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ORIGINAL ARTICLE

Mathematics education research embracing arts and sciences

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Abstract As a young field in its own right (unlike the ancient discipline of mathematics), mathematics education research has been eclectic in drawing upon the established knowledge bases and methodologies of other fields. Psychology served as an early model for a paradigm that valorized psychometric research, largely based in the theoretical frameworks of cognitive science. More recently, with the recognition of the need for sociocultural theories, because mathematics is generally learned in social groups, sociology and anthropology have contributed to methodologies that gradually moved away from psychometrics towards qualitative methods that sought a deeper understanding of issues involved. The emergent perspective struck a balance between research on individual learning (including learners' beliefs and affect) and the dynamics of classroom mathematical practices. Now, as the field matures, the value of both quantitative and qualitative methods is acknowledged, and these are frequently combined in research that uses mixed methods, sometimes taking the form of design experiments or multi-tiered teaching experiments. Creativity and rigor are required in all mathematics education research, thus it is argued in this paper, using examples, that characteristics of both the arts and the sciences are implicated in this work.

1 Introduction

'Beauty is truth, truth beauty,'—that is all
Ye know on earth, and all ye need to know (Keats, 1820/1953, p. 234).

As reflected in his famous closing lines to "Ode on a Grecian urn," John Keats had a deep sense of the extent to which the arts and the sciences are intertwined in the human psyche. As mathematics education researchers with interest in improving the experiences of those learning and teaching mathematics, we are involved with human beings in all their complexity. The *beauty* of aesthetic experience and the affective issues that accompany this experience or its absence, are counterbalanced and intertwined with the need for mathematical *truth*. "Don't force it! Maths just won't be forced. That's the beauty of it, that's its beauty: where it stands strong against this forcing things into it that don't have any place for it at all," emphasized Mr Blue, in pointing out an error of reasoning to the boys in his grade 12 mathematics class (Presmeg, 2006b, p. 20). Thus I shall argue in this position paper that despite the inevitable fashions that influence modes of research, the *humanism* of our endeavor necessitates the implication of aspects of both the arts and the sciences in investigating issues of mathematics education.

As a means of summarizing where we were a decade ago, I shall revisit a vignette that I described for the International Commission on Mathematical Instruction (ICMI) study, "What is research in mathematics education and what are its results?" in 1994 (Sierpinska & Kilpatrick, 1998). I shall view this vignette in the light of some of the directions taken in our field since this ICMI study, with the lens of aspects of the arts and the sciences as a focus for attention. Throughout this paper, the *arts* and the *sciences* are taken broadly as ways of viewing the world. I wish to highlight both the creative features of the arts, as epitomized in poetry, painting, and creative writing, and the humanistic features of fields that relate to human beings in

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all their complexity. At the same time, I acknowledge and celebrate the rigor and certainty (albeit contingent) of the methods of the sciences. The main thrust of the argument is that these contrasting aspects are not necessarily mutually exclusive, and that an integrated, unified whole is possible in mathematics education research—as it is in individual human beings—with an appreciation of the strengths, limitations, and purposes of each facet on its own terms, but in relation to the whole.

2 A vignette

In the 1990s, I taught a course on informal geometry to students at The Florida State University who are prospective middle grades and high school mathematics teachers. One goal of the course was to introduce the students to ways that manipulatives and real world experiences might undergird the learning of geometry in grades 5–8. In the first week I asked them to bring or wear to the next class, something that had geometry in it, and to come to class prepared to tell why they had chosen that particular item and to talk about its geometry. In an interview, one of the students, Dena (who wanted to teach algebra rather than geometry), told me about her reactions to this task, as follows (Presmeg, 1998a, pp. 57–58).

> *Dena*. I noticed when you said, for us to bring something to class or wear something that had geometry in it, for a little while I was having a difficult time, because, everything I picked up had geometry in it. And, I said, maybe there's something I misunderstood about the directions. Y'know.
>
> *Interviewer*. In fact, even just the shape of a piece of clothing, any clothing.
>
> *Dena*. Yeah. Anything, has geometry in it. So, for a little while I was confused. I didn't know what to bring to class, until, until I realized that, everything is going to have. I said to myself, everything, of course everything is going to have geometry to it because, y'know, anytime… You're going to make a desk. I mean, you draw, y'know. Your plans, for making the desk, involves geometry. And everything, that is, just everywhere. I think that geometry is taught as something abstract, sketching things with proofs and rules and, not as very, everyday.

