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Toward networking three theoretical approaches: the case of social interactions

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Abstract We rely on discussions initiated at CERME4 and continued at CERME5 to compare, contrast and combine in a coherent way different theoretical frameworks currently used in mathematics education, with the eventual aim of networking between theoretical approaches. Specifically, we chose for this purpose the theory of didactic situations (TDS), the nested epistemic actions (RBC+C) model for abstraction in context (AiC), and the theoretical approach of interest-dense situations (IDS). As an example, we focus on how each of these frameworks is taking into account social interactions in learning processes. We identified not only connections and contrasts between the frameworks but also additional insights, which each of these frameworks can provide to each of the others. We also present some methodological reflections about the process of networking different theoretical approaches.

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1 Introduction

Learning processes are at the center of interest of mathematics education as a scientific endeavor. They are very complex, taking place in a multi-faceted environment, with many aspects interacting and influencing the process. The different theoretical frameworks used today in the field of mathematics education offer different ways for approaching learning processes and for taking into account environmental conditions and influences on these processes. No single framework is able to provide a full understanding of the complex phenomena at stake, but combining their respective insights in an efficient way is far from trivial. Each theoretical frame obeys its own logic and has its own coherence. It looks at the educational reality through a specific lens, without the ambition of developing a holistic view, a sine qua non condition for efficiency and operationality. Trying to combine theoretical perspectives thus presents the researcher with unavoidable problems of coherence and compatibility. It is a crucial question for mathematics education, how to cope with these problems of coherence and compatibility in order for the diversity of existing approaches to support our understanding of teaching and learning processes, and in order for research to give more effective assistance to the teachers who have to handle their complexity.

With the aim of making progress toward answering the question, how to cope with these problems of coherence and compatibility, it is certainly of interest to compare and contrast different approaches in order to identify possible connections between theoretical approaches, develop complementary or dialectical theoretical views, investigate when and why theoretical approaches contradict each other and, in the long run, establish a network of theoretical approaches (Bikner-Ahsbahs and Prediger, 2006).

The idea to compare, contrast and combine different theoretical frameworks was presented (see for example, Artigue et al. 2006b; Kidron, 2006) and discussed in the working group on theoretical perspectives in mathematics education at the fourth Congress of the European Society for Research in Mathematics Education in 2005 (Artigue et al. 2006a). The analysis presented in this paper is influenced by the discussion and views expressed in that working group and constitutes a theoretical attempt at comparison of three theoretical frameworks: the theory of didactic situations (TDS) (Brousseau, 1997), the nested epistemic actions (RBC+C) model for abstraction in context (AiC) (Schwarz et al. 2008), and the theory of interest-dense situations (Bikner-Ahsbahs, 2003). The aim of the present paper is to compare, combine and contrast these three theoretical approaches. We provide a concrete example in which we observe how networking permits to deepen the analysis of a given situation by a combined use of the three different theoretical frameworks. As an example to talk about networking we decide to exhibit, compare and contrast how social interactions, a phenomenon which is more and more considered as an essential dimension of mathematics learning processes, are taken into account by these different theoretical frameworks.

In the next section, the three theoretical approaches are presented. For each framework, we discuss the role of social interactions. The following section is the main section of the paper. In it, commonalities and contrasts are noted, and it is analysed what each framework may have to offer to the others, with respect to the role of social interaction. In the section that follows we illustrate this analysis with a concrete example. Finally, the concluding section presents a wider perspective on the potential benefits and difficulties of networking between theoretical approaches, and some methodological reflections about the process of networking.

2 Social interaction in three frameworks

2.1 Social interaction in the theory of didactic situations

In order to understand the way social interactions are dealt with by the theory of didactic situations (TDS; see Warfield, 2006, for an excellent entry level description), it is necessary to return to the origins of this theory and to the essential role that design has played in its development. As recalled by Perrin-Glorian in her analysis of the historical development of the theory (Perrin-Glorian, 1992), TDS's first aim was to lay the theoretical foundations for what Brousseau called at the time (in the late 1960s) an *experimental epistemology*. This contributes to explain the central role given in this theory to the situation, seen as a system involving three different components in interaction: students, a teacher and some piece of mathematical knowledge.[1] According to the TDS, students' learning results from interactions taking place within such systems and is highly dependent on the characteristics of these. The theory aims at understanding these dependences and also at determining conditions for their optimal functioning. In the TDS, mathematical knowledge is supposed to emerge first as means for action through models that can remain implicit, but it cannot develop without the building of an appropriate language (here the term language has to be understood in a very wide sense), and has then to become part of a fully coherent body of knowledge. These different steps rely on three different dialectics, the dialectics of action, formulation and validation (Brousseau, 1997), which require different types of situations. Another important point is that, according to the TDS, significant mathematical learning cannot be achieved if the students' work is too much dependent on the teacher. This basic assumption is embedded in the TDS through the notions of *adidactical situation* and *milieu*.[2] In a-didactical situations (an ideal type) students are expected to be able to test, reject, progressively adapt and refine their models and solutions thanks to the potential offered by the milieu of the situation in terms of action and feedback, without relying on the teacher's guidance, and without trying to guess the teacher's expectations.

In essence, the central object of the theory, the situation, incorporates the idea of social interaction. To each of the dialectics mentioned above is associated a particular type of game, and the games associated to the dialectics of formulation and validation cannot be conceived as games played by an individual learner. These are necessarily more collective games involving at least groups of learners, if not the whole class. The notion of situation of communication often associated with the dialectic of formulation, for instance, attests to this characteristic. In such situations, students are asked to send messages to other students allowing them to perform a given task, for instance

[1] In this part, the term "situation" has to be understood with the meaning it is given in the TDS, that is to say as explained in this sentence as a system involving three different components in interaction. In the TDS, different distinctions have been progressively made, and among these the distinction between a-didactic situations where interaction between students and knowledge can function without teacher intervention, and didactic situations where teacher intervention plays an essential role in this interaction.

[2] The a-didactic milieu was initially defined by Brousseau as the system with which the student interacts in the adidactic game. It generally includes material and symbolic artefacts, and other students. In Warfield (2006), the milieu is more globally defined as "all the pertinent features of the students' surroundings" regarding his (her) interaction with mathematical knowledge.

reproducing a geometrical figure, and the pair wins if the task is performed successfully, that is to say if the linguistic choices of the producer of the message support an effective mathematical communication. In situations of validation, the idea of mathematical truth emerges as a social construct through debates about mathematical assertions.

