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ORIGINAL ARTICLE

Exploring strategy use and strategy flexibility in non-routine problem solving by primary school high achievers in mathematics

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Abstract Many researchers have investigated flexibility of strategies in various mathematical domains. This study investigates strategy use and strategy flexibility, as well as their relations with performance in non-routine problem solving. In this context, we propose and investigate two types of strategy flexibility, namely inter-task flexibility (changing strategies across problems) and intra-task flexibility (changing strategies within problems). Data were collected on three non-routine problems from 152 Dutch students in grade 4 (age 9–10) with high mathematics scores. Findings showed that students rarely applied heuristic strategies in solving the problems. Among these strategies, the trial-and-error strategy was found to have a general potential to lead to success. The two types of flexibility were not displayed to a large extent in students' strategic behavior. However, on the one hand, students who showed inter-task strategy flexibility were more successful than students who persevered with the same strategy. On the other hand, contrary to our expectations, intra-task strategy flexibility did not support the students in reaching the correct answer. This stemmed from the construction of an incomplete mental representation of the problems by the students. Findings are discussed and suggestions for further research are made.

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1 Introduction

Problem solving is considered the most significant cognitive activity in everyday and professional environments (Jonassen 2000). An attribute which is considered integral to the problem solving process is strategic behavior (Polya 1957; Schoenfeld 1992). Another important characteristic of problem solving is that people are able to work in a flexible way and can modify their behavior according to changing situations and conditions. In fact, a person's flexibility determines to a large degree how well he or she can cope with a new situation. As Demetriou (2004) emphasized, more flexible thinkers can develop more refined concepts that are better adjusted to the special features of the environment and produce more creative and appropriate solutions to problems.

What is true for solving problems in general also applies to mathematical problem solving. Numerous studies in mathematics education (e.g., Pape and Wang 2003; Verschaffel et al. 1999) hold strategy use central to processing mathematical problems. A well-documented finding is that success in solving a mathematical problem is positively related to the students' use of problem solving strategies (Cai 2003; Kantowski 1977). In mathematics education, though, students continuously face new situations and new problems (Stanic and Kilpatrick 1988), which require them not only to know and apply various strategies, but also to be flexible (e.g., Baroody 2003; Silver 1997). What they have learned in one situation and what applies to one problem, will not necessarily fit another situation or be appropriate for another problem. As a result, in the mathematics education community considerable research has been devoted to studies on strategy flexibility. However, most of these studies focused on children's strategies related to arithmetic concepts and skills (e.g., Baroody 2003; Beishuizen, Van Putten and Van Mulken 1997). Less attention has been devoted to the study of flexibility in using heuristic strategies in mathematical non-routine problem solving (e.g., Kaizer and Shore 1995), especially among primary school children. More information is needed to understand how flexibility in using heuristic strategies occurs in non-routine problem solving and how it is associated with performance.

This paper aims to give insight into the strategy use and strategy flexibility of high achievers in primary school mathematics in non-routine problem solving. The theoretical value of our study lies in that it may contribute to the formation of an operational definition of strategy flexibility in non-routine problems, as it proposes and explores two distinct aspects of strategy flexibility in students' problem solving behavior: strategy flexibility *between* different problems and *within* a problem. Furthermore, the study may clarify the interrelations of strategy use and strategy flexibility within problems and across problems with problem solving success. From a practical point of view, knowledge about the above may contribute to the interpretations of individual differences in problem solving and provide suggestions on how to support student development in solving non-routine problems.

In our study, we interpret and use the terms 'non-routine problem' and 'problem' interchangeably, on the basis of Schoenfeld's (1983) definition of a problem, that is, as an unfamiliar situation for which an individual does not know how to carry out its solution. In other words, he or she is unable to solve the situation comfortably using routine or familiar procedures (Carlson and Bloom 2005). Furthermore, the term 'strategies' refers to problem solving strategies or heuristics, in the sense given by Schoenfeld (1992) and Verschaffel et al. (1999), such as drawing a picture, making a list or a table or guessing and checking.

2 Theoretical framework

2.1 Flexibility

The term 'flexibility' has been extensively used by researchers in the field of cognitive and developmental psychology (Demetriou 2004; Krems 1995) on the one hand, and of mathematics education (Krutetskii 1976; Verschaffel, Luwel, Torbeyns and Van Dooren in press), on the other hand.

According to Demetriou (2004) flexibility refers to the quantity of variations that can be introduced by a person in the concepts and mental operations he or she already possesses. Krems (1995) defines cognitive flexibility 'as a person's ability to adjust his or her problem solving as task demands are modified' (p. 202). Mathematics educators highlight the educational value of recognizing and promoting flexibility in children's self-constructed strategies and have developed and implemented instructional materials and interventions planned for the improvement of such flexibility (e.g., Freudenthal 1991; Torbeyns, Desmedt, Ghesquière and Verschaffel 2009).

A term that is closely related to flexibility is 'adaptivity'. An in-depth analysis by Verschaffel et al. (in press) of how these terms are currently used in the literature, suggests that the term 'flexibility' is primarily used to refer to using multiple strategies and switching between them, while the term 'adaptivity' emphasizes the ability to consciously or unconsciously select and use the most appropriate approach for solving a certain mathematical item or problem, by a particular person, in a given socio-cultural context. In the present study, which we see as the start of a program to investigate flexibility and adaptivity in non-routine problem solving, we focus on flexibility in strategy use.

In the light of the above, we consider strategy flexibility as the behavior of switching strategies during the solution of a problem, i.e., intra-task strategy flexibility, or between problems, i.e., inter-task strategy flexibility. This is a broad operational definition of flexibility, which includes different patterns of changing strategies. It is noteworthy that, in this definition, we do not connect strategy flexibility with the appropriateness of the problem solving strategies. Connecting strategy appropriateness with strategy adaptivity in problem solving would be an interesting topic to investigate. However, this is not our focus in the present study.

