Jim Kaput (Hegedus & Lesh, 2008). Hoyles and Noss (2008), e.g., stress that "Kaput's abiding focus was on the hegemonic role of algebraic expression as a means to express mathematical structure in pedagogical settings" (p. 89). Kaput's view was also shaped by the emergence of computers and, at the same time, influenced software designers (Tall, 1991). Jere Confrey (1991), for instance, the main developer of Function Probe, emphasized the role of multiple representations as a means to show, among other things, that algebraic expression was just one among many possible representations (Confrey, 1994). In this kind of environment, a considerable amount of research was developed in the teaching experiment tradition (Cobb & Steffe, 1983; Steffe & Thompson, 2000), in which students' actions with software could be closely observed and models constructed about the way they were thinking. I was a member of the Function Probe design team and was able to observe the aims of the design team when they created a new item on the toolbar menu (Borba, 1993). I was able to observe how the design impregnated in a tool shaped the way students think about functions. For instance, the fact that this software allowed for direct manipulation on the graph made it possible to challenge the predominance of algebra discussed by Jim Kaput. Students transformed graphs of parabolas and then studied what kind of change such a transformation would generate in tables and in algebraic representations of the function being studied. At the same time, students would shape the software tool in ways unanticipated by the design team (Borba, 1993). The use of the edge of the screen or the imagination of features which were not present in the toolbar menus of the software shaped the software in ways that were appropriate for the user and also influenced the design of the next version of the software. It was possible to conclude based on the study that:

> Multi-representational approaches and the relationships among representations shape students' views of function just as a "monolithic" algebraic view or the "unidirectional view" (algebra → graph) do, though in different ways. In this sense, the paper and pencil environment and the algebra → graph approach that have reigned in the study of transformations influence the constructions students make of transformations of functions, just as a multi-representational approach in a computer environment does. But students are active in

> the construction and reconstruction of representations and therefore they do not "swallow" the pre-fabricated representation of others. Instead they "shape" the purposefully designed representations in Function Probe in accordance with their experience.
> The cycle between representations and students' constructions could be endless in this "inter-shaping" relationship. This cycle might take different directions as a student finds new relationships that allow him/her to "see" something s/he could not see before, when a new representation is created or new relationships among and within representations are created, or when a new medium is introduced into the process. Since representations are always associated with a kind of medium, it can be said that media are also part of the "inter-shaping" relationship (Borba, 1993, p. 8).

Intershaping relationship (Borba & Confrey, 1996) was the expression created to characterize the shaping of students' cognition by a software tool and the way the students appropriated the tool for their own purposes. This relationship between tools and ways of knowing has recently been described using expressions such as instrumentation and instrumentalization (Guin, Ruthven & Trouche, 2005), and has also had consequences for pedagogy. The availability of software, and epistemological perspectives, such as the one described above, have resulted in changes in curriculum, as documented by authors such as Moreno-Armella, Hegedus, and Kaput (2008) as they looked back at changes in curriculum that were related to the availability of software.

The notion of intershaping relationship helped make the mutual influence of software and cognition visible, but it also helped to shed light on the role of other tools, such as paper-and-pencil. Borba (1993) related his findings to oral mathematics found in slums in Brazil and normal algebraic developments to paper-and-pencil. Since 1993, these ideas have been developed drawing on the work of Pierre Lévy (1993) and Oleg Tikhomirov (1981).

In Lévy's discussion of the history of media, he identifies three main eras. From this perspective, orality, writing, and informatics are qualitatively different ways of extending our memory, and once they are put into practice, they also help to shape humans. For instance, orality shapes the knowledge that is impregnated in the myths of societies who have not developed writing. Borba and Villarreal (2005) have proposed that research in ethnomathematics would acquire new insights, and the mathematics produced by certain cultural groups could gain new status, if their mathematics were seen as shaped by a media such as orality rather than characterized as lacking writing.