In addition, even with respect to the dialectic of action, an analysis of some paradigmatic situations such as the "Race to 20" or the "Enlargement of the puzzle" (Warfield, 2006, pp. 19–21, 55–57) shows that some organization of social interaction is constitutive of the design. For instance, in the Race to 20, the action phase in the original scenario is based on a succession of plays involving pairs of students. For enlarging the puzzle, students work in groups; first, each student in a group is in charge of the enlargement of a specific piece of the puzzle, and then they have to put all these pieces together to build the enlarged puzzle; usually, they discover that this does not work and, discussing the strategies they have used, they have to understand why. Social interactions thus play an essential role in the adidactic functioning of situations, that is to say in making a given piece of mathematical knowledge appear as the means of producing winning strategies through the interactions of the students with a certain *milieu*. As stressed by Warfield (2006), the adidactic milieu includes material and symbolic artefacts, but also other students.

Another point is that, in the TDS, the conceptualization of social interactions is not limited to interactions between students but also includes the teacher. Very early, this has been embedded in the theory through different notions. The main one is certainly that of *didactic contract*, understood as the system of reciprocal expectations (both explicit and implicit) between the teacher and the students as regards mathematical knowledge. The notions of *devolution* and *institutionalization*, central in the TDS, denote the processes organizing the distribution of mathematical roles in situations. An adidactic situation, indeed, can only exist if students forget for a while that the mathematical task posed to them has been prepared by the teacher with a specific didactic goal, and if they accept the responsibility of solving the task with their mathematical means and knowledge. According to the TDS, this delegation of mathematical responsibility to the students requires a specific action from the teacher called *devolution*. Conversely, the teacher has the responsibility to link the knowledge built by the students in adidactic situations with the intended institutional knowledge and to decontextualize it. This process is called *institutionalization*.

Social interactions are thus a central focus in the TDS, both interactions between students and student–teacher interactions. In engineering designs built according to the theory, particular attention is paid to the ways the organization of these social interactions can support adidactic adaptations through the creation of a milieu offering rich enough potential for action and retroaction. But conditions for a productive adidactical functioning are not so easily satisfied in standard contexts, and in classrooms, even experimental classrooms, most often, adidactic and didactic episodes tightly intertwine. Even if the same conceptual tools can be fruitfully used, as attested for instance by the special issue recently published by Educational Studies in Mathematics (Laborde and Perrin-Glorian, 2005) or by our own research work (Artigue et al. 2006b), the analysis becomes more complex. The extensive use of the TDS for analyzing and understanding the functioning of ordinary classroom situations in the last 15 years has thus led to specific theoretical developments concerning the notions of didactic contract and milieu (Bloch, 2002; Brousseau, 1997; Margolinas, 2004), and the characterization of practices developed by teachers in order to conciliate ordinary classroom constraints and institutional expectations in terms of mathematical responsibilities to be given to the students. This is for instance the case with the Interactive synthesis discussion[3] practice (Hersant and Perrin-Glorian, 2005).

Beyond these evolutions of the TDS, it is worth noticing that the increasing attention paid to ordinary classrooms has also lead to the development of hybrid constructions[4] combining concepts from the TDS and from the theory of didactical transposition due to Chevallard (1985) or from its extension in terms of anthropological theory of didactics (ATD; Chevallard, 1992, 2002), familiar to most users of the TDS. Regarding the analysis of social interactions, a good example of such constructions is provided by Sensevy et al. (2005) who combine the use of the TDS and of the notions of *mesogenesis* and *topogenesis* coming from the theory of didactical transposition. As they recall in the article just mentioned, mesogenesis "describes the process by which the teacher organizes a milieu with which the students are intended to interact in order to learn", while topogenesis "describes the process of division of the activity between the teacher and the students, according to their potentialities". They show how these notions can be used together with those of adidactic situation and didactic contract in order to understand teacher-student interactions

[3] The Interactive synthesis discussion is an intermediate practice consisting of problem solving sessions where students work in small groups followed by whole class discussions. In these discussions, specific techniques are used by the teacher for ensuring the progression of knowledge beyond what has been produced by students in the adidactic phase of group work, while giving them, collectively, some mathematical responsibility.

[4] Note that these hybrid constructions can be interpreted as the result of networking between TDS and ATD, networking being in that case facilitated by the cultural proximity of these theories.

and the ways these are affected by the mathematical knowledge at stake. Asking two different teachers to carry out Race to 20 lessons but giving them complete freedom in the organization of these lessons, the authors create an intermediate object between a lesson design piloted by the TDS and an ordinary classroom lesson, especially appropriate for such a study.

These combinations between the TDS and the ATD insert the analysis of social interactions proper to the TDS into a larger perspective. The basic object of the ATD is indeed the notion of institution, and the hierarchy of levels of determination introduced more recently into the theory tends to relate the understanding of what happens locally in a classroom about a specific mathematical topic to characteristics and constraints situated at the more global levels of the educational system, the society, the culture and even the civilization.

2.2 The AiC approach: social interaction as a component of context

The dynamically nested epistemic actions model of abstraction in context proposed by Hershkowitz et al. (2001) provides researchers with a tool for the analysis of processes of abstraction, where abstraction is defined as vertical (in the sense of Treffers and Goffree, 1985) reorganization of knowledge. A main aim of the model is to get insight at the micro-level into processes of learning by means of progressive abstraction, over of several lessons, while taking into account the contexts in which these processes occur, such as classrooms, tasks given to the students, and available technology.

According to the model, the genesis of an abstraction passes through three stages. The first stage consists of the emergence of a need for a new construct; the need may arise out of the design of learning, out of the student's interest in the topic or problem under consideration, or out of combinations of these; the student may be aware or not of the need, but without need, no process abstraction will be initiated.

The second stage constitutes the core of the model and of the process of abstraction, namely the emergence of a new construct. The associated process of knowledge construction is expressed in the model by means of three observable and identifiable epistemic actions, *Recognizing, Building-with, and Constructing* (whence *RBC*). The model suggests constructing as the central process of mathematical abstraction. Constructing refers to the first use of a new knowledge element and is largely based on vertical reorganizing of existing knowledge constructs in order to create a new one. Recognizing takes place when the learner recognizes that a specific knowledge construct is relevant to the problem she or he is dealing with. Building-with is an action comprising the combination of recognized knowledge elements, in order to achieve a localized goal, such as the actualization of a strategy, or a justification, or the solution of a problem. Building-with subsumes recognizing previous knowledge constructs. Constructing is composed of recognizing and building-with actions relating to previous constructs; not infrequently, constructing includes lower-level constructing actions. In other words, recognizing is nested in building-with; building-with, recognizing, and possibly lower-level constructing actions are nested in constructing.

Since constructing refers to the first instance of a learner's using or becoming aware of a construct, one may assume this construct to still be rather fragile for the learner. The third stage then consists of the progressive *Consolidation* (whence RBC+*C*) of knowledge constructs by means of recognizing, and building-with them during sequences of activities, which may include problem-solving activities, reflection, as well as further constructing actions. Research on consolidation shows students' increasing self-confidence and flexibility in recognizing and building-with the construct, as well as increased awareness, and linguistic precision referring to it.