2.2 Mathematical problem solving

2.2.1 *What counts as a problem?*

A (non-routine) problem appears when an individual encounters a given situation, intends to reach a required situation, but does not know a direct way of accessing or fulfilling his or her goal. A central issue is the problem solver's ignorance with respect to a solution method (Mayer 1985). In contrast with routine problems which involve the application of routine calculations, non-routine problems do not have a straightforward solution, but require creative thinking and the application of a certain heuristic strategy to understand the problem situation and find a way to solve the problem (Pantziara, Gagatsis and Elia in press). Therefore, non-routine problems are considered more complicated and difficult than routine problems. However, Polya observed that, although routine problems can be used to fulfill particular didactical functions of teaching students to apply a certain procedure or a definition correctly, only through the careful use of non-routine problems can students develop their problem solving ability (Stanic and Kilpatrick 1988).

In our study, we concentrate on non-routine problems which involve interrelated variables and require the understanding that a change in one variable affects the other variables. Since the participants in this study—fourth-graders—do not have any algebraic tools at their disposal, they cannot apply a routine algebraic method, but have to confront these tasks by a heuristic or problem solving strategy, such as trial-and-error or systematic listing of possible solutions. A more detailed description of the problems is given in Sect. 4.2.

2.2.2 *Strategy use in problem solving*

Problem solving strategies constitute a fundamental aspect of mathematical thinking (Schoenfeld 1992). Previous research has shown that students' use of heuristic strategies was positively related to performance on problem solving tests, but the effects were only marginal (Kantowski 1977). The results of more recent studies have provided stronger evidence for the use of heuristic strategies as a means to enhance problem solving. Specifically, the problem solvers' ability to try possible solution approaches and to assess the likely outcome of each one has been found to play an important role to their efficient decision making and problem solving success (Carlson and Bloom 2005). Altun and Sezgin-Memnun (2008) have found that among mathematics teacher trainees, the strategies used had a dominant and decisive role in determining success on a problem.

Several mathematical problem solving strategies can be introduced in primary or middle school mathematics teaching, such as: guess–check–revise, draw a picture, act out the problem, use objects, choose an operation, solve a simpler problem, make a table, look for a pattern, make an organized list, write an equation, use logical reasoning, work backward (Charles, Lester and O'Daffer 1992).

Although many students do not spontaneously use valuable heuristic strategies when dealing with unfamiliar complex problems (De Bock, Verschaffel and Janssens 1998; Schoenfeld 1992), a number of researchers have found evidence that primary school students are capable of using strategies (like the ones noted above) to solve problems (e.g., Cai 2003; Follmer 2000). Moreover, Verschaffel et al. (1999) found that problem solving instruction addressed to fifth-grade students contributed to the improvement of their ability to solve mathematical application problems and to use problem solving strategies. Follmer's (2000) findings showed that in the fourth-grade, the instruction on non-routine problems had a positive impact on students' use of cognitive strategies and their awareness of how they solved the problems.

Besides heuristic (cognitive) strategies, solving a problem also requires metacognitive strategies. Metacognitive strategies involve self-regulatory actions, such as decomposing the problem, monitoring the solution process, evaluating and verifying results (Schoenfeld 1992; Verschaffel et al. 1999). These strategies play a crucial role in achieving problem solving success (Schoenfeld 1992; Carlson and Bloom 2005). However, a number of studies have shown that students display deficiencies in applying these strategies in their solution efforts (Schoenfeld 1985; 1992).

2.3 Strategy flexibility and problem solving

Strategy flexibility appears to be strongly interconnected with problem solving activities and performance. On the one hand, from a developmental perspective, developing and excelling in problem solving is to a considerable extent a function of increase of flexibility. On the other hand, from a differential point of view, individual differences at the same age level result from variations in flexibility, which enables the individual to carry out strategy alternations on the basis of the requirements of particular problems (Demetriou 2004). Given that problem solving performance is improved when task requirements and problem solving methods are coordinated (Krems 1995), flexibility in strategy use may significantly contribute to success.

Martinsen and Kaufmann (1991) distinguished between solvers who extended or persevered with the use of a particular problem solving strategy ('assimilators') and solvers who varied their problem solving strategy more frequently ('explorers'), even when a shift was not essential. Flexibility, however, especially characterizes competent students (Shore and Kanevsky 1993). Kaizer and Shore (1995) compared the flexibility of solution strategies between mathematically competent and less competent 11th-grade students on mathematical non-routine word problems which were drawn from Kruteskii's (1976) work. A distinction was made between verbal–logical methods, visual strategies and trial-and-error procedures. Across the problems, students employed different strategies. Competent students switched primarily between verbal–logical and visual methods, whereas less competent students alternated equally between verbal–logical and visual strategies, or between verbal–logical methods and trial-and-error.

Flexibility can be activated not only across problems, but also within a problem. Krems (1995) suggests that a type of mechanism that is important for flexible strategy use in solving a problem is modifying strategies. A flexible problem solver can modify strategies to correspond with alterations in resources and task requirements and can use several different techniques to find an answer. With development, thinkers become more competent in observing each of the components of a problem separately and reconstruct their structure (or relations) according to the plans or objectives of the particular situation (Demetriou 2004). As the person's knowledge of the problem's components (mental representation of the problem) becomes more complete and interconnected, he or she can more easily invent and use strategies, determine the most efficient solution path for the intended goal or model and flexibly determine alternative solutions (Baroody 2003; Demetriou 2004). Kaizer and Shore (1995) suggest that alternative solution strategies in problem solving may occur when the students experience difficulties with a problem or at any stage of the problem solving process. Nevertheless, Muir and Beswick's (2005) study has shown that most sixth-grade students were not capable of reflecting on the appropriateness of the strategy they had chosen, or display any inclination to use an alternative strategy, even when the initial strategy was not working. Furthermore, students were unwilling to attempt to confirm the appropriateness of the answer using an alternative method.