Dena's recollections of her high school geometry experiences were negative ones. "I didn't like it at all!" she concluded.

It is important to note that in this vignette Dena is using the word "abstract" as a placeholder for the rote, and for her meaningless, way that she learned school geometry. This usage in no way implies that abstraction is unimportant in mathematics education. On the contrary, I believe that, along with generalization, it is essential in meaningful teaching and learning of mathematical content, a difficult and deep topic that I have addressed in more detail elsewhere (Presmeg, 1997b, 2008).

Implicit in this episode are several points that are relevant to the emergence of mathematics education as a field in its own right, separate from but not unrelated to other disciplines such as mathematics, psychology, sociology, philosophy, linguistics, history, and anthropology. It is significant that in coming of age, mathematics education research broke away from its primary reliance on psychometric research and emulation of the hard sciences. After all, in the complex worlds of human beings learning mathematics in group settings, all aspects of the arts and the sciences that might have bearing on the improvement of this learning are relevant.

2.1 The many fields implicated in learning and teaching mathematics

Firstly, the disciplines of mathematics in its research aspect and mathematics education research are related by their common interest in mathematics. However, these fields differ substantially because their subject matters and goals are different. The subject matter of research mathematicians is the content of mathematics, and without this content there would be no mathematics education. On one level, because mathematicians teach, they are also engaged in mathematics education. However, in mathematics education it is the complex "inner" and "outer" worlds of *human beings* (Bruner, 1986), as they engage in activities associated with learning of mathematics, that form a primary focus of the enterprise, and therefore also of its research. Dena's agonizing over the nature and boundaries of geometry is fruitful and provocative subject matter to a mathematics education researcher interested in the teaching and learning of geometry. The avenues along which this research may lead depend not only on the data, but also on the interests and interpretations of the researcher. The tendency of such hermeneutic research to use progressive focusing rather than pre-ordinate design (Hartnett, 1982) makes this kind of research as interesting as a mystery story, even if the mystery is to some extent self-created. In this respect, mathematics education research may have elements in common with mathematics research. Certainly, the humanism of the arts and the rigor of the sciences are implicated in both, despite their different goals.

A second point is that the inner and outer worlds of a student relate to concerns of the disciplines of psychology and sociology, respectively, and to the interactions between their elements. A balance between elements of these two disciplines is required in mathematics education, as witnessed in 1990s debates on the necessity of steering a course between Piaget and Vygotsky, representing individual and social aspects of learning, respectively, in constructing theory for mathematics education research (Confrey, 1991; Ontiveros, 1991). It is significant that Confrey believed that neither Piaget's nor Vygotsky's theory alone was adequate to model the complex processes of human learning. She suggested that proposing an interaction between the two lenses would necessitate significant changes in both theories.

Confrey's analysis prefigures the point—well expressed by Cobb (2007)—that in the face of incommensurable theories one way of proceeding is to find out how practitioners in the discipline of the parent theory view the canons of their research. This perspective enables the mathematics education researcher to bring a broader vision to the construction of home-grown theories that will be useful in addressing problems of mathematics education. Cobb (2007) explained the benefit of this attitude as follows.

> The openness inherent in this stance to incommensurability has the benefit that in coming to understand what adherents of an alternative perspective think they are doing, we develop a more sensitive and critical understanding of some of the taken-for-granted aspects of our own perspective (p. 32).

In the creativity literature it has long been a well-accepted principle that new views may be garnered by *making the familiar strange*, and by *making the strange familiar* (e.g., De Bono, 1970). However, Cobb (2007) went much further than that. He compared four theoretical perspectives that have been influential in mathematics education research. The first of these is experimental psychology, whose methodologies have been advocated again—as in the 1950s and 1960s—by funding agencies in the USA recently as the only form of *scientific* research in mathematics education (US Congress, 2001). Next is cognitive psychology, viewed from the *actor*'s perspective rather than the *subject*'s. The final two are Vygotskian sociocultural theory, and distributed cognition. In comparing these four perspectives with regard to their characterization of the individual learner, and in their usefulness for design research in mathematics classrooms, Cobb came to the balanced conclusion that each perspective has merits *for certain purposes*, but not necessarily for designing effective mathematics teaching. In his view, scientific randomized experiments are useful to and serve the administrative and political purposes of policy makers. He makes a strong case that insistence on the hegemony of scientific research in the form of randomized statistical experiments would be short-changing the community of classroom teachers of mathematics. As he shows clearly, all theories are based on philosophical premises, although those advocating a particular stance may not acknowledge the limiting effect of these choices. This analysis suggests that although the scientific and the humanistic aspects of mathematics education research are both legitimate and integral to the enterprise, they address different questions, have different purposes, and are useful to different stakeholders.