The RBC+C model is empirically based. It has been developed and validated in a sequence of research studies showing great variety in terms of mathematical contents, ages of students, and social contexts in which learning took place. For more detail, we refer the reader to the review by Schwarz et al. (2008) and to the relevant research literature mentioned there.

In the AiC approach, contextual aspects are considered to be determining and integral factors of the learning process. Context is regarded in a wide sense, comprising historical, physical and social context. Historical context includes students' prior learning history, physical context includes artefacts such as computers and software, and social context refers to the opportunities, kind and frequency of interaction with peers, teachers and others.

Dreyfus et al. (2001) studied processes of abstraction and social interactions in parallel, and in conjunction. Pairs of students were led to discover a surprising numerical pattern and then asked to justify it. The students were thus collaborating on a task with potential for abstraction; more specifically, the intended constructs were (a) conceiving algebra as a tool for justification and, nested within (a), (b) an algebraic technique.

The researchers independently undertook a cognitive and a social analysis of the interview protocols, with the aim of comparing them. The cognitive analysis used the RBC epistemic actions, and allowed to generate diagrams showing episodes of the constructing processes. The social interaction analysis used common categories such as explanation, query, and agreement, as well as diagrammatic reference of each utterance to previous utterances. It allowed generating diagrams showing blocks of interaction. A main result of the research was that the cognitive and social diagrams show essentially the same blocks.

The other main result was the identification of patterns of interaction likely to support abstraction:

- Coherence is a characteristic of interaction that strongly favors abstraction; similarly, lack of coherence inhibits abstraction. Coherence is taken in the sense of sharing a common motive for an activity; in our case, the motive was to arrive at the (algebraic) justification;
- Symmetric argumentative interactions are likely to lead to construction of knowledge;
- In asymmetric interaction, with one student leading the other, combining guidance with (self)-explanation is particularly fruitful for abstraction.

In more recent work, Hershkowitz et al. (2007; see also Hershkowitz et al. 2006), investigated ways in which the common basis of knowledge of a group of students emerges from the individual students' constructing of knowledge through interaction, and as such enables the group to continue to construct further knowledge. The epistemic actions were observed within a larger continuum of activities to study the consolidating processes of the abstracted construct. Cognitive and interactive processes of constructing knowledge were investigated as a single process. This provided insight and understanding of the ways by which knowledge is abstracted by a group.

Methodologically, the data were considered as "stories" taken from the activities of two groups of three students each, from classrooms in different schools, on problems from an elementary probability unit. These stories use the epistemic actions R, B and C to exemplify flows that describe how shared knowledge was constructed out of the individual knowledge. The study showed that the shared knowledge of the group is characterised by its diversity, each partner expressing her own way of constructing a piece of knowledge. Yet all three-group members may benefit from this multifaceted shared knowledge in their common work, when going on to new constructs and/or consolidating constructs in follow-up and assessment activities. As in the earlier study, different patterns of interactive constructing were identified:

- In story 1, one student acted as the source for the construct and, in a very intensive series of questions and requests for clarification, supported the constructing process of a second student (asymmetric, guidance). In a further interactive phase, both these students supported the third, and thus the three students in the ensemble shared the constructed knowledge.
- In story 2, the two students co-constructed in interaction, and the knowledge was shared by both of them. A third student, objecting to her colleagues' shared construct (argumentative), constructed a unique strategy to solve the same problem.
- In story 3, the shared knowledge of three other students was constructed in a process of three cycles, from a shared awareness of the need for a construct (coherence), via denial of the correct construct (argumentative), to constructing the shared construct by all three students as an effect of the teacher's demonstration.

In a parallel line of research the role of teacher–student interactions in the construction of knowledge is being investigated through the lens of AiC (Schwarz et al. 2006). In summary, the cognitive development of peers learning together in groups and classrooms is closely linked to the interaction among peers and with the teacher; processes of constructing knowledge and patterns of social interaction strongly influence each other and analyzing them in parallel or as a single process serves to specify, detail and explain processes of knowledge construction.

2.3 Social interactions in interest-dense situations

In the project "Interest in mathematics between subject and situation" (Bikner-Ahsbahs, 2003, 2005), social interactions are not regarded as part of the learning environment but as basis, which constitutes learning mathematics itself. In this approach learning is assumed to be a social event in which mathematical knowledge is created through social interactions as part of the interaction space, and the participants align their behavior with the behavior of the other participants. A main assumption is that a thing in the world is closely related to a person's interpretations about this thing. That means: people behave towards a thing according to their meanings about it; meanings are created through interpretations within social interactions with other persons and can be changed during processes of negotiation (Blumer, see Wagner, 1999, p. 32). Analyzing scientifically in this sense means reconstructing the social processes by re-interpreting the interpretations according to the research question.

In the project mentioned above, so-called Interest-Dense Situations (IDS) were investigated in the classroom discourse of a sixth grade class with the teacher during half a year. Its result is a theoretical brick regarded as a contribution to the development of an interest theory, which is able to describe and explain the development of interest in mathematics. One source of interest development is the experience of interest activities in mathematics classes. Interest-dense situations are situations in mathematics classes in which students experience how interest-based activities are shaped by their classmates and themselves. These situations provide opportunities to act in an interest-based manner. Hence, interest-dense situations are situations which foster learning mathematics with interest. They consist of an epistemic process, begin with a mathematical problem or question and are closed as far as the mathematical theme is concerned. They are defined by three features: Within an interest-dense situation students get more and more intensively involved in the mathematical activity (*involvement*), they progressively construct further reaching mathematical meanings (*dynamic of the epistemic process*) and the activity leads them to highly regard the mathematics at hand (*mathematical valence*). The first task was to identify interest-dense situations within all class discussions of 89 lessons. This was far from easy. A lot of lessons were observed which showed only one or two of the features above but only 18 lessons contained interest-dense episodes; all of these were far away (in time) from tests.

The aim of the project was to reconstruct the conditions, which foster or hinder the emergence of interest-dense situations. The basic view was provided by the perspective of social interactions; building upon these, a profound analysis from the perspective of the epistemic processes and an analysis from the perspective of constructing mathematical values were carried out. Two methodical principles were used: reconstruction of the learning process while progressively comparing the social interactions, the epistemic process and the value construction within one case, and comparison of the learning processes among the cases. Comparison within and among the processes led to an ideal-type description of the genesis of interest-dense situations. Nine of the interest-dense situations occurred ad hoc due to a sudden utterance or question of one student. The other nine were socially generated so that processes of genesis could be reconstructed by their analyses. During interest-dense situations the teacher does not behave according to his own content-specific expectations towards the solution of the problem: he does not behave in an *expectation controlled* but in a *situation controlled* way. This means the teacher focuses on the students' utterances, he anticipates mathematical ideas, concepts, rules from the students' viewpoint and the direction, in which the social construction of meanings is about to develop. He supports the students in presenting their own mathematical views and gives assistance in the use of comprehensive words. The teacher will not usually evaluate, he rather poses questions to better understand the students' ideas. The students comment, change and state more precisely. Processes of this kind can only be sustained if the students do not orient themselves according to the assumed content-specific expectations of the teacher, but rather behave *expectation independent*. In these cases, the social interaction is oriented towards the mathematical content and not towards reproducing the teacher's expectations.