Lévy (1993) analyzed writing as a way of extending memory. Writing and the linearity that it allows helped to condition the way humans think. Reasoning could be "frozen" into a book in a different way than it could in a myth. Reasoning could become linear, and demonstrations in mathematics could become more detailed, not only with the availability of writing but also with its popularization (Lévy, 1998). The availability of cheap paper and printing devices in the last 300 years made books more affordable for a larger number of people.

Computers brought a qualitatively different kind of memory extension that not only made it possible to store a larger amount of information in less space, but also made new forms of discourse easily accessible. Multimodality (Kress, 2003) manifested in hypertext changed the nature of language, as it combined figures, video, and animations with regular text. More recently, with the availability and popularization of the Internet, computers have acquired another feature that has changed the nature of memory extension: interactivity. More than any other medium before it, the Internet has combined new forms of discourse, even greater storage capacity, and communication capability. Computers have become media! One may argue that multimodality was not born with the Internet. For example, the way gestures are combined with language in our everyday communication is a pre-Internet example of multimodal discourse. Another example, as discussed in this paper, is the way function and geometry software has changed the nature of investigation in the classroom. It has increased the role of visualization considerably, creating a multimodal communication in the mathematics classroom that combines graphs, algebra, tables and manipulation of figures in a qualitatively different way compared to the way graphs and tables were used when the blackboard was the only medium (Borba & Villarreal, 2005). The Internet as it is understood today has again transformed this medium named computer. Multimodal discourse can now easily incorporate pictures, movies, and music in a way that was not possible before. This new phenomenon, which is analyzed by authors such as Hughes (2007), among others, offers new possibilities for mathematics education. Combining these new possibilities with what is currently available in mathematics education is a necessity today in the same way that it was for some of the "visionaries" who began transforming the mathematics classroom with activities for software such as Logo and dynamic geometry software (see Kieran & Yerushalmy, 2005; Laborde et al., 2006, for a review on this issue).

Outside the field of mathematics education, a Vygotskian psychologist was also envisioning the future of the relationship between cognition and computers. In a paper ahead of its time, written in 1972, Oleg Tikhomirov (1981) described the qualitative difference of informatics compared to regular language. Although personal computers, fast computer chips, and the Internet were yet to be invented, he proposed that computers could influence cognition differently than regular language. As a former student of Vygotsky, he established a parallel with his work and proposed that informatics would reorganize human thinking, not substitute or juxtapose it.

He strongly argued that the view that computers can substitute human thinking, or that computers and humans have separate and distinct (juxtaposed) roles in cognition, could not hold unless cognition is seen as divided into small fragmentary pieces that are value-free. If cognition is considered to be information-processing, one can imagine computers substituting humans in carrying out this process of dividing thinking into small pieces and manipulating them. However, if cognition is seen as involving complex heuristic processes, as well as other variables such as values that drive the process, it is not possible to conclude that computers substitute humans. The possibility that computerized devices do not interfere in thinking is also ruled out. As a follower of Vygotsky, Tikhomirov was already aware of the way language interacts with thinking, and he proposed that computers interact with thinking.

He proposed that computers, with their qualitatively different way of extending human memory, reorganize thinking in a way that language cannot. Tikhomirov's (1981) predictions regarding the way computers would influence thinking were made at the time when computers were room size. With the arrival on the scene, in the 1980s, of personal computers and interfaces such as keyboards, mouse, and screens, and their relative ubiquity in our lives today, it can be argued even more strongly that computers reorganize thinking.

Our research has been informed by such a view and has contributed to its further development. The interactions made possible by the interfaces mentioned above, and those made possible by the Internet, have changed the nature of communication and the very notion of what being human means. It is very hard to think about humans without electricity at the turn of the last century, and it would be very hard to think about humans today without computer technology impregnating those thoughts. Hunting is no longer an essential part of being human, but searching for information is becoming more and more a part of the collective notion of human that is evolving at the beginning of the twenty-first century.

We have been taking these ideas into the epistemological debate, proposing that knowledge is produced by collectives of humans-with-media (Borba, 1999; Borba and Villarreal, 2005). This notion stresses the fact that every technology of intelligence (Lévy, 1993) interacts with human thinking, and that knowledge has been shaped historically by humans and by the media being used. Knowledge is historically bounded because both humans and media, co-actors in its production, are historically situated.