A third point implicit in Dena's pondering in the initial vignette is that philosophy is ubiquitous in all questions which are of concern to mathematics education researchers. The nature of geometry is an ontological issue, while how it was taught in Dena's school experience relates to issues of epistemology. Both components are essential in mathematics education theory building, since one's beliefs about the nature of mathematics and mathematical knowledge are the 'spectacles' through which one looks at its teaching and learning. These ontological and epistemological issues are still being debated in research that concerns the beliefs of teachers and students regarding the nature of mathematics and its teaching and learning (Leder, Pehkonen, & Törner, 2002).

Tension between the view that "Everything is mathematics" (as Dena expressed it, "Everything is going to have geometry to it"), and the rigorous mathematical position that "Only formal mathematics is valid", was well expressed by Millroy (1992) in her monograph on the mathematical ideas of a group of carpenters, who did not consider their practice to involve *mathematics*. This tension still plays out in mathematics classrooms. On the basis of her research results, Millroy argued strongly for the broadening of traditional ideas of what constitutes mathematics. She wrote, "We need to bring nonconventional mathematics into classrooms, to value and to build on the mathematical ideas that students already have through their experiences in their homes and in their communities" (p. 192). Steen's (1990) view of mathematics as the science of pattern and order opens the door to this lifting of the limiting boundaries of mathematics. Millroy's recommendation is consonant with those in the National Council of Teachers of Mathematics (NCTM)'s (2000) recent calls for connected knowledge in mathematics education. A related point is that a "mathematical cast of mind" may be a characteristic of students who are gifted in mathematics (Krutetskii, 1976). This mathematical cast of mind enables these students to identify and reason about mathematical elements in all their experiences; they construct their worlds with mathematical eyes, as it were. But unless teachers are aware of the necessity of encouraging students to recognize mathematics in diverse areas of their experience, only a few students will develop this mathematical cast of mind on their own. Many more will continue to regard mathematics as "a bunch of formulas" to be committed to short term memory for a specific purpose such as an examination, and thereafter forgotten (Presmeg, 1993).

The foregoing sets the scene for a fourth point which emerges from these considerations, namely, the links which mathematics education research has been building with various branches of anthropology, particularly with regard to methodology and construction of theory. Millroy's (1992) study was ethnographic. Entering to some extent into the worlds of Cape Town carpenters in order to experience their "mathematizing" required that Millroy become an apprentice carpenter for what she called an extended period, although the four-and-a-quarter months of this experience might still seem scant to an anthropologist (Eisenhart, 1988). But the point is that the ethnographic methodology of anthropological research is peculiarly facilitative of the kinds of interpreted knowledge that are valuable to mathematics education researchers and practitioners. After all, each mathematics classroom may be considered to have its own culture (Nickson, 1992). In order to understand the learning, or, sadly, the prevention of learning which may take place there, the ethnographic mathematics education researcher needs to be part of this world, interpreting its events for an extended period, and then documenting the culture of this world, making the familiar strange and the strange familiar while walking the tightrope of being in but not totally of the world that is observed. In this kind of research the humanities are implicit.

3 Recent trends in research foci and methodologies

While the history of mathematics goes back several millennia, mathematics education as a field of study in its own right is barely half a century old (Sierpinska & Kilpatrick, 1998). The oldest fully international journal in this field, *Educational Studies in Mathematic*, a few years ago celebrated its 50th volume. (The journal was founded by Hans Freudenthal in 1968. *Journal for Research in Mathematics Education* was started shortly thereafter.) All of the emphases identified in the foregoing section are still relevant to mathematics education research (Lester, 2007). However, in the last decade there have been some developments that emphasize the integrated nature of all the aspects of being human that play out in the learning of mathematics. I shall mention just a few of these trends here. The recent work of Luis Radford and his collaborators epitomizes two such strands, namely, an expanding emphasis on semiotics as a theory for mathematics education research, and the place of gestures, not as an adjunct but as part of an integrated semiotic system for learners to make sense of mathematical concepts (Radford, Bardini, & Sabena, 2007). Radford et al. use a "semiotic-cultural" theoretical framework as a lens for interpreting the learning taking place in a micro-analysis of a video segment in which a group of three grade nine students are trying to generalize the pattern in a sequence of geometrical figures. The video technology is indispensable in this fine-grained work, because the researchers aim to document the role of their gestures as semiotic means for students to grasp the ways that they are seeing the patterns, not merely for the purpose of communication, but in order to reify these patterns and give them meaning. This research emphasizes the integrated nature of human learning. The rigor of the careful documentation certainly has scientific qualities, while the humanities are implicit in the goals and methods of the investigation. Another recent indicator of the significance of attention to the whole learner rather than an emphasis on cognition, is evident in research that addresses the mathematical *identities* of learners, and the way in which culture and experience shape these identities (Sfard & Prusak, 2005).