The interaction structure, which is shaped this way is very fragile. If suddenly the teacher behaves in an expectation-controlled way, a conflict can arise because the students resist the teacher's expectations. In this case, either the public conflict disturbs the epistemic process, or the teacher changes his behavior. The interaction process can go on if the teacher's and students' behaviors are not deeply related to each other. In this case each takes keywords from the other's utterances as starting-points; for instance, the teacher tries to offer help by posing questions although the student does not need any; the student might pretend to accept help by saying "yes" or "alright" but continues along his/her own ideas.

If, on the other hand, *expectation dependent* student behavior meets *situation controlled* teacher behavior, students filter the teacher's utterances in order to find out what the teacher wants them to say, and the teacher takes the students' utterances as an expression of their thinking process. Interactions of this kind look aimless; they do not have a common basis of orientation.

In most of the non interest-dense situations we find a very stable interaction structure in which the teacher arranges his behavior according to his content specific expectations, gives hints and poses constraining questions (*expectation controlled*) and the students try to use these hints to reproduce what the teacher wants to hear (*expectation dependent*). These interaction processes look like guessing games, which do not permit to concentrate on deepening the understanding of the mathematical content. They are easy to manage and this might explain why they occur often and proceed routinely. All the participants know that the problem is solved when the teacher's expectations are reproduced. This could be an explanation of the stability of such interaction structures.

If the teacher abstains from his/her expectations and this meets expectation independent student behavior a fragile interaction structure is shaped. This social structure fosters the dynamic of the epistemic process and is connected with valuing the mathematical content at hand in interest-dense situations.

Analysis from the epistemic point of view shows that the epistemic processes in interest-dense situations are built by three different epistemic actions: collectively gathering and connecting mathematical meanings, and structure seeing. A group of students gather mathematical meanings if the students in the group gather single units of a mathematical content like examples, counter examples, ideas, formulas, ... They collectively connect mathematical meanings if they, as a group, put pieces of knowledge together to make sense of connections. Structure seeing means perceiving a pattern or a rule which is tested, proved, verified, validated or confirmed. These three kinds of actions shape ideal type processes approximately found in the data. The dynamic of the epistemic process in these ideal type interest-dense situations is organized differently:

- In a three-step process gathering meanings provides examples and ideas first, connecting them leads to more insight in a second step and allows structure seeing in a third one. Students stay as long as they need on the first two steps until structure seeing is possible.
- In a spiral process, gathering and connecting meanings shape a spiral structure. Gathered meanings are connected immediately. This might initiate a process of gathering and connecting mathematical meanings based on the process before, and so on. This way mathematical ideas are worked out progressively as far as possible until structure seeing occurs and a new spiral process may start.
- In a confluent process, students work separately in a first phase. During the second phase the results are presented and connected with each other as far as possible. This provides the basis for structure seeing. Gathering, connecting and structure seeing are here intertwined.

Within interest-dense situations neither the teacher nor the students take the upper hand. The progression of gaining mathematical meanings is due to a fruitful interplay between the students' concentration on their own epistemic process and the teacher's concentration on the students' epistemic process, both focusing on the construction of mathematical ideas which might initiate additional involvement of the students and the teacher. Within this interplay, gathering and connecting mathematical meanings take place until the social interaction space is saturated with them and structure seeing occurs.

The whole process is pushed by the common goal to produce commonly valuable mathematical ideas. The students try to construct own ideas, the teacher provides a situation that makes finding valuable mathematical ideas possible. This way social interactions generate the emergence of mathematical knowledge in interest-dense situations.

3 Mutual benefits and additional insights offered to each other by the frameworks

In the last decade, the extension of focus in mathematics education from individual students' mathematical conceptions to social interactions among students and between students and teacher has become a general trend. As set forth in the previous sections, the three frameworks considered in this paper agree on the importance of social interactions for learning processes. Indeed, in the TDS by essence, the central object of the theory, the situation, incorporates the idea of social interactions. In IDS, social interactions are regarded as basis, which constitutes learning mathematics itself. And in AiC, processes of constructing knowledge and patterns of social interaction strongly influence each other.

Nevertheless, even if there seems to be an agreement between the theoretical approaches on the importance of social interactions, there are great differences in the ways in which the theoretical frameworks take social interaction into account. For example, in the TDS and IDS learning situations are central objects while in the AiC approach the focus is on the learner or an interacting group of learners. Moreover, experimental studies carried out in the two first perspectives generally concern classroom situations or at least some kind of institutional design while experiments using AiC consider a greater diversity of learning situations inside or outside the classroom. The AiC approach was used for example as the theoretical perspective in a research study on the learning processes of highly structured, advanced mathematics by a solitary learner (Dreyfus and Kidron, 2006).

Social interactions are also viewed differently by the TDS and IDS. In IDS, social interactions constitute the epistemic process. Thus, knowing is an outcome of the social processes in which a group of students struggle with a mathematical problem. An interaction structure which is shaped by the teacher and the students supports the emergence of these situations. In the TDS, the conceptualization of social interactions includes interactions between students and also between students and teacher. Social interactions between students are viewed as a contribution to the learning potential of the adidactic milieu. Social interactions between teacher and students are approached through the notions of didactic contract, devolution and institutionalization that structure the links between the adidactic and didactic models of situations. In the TDS, great attention is indeed paid to two crucial roles of the teacher: having the students take the responsibility for the mathematics when solving proposed tasks (devolution process), and conversely, linking what has been achieved by the students in the research phase to the official intended knowledge (institutionalization process) (Artigue et al. 2006b).

In a more general way, the different views the three theoretical approaches have in relation to social interactions force us to reconsider these approaches in all their details. The reason for this is that the social interactions, as seen by the different frameworks, intertwine with the other characteristics of the frameworks.

In order to compare, contrast and combine the three theoretical perspectives, it is not sufficient to note commonalities or contrasts. We are interested in examining, what insights each framework can offer to the two others in relation to the way this specific framework views the social interactions. Moreover, the specific aspects of one framework can be viewed in terms of the others and this re-viewing might bring mutual analytic benefits. Investigating these mutual analytic benefits is the core of this section, and this paper.