3 Research questions and predictions

The present study aims to contribute to our understanding of strategy use and strategy flexibility in non-routine problem solving. Our research questions, which refer to primary school high achievers in mathematics, are distinguished into two thematic groups. The first group of questions is concerned with strategy use, while the second group refers to strategy flexibility. For each question, a prediction has been formulated on the basis of the theoretical background presented above and the setting of the study.

3.1 Strategy use

Question 1. To what extent do the students apply strategies to solve non-routine problems?

Prediction 1. We expect that only a small number of students will use strategies to solve non-routine problems. Previous research has shown that many students do not spontaneously use heuristic strategies to tackle unfamiliar complex problems (De Bock et al. 1998; Schoenfeld 1992). This hypothesis is also based on the setting of this study, which cannot be ignored, since the current Dutch textbooks in mathematics include only a small set of non-routine problems (Kolovou, Van den Heuvel-Panhuizen and Bakker 2009).

Question 2. What is the relation between strategy use and answer success?

Prediction 2. De Corte (2007) suggests that, while heuristic strategies do not guarantee a correct solution, they significantly strengthen the potential of providing one. Some possible strategies that could be used in solving the problems of this study are: Trial-and-error, systematic listing of possible solutions, calculating extreme values. These strategies vary in their cognitive demands, thus the use of each one of them may have a different effect on problem solving success. The trial-and-error strategy, for example, is a strategy without high cognitive demands, which is commonly used in mathematics classrooms and in everyday life. Stacey (1991) characterizes trial-and-error as an intuitive strategy that everyone can use. In applying the trial-and-error strategy, one has to try possible solutions and compare the results with the intended results. If a match does not occur, then one has to try other solutions by adopting the processes used according to the requirements of the task. Students may be more skillful and experienced in this strategy than in the more sophisticated and less familiar ones. Thus, we expect that the trial-and-error strategy is more likely to lead to success than the other strategies.

3.2 Strategy flexibility

Question 3. To what degree does inter-task strategy flexibility occur among the students?

Prediction 3: We anticipate that only a few of the students will demonstrate strong inter-task strategy flexibility, that is, modify strategies in every problem situation. To be able to show inter-task strategy flexibility, one should have a rich repertoire of strategies. Thus, we ascribe prediction 3 to the Dutch mathematics curriculum, which does not substantially contribute to the development of students' repertoire of heuristics (Kolovou et al. 2009) that would enable them to flexibly use a variety of strategies.

Question 4. To what degree does intra-task strategy flexibility occur among the students?

Prediction 4: As a person's mental representation of the problem becomes more complete and interconnected, he or she can more easily invent and use strategies, determine the most efficient solution path for the anticipated goal and flexibly establish alternative solutions (Baroody 2003; Demetriou 2004). Understanding or building a mental representation of a (non-routine) problem depends on the existing cognitive schemes of the solvers (Mayer 1985). Thus, taking into account Dutch students' marginal learning experiences with non-routine problems at school (Kolovou et al. 2009), we expect that the majority of the participants who will make a solution attempt for a problem, will use mainly one strategy and fail to consider alternatives even if they encounter difficulties in the problem solving process. In other words, we anticipate that students will rarely demonstrate intra-task strategy flexibility when solving the problems.

Question 5. Are there any differences between students who show inter-task flexibility and students who use a single strategy across the problems in their problem solving performance?

According to Demetriou (2004) individual differences in performance at the same age level are closely related to variations in flexibility, which enables the individual to alternate between strategies on the basis of the requirements of particular problems. On the basis of the above, we formulated the following prediction:

Prediction 5: We expect that students who flexibly switch strategies across problems will outperform students who persevere with the same strategy.

Question 6. Are there any differences between students who show intra-task flexibility and students who use a single strategy within problems in their problem solving performance?

Prediction 6: We anticipate that students who use different strategies when solving a problem will exhibit greater problem solving performance than students who use a single strategy within problems.

The above prediction is based on a previous research finding that the solvers' ability to try different solution strategies for a problem contributes to their efficient decision making and problem solving success (Carlson and Bloom 2005).

4 Methods

4.1 Participants

A total of 152 high achieving students (97 boys and 55 girls) in grade 4 (9–10 years of age) from 22 different schools in the Netherlands were examined. The students belonged to the top 25% ability range in mathematics, and were selected by their teachers on the basis of their mathematics score. In most cases this was the students' mathematics score on the CITO Student Monitoring Test. In some cases the so-called DLE score was used. In the data preparation, the CITO scores were converted into DLE scores.[1]

4.2 Tasks

Three non-routine problems were given to the students.[2] Instructions involved an explicit request for showing the solution strategy.

The three tasks were:

1. Angela is 15 years now and Johan is 3 years. In how many years will Angela be twice as old as Johan? (age problem)
2. Liam has tokens of value 5 and 10 only. In total he has 18 tokens. The total value of these tokens is 150. How many tokens of value 5 does Liam have? (coins problem)
3. In a quiz you get 2 points for each correct answer. If a question is not answered or the answer is wrong, 1 point is subtracted from your score. The quiz contains ten questions. Tina received 8 points in total. How many questions did Tina answer correctly? (quiz problem)

A major characteristic of these problems is that they do not have a straightforward solution, but require a good understanding and modeling of the situation, that is, recognizing how different variables covariate. It is evident that a person who knows elementary algebra might use this knowledge to find the answer to the problems. The third problem, for example, could be tackled by solving the equation $2x - 1(10 - x) = 8$. But, as fourth-graders have not yet learned such techniques, they have to coordinate several pieces of information and use other strategies, such as systematic listing of possible solutions or trial-and-error, to solve the problems. These features make the tasks non-routine problems for the students. On the other hand, despite their complexity, the problems are accessible, as they involve small numbers and do not entail difficult calculations.

[1] CITO (Central Institute for the Development of Tests) provides Dutch schools with standardized tests for different subjects and grade levels. One of the CITO Tests is the Student Monitoring Tests for Mathematics. The DLE Test (Didactic Age Equivalent Test) is a different instrument published by Eduforce that teachers can use to measure their students' development in a particular subject.