In recent years it has become acceptable in mathematics education research to use a methodology of mixed methods (Johnson & Onwuegbuzie, 2004), in which the scientific rigor of statistical research is perceived as complementary to the intuitive insights that are possible in fine-tuned qualitative research. Each addresses different questions, and serves different functions. In mixed-methods research, going beyond the significance for different stakeholders that Cobb (2007) identified, an investigation may address the details of some educational phenomenon and attempt to generalize by identifying, for instance, how widespread the phenomenon is. Johnson and Onwuegbuzie present an eight-step process for conducting such research, which they consider superior to mono-method research. As more mixed-method investigations appear in mathematics education research it will be interesting to see whether they have significance for both the groups identified by Cobb (2007)—policymakers and administrators, as well as classroom teachers of mathematics. What counts as "good" educational research? Hostetler (2005) encourages researchers to move beyond questions of qualitative and quantitative paradigms, and to consider the ethical and moral values entailed in research methodologies.

The foregoing account downplays the contestations that accompany changes in any field, and these have certainly been present in the changing paradigms of mathematics education research too (Sriraman, 2007; US Congress, 2001). In what follows, I shall use a first-person-singular account of my own experiences (characterized as a war between the arts and the sciences in my own nature) in parallel with a narrative description of some elements of the changing field of mathematics education research during the last four decades. The different and sometimes conflicting voices in these accounts find a rationale in some elements of hermeneutics and phenomenology, which are addressed briefly in the next section.

4 A conceptual framework for a narrative account

To some extent a first-person narrative account finds conceptual underpinnings in a hermeneutic-phenomenological theoretical framework such as that used by Roth (2008) to justify his personal voice in analyzing editorial power and its role in authorial suffering in science education research journals, exacerbated by the demands of promotion and tenure processes in academia. As he points out (citing Ricoeur and Latour), this framework acknowledges and celebrates the importance of both scientific explanation and personal understanding in interpretation. Thus it is also an appropriate framework for an account that compares personal history and the history of a field, and that posits complementary roles for humanistic and scientific elements in both. The phenomenological dimension draws on lived experience, whereas the hermeneutic aspect relates to the interpretation of parallels between this personal experience and the changing modes of research in mathematics education. These interpretations can never be considered as complete. As Brown (1997) pointed out,

> In emphasizing that mathematics only ever comes to life in human exchanges we highlight [the] self-reflexive dimension. For Derrida, meaning is always in the future, always 'deferred', there is never a closure to a story because this story can always be extended (for example, 1992). … We can always explore further and revise the meanings we have created. The meaning we derive is always contingent. … I cannot disentangle things independently of my history (pp. 70–71).

Thus I undertake to analyze movements in the field of mathematics education research in conjunction with my own history as a mathematics education researcher. This hermeneutic-phenomenological position resonates with that of Peirce (1992, p. 313) in his construct of *synechism*, "the tendency to regard continuity… as an idea of prime importance in philosophy," the startling notion that knowledge in its real essence depends on future thought and how it will evolve in the community of thinkers.

In the following contingent account, I describe how the "war" between the arts and the sciences in my nature during my teenage years was reconciled to an integrated whole in the conduct of contemporary mathematics education research. I suggest that it is possible for the corresponding "war" between scientific and humanistic elements in the field of mathematics education research to find integration in recent unifying trends that see both quantitative and qualitative methodologies as valuable, although serving different purposes and having different goals.

5 The arts and the sciences—at war?

When I was a teenager, a senior in high school, I read Sir James Jeans' books about the universe, and I was also particularly inspired by the life and work of Marie Curie, who was a dedicated woman in the man's world of the hard sciences at the end of the nineteenth century. I was also intrigued by the incomparable life and work of Albert Einstein (1970, 1973, 1976, 1979). At that point it seemed that the arts and the sciences were at war in me, because I was attracted to both and choosing a career was difficult. At last, decades later, I "came home" to mathematics education research, which included elements of both of these two sides of my nature.