This view of how knowledge is produced can inform changes in the classroom, especially if it is taken into account that students arrive at school at different levels, with no "accent" as far as use of computers and the Internet is concerned. As the reader will see, the "accent" metaphor will be discussed further later in this paper.

Many social psychologists have discussed how the idea of an individual knower is problematic as it does not consider the role of other people and overemphasizes the role of the individual. These psychologists, strongly influenced by Vygotsky, see the role of other artifacts as mediating tools. As discussed in detail in Borba and Villarreal (2005), the Piaget × Vygotsky debate in the United States in the early 1990s had a great influence on the notion of intershaping relationship, proposing that tools were shaped by users while at the same time helping to shape the thinking of the users. Further contact with authors such as Tikhomirov and Lévy helped us to see that considering tools as only mediators was a way of maintaining the human as the basic unit of knowledge production. Some of us are building an alternative perspective of knowledge in which the very notion of what human means is impregnated by technology. Technology does not exist without humans, and the notion of human commonly held, at least in modern history, does not exist without technology as well.

The notion of humans-with-media expresses this mutual dependency and calls attention to the following: that the way knowledge is produced, and whether different results are considered true or not at a given moment, depend on social agreements. For mathematicians such as Bicudo (2002), a demonstration in mathematics is not the result of pure logic but rather must be agreed upon by many mathematicians in order to be considered a valid result in mathematics. In Borba and Villarreal (2005), media, such as orality, paper-and-pencil, and computers, are also seen as co-actors in different ways of validating different results in different communities. In particular, we show how paper-and-pencil are co-actors in demonstrations. In the mathematics classroom, many of us in our research group GPIMEM have shown how different pieces of technology have become co-actors in the production of knowledge, in the way communication happens, and so on. In this paper, I will use the construct of humans-with-media to show how the Internet may "deconstruct" the structure of curriculum if the Internet becomes part of the mathematics classroom. If the kinds of problems that traditionally compose the mathematics curriculum become obsolete, what might be next? What kinds of activities might replace them?

## 3 Problem and media

The above discussion about the notion of humans-with-media alters the way computers impact the production of knowledge and the way people collaborate (Borba & Villarreal, 2005; Borba & Zulatto, 2006). I believe that the very notion of problem is transformed by media. Saviani (1985) and Borba (1994) stated some time ago that a problem has both an objective and a subjective part. The objective part is an obstacle that emerges within the life experience. However, an obstacle can pass unnoticed by one person and provoke considerable interest in another, and this is what I refer to as the subjective part. I believe it is time to expand this discussion to emphasize that a problem also depends on the medium that is available in a given collective of humans-with-media. Studying function using only paper-and-pencil is one thing, and with a graphing calculator another. Many who studied function without access to computers could consider an obstacle, in the sense proposed above, to be "graph $y = x^2 - 5x + 6$".

Of course, this is hardly a problem for collectives that include graphing software, since there is no obstacle to be overcome. As is well known, a 'typical problem' for collectives of humans-with-graphing-calculators is to explore relationships of graphs, tables and algebra as "$a$" varies in the equation $y = ax^2 + bx + c$, keeping "$b$" and "$c$" constant. I do not see problems as being inherently good or bad, but rather as conditioned by different media available. What is perceived as an obstacle, and what provokes interest, also depends on the media available. Subjectivity and objectivity are merged even closer by the notion of humans-with-media and the way the notion of problem can be transformed by it. In this example, there is another aspect to be considered: problems like the one above, with open possibilities of answers, are more likely to challenge the students if media, like graphing software, are available.