3.1 TDS and AiC

The categories of analysis of the AiC framework are clearly different from those of the TDS. As stressed above the two approaches do not focus on the same objects but on the learner and the situation, respectively. As regards the development of mathematical knowledge, they also use different categories: AiC approaches abstraction through three types of epistemic actions: recognizing, building-with, constructing; TDS distinguishes between three functionalities of mathematical knowledge: for acting, for communicating, for proving, which serve to organize the development of students' conceptualizations through appropriate situations. Thus the a priori analysis of the TDS accords high importance to the mathematical problem at stake, and the nature of the relationships with mathematical knowledge that the students can develop interacting with the milieu and their peers.

Due to its focus on the learner, it might seem that the epistemic actions in the RBC model are described independently of the characteristics of the contextual components that make them possible. In reality, however, contextual aspects in AiC are determining and integral factors of learning processes. That is why this framework is called a model of abstraction *in context*. Studies within the AiC perspective analyze the influence of patterns of social interactions on the processes of constructing knowledge by the learner. Moreover, on-going research studies within the AiC framework deal with the general question of the influence of contextual arrangements on different patterns of epistemic actions (e.g., Kidron and Dreyfus, 2007). At the same time, this kind of analysis contributes to the development of the analytical nature of the AiC approach.

3.1.1 Additional insights offered by TDS to AiC

The AiC approach, as a research methodology, is used with task sequences that have been designed with well-defined conceptual learning objectives in mind. However, it does not proceed from a design phase nor does it impose the kind of a priori analysis that is an essential methodological tool in TDS. The AiC approach could be enriched with the idea of developing a systematic a priori analysis, as is the case in the TDS. It would allow the researchers to better take into account, from the beginning, some of the contextual arrangements and the influence these can have on epistemic processes.

According to Hershkowitz et al. (2001), the genesis of an abstraction originates in the need for a new structure. In order to initiate an abstraction, it is thus necessary (though not sufficient) to cause students' need for a new structure. We may attain this aim by building situations that reflect in depth the mathematical epistemology of the given domain. This kind of epistemological concern is very strong in the TDS, and the notion of fundamental situation has been introduced for taking it in charge at the theoretical level. It could be helpful for AiC.

3.1.2 Additional insights offered by AiC to TDS

When TDS is used with a design perspective, situations are often modeled in terms of games, and in that case the winning states of these games must be clearly identifiable. It is expected that the students can tell if they have reached a winning state, in order to favor adidactic adaptations over adaptations piloted by the didactic contract. It is also expected that students, at least in their great majority, be able to reach such a winning state with pair interactions but without substantial help of the teacher. The situation is different in the AiC approach: the accent is not on the design of situations obeying the characteristics of adidactic situations recalled above; the task can be an open exploration task and the "end of the game" might be not very clear. But, as shown by AiC research, even so, it might be a situation offering a rich learning potential, and this vision can be helpful for TDS, especially when TDS is used for analyzing ordinary classrooms situations, which is more and more frequent.

In the AiC approach, the focus is on the learner or the group of learners. The identification of constructs in the AiC perspective enables the researcher to identify details of the constructing process. Even if the intended theoretical element, the "end of the game" has not been reached or has been reached only partially, the evolution of the process of construction and its connections with contextual aspects is important in itself. Such a detailed vision can offer complementary insights to those usually reached with the TDS for identifying the evolution of students' mathematical knowledge in the a posteriori analysis, and for becoming aware of some subtle constructions that could not be anticipated in the analysis a priori.

3.2 TDS and IDS

Close connections between TDS and IDS are less difficult to identify than between TDS and AiC. In the two approaches, learning situations and classrooms are given a central role. The characteristics of interest-dense situations and adidactic situations seem rather close, and the distinction made between student behavior according to its dependence or not on teacher's expectation in interest-dense situations can be easily interpreted in terms of didactic contract. Nevertheless the two theoretical frames do not simply overlap. Social interactions are given in IDS a more fundamental role than in the TDS. As pointed out above, they constitute the epistemic process, which is not the case in the TDS.

The combination of the TDS and the theory of didactic transposition has led to the notion of mesogenesis which "describes the process by which the teacher organizes a milieu with which the students are intended to interact in order to learn". This notion puts the accent on the dynamic character of the milieu, and the role the teacher plays in piloting this dynamic. Considering this process when analyzing situations could certainly help characterize conditions on situations for making them reasonable candidates for interest-dense situations. Within the framework of interest-dense situations such situations in everyday classrooms are identified and investigated in order to find conditions, which hinder or foster their emergence and describe their emergence as ideal types. There is an underlying social contract, which seems to allow or forbid the emergence of interest-dense situations.

3.2.1 Additional insights offered by IDS to TDS

Regarding the whole process and its outcomes as constituted by social interactions, the theory of IDS could offer TDS a micro-ethnographic approach, which allows to describe in detail, how the emergence of adidactic situations or adidactic phases in ordinary situations and its underlying social contracts are hindered or fostered.

3.2.2 Additional insights offered by TDS to IDS

Through the notions of didactic contract, adidactic situation and fundamental situation, the TDS offers another perspective to reflect on social contracts, on the dynamics of the epistemic process, and on the building of situations reflecting in-depth the mathematical epistemology of a given domain. This last aspect might be very beneficial, especially if there is an intention to extend the project of IDS from elementary to advanced mathematical thinking.

3.3 AiC and IDS

The focus of AiC are the epistemic actions, hence the epistemic process and its outcomes. Social interactions belong to the context. As has been pointed out above, analysis shows that social interactions are strongly related to the epistemic process: the epistemic process and the social interactions build the same blocks.

In IDS, social interactions shape the epistemic process; thus, knowing is an outcome of the social processes in which a group of students struggles with a mathematical problem: coming to know is part of social interactions in a classroom discussion. IDS research tries to find patterns which establish the whole situation. All interest-dense situations seem to be coherent, in the terms of AiC, and thus have a high potential to lead to constructing.

In both frameworks, epistemic actions are used but their genesis processes are different. The two models can be regarded as useful analytical tools for different but related purposes.

Investigating the epistemic processes in more detail might lead to mutual benefits for the two frameworks. For example, the following questions might be of interest: "What are the deeper reasons that the same methodological tools, namely epistemic actions, are useful for both, interest-dense situations and construction of knowledge? Are there (other?) epistemic actions that might be appropriate for investigating both, interest-dense situations and knowledge construction?"

3.3.1 Additional insights offered by AiC to IDS

AiC deals with contextual influence. The influence of additional components of context, in addition to the social interaction component, might also be of importance in the framework of IDS. As part of the context, the nature of the mathematical topics in the given domain could be considered. Taking into account that some constructions are fragile, the issue of consolidation might also be important for IDS research. This may help answer the question, under what conditions students are able to use (build-with) the knowledge constructed in interest-dense situations, in new situations, which are not necessarily interest-dense.