Albert Einstein was a visualizer, and his mental imagery was the rich source of his creative insights (Holton, 1973; Schilpp, 1959). In my first career as a high school mathematics teacher, I noticed that there were students in high school mathematics classes who were visualizers, as I knew from the exceptionally high spatial scores they were achieving on the battery of tests they were doing for vocational guidance—and they were achieving poorly in mathematics, as had Einstein in the restrictive environment of the *gymnasium* he attended in Munich before moving to Switzerland. The question of *why* demanded further investigation. Thus the following central research goal, as it concerned mathematics education, became the topic of my doctoral research (Presmeg, 1985):

> *To understand more about the circumstances which affect the visual pupil's operating in his or her preferred mode, and how the mathematics teacher facilitates this or otherwise.*

The research was exciting, absorbing, and full of surprises. In keeping with the phenomenological stance I am adopting, I see parallels between my experience in this investigation and the field of mathematics education research itself, which was starting to emerge as a field of study in its own right.

As suggested in the opening section, initially the study of problems in the learning of mathematics was a small subset of the wider realm of the concerns of psychology. With respect and admiration for the relative certainty of results obtained by researchers in the hard sciences, in which empirical investigation was used to confirm or disconfirm theory, early researchers in mathematics education (especially in the 1960s and 1970s) tried to emulate this research. Psychometric research was the only genre of research in mathematics education that was considered worthy of the name. Of this period, the Soviet psychologist Krutetskii (1976) wrote as follows:

> It is hard to understand how theory or practice can be enriched by, for instance, the research of Kennedy [in 1963], who compared, for 130 mathematically gifted adolescents, their scores on different kinds of tests and studied the correlation between them, finding that in some cases it was significant and in others not. The process of solution did not interest the investigator. But what rich material could be provided by a study of the process of mathematical thinking in 130 mathematically able adolescents! (p. 14).

Indeed, it was lamented that mathematics education research was having little impact, in fact appeared to be irrelevant, in mathematics teachers' classroom practices. Research as epitomized in "Aptitude-Treatment Interaction" studies (ATIs) seemed to have little impact or relevance in mathematics classrooms. The question of relevance is still an issue in mathematics education research, but more recent developments in this growing field as it embraces mixed methods and welcomes teachers as researchers (Kemmis, 1999) may have the capacity to address this issue.

In the early 1980s, when I was engaged in my doctoral research, qualitative, hermeneutic research under banners such as "illuminative evaluation" (McCormick, Bynner, Clift, James, & Brown, 1977) was starting to be viewed as legitimate in mathematics education because it could address questions about details of teaching and learning that were inaccessible to purely statistical research. My study involved both quantitative and qualitative methods, but it was the fine grain of transcribed interview data that enabled the insights into *why* some students who liked to visualize were not achieving their potential in mathematics. At about the same time, research carried out by teachers in their own classrooms (later widely accepted as "action research", e.g., Ball, 2000) was gaining currency. It was recognized that methods from other disciplines might need adaptation to the particular requirements of mathematics education research, but that there was a rich variety of methodologies that could be valuable. In the last three decades, mathematics education journals and conferences have proliferated, and universities internationally have established programs in mathematics education, housed either in schools of education or more rarely in mathematics departments. These changes accelerated in the 1990s. In a search for identity in its own right (Sierpinska & Kilpatrick, 1998), mathematics education and its research became recognized as a legitimate field, distinct from, yet informed by, the disciplines of mathematics, psychology, sociology, anthropology, philosophy, and even linguistics (Sfard, 2000; Dörfler, 2000). Mathematics education, as a human science, embraces human concerns as well as the need for abstraction and rigor. Various qualitative research methodologies adapted from the humanities became recognized as legitimate in addition to the previously dominant psychometric paradigms. In particular, following Bishop's (1988, 2004) seminal work, there was increasing recognition of cultural and social aspects of the classroom learning of mathematics, complementing the psychological emphasis of cognitive theories of learning. Despite some movements that resisted the changes (cf. the "math wars" in the USA), in this field there is no need for war between the arts and the sciences—both are important. I have come home!

6 Creativity in the arts and in the sciences: mathematics education creativity spanning both

As mentioned, the heart of Albert Einstein's immensely creative thought was his capacity to visualize (Schilpp, 1959). Mathematics has an obvious visual component, not only overtly, as in geometry or trigonometry, but also in the mental imagery that by self-report enhances the thinking of many creative mathematicians (Sfard, 1994). Why, then, were there visualizers in high school mathematics classes who were finding this subject so difficult that they were obtaining failing grades in examinations (Presmeg, 1985)?