In the schools, however, even though calculators or software are allowed in the classroom, their use is not likely to be permitted during examinations. In particular, access to the Internet may not be allowed based on arguments such as: students can find the answers to the problems given; students may get distracted; or it will favor students who know how to navigate better on the Internet. The above arguments are just possibilities, based on similar arguments that were presented in the past when calculators and software were first introduced as actors in the mathematics classroom. However, they are also based to some degree on current practices: some of the few universities that do accept use of the Internet in class do not accept them in exams. But in an attempt to look into the future, the question can be raised: What will become of the notion of problem if the Internet is fully allowed in the classroom? What kind of problems will be posed to collectives of humans-with-Internet?

It is very likely they will be quite different from those posed to collectives that include only paper-and-pencil as media, not only because the answer can be easily found on the Internet, but because the Internet is also shaping the way humans organize and see things. Searching for something on the Internet can be more motivating for kids who belong to the "Internet generation". As I have discussed elsewhere (Borba, 2004), more and more kids do not see the Internet as something new; they were born with it, and they have no 'accent' when speaking and listening to Internet language. There is a new language being generated on the Internet, and most teachers learned this language when they were no longer children, and probably speak it with a strong accent, it not being their "mother tongue". "Soon generations will be arriving in the schools, and next the universities, who have no accent, who, born, raised, and impregnated with 'Internet-ized', hyper-media texts, will expect educators to speak their language" (Borba, 2004, p. 314). So I believe problems like those given in traditional paper-and-pencil environments will have to be, at a minimum, transformed into questions such as "Why does the graph of $y = x^2 - 5x + 6$ look like this?"

The literature already analyzed in this paper (e.g., Kieran and Yerushalmy, 2004; Laborde et al., 2006) regarding the use of software in the classroom has provided considerable evidence regarding how the nature of a given task may change once these software become part of a collective of humans-with-media. It is time for all of us to start thinking about problems that would be appropriate for classrooms characterized by intensive presence of the Internet. Although not all mathematics classrooms currently have access to computers, for different reasons, it is reasonable to admit that hardly any classroom would have access if the early visionaries had not started thinking about how to use computer technology in the classroom a long time ago.

One way of thinking about possible mathematical problems in a classroom where students are allowed to use the Internet is to transform the elaboration of a problem into a problem itself. Proposals of this type have been developed in Brazil and elsewhere under the titles of project work or modeling. In such pedagogical approaches, students are invited to choose a theme to study, to generate a question to be investigated, and elaborate an answer. Students are, in this case, exploring part of the same process experienced by researchers. The Internet, in this case, is well-suited to such approaches, and many researchers in Brazil, such as Diniz (2007), Borba and Malheiros (2007), and Borba and Villarreal (2005) have documented and classified ways in which students used the Internet to develop their projects. The Internet allows students to investigate themes that are not yet available in books. For example, students elaborated a project about mad cow disease when it first arose in England, and some used the Internet to compare findings from their mathematical modeling activities with those developed by professional researchers. Students who work during the day and attend classes in the evening have used the Internet as a means of communication, as they have very little free time to meet outside the classroom. In our research group, we have also discussed what kinds of mathematics emerge from these projects, which are often interdisciplinary. A pedagogical approach that emphasizes students' choice of the theme to be studied and breaks from traditional didactical contracts in the classroom may be suitable for a classroom that embraces the Internet.

In his research, GPIMEM member Diniz (2007) details the way a group of students that has chosen termites as their theme turns first to books and then to the Internet as their main source of data. They also use the Internet to contact professors and display their results. In a study conducted by Malheiros (2004), another aspect of the use of the Internet in modeling was identified: students used it to verify whether or not the results of a "short-quick" biological experiment they performed were in agreement with more thorough scientific experiments. Discrepancies were found, and they discussed possible explanations in their project reports.

Both studies mentioned above were conducted in classes where the 30% of the final grade was based on assessment of the modeling project. Thus, although the Internet was used mainly outside the classroom, it gained partial admittance into the classroom. Another example of an approach being used that illustrates how the Internet is making inroads into the classroom, and its potential for mainstream classroom use in the future, is digital mathematical performance (Gadanidis, 2006; Gadanidis & Borba, 2008). The educational activity involving use of the Internet allows young people to combine exploration of the arts and mathematics, and also engages them in the role of performers.