[2] The original versions of these problems have been developed for the World Class Tests. In 2004, Peter Pool and John Trelfall from the Assessment and Evaluation Unit, School of Education, University of Leeds who were involved in the development of these problems asked us to pilot them in the Netherlands.

4.3 Procedure

As only about a quarter of each fourth-grade class participated in the study, the tasks were administered to these students either by their teacher or by another member of the educational staff of the school. These tasks were administered in the middle of the school year, during regular school hours in a quiet place at school. Students were given enough time to finish all three tasks. The teachers were instructed that students must work on their own and no assistance could be given to them. Students were not allowed to use a calculator and were instructed that if they needed to do a calculation, they could use the test sheet.

4.4 Strategy analysis and scoring

As already noted, students were asked to write down their solution process for the problems. For the analysis of the students' responses, a coding scheme was formulated[3] for each problem. After the student work on the three problems was coded, we asked a judge to do a second coding on the responses of 20 randomly chosen students for one problem, which involved 400 dichotomous codes.[4] The inter-rater reliability was measured with Cohen's Kappa (0.83) from which we concluded that the coding scheme was reliable.

In the present study we concentrate on the codification and analysis of the strategies which were visible on the students' test sheets. The strategies that we identified in students' test sheets for the three problems, their explanation and variables' names are presented in Table 1. Examples of these strategies are illustrated in Sect. 5.2.4.

As for the correctness of the answers, each correct answer on a problem was scored as 1, and each wrong or no answer as 0. An answer was assessed as correct when the accurate numerical result was written on the test sheet.

5 Results

The results are organized into two subsections, which correspond to the two thematic groups within the research questions. That is, we will first present our findings about strategy use and then we will turn to strategy flexibility within and across problems.

Table 1 Coding scheme for the strategies across the three problems

	Strategies					
Category	Repeating information	Trial-and-error	Systematic listing	Calculating-an-extreme	Proof-or-check	Halving the number of coins or value[a]
Variable name	Rep	TE	Sys	Extr	PoC	Half
Explanation: the response involves…	Repeated information from the problem formulation	Two or more trials and the last one is the given answer. The steps are not of the same size each time and the 'movements' of the trials do not need to go in one direction	A systematic listing strategy which entails at least three elements (including the final answer). The step size is stable (mostly 1) and the 'movement' of the list goes in one direction	Use of extreme values, e.g., in the coins problem the students start with calculating the value of 18 5-cent coins, or 18 10-cent coins, or calculate how many 5-cent coins or 10-cent coins are in €1.50	A kind of proof of the result or a check of the calculation. Students explain that the given answer is correct (according to them)	The assumption that there were 9 10-cent coins and 9 5-cent coins, i.e., halving the total number of coins or the total amount of money

[a] This strategy is applied only in the coins problem

[3] The coding scheme was developed by two of the authors, Marja van den Heuvel-Panhuizen and Angeliki Kolovou, and our Freudenthal Institute colleague Arthur Bakker.

[4] This control coding was done by Conny Bodin-Baarends who was involved in the data collection, but did not participate in the development of the coding scheme.

5.1 Strategy use

5.1.1 Students' strategies

The non-routine problems turned out to be difficult for the majority of students despite their high general mathematical ability as measured by the CITO Student Monitoring Test and the DLE Test. Only 35 students out of 152 (23%) provided a correct solution to the age problem, 61 students (40%) solved the coins problem correctly and 30 students (20%) succeeded in the quiz problem. The relatively lower success rates at the age and the quiz problem are probably due to their higher complexity in comparison to the coins problem. The two problems required the understanding and coordination of some additional data components and relations. The age problem entailed understanding the same change in time for both children's ages, while the quiz problem involved the coordination of two variables which changed in two different directions, one increasing and another decreasing.

Table 2 shows the percentages of the students who used each of the five strategies per problem and the strategy of halving the total number of coins or the total amount of money in the coins problem. Although repeating information helps students confirm that they use the data given, only a small number of students repeated the problems' information components. The quiz problem appears to invite to do that more often (10%) probably because of its high complexity and longer statement. The calculating-an-extreme strategy was hardly used (1–7%). More commonly used strategies were—what we have called—proof-or-check (15–32%), trial-and-error (P1: 8%, P2: 16%, P3: 7%) and systematic listing (P1: 15%, P2: 3%, P3: 11%). As for the 'halving strategy' in the coins problem, it was used only by four students (3%). In general, traces of strategies were only found on about half of the test sheets. These findings provide evidence to prediction 1 suggesting that only a small number of the students could spontaneously apply strategies to tackle the problems.

5.1.2 Successfulness of students' strategies

To determine the strategies that can be considered successful in the solution of the three problems, we performed the implicative statistical method using the computer software CHIC (Classification Hiérarchique, Implicative et Cohésitive) (Bodin, Coutourier and Gras 2000). This method of analysis determines the implicative relations between variables (Elia, Panaoura, Gagatsis, Gravvani and Spyrou 2008; Gras, Suzuki, Guillet and Spagnolo 2008), which give a statistical meaning to expressions such as: 'If we observe the variable *a* in a subject, then in general we observe the variable *b* in the same subject'. The underlying principle of the implicative analysis is based on the quasi-implication: 'If *a* is true then *b* is more or less true'. The implicative diagram represents graphically the network of the quasi-implicative relations among the variables of the study. Figure 1 shows the implicative relations among the variables of students' strategies (see Table 1) and their success on the three problems.