3.3.2 Additional insights offered by IDS to AiC

Looking at interest-dense situations as providing motivation for in-depth knowledge construction provides an analytic tool for investigating the emergence of the need for a new structure in AiC in terms of the motivation of the learner rather than in terms of design. Since it is based on epistemic actions as well, this analytic tool may be eminently suitable to be combined with the RBC epistemic actions. The perspective of interest-dense situations, its epistemic actions, and its background theory might enrich the analytic nature of abstraction in context including the view of its social constitution.

4 A concrete example

In this section, we demonstrate the general ideas developed in the previous section by means of a concrete example. The data set from which this example is drawn was provided by means of a video recorded during a lesson forming part of a long-term project (Castagnola, Dané, Impedovo, Paola and Tomasi, 2005), in which Italian students are introduced to the fundamental concepts of calculus at the beginning of high school.

In the specific lesson, a technological artefact (Cabri géomètre plus) was used in order to mediate the constructions of students' meanings for exponential growth. A pair of students, Gabriele and Ciro, is working for about 1 h on a sequence of three worksheets: in the first, the students explore the variation of the function $x \to 2.7^x$, mainly on the basis of its graph. In the second worksheet, the students use Cabri in order to explore the effect of varying the basis of an exponential function. In the third worksheet, the students investigate how the slope of the line tangent to the graph of the function $x \to a^x$ in the point of abscissa x changes with x. The video shows the students' explorations in three episodes corresponding to the worksheets. At the end of the third episode, the students realize that exponential growth is directly proportional to the value of the function itself. During the entire session the role of the teacher seems limited: he interacts just a little with the pair of students, mainly during the third episode.

In collaborative work that preceded the preparation of this paper, the authors and other colleagues were using, comparing, combining, and contrasting a number of different theoretical approaches while attempting to apply them to this common data set. In this paper, we deal only with the three frameworks described in the previous section, and only with social interaction aspects of learning processes. Each of the authors described and commented on the video from the point of view of the framework she or he was most familiar with. By means of these three analyses, we aim to exemplify the positions of the three frameworks in relation to the subject of social interactions, and the insights offered by each framework to the others.

4.1 The positions of the three frameworks on the role of social interactions as seen in the analysis of the video

A common reaction in the three analyses was a claim that the video did not provide the data required in order to do the appropriate analysis from the point of view of the specific theoretical framework. The interesting point is that the researchers using different theoretical lenses did not miss the same data.

Investigating for each framework the information that the researchers claim that they miss in the data offered by the video, we learn about the existing positions of the frameworks on the subject of social interactions in learning processes.

4.1.1 The TDS lenses

The TDS researchers miss information about the context of the learning experiment, about which grade the students are in, the students' mathematical and technological background, about didactic goals of this particular session and its place and role in the teaching of exponential functions. They explain that social interactions as reported in the video are essentially social interactions between the two observed students and this does not reflect the richness and complexity of the social interactions usually at stake in learning processes at schools. Moreover, they add that the learning episode reported by the video does not permit to access the institutionalization phase, which plays a crucial role in the learning process according to the TDS theory. They wish to know more about the didactical goal of the teacher, and especially more about the role of the described situation in the construction of knowledge "in the long run". They wish to know if the "nearly isolated" functioning of the teacher as described in the video is usual and if this is the case, how the institutionalization phase is taking place.

In their effort towards an a priori analysis, the TDS researchers try to anticipate what could have been the sharing of mathematics responsibilities between the teacher and the students. They claim that this could lead to a rather different didactical construction than the construction observed here, where the teacher would be given a more important role, and some local institutionalization would take place. They would have expected the teacher to exploit the students' exploration phase, trying to make common reflections with the whole class.

Pointing on what they miss in the data as described in the video, the TDS researchers demonstrate how the central object of their theory, the situation, incorporates the idea of social interaction and that the conceptualization of the social interactions is not limited to interactions between students but also includes the teacher.

4.1.2 The IDS lenses

The IDS researcher observes that the data shows a group of two students who work at the computer, mainly without the teacher, and that usually the data in IDS analyses consists of class discourses with the teacher but without the computer.

Social interaction in the sense this notion was used in previous IDS research takes place only at the end of the session. The earlier episodes show social interaction as pair interaction with separate roles. Nevertheless, taking the whole process as given, the IDS researcher was able to carry out an analysis by broadening the central concept of epistemic action in order to be able to include the specific aspects of the situation at hand.

Reconstruction of the case through progressively comparing the social interactions and the epistemic process of scenes within the episode led to the central result that the students become progressively involved into the epistemic process

- after they have overcome extrinsic obstacles like how to open a new file or what measure unit means,
- as they were able to raise a question which could not be easily answered,
- while they wanted to write down their results and
- when they started to explain their ideas to the teacher at the end of the episode.

However, comparison among learning situations in order to construct an ideal type description about the social interactions and the included epistemic process was not possible. For this purpose, the IDS researcher would need more situations from other pairs of students of the same class in order to reconstruct how interest-dense situations are generated in this class and how the teacher encourages (or blocks) this genesis. Other situations from the class are also necessary to find out how valuing mathematical contents takes place and what kind of common goals and underlying attitudes towards mathematical values the teacher and the students produce during social interactions.

4.1.3 The AiC lenses

Both TDS and IDS researchers require more information about classroom situations. This was not the case for the AiC researchers. The reason is that for TDS and IDS learning situations are central objects, while for AiC the focus is on the learner.

The AiC researchers missed a teacher-guided discussion. A similar claim was formulated by the TDS researchers, but the reasons for this claim were different. The AiC researchers explain that in such a teacher-guided discussion the concepts are worked out, so that constructing can be rounded off. They add that this could possibly also have been done by a much more structured worksheet. Their focus is on students' processes of constructing the knowledge under consideration. The missing information is required towards their analysis of students' emerging knowledge constructs.

The AiC researchers do not deny the importance of the social aspects. Social interactions are seen as a part of the context. The AiC researchers are interested in the contextual influence on the different patterns of epistemic actions. In this study, they claim that the limited data did not permit such an analysis.

In the following, we observe that in spite of the "limited" data offered by the video, TDS researchers did succeed in analyzing some contextual influence on the epistemic actions. This was made possible by the TDS "a priori" analysis. Such an a priori analysis might enrich AiC analysis. Mutual benefits between the different frameworks as seen by the analyses of the video are the subject of the next subsection.