Authors such as Boal (1979), an important Brazilian playwright, introduced alternative forms of theater and performing arts that emphasize the end of the separation between actors and audience. In this conception of theater, these roles, which are traditionally well-defined, become blurred. Actors emerge from the audience; audience and actors share the same space. Actors perform as traditionally expected, but spectators can also influence the events of the play, and even perform. We have drawn on Boal's work for inspiration to think about digital mathematics performance. In this way, the rapid communication made possible by the Internet, and the possibility of everyone "publishing" their work, can be used as means to invite students to become more active in the learning process. In this scenario, students would be encouraged to produce mathematics as they do art; they would perform on the Internet, expressing their understanding of mathematics in a digital mathematical performance.

Another source of inspiration for the notion of digital mathematics performance is Kress (2003), who has emphasized that radical changes are taking place with the increasing ubiquity of multimodal communication in the virtual world, which joins writing, pictures and animated videos. In such a perspective, multimodal discourse can be used to develop mathematical ideas that can be posted on the Internet. The reader can view many samples of the types of projects that students and/or teachers can produce at http://www.edu.uwo.ca/dmp. I have borrowed the ideas of Boal and Kress to build the argument that a new kind of problem—which exploits the resources offered by the Internet—can transform the mathematics classroom. With function and geometry software, students were not yet able to associate school mathematics with other parts of their lives, but this would now be possible if the Internet were allowed in the classroom. Just as software did not imply an end to paper-and-pencil; paper-and-pencil and geometry and function software can act together with the Internet in the mathematics classroom.

In a more artistic fashion, the parallel lines poem, http://publish.edu.uwo.ca/george.gadanidis/parallel/, proposes ideas that challenge the predominance of Euclidean geometry in our way of seeing parallel lines. Digital mathematical performances have been used as an educational tool in contexts ranging from elementary school to teacher education. Also, Gadanidis and Borba (2008) emphasized several dimensions involved in the investigation and creation of digital performances. The multimodal design of those virtual objects involving mathematics and arts highlights the role of the Internet in mathematical thinking.

Some examples of digital mathematical performances are shown in Fig. 1.

This animation of mathematics is deeply influenced by the plasticity of digital technology provided by the Internet. But another important role of the internet is that, like in Boal's theater, the difference between actors and spectators may become blurred. Students and teachers who sing about numbers, or who create flash animations and publish them in Internet environments, are also becoming actors. In Fig. 1 (left), e.g., animations are performed on the Internet "to show" how the series resulting from the addition of even numbers is equal to $n^2$. In Fig. 1 (right), musicians and mathematics educators sing mathematics songs to fourth grade students, with poetry written by George Gadanidis, which is then published on websites. Students' poetry is also turned into songs and published in websites.


**Fig. 1** Examples of digital mathematical performances available on the Internet at http://www.edu.uwo.ca/dmp. (*Left*) *L-Patterns*: sequence and series of numbers. (*Right*) *I am too*: quadrilaterals. Digital performances created by George Gadanidis

Theater and cinema have conditioned different kinds of spectators, and now the Internet may be creating what Boal had dreamed of: an active "spect*actor*". Theater allows two-way interaction in a way that cinema does not. In the first case, actors can tease the audience or even invite them into short parts of the play. In theater, each person has a standpoint, seeing the stage from a different perspective. In cinema, it is hard to think about teasing in the same way possible in theater, and although the seats in the theater are different, everyone who is at the movie theater sees the big screen in the same way. This does not mean that cinema is worse than theater or vice versa, but that the way different structures of the means of communication, the medium, opens and closes possibilities.

The ideas of Boal and Kress about blurring the line between audience and actors and about multimodal communication, respectively, are consistent with the notion of humans-with-media. Different collectives generate different kinds of knowledge and transform the very notion of problem. A problem where the main medium used is paper-and-pencil is likely to result in one solution or a demonstration. A problem approached using geometry software is likely to emphasize conjecturing and trial-and-error before arriving at a demonstration. Since answers and demonstrations for many problems are now easily found on the Internet, it is unlikely that finding answers will be the focus of problems posed to collectives of humans-with-Internet.