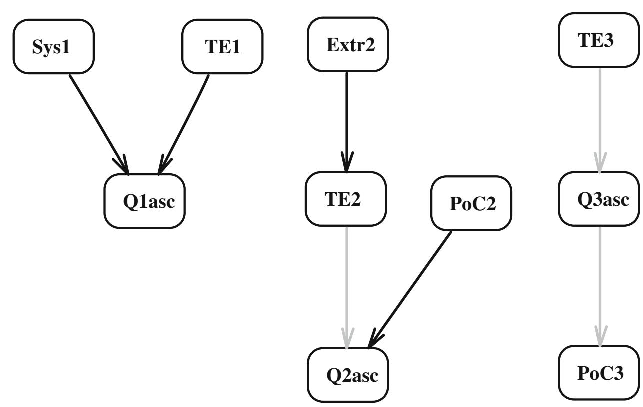

Fig. 1 Implicative diagram of strategies and correct answers to the problems. Notes: (1) *Q1asc*, *Q2asc* and *Q3asc* refer to students' success on the three problems respectively, (2) the numbers *1, 2 and 3* next to the variable names correspond to the three problems respectively, (3) the *thickness of the arrows* is a function of the strength of the implications, whose estimated probabilities are 95 and 90% respectively

Table 2 Students' strategy distribution for each problem

Strategy	Age problem		Coins problem		Quiz problem	
	f^a ($n = 152$)	%	f ($n = 152$)	%	f ($n = 152$)	%
Repeat	7	5	4	3	15	10
Trial-and-error	12	8	24	16	10	7
Extreme	2	1	7	5	10	7
Systematic listing	22	15	5	3	16	11
Proof-or-check	22	15	36	24	49	32
Halving coins/value	–	–	4	3	–	–

[a] Students could use more than one strategy (see Sects. 5.2.2 and 5.2.4)

Three groups of implications were found, each of which links the variables of a different problem. First, systematic listing and trial-and-error strategy appear to be crucial for solving the age problem. Second, success on the coins problem is found to be a function of using primarily the strategy of giving a proof of the result or checking the calculation, and secondly the trial-and-error process. Students who used the strategy of calculating an extreme in the coins problem employed also the trial-and-error process in their solution. Third, using trial-and-error is the major strategy that leads to success in the quiz problem. An important process feature of successful students in the quiz problem is their effort to prove or check their answer. A noteworthy commonality among the three groups of implications is the link between the trial-and-error process and the problem's solution, indicating that using the trial-and-error strategy implied success in all three problems.

The implicative analysis provided evidence for prediction 2 suggesting that the trial-and-error strategy would have a strong potential to lead to success. Specifically, it is the only strategy that led the students to a correct solution for all three problems. Thus, students used the trial-and-error strategy more widely and competently in the problems, independent of the situation or the numbers involved. In the age problem, systematic listing was equally successful. In the coins problem explaining and checking the correctness of a given solution was found to be more successful than trial-and-error. A hypothetical explanation might be that students mentally used a strategy, such as trial-and-error or systematic listing, and reached an answer, which they just checked in the end, without writing down their strategy.

5.2 Strategy flexibility

5.2.1 *Inter-task strategy flexibility*

Having three problems in total means that there are three different pairs of problems (1–2, 1–3 and 2–3) in which the students can show a difference in strategies. A difference or a change in strategies between each set of two problems was considered to occur if the strategy or the strategies used were not the same between the problems. It should be noted here that this method of analyzing inter-task flexibility does not take into account the effects the order of the problems may have on students' strategies. This is a methodological limitation of our study that should not be disregarded when discussing inter-task strategy flexibility.

A total of 51 students (34%) demonstrated inter-task strategy flexibility. Most of these students ($n = 27$, 18%) changed strategies in only one pair of problems, while they did not exhibit any traces of strategy use in the other problem (score: 1). An equal proportion of students alternated between strategies in two problem pairs (score: 2; $n = 12$, 8%) and three problem pairs (score: 3; $n = 12$, 8%) respectively. The frequencies and percentages in strategy alternations are shown in Table 3. These findings lend support to prediction 3 which suggests that only a small number of students would display high levels of inter-task strategy flexibility, that is, change strategies in every problem situation. A larger number of students, though, had a low score in inter-task strategy flexibility. It is also noteworthy that only 11 students (7%) were found to persevere with the use of a single strategy across the problems.

Table 3 Distribution of students' inter-task strategy flexibility scores

Score of changes in strategy use over the three problem pairs	f ($n = 152$)	%
Non-applicable	90[a]	59
0	11	7
1	27	18
2	12	8
3	12	8

[a] These students did not exhibit strategy use in more than one problem

5.2.2 *Intra-task strategy flexibility*

Regarding the *intra-task strategy flexibility*, only 36 students (24%) alternated strategies in one or more problems. Specifically, 28 students (18%) displayed intra-task strategy flexibility in one problem, 7 students (5%) in two problems and 1 student (1%) in all three problems. Table 4 shows the frequencies and percentages of students who used a particular number (0–3) of strategies per problem. A similar pattern appears in all three problems. Only a small number of students, i.e., 6, 18 and 21, switched between two or three strategies, in each of the three problems respectively. A single strategy was employed by a considerably larger number of students in each problem, while the vast majority of the students did not apply or did not report on strategies, as shown in their test sheets. These findings verify prediction 4 stating that intra-task flexibility would be rarely detected in students' solutions.

5.2.3 *Inter-task strategy flexibility and success*

To explore the differences between students who showed inter-task strategy flexibility and students who used a single strategy across the problems on their problem solving performance we used the t-criterion for independent samples. The results provided evidence for prediction 5, since students who switched strategies across the problems ($\bar{x} = 1.35$; SD $= 0.82$; $n = 51$) performed significantly better in problem solving ($t = 2.70$, $p < 0.01$) than

Table 4 Distribution of students' intra-task strategy flexibility scores per problem

Number of strategies	Age problem		Coins problem		Quiz problem	
	f ($n = 152$)	%	f ($n = 152$)	%	f ($n = 152$)	%
0	93	61	92	61	75	49
1	53	35	42	28	56	37
2	6	4	16	11	19	13
3	–	–	2	1	2	1

Table 5 Students' mean performance and standard deviation in problem solving per inter-task strategy flexibility score

Score of inter-task strategy flexibility	$\bar{x}$	SD	n
0	0.64	0.67	11
1	1.15	0.79	27
2	1.58	0.67	12
3	1.58	1.00	12

students who persevered with the same strategy across the problems ($\bar{x} = 0.64$; SD $= 0.67$; $n = 11$).