4.2 Insights offered by each theory to the others as seen in the analysis of the video

4.2.1 TDS and AiC

In TDS the a priori analysis has a role of reference and of revealing the didactical phenomena. Quite often, the hypotheses made in the a priori analysis are not completely confirmed in reality. Differences are observed between what was expected in the a priori analysis and what actually happened. These differences are especially important for a deep understanding of the learning situation, if we wish to use TDS words. These differences might also enrich the in-depth analysis of processes of knowledge construction with the AiC lenses. We illustrate this claim with the video analysis: the TDS analysis of the video points at the different roles played by the two students and the resulting consequences in terms of "milieu". The TDS analysis draws our attention to the interesting fact that two different students facing a priori the same objective milieu can interact very differently with it. In fact they do not interact with the same milieu: from the very beginning, Gabriele manipulates the mouse, works with the computer while Ciro works with paper-and-pencil. Gabriele works essentially in the graphical mode while Ciro works in the algebraic mode.

The TDS analysis considers the cognitive effects of this different sharing of role observed in the computer environment. For Gabriele, the interactions with the "milieu" are as could have been expected in an a priori analysis. That is not the case for Ciro who did not interact directly with the machine. An algebraic register took an active role, which would have been unexpected in an a priori analysis.

We illustrate this observation with excerpts that relate to the way the students work with the two first Cabri worksheets as written in an English transcript of the video.

The TDS researchers prepare an a priori analysis of the two situations that relate to the first two Cabri worksheets.

In the first situation, the students were asked to explore the function $x \to 2.7^x$ from its graphical representation. The students were asked to study how y varies when x varies and then to observe what happens when x is negative. In the second situation, the students were asked to use the second Cabri worksheet to explore the notion of basis of an exponential function. This aim was translated in the TDS a priori analysis into an understanding how the value of the basis of the exponential influences the way the curve looks or, in other words, into an understanding of the link between the value of the basis and graphical characteristics of the curve. Some conjectures were prepared by the TDS researchers in relation to the anticipated interactions of the students with the milieu in the two situations.

For Gabriele, these anticipated interactions with the milieu were observed as expected in the a priori analysis. For example, one conjecture in the a priori analysis that relates to the first situation was that when x decreases with negative values smaller and smaller (x is moving to the left) y is approaching more and more the value zero and then obtains this value.

This conjecture was expressed by Gabriele

(40) *This is approaching zero.*

(109) *For the negative x's the function decreases up the point... up to this point here where it's zero.*

The conjecture leans on an observation of the Cabri worksheet. It seems that in order to find the value where the function is zero, Gabriele moves the point x from the right to the left, and this gives the "feeling" that the values of $f(x)$ decrease.

Ciro's reactions were different.

For example, one conjecture in the a priori analysis that relates to the first situation was that "when x approaches 0, y approaches 1 and possibly for x equals 0, y equals 1, even if this fact is not directly observable". Ciro's first conjecture relates to the value at zero in terms of equality.

(3) *It's 1 at 0.*

This conjecture was not a result of a graphical observation: Ciro adds an algebraic proof:

(5) *2.7^0, a number to the 0 gives 1.*

Ciro also claims that

(9) *If we replace 2.7 by 1 we obtain a straight line.*

This conjecture was anticipated in the a priori analysis of the second situation with the second Cabri worksheet in which the students could manipulate different values of the basis a and reach different conclusions for $a > 1$, $a < 1$, and also for $a = 1$ in which they could realize that the curve is a horizontal line. But this manipulation was not possible with the first worksheet which the students followed at that time. Indeed, as we can read in the following excerpt Ciro's claim was a result of an algebraic register:

(23) *While x is changing, even if x is 100, however, 1^{100} is 1.*

Ciro pronounces another conjecture:

(90) *The value of $f(2)/f(1)$ where $f(x) = 2.7^x$ is 2.7!*

It seems that once more he uses the algebraic register. He does not manipulate the computer and in any case the "verification" with the computer does not give values of $f(2)$ and $f(1)$ which permit to obtain precisely 2.7 as a ratio.

All these excerpts demonstrate that the two students did not interact with the same milieu.

Such an a priori analysis, as demonstrated above, which is an essential methodological tool in TDS, might enrich the AiC analysis of the learners' epistemic actions. It will permit to better take into account from the beginning some of the contextual factors and their influence on epistemic processes.

The task proposed to the students in the video session is not the solving of a mathematical problem as usual in tasks designed using TDS or AiC, but an exploration task. For the TDS researchers this might affect the analysis as they cannot model the situation as a game whose winning states could be clearly identifiable. Thus the question of the "end of the game" is raised and also the question of the role-played by the didactic contract in that respect. For the AiC researchers, the lack of a specific mathematical task creates the difficulty of defining the specific mathematical constructs on which the analysis could focus.

Nevertheless, the AiC analysis of the video demonstrates that even though the "end of the game" is not well defined, the situation does offer a rich learning potential. In fact, using the data offered in the video the AiC researchers aimed to identify knowledge constructs and this identification helped them to see the details of the constructing process. As an example, they observed the transition from the geometric representation of the derivative as a tangent, a local construct, to the more global view as expressed by Gabriele while manipulating the third Cabri worksheet:

(349) *well, if you take it... if you take it with a very large zoom... you can approximate it with many small lines*

(351), (353) *Such lines may have slope that increase*

This view of the exponential function which can be approximated by many small line elements, whose slope increases with x, marks the transition from a set of points to the graph of the exponential function—the transition from discrete to continuous. This transition was observed by means of verbalization with a language more and more precise during the constructing process.

These details of the constructing processes help towards an awareness of subtle constructions that might not be anticipated in the TDS a priori analysis.

4.2.2 AiC and IDS

AiC researchers are interested in epistemological, cognitive and social factors towards their qualitative analysis of processes of knowledge construction. While AiC researchers may consider individual and social aspects of knowledge construction, IDS researchers focus on interactions between students and between students and teacher, since they consider knowledge as being socially constituted.

The transcript of the video contains three episodes. It is interesting to note that AiC and IDS researchers decided separately to focus mainly on the same part of the third episode. Their analysis, however, is different.

The AiC analysis highlights the identification of constructs and their specifics like “the geometrical representation of the derivative as a tangent”, “the asymptotical behaviour of the function”... The AiC analysis highlights the evolution of thinking: the passage from a construction to another, from a local construct “geometrical representation of the derivative as a tangent” to a global view “the exponential function which can be approximated by many small lines which have an increasing slope”. The analysis deals with the fact that constructions are nested. It also deals with the way students “build-with” earlier constructs, for example during further constructing actions. In the following, we describe a short intervention of the teacher at the end of the episode and his challenging question. The teacher’s intervention is used by the AiC researchers to observe a consolidation phase for the couple of students.

In this episode we observe the students’ construction of knowledge that in the case of the exponential function, the growth percentage of the y’s remains constant.