Thus, I believe that digital mathematical performance, or digital performance, may be an alternative for schools that are open to using the Internet for teaching and learning (Gadanidis, 2006). It is possible to imagine a scenario in which students will gather to create a performance and rather than to solve problems whose answers are easily found on the Internet. Maybe generating performance, in the sense of performance arts, will be the goal of a school, in contrast to the views of some schools that are only concerned with performance in the radically different sense of measuring students' and teachers' achievement through some quantitative criteria. If traditional classrooms are seen as having a stage occupied by the teacher who is delivering a monolog, with students as the spectators, how can different media contribute to different kinds of performance in the "classroom arena"?

Modeling and digital mathematical performance can, therefore, represent alternatives for "problems" for students to deal with if the Internet becomes ubiquitous in our schools. Moreover, as mentioned before, I propose a change in curriculum and substantial changes in the didactical contract, as students will not likely to be waiting for the teacher to initiate the teaching activity, and they will be designing projects or developing activities that resemble research. This is one reason why it is not possible to predict whether the Internet will or will not be accepted in the classroom. Once more, however, it should be remembered that it is important to create alternatives for its use in the classroom, looking ahead to the future.

But again, the different roles that students and teachers play, and the very notion of multimodal discourse, despite being more relevant to environments in which the Internet is a major player, are not limited to such environments. Reflecting on the way the Internet may transform traditional classrooms helped me to think about how other types of interfaces play different roles in other types of classrooms. For instance, geometry software also transformed the problems posed in the classroom. As with function software, they invite students to experiment, value the activity of trial-and-error, and at the same time, according to some (Laborde, 1998), allow demonstrations to be enriched by analysis of particular cases. In the references related to dynamic geometry already analyzed in this paper, there is considerable evidence showing how these tasks are being designed and transformed. As already mentioned, the use of such tasks varies considerably in different countries, and in different classrooms in the same country.

The extensive reviews analyzed in this paper support the idea that collectives of humans-with-geometry-software provide students with the experience of experimentation without overriding traditional roles of paper-and-pencil, such as their role in demonstration (Borba & Villarreal, 2005). There are classrooms in which both media co-exist, in what can be seen as one more case of Lévy's (1993) idea that one medium generally does not replace the one it competes with, but rather incorporates it and, I would add, transforms it.

The use of software in the classroom changes the status of visualization in the classroom, as well. The "flashier", more dynamic and interactive process of drawing graphs of functions and different geometric figures is becoming more common in the classroom. The discourse that makes this process possible is, of course, multimodal. The dynamic of figures was connecting the world of school, traditionally associated with writing, to the experience most were already having with television, and that many others are starting to have with the Internet, as well. Kerckhove (1995), based on substantial empirical research, described how television influences the way one thinks and how it sometimes reaches our intelligence even before it is possible to reflect on some of the images being shown. Dynamic geometry and function software are not television, but they are also seen in a screen, and the images are perceived very strongly, as most advocates of its use have shown, as well as those who criticize the misconceptions generated by the images (see Ferrara, Pratt & Robutti, 2006). Experimentation and visualization have invited body language, in particular gestures, to play an important role in mathematics education (Arzarello & Robutti, 2004; Borba & Scheffer, 2004). Like our arguments regarding the Internet, it can be said that this software has already initiated the creation of a multimodal discourse.

Internet, software, paper-and-pencil, and orality are media that not only express ideas, but also shape ideas and language. It is in this way that I see them as actors in the collective of humans-with-media. Graphing and geometry software brought the first generation of multimodal discourse into mathematics, and I believe that approaches like modeling have been paving the way, prior to the arrival of the Internet, for the belief that students should participate in the design and development of projects, rather than limiting their activities only to solving problems. Fig. 2 can show static views of students working with a graphing calculator, discussing graphs on the blackboard, pointing and performing other kinds of gestures, using PowerPoint to present part of their project, the teacher pointing, explaining and debating with students.