Moving a step forward, we examined whether students' performance varied as a function of the inter-task strategy flexibility score. As illustrated in Table 5, using analysis of variance with performance as the dependent variable and score of inter-task strategy flexibility as the independent variable showed that problem solving performance was *generally higher* when the inter-task strategy flexibility was higher.

Post hoc analysis showed that there were statistically significant differences among the mean performances of the students who used the same strategy across the problems (score: 0) and the students who changed strategies in two (score: 2) or three (score: 3) problem pairs. Students who exhibited high scores of flexibility, that is 2 or 3, performed equally well.

5.2.4 Intra-task strategy flexibility and success

To explore the differences between students who showed intra-task strategy flexibility and students who used a single strategy within the problems on their problem solving performance we used the *t*-criterion for independent samples. The results showed that the difference in performance between students who used various strategies ($\bar{x} = 1.17$; SD $= 0.85$; $n = 36$) and students who used a single strategy ($\bar{x} = 0.95$; SD $= 0.78$; $n = 73$) within problems was not significant ($t = -1.36$, $p = 0.18$). This equivalence in performance deviates from prediction 6, that is, students who use different strategies would outperform students who use only one strategy in the solution of a problem. This is an interesting finding which motivated us to carry out a qualitative analysis of the responses of the students who showed intra-task strategy flexibility.

The purpose of the qualitative analysis is to understand why the intra-task flexibility did not support the students in finding the correct answer. For this analysis we selected the quiz problem (see Sect. 4.2), because it had the lowest proportion of correct answers of the three problems when multiple strategies were used. More specifically, out of the 21 students who applied more than one strategy only six came up with the correct answer.

Table 6 shows all the combinations of strategies that were used by the students in the quiz problem. The first thing that stands out is that repeating is the most frequently used strategy. In total, we found that the repeat strategy was applied ten times in combination with other strategies, which is the highest frequency among all problems, as already noted in Table 2. This probably reflects that the students had difficulties in grasping or keeping in mind the complex data structure of the problem.

In seven cases the repeat strategy was followed directly by proof-or-check. In all these cases the students came up with an incorrect solution. This is remarkable because proof-or-check either on its own or in combination with other strategies is mostly connected with finding the correct outcome. Figures 2 and 3 show two examples of students who combined the repeat strategy with the proof-or-check strategy.

Table 6 Distribution of the applied strategy combinations in the quiz problem and correctness of the answers

Applied strategy combinations	All (correct and incorrect answer) f	Correct answer f
Rep, PoC	7	–
Extr, Sys	4	2
TE, PoC	3	2
Extr, PoC	2	1
Rep, Sys	2	–
Sys, Extr	1	–
Extr, TE, PoC	1	–
Rep, TE, PoC	1	1
Total	21	6

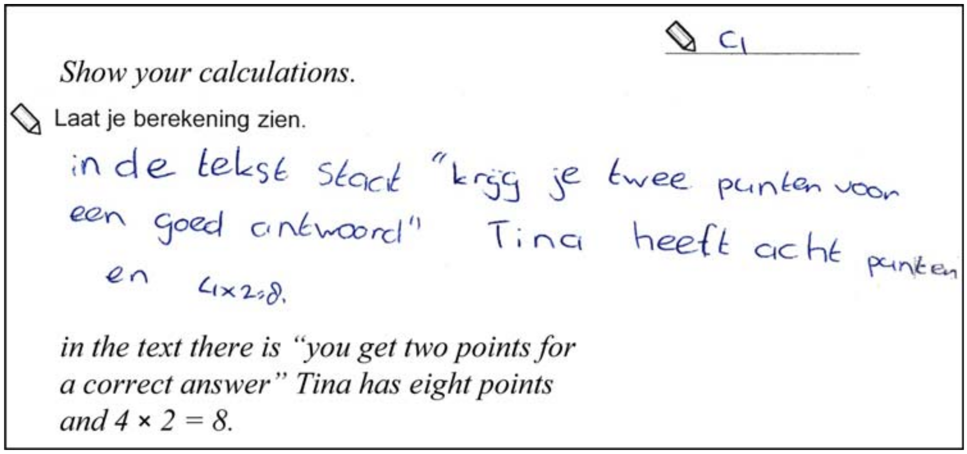

Fig. 2 Example of a combination of repeat strategy with proof-or-check strategy in which student neglects incorrect answers

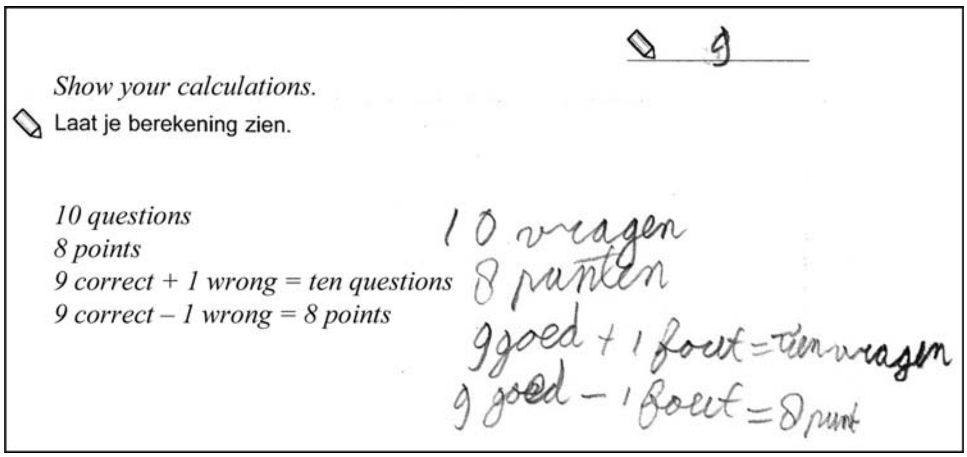

Fig. 3 Example of a combination of repeat strategy with proof-or-check strategy in which 1 point is given to the correct answer

These examples make clear that when the students repeated the problem information they did it in an incomplete, fragmented way. As a consequence, the students applied the proof-or-check strategy without taking into account all the problem information, therefore their results were incorrect. Specifically, the student whose work is shown in Fig. 2 completely disregarded the incorrectly answered quiz questions, while the student in Fig. 3 assumed that for a correct answer only 1 point is added to the score. Therefore, this student found that there were nine correct answers and one incorrect answer.