The purpose of my doctoral research was to investigate the strengths and limitations of visual processing in mathematics in a classroom context at senior high school level, and to investigate the effect on learners who are visualizers of the preferred cognitive modes, attitudes, and actions of their mathematics teachers. (For a fuller account, see Presmeg, 2006a, b.) Selection of students and teachers required the development of a new mathematical processing instrument to measure preference for visual thinking in mathematics. I still use this instrument to understand more about the visualization styles of students in my classes. On the basis of the preference for mathematical visualization (MV) scores obtained using this instrument, 13 mathematics teachers were chosen to represent the full range of scores available. In the senior classes of these teachers, 54 visualizers (23 boys and 31 girls) were chosen from 277 high school students. Visualizers were taken to be those who scored above the median score for this population, on the preference test.

The research methodology included participant observation in the classes of the teachers over an eight-month period, and tape-recorded interviews with teachers and students, as well as sparing use of non-parametric statistics to identify trends in the data from the visualization instrument. As a framework for observation in lessons, 17 classroom aspects (CAs) were identified that the literature suggested were facilitative of formation and use of visual imagery in mathematics. The teaching visuality scores obtained by triangulation of viewpoints (teacher's, students', and researcher's) on the basis of the CAs were only weakly correlated with the teachers' MV scores from the preference instrument. It made sense that a good teacher who feels little need of visual supports might recognize the need of mathematics learners for more of these supports. After item analysis and refinement of the CAs, teaching visuality scores divided the teachers neatly into three groups, namely, a nonvisual, a middle, and a visual group according to their styles of teaching. Analysis of 108 transcripts of lessons revealed 45 further classroom aspects that differentiated the three groups of teachers, and that suggested that the visual teachers manifested traits associated with creativity, such as use of humor in their teaching. (Einstein had a marvelous sense of humor—see Dukas & Hoffmann, 1979.)

One of the biggest surprises in this research was that it was the teaching of the middle group of teachers, not the visual group, which was optimal for the visualizers in the study. All the difficulties experienced by the visualizers in their learning of mathematics related in one way or another to the generality of mathematical principles. An image or a diagram, by its nature, is one concrete case, and students need to learn how to distinguish the general elements from the specific ones in learning mathematics. Visual teachers, who had mastered these distinctions, were not cognizant of the difficulties experienced by their students. In my data, there were two ways in which a mental image or related diagram could represent generalized mathematical information. Firstly, the image itself could be of a more general form, which I designated *pattern imagery*. Secondly, a concrete picture (mental or represented on paper or a computer screen) could be used *metaphorically* to stand for a general principle. This latter result of this research led me to the fascinating study of the use of metaphor and metonymy in mathematics education, during the decade of the 1990s (Presmeg, 1992, 1997a, b, 1998b). However, I also became involved in another compelling research agenda, which I shall describe in the next section.

It is noteworthy that the need for rigor, including equivalents of validity and reliability, respectively, was never absent in the qualitative research paradigms that were gaining ground in the 1980s. But the pendulum swung too far away from the previous quantitative paradigm in the 1990s, occasioning a necessary backlash in the 2000s from proponents of statistical methodologies—suggesting that in this field the war was not yet over.

7 Different bridges: semiotic chaining linking mathematics in and out of school

In the last two decades, two strands of significance have been developing in the mathematics education research community. On the one hand, there have been increasing calls that teachers facilitate the construction of *connected* knowledge in mathematics classrooms (National Council of Teachers of Mathematics, 1989, 2000). These connections entail not only the linking of various branches of mathematics that have been taught as separate courses at high school level, but also the linking of classroom mathematics with other subjects in the curriculum. And particularly, the importance is stressed of linking school mathematics with the experiential realities of learners. On the other hand, the importance of symbolizing and discourse in the teaching and learning of mathematics has come to the fore (Cobb, Yackel, & McClain, 2000), along with recognition of the significance of sociocultural aspects of the learning of mathematics (Bishop, 1988).