Figure 2 came from a mathematics class for first year Biology majors in which students experience the use of graphing calculator and modeling, together with more traditional use of the blackboard and solution of exercises in the classroom. Considering these from a performance perspective, in the way that Gadanidis and Borba (2008) are attempting to develop, it can be seen that, in the face-to-face classroom, the possibility exists for actor (teacher) and audience (students) to play out some versions of Boal's Theater of the Oppressed, in which actors and spectators change roles and are transformed into spect*actors*. It is clear that changing the didactical contract or giving more voice to students is neither a casual implication of the use of technology in general, nor of the use of the Internet in particular. What is being argued in this "research of the future" is that if students engage in project work, and if there are no clear correct answers, the student–teacher relationship is likely to change.

Skovsmose and Borba (2004) proposed that research should not be concerned only with what is happening in the classroom, but also with what is not happening. Focusing research only on current practices in the classrooms can generate a conservative agenda in which researchers look only at things that are there. However, imagination—an imagined situation—can also be researched. Imagining mixed-race classrooms in South Africa should have been on the (mathematics) education agenda before the end of apartheid. Of course, it was not easy to do so before the end of the racist regime, so it may have been necessary to create arranged situations. An arranged situation is therefore an intermediate situation between the current situation and the imagined situation.

Arranged situations may be necessary to investigate modeling and technology in classrooms in universities and societies where the problem is traditionally given by the teachers, and where availability of technology is increasing slowly. Modeling challenges the current situation and presents alternatives for a classroom in which students have a voice in choosing what will be studied. Students are using the Internet already outside the classroom for gathering material for their projects (see Diniz, 2007, for instance), but even in studies such as these, the Internet is not still fully admitted into the classroom and is not actually even available, for the most part. Digital mathematical performance is a product of Gadanidis' (2006) imagination, and arranged situations are being created to conduct empirical research related to this notion.

The goal of this paper was to imagine possible scenarios for a classroom that is saturated with the Internet and where technology is allowed in all instances of education. In these scenarios, the Internet would not be forbidden from day 1 of class to the final exam, in the same way that a traditional wristwatch is not. If this comes to pass, it is unlikely that problems from a regular textbook will be at center stage, as argued earlier, due to students' lack of "Internet accent" or because answers will be easily found on the Internet. The notion of "problem" may be transformed in a similar fashion as it was when geometry and function software become incorporated into educational environments.

But maybe there will be no problems, and the task will be to find problems and issues in this information-saturated environment, which is a way that a classroom with Internet can be described. This scenario can be similar to some of


**Fig. 2** Different "performance" in the mathematics classroom

the modeling examples presented in this paper. Students may conduct searches on themes and develop their search in the virtual world, but with the major difference that the entire class curriculum will be permeated by the Internet. Looking for problems, and discussing solutions for them and generating new problems may be part of this educational scenario. Modeling projects will be published on the Internet, and communities of interest will be formed in virtual space, and not necessarily according to geographical or university divisions.

Another scenario for an Internet-saturated classroom is one in which digital mathematical performance is widely used. The convergence of mathematics, performance arts and digital technology with the support of the Internet is one possibility for a classroom in which the routine task-solution-assessment may not occupy central stage. Students thinking about mathematics as art and developing mathematical poems may be at the core of learning mathematics in this kind of classroom. Digital mathematical performance is usually displayed in a combination of video, animations, algebraic expressions and mathematical graphs. This combination can be seen as a new language for mathematics education that can become more powerful as the power of Internet connections increases. Digital mathematics performance developed by students may be an alternative to the usual model of having students looking for answers to problems that are already known.

A synergy exists between digital mathematical performance and modeling. So it is possible that a third scenario could be the use of both approaches together if the Internet is allowed to play an important role in the mathematics classroom. The Internet actually increases the possibilities of modeling, as it allows for the broad range of themes that students may express interest in, as discussed in Borba and Villarreal (2005) and Malheiros (2004). The Internet and user-friendly browsers are inherent parts of digital mathematical performance. I believe that as access to the Internet increases, and user-friendly tools become more available, possibilities for collaboration and constant change in pieces of digital mathematical performance may become even more real. Modeling and digital mathematical performance are alternatives for face-to-face classrooms, but also for "online classrooms".