Among the students' responses, eight test sheets showed the use of the extreme strategy. Four times the extreme strategy was followed by a few systematic trials until an answer was found. The students that applied this combination of strategies in the quiz problem, assumed firstly that all ten answers were correct, resulting in a score of 20 points. Using the ten correct questions as a starting point, the students made a double list with the number of correct questions and the total points. Each step was usually one correct question less, until the score of 8 points was attained. What can go wrong in this process though is that the student may focus only on the points of the correct answers and disregard the penalty points for wrongly answered questions. Figures 4 and 5 depict the work of students who used the combination of extreme strategy and systematic listing strategy. The student in Fig. 4 found the correct answer to the problem, while the student in Fig. 5 came up with an incorrect answer.

6 Discussion

The present study examined strategy use and strategy flexibility of high mathematical achievers in grade 4 when solving non-routine problems, which involved the co-variation of different variables. The relationships of strategy use and flexibility with success on these problems were also investigated.

The use of heuristic strategies in students' solutions was poor, despite students' high mathematical competence. This result is in line with previous studies' findings suggesting that heuristics are rarely used by students when confronted with unfamiliar complex problems (De Bock

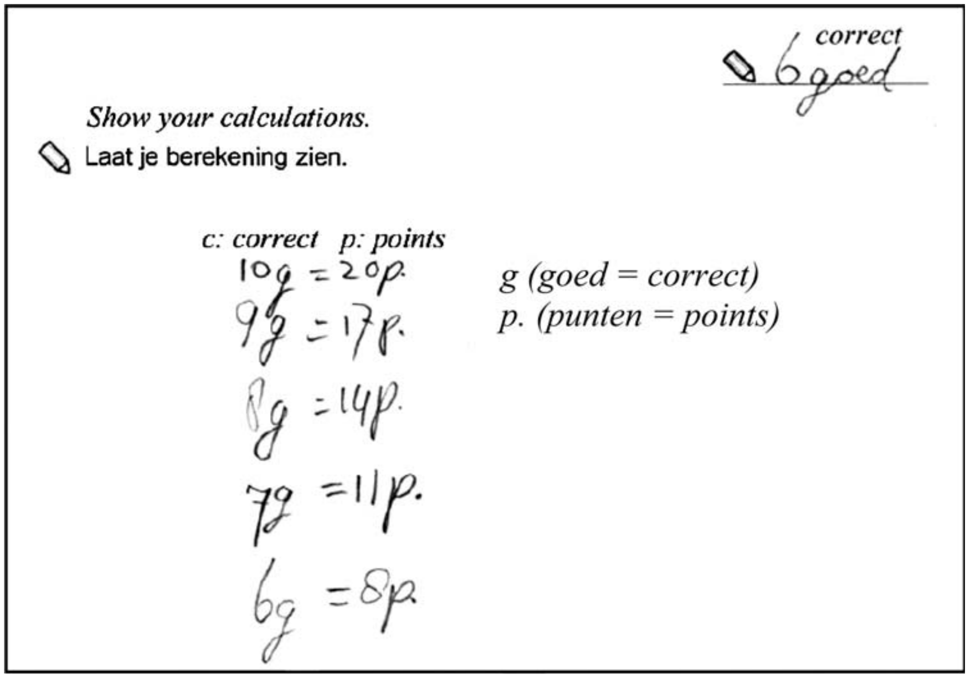

Fig. 4 Example of a combination of extreme strategy with systematic listing strategy resulting in correct answer

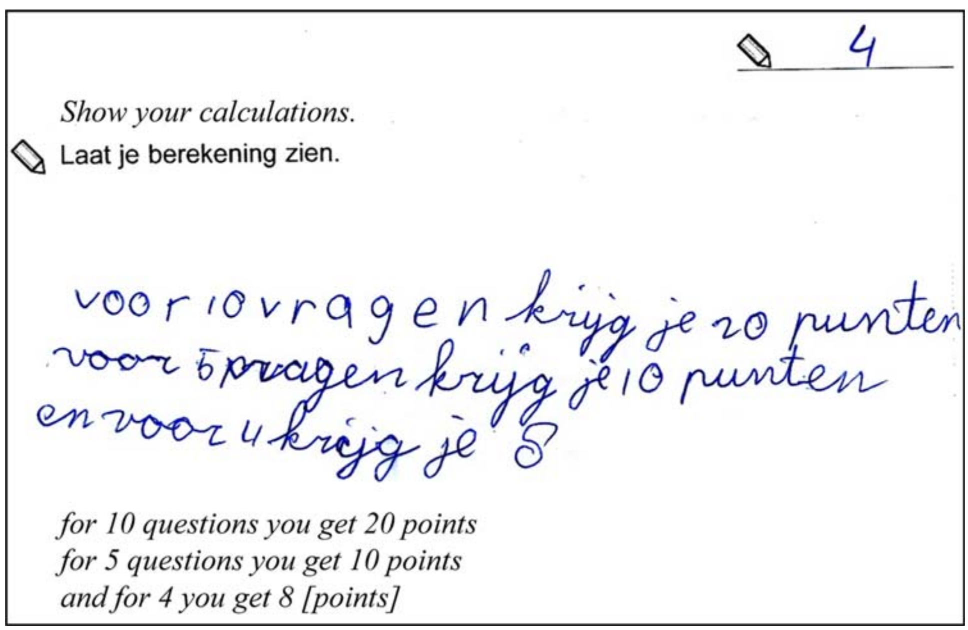

Fig. 5 Example of a combination of extreme strategy with systematic listing strategy resulting in incorrect answer

et al. 1998; Schoenfeld 1992; Verschaffel et al. 1999). The marginal place of non-routine problems in the Dutch mathematics textbooks could offer an explanation for this result (Kolovou et al. 2009).