I set out to link these two significant strands by exploring answers to the following question: *How can teachers use semiotic theories to help them facilitate the construction of connections in the classroom learning of mathematics?* In particular, semiotic chaining presented a fruitful method of bridging the formal mathematics of the classroom and the informal out-of-school mathematical experiences of learners. The significance for mathematics education of theories originating in linguistics was becoming apparent to me. At first in this research I used chaining of signifiers based on Lacan's inversion of Saussure's dyadic model of semiosis (Saussure, 1959). I investigated how teachers and graduate students could use these chains to link the cultural activities of learners with mathematical principles. Working with two research assistants and a doctoral student, Matthew Hall, we interviewed students and taught teachers to build such chains and use them in the mathematics classroom (Hall, 2000). There was the potential for the celebration of diversity and equity. We had some success, but the research suggested the need for a more complex model, because not just signifiers and signifieds, but *interpretation*, were endemic in the activities. Thus I was led to development a nested model of chaining based on the triadic theory of Peirce (1992, 1998). Some of his many constructs illuminated the research, like searchlights, and I am still excited and involved in the exploration of the repercussions of this work. Many instances of the potential of semiotic chaining to foster connected knowledge of mathematics illustrated its significance (e.g., Presmeg, 2006c), and the research is ongoing. Recently, I have been using a triadic Peircean lens to investigate ways that students connect, or fail to connect, the various registers (Duval, 1999) of school trigonometry (Presmeg, 2006b).

There are clearly intertwined elements of the arts and the sciences in this mathematics education research. In the wider field of research methodologies accepted as useful in the twenty-first century, a renewed interest in statistical research to counter the pendulum swing of the 1990s is evident. It will be a pity if another (counter) pendulum swing prolongs the war, because both qualitative and quantitative methodologies have a role to play in the complex field of research on the teaching and learning of mathematics.

In the next section I invoke Habermas's (1978) *knowledge-constitutive interests* to argue this case further.

8 Knowledge-constitutive interests invoking arts and sciences

Using Ewert (1991) and Grundy (1990) as sources, in Fig. 1 I have summarized the three types of knowledge and their philosophical bases posited by Habermas (1978). This triad comprises not merely three different ways of looking at knowledge, but three different ways of characterizing what *counts* as knowledge. It is beyond the scope of this paper to discuss Habermas's theory in depth. (Interested readers should consult the original sources.) In this paper I shall use this summary to argue that there is room in mathematics education research for all three kinds of knowledge.

	Technical	**Practical**	**Emancipatory**
Social media:			
	labour	*interaction*	*power*
Conditions for the three sciences:			
	empirical-analytic	*hermeneutic*	*critical*
→ procedures for basic activities:			
	control of external conditions	*communication*	*reflection*
Trichotomous division between sciences:			
	natural science	*cultural science*	*critical science*
Forms of knowledge:			
	instrumental rationality	*subjective meaning*	*critical theory*
Philosophical basis:			
	positivism	*phenomenology*	*critical theory*
**			
Eidos and disposition:			
	specific, definable ideas - techne (skill)	*the Good - phronesis (judgement)*	*liberation - critique (critical*
community)			
Action and outcome:			
	poietike	*practical action*	*emancipatory*
action			
	→ product	*→ interaction*	*→*
praxis			
**			

Fig. 1 Three Knowledge-constitutive Interests

Of Habermas's three types of interests that constitute knowledge, it is the technical one that epitomizes knowledge in the hard sciences. Literary creativity and research are examples of the seeking for knowledge of the second type, in which interpretation of the human condition is paramount. The enterprise seeks to understand that condition, but not necessarily to change it. The critical reflection called for in the third category, by way of contrast, has the goal of changing the human condition in some way—hence its designation as emancipatory. In contemporary mathematics education research, examples are found of all three types of interests. In broad categories, the *technical* interest is ongoing in large-scale statistical studies, the *practical* interest is evident in hermeneutic studies that aim for understanding of the mathematical thinking of individual students or small groups of students, and the *emancipatory* interest is apparent in studies that address issues of social justice and critical issues such as access to the study of mathematics. It is beyond the scope of this paper to characterize the landscape of mathematics education research in detail, but the following are examples of research in each of these three categories.

As an example of research in the first category, the investigations of Gagatsis and his co-researchers at the University of Nicosia seek new knowledge of issues in the teaching and learning of mathematics through the statistical investigation, using large samples, of such topics as "Students' improper proportional reasoning" (Modestou and Gagatsis, 2007), or "Exploring young children's geometrical strategies" (Gagatsis, Sriraman, Elia, & Modestou, 2006). Because it is not feasible to assign children randomly to the classes in these studies, the studies may be characterized as of pseudo-experimental design. The methodology enables group trends and relationships to be uncovered, without seeking to ascertain the reasons *why* these trends and relationships are significant. In-depth investigation of the question of "Why?" would entail research in the second category. In my own research on visualization, the construction and validation of an instrument for preference for visualization involved interests in the technical category: validity and reliability were established using non-parametric statistics (Presmeg, 1985). Large samples showed that there was no statistically significant difference between the boys and the girls with regard to their preference for visual thinking in mathematics; however, there was a significant difference between the preference for visualization of the teachers in this part of the study, and their students, who needed far more visual supports than they did.