It is very likely that these two scenarios for Internet-saturated classrooms will be combined with educational practices being used today, but it was important to consider them independently and apart from other ways of developing curriculum as a means of trying to look into the future.

## 4 Looking into the future of education

Banks have accepted computer technology in a way that has transformed banking. Industry, in general, has accepted technology. In particular, the tourist industry has incorporated the Internet and computer technology as a means of attracting more people to beach resorts or hotels near the sites of scientific conferences. Conference organizing may be considered by some as part of the tourist industry and by others as part of academic activities. It no longer seems reasonable to organize a conference without a webpage to advertise, report updates, and provide some means of paying conferences fees. If organization of educational conferences is considered as being part of education, it is possible to conclude that technology has been accepted, to a certain extent, in the field of education. If one focuses more closely on the administration of education, it is possible to find that student registration, diplomas, and grading are also impregnated by technology. But this does not mean that the classrooms are fully permeated by them.

Schools, understood in this paper in a broad sense that encompasses all levels from kindergarten to graduate school, seem to be resistant to technology. Memorization is still largely required in classrooms, and even more in tests. National tests and entry exams also depend largely on memory, structured by orality, and paper-and-pencil as the main media. In mathematics education, simulation is still a second-class citizen. Many of us who use technology in basic education or introductory courses in college do not seem to know how to incorporate technology into the teaching of topology and analysis; and, of course, demonstrations do not seem to have been transformed by more than 20 years of research on the use of technology in classrooms. There is no evidence that all the research in mathematics education has contributed to widespread use of software in the mathematics classroom.

So it is not impossible to bar the Internet from the classroom, or to delay its admittance. Wristwatches seem to be the only technology containing microchips that are used without restriction in the classroom. No one thinks that students who look at watches to check the time are cheating. But looking at a result or some information on the Internet would be considered cheating. Universities that are on the cutting edge of technology use, such as UOIT[3] in Canada, still have problems in admitting technology into the testing process. Smartboards are ubiquitous, and students receive a new laptop at the beginning of every school year, but laptops are barred from the exams. In the class I teach at UNESP,[4] technology is fully used in class, and is used by the students to present their projects, but it is barred from the exams, which compose approximately 70% of the final grade. It is not easy to implement something that would appear to encourage cheating for students and for the community in general. As a teacher, I still do not feel fully confident preparing test items using software, and I have only begun to think about how to use the Internet in tests.

In this paper, I intended to show some possibilities for using the Internet in the current mathematics classroom and to imagine how it can be used in the future if the Internet becomes fully accepted in the classroom. If collectives that include the Internet are allowed in schools, a need to design tasks for collectives of humans-with-Internet will emerge. Two scenarios were presented, one that includes projects like those developed in the modeling tradition, and those that adapt innovative ideas proposed by the digital mathematical performance projects. Modeling (see for instance, Barbosa, Caldeira & Araújo, 2007; Borba & Villarreal,

[3] University of Ontario Institute of Technology.

[4] Sao Paulo State University.

2005) is a consolidated trend in mathematics education and has been transformed by the use of technology; at the same time, it illustrates one of the possibilities for introducing the Internet into education. Digital mathematical performance (Gadanidis & Borba, 2008) is a brand new attempt to bring arts together with education, in which the Internet plays an active role—a role that can often be taken for granted, or be considered natural, as occurred when electricity and blackboards were introduced in the nineteenth and twentieth centuries in most schools in Europe and other "developed" parts of the world.

The distance between students' everyday experiences and their experiences in school may diminish if the Internet becomes accepted in the schools. It is very possible that, if we bring the Internet into the classroom, we will bring more technology into schools. We may start to think that maybe it is time to bring schools into technology. Alternatively, we may think that the classroom may possibly be dissolved into the Internet!

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