However, strategy use in this study was assessed on the basis of what was visible on the work space of the students' test sheets. Since only about half of the students made use of the work space when solving the problems, another explanation could be that students had difficulties in writing down their thinking. The tendency not to write down one's reasoning is a general attitude, as other studies showed similar results in different mathematical tasks (Doorman et al. 2007). For example, students might believe that it is better not to use the paper, because solving the problems mentally indicates a higher level of mathematics. Moreover, students probably have not learned to organize the data and write down the solution steps to support their thought process. This is especially true for high achievers who are not accustomed to use scrap paper when they deal with tasks during regular mathematics class work (Doorman et al. 2007).

Furthermore, our study provided us with some new insights concerning the successfulness of strategies. Findings showed that the trial-and-error strategy, although not very advanced, was the most broadly successful strategy. An explanation that can be given is that when students are not explicitly taught any heuristic strategies, trial-and-error may be the only strategy they can use, as it does not entail high cognitive demands and it is widely used in a variety of mathematical and everyday situations. Thus, students are more experienced and competent in using this strategy rather than other strategies. Systematic listing of possible solutions and proving or checking the answers were the other two strategies that had the potential to lead to success in some problems. It could be interesting for future studies to examine if the pattern between heuristic strategies and problem solving success changes when students receive systematic strategy training in non-routine problem solving.

One of the most important contributions of this study is the introduction and exploration of a new operational conceptualization of strategy flexibility in non-routine problem solving. Two distinct aspects of strategy flexibility were identified and examined: intra-task strategy flexibility and inter-task strategy flexibility. Students were not very often found to show traces of strategy flexibility either between or within the problems. This can be attributed to the problems' novelty and complexity, which may have hindered the flexible change of strategies by the students.

In concern to inter-task strategy flexibility, however, the number of students who changed strategies across the problems was larger than the number of students who persevered with the same strategy. This finding lends support to previous studies' findings which distinguished between solvers who extended or persevered with the use of a particular strategy and solvers who varied their strategy more frequently (Martinsen and Kaufmann 1991). The larger number of flexible students may be attributed to the participants' high mathematical competence (Shore and Kanevsky 1993).

As regards intra-task strategy flexibility, the majority of the students who made a solution attempt, used mainly one strategy and failed to consider any alternative or complementary strategies. This finding is in line with the study of Muir and Beswick (2005) which provided evidence for students' lack of any inclination to try an alternative strategy even when frustrated by their weakness to proceed in their solution. This inflexible behavior can be seen as an indication of students' deficiency to reflect on the appropriateness or adequateness of their initially chosen strategy and to use an alternate or complementary strategy that could lead to the correct answer.

The results of the present study showed that students who displayed inter-task strategy flexibility were more successful problem solvers than students who persevered with the same strategy between the problems. Furthermore, higher scores on problem solving performance were found with higher inter-task strategy flexibility. A possible explanation is that students who displayed inter-task strategy flexibility did not only possess more strategies, but could understand the rationale of these strategies, and therefore flexibly modify the procedures so that they were successfully used in different contexts (Baroody 2003). This result has a practical implication for the teaching of non-routine problem solving in primary school. When solving different non-routine problems, even of a similar structure, it could be useful and effective (and probably less frustrating) for the students of this age to use multiple strategies. Yet more research is needed to find didactical methods to develop inter-task strategy flexibility in non-routine problems and to explore the impact of these types of instruction on problem solving performance.

Surprisingly, students who changed strategies within the problems were equally successful with the students who applied only one strategy for the solution of the problems. This finding suggests that intra-task strategy flexibility did not support the students in reaching a correct answer. A qualitative analysis of the intra-task strategy flexibility showed that comprehending the problem situation intervened with the solution strategies and therefore influenced the correctness of the answer. Specifically, when the students ignored or altered a part of the problem information, no matter how flexible they were in strategy use, they were not able to reach the correct answer. This means that knowledge and flexible use of multiple strategies could not lead to success unless sufficient understanding of the problem was achieved. This finding is in accord with Mayer's (1985) view that the construction of a complete mental representation of a problem is essential for a successful solution. A practical implication for problem solving instruction that may be deduced is that teachers could provide support to their students, so that they can master skills of sense making and organizing the information given in a non-routine problem, before rushing them to make decisions about the problem solving strategies.

Despite the fact that this study gives evidence for a number of conclusions, we need to emphasize that it is only an initial attempt toward the exploration of strategy flexibility in non-routine problem solving. Thus, further research is necessary before we can draw more firm and generalizable conclusions. First, the study included only three non-routine problems of a specific kind. If we want to have more evidence for students' strategy use and flexibility in non-routine problem solving, we need to investigate these aspects of behavior with various types of non-routine problems. A second issue of discussion is the limited number of students involved and their specific characteristics (fourth-grade high achievers from the Netherlands). To find more robust evidence for the findings of this study more students should be involved in the data collection. Future research could also investigate whether these findings about strategy use and flexibility vary with age, ability and setting, by examining students of different grades, mathematical abilities, and educational systems. A final issue that needs further deliberation is how to measure students' strategy use and flexibility. In our study we focused on what was visible on students' test sheets. Students' inner thinking was not analyzed. In future studies more qualitative techniques could be used to collect data about students' cognitive processes, especially when alternating or maintaining strategies within or across problems.

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