Again, the question of *why* was deferred to Habermas's second category. Insights into the difficulties and strengths of visualization in teaching and learning mathematics came from interpretive research involving a whole school year of classroom observation and interviews with 54 high school "visualizers" and their 13 mathematics teachers. All of the problems experienced by these learners related in one way or another to the need for mathematical abstraction and generalization, as indicated in an earlier section of this paper. Whereas this kind of research provided insights, it did not have the overt goal of changing classroom practice, although teacher awareness of the results might in fact result in "practical action"—*praxis*—in the classroom (Grundy, 1990). Emancipatory interests, in contrast, have the goal of praxis.

Examples of research involving emancipatory interests can be found in the chapters of the monograph on *International perspectives on social justice in mathematics education* (Sriraman, 2007). After a useful historical introduction to issues of social justice by the editor, Sriraman, several of the chapters describe projects that in one way or another attempt to address the issues of equity that are implicit in social justice applied to mathematics education. For instance, Merrilyn Goos, Tom Lowrie, and Lesley Jolly describe a framework for analyzing key features of partnerships amongst families, schools, and communities in Australian numeracy education. Iben Maj Christiansen contributes a thoughtful and exploratory chapter based on her experiences introducing mathematical ideas to university students in South Africa and Denmark, through social data that highlight inequity. Her analysis leads her to the startling question, "Does our insistence on these 'critical examples' end up being 'imposition of emancipation'?" Tod Shockey contributes the positive influence of a culturally appropriate curriculum for Native Peoples in Maine, USA. Libby Knott explores issues of status and values in the professional development of mathematics teachers in Montana, USA. Eric Gutstein provides a companion piece to his recent influential book on social justice in a Chicago school classroom (Gutstein, 2006). These chapters and others have the more or less explicit goal of changing praxis in mathematics education. Although the monograph also contributes useful empirical and theoretical ideas to the ongoing conversation about social justice in mathematics education (practical interest), its emancipatory interest places it squarely in Habermas's third category. My own research on ways that teachers may incorporate the cultural practices of students in their classes into the praxis of school teaching and learning of mathematics also embraces this category to some extent (Presmeg, 2006a).

9 Final thoughts

Although I am positing a balance among Habermas's categories, and the necessity of embracing all three interests in various aspects of the complexities of mathematics education and its research, Habermas in his formulation suggested a movement in the direction of the critical theory component (Brown, 1997). Brown described succinctly the educational implications of movement towards the emancipatory interest, as follows.

> If we were to follow Habermas in defining more 'emancipatory' forms of educational practice we would need to differentiate more clearly between *teacher's intention* and *significance for the student* and stress the developing critical powers of the individual student. Such moves towards emphasizing interpretive aspects of mathematical activity, however, inevitably result in placing less stress on the conventional categories of mathematics, as may be represented in the teacher's input or school curriculum. … In doing this we may hope to achieve a style of teaching which enables students to critically examine the purpose and scope of the mathematics they meet, while at the same time recognizing its grounding in their personal experience (pp. 97–98, his emphasis).

It is my contention in this paper that it is not necessary to abandon the "conventional categories" of mathematics in striving for students' individual critical thinking and personal interpretation. Of the three categories of Habermas's (1978) knowledge-constitutive interests, the technical one pertains to the sciences, whereas the practical and emancipatory belong to the concerns and complexities of human life and its interpretation, to the integrated thoughts and feelings of human beings. The discipline of mathematics itself, with its inexorable logic and *instrumental rationality*, resides as a content domain in the technical category, although the creative domain of mathematicians doing research in mathematics might arguably relate better to the *subjective meaning* of the practical category. In contrast, because the teaching and learning of mathematics are practices engaged in by human beings, subjective meaning is all-important if mathematics is to be learned meaningfully, and *critical theory* relates to the improvement of this teaching and learning in mathematics classrooms. However, the content of mathematics with its historically constituted canons is the subject of this teaching and learning.

Thus I argue that both the sciences and the arts are inevitably implicated in mathematics education, whose research also requires the full gamut of methodologies available in the arts and the sciences.

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