The teacher asked them a challenging question:

(360) *Does it surprise you, the fact that the function crushes on the x-axis? Here [x < 0] it seems that the function increases not much and here [x > 0] it increases very much. Does it surprise you such a type of increasing with the ratio being constant?*

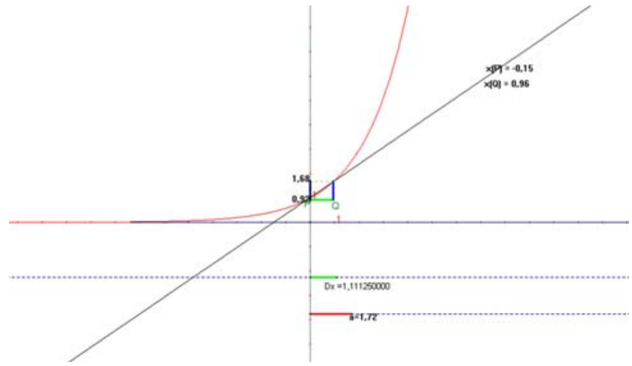
The students’ self-confidence, their resistance to challenges were observed in their answers.

(362) *Sure, because before, the numbers are small and with small numbers the ratio is always between nearer points.*

The researchers also noticed that the students’ language is more and more precise while they consolidate earlier constructs.

(368.1) *If the dx increases again, the line passes through P and Q and is almost constant, it becomes almost a tangent... This is because if we take a very big zoom, we can approximate the exponential function with many lines with an ever increasing slope.*

(368.2) *Then if the point P is very close to zero, this line approximates very much the exponential function. Also, here where the numbers are very small, it increases very little, almost like a straight line.*



The focus of the IDS analysis is different: it is not only on the evolution of the precision of language but above all how this is generated. The IDS analysis makes it clear that during the process the students construct further reaching meanings about the exponential functions but the analysis does not specify these constructs in detail. Instead, there are observations of the way students get excited about the process of constructing deeper meanings, how they get involved in the activity and how they encourage each other: For instance, the analysis shows two different kinds of obstacles; those in which the students do not get involved (which are avoided or overcome with the help of other students or the teacher) and obstacles in which the students become involved. Their contrasting behaviors indicate the students’ interest in “constructing new mathematical meanings” as they tried to understand and describe how a tangent of the exponential function is produced by the electronic learning environment.

(287) *Ciro: look it ... slowly ... slowly it seems that ... I do not know like saying tangent*

(288) *Gabriele: eh ... yes*

(289) *Ciro: It seems that it touches it, let’s go let’s go let’s go*

Ciro describes what happens and interprets it (287). He encourages Gabriele to continue drawing the mouse at the computer, and seems to get excited (289).

The situation consists of three episodes. Mainly gathering meanings and overcoming extrinsic obstacles are observed in the first episode, mainly connecting meanings and deepening insight through writing down the findings are observed in the second episode, finally seeing structures, describing, explaining and validating them occurs during the third episode. The three episodes organize a three-step situation. Furthermore, the IDS analysis draws our attention to the increasing involvement of the students, which becomes more intensive when the teacher is included in the discussion: the fact that the teacher shows interest in their mathematical views involves the students even deeper.

When we read the two analyses, the different focus of attention is clear. Interestingly, the two teams chose the same part of episode for their main analysis. In this part of the episode, the AiC analysis recognized deep construction of knowledge. It seems that among interest-dense situations there are situations with deep construction of knowledge as well. Even so, IDS analysis could benefit from other components of context such as the epistemology of the given mathematical domain as it is used in the AiC analysis. It could benefit also from the cognitive factors as used in the AiC analysis of epistemic actions.

On the other hand, using the IDS analysis, the AiC team gains an interesting additional perspective of the relation between the motivation for a new construct and the process of knowledge construction. This is especially important because the need for a new structure might be seen as a first step towards abstraction.

4.2.3 IDS and TDS

The above description of the IDS analysis of the video, the entire learning situation in which the social interactions are highlighted including, for example, the way the teacher shows interest in the students' mathematical views, might offer TDS an approach which allows to describe, how the emergence of adidactic situations or adidactic phases are encouraged, and how the teacher manages the devolution process necessary for adidacticity.

Vice versa, we show in what follows how the a priori analysis of TDS offers another perspective to IDS to think about the building of situations reflecting in-depth the mathematical epistemology of a given domain and the consequence of such reflection on the analysis of the social interactions.

The IDS analysis of the video mentions that the activity is divided between the two students: Ciro reads out the tasks on the worksheet, Gabriele, working directly with the computer, performs according to Ciro's instructions and then describes what he sees on the screen.

The IDS analysis questions whether the group acts as a unity and what role the computer plays. Detailed answers to these questions are offered by the TDS analysis of the students' role sharing in the computer environment. The important point we want to make clear is that the awareness of these specific interactions between the two students was possible by means of the a priori analysis, which took into account the mathematical epistemology of the given domain. The fact that an algebraic register took an active role, which was unexpected in the a priori analysis reveals important details about the interactions between the two students.

The results of the analyses showed differences in relation to two crucial characteristics of social interactions in the teaching–learning experience: Interaction between the two students, and the role of the teacher. The TDS lenses, by means of a priori analysis and of the identification of the characteristics of the interaction with the milieu that each student develops, reveal to the readers the role sharing of the two students in the computer environment, and the cognitive effects of this role sharing—a fact which was not clear in the other analyses. On the other hand, AiC and IDS lenses reveal to the readers that the role of the teacher was not as limited as it seemed at first glance. AiC lenses reveal this fact by focusing on the learners' epistemic actions (the consolidation phase) and IDS lenses reveal it by observing students' increasing involvement and their progression of insight.

Being intrigued by the differences in the analyses in these two crucial aspects of social interactions, we wanted to know more about the teacher's views and aims in this experience. The teacher was asked to answer a sequence of written questions. We report some excerpts of his answers.

(1) In relation to student interactions:

I hope students interact actively; that they listen to the words of other students... In general, I don't like that Ciro uses only paper and pencil and Gabriele uses only the PC. This subdivision of the role may be useful in order to reach the final result in a short time, but it may be an obstacle for the process of construction of meaning.

The obstacle was very clear in the TDS analysis. However, the analysis from the IDS perspective in the last episode shows that the two boys work deeply together when the dynamic of the epistemic process increases, for example they become deeper involved in the process of constructing mathematical meaning when they write down their findings.

(2) In relation to the role of the teacher (the teacher was asked how he decides when to get involved with a pair of students):

I enter a working group if the students call me. ...At other times, I enter because I realize that students have very good ideas that need to be treated more deeply. ...I try to work in a zone of proximal development. The video analysis and the attention we paid to gestures made me aware of the so called "semiotic game" that consists in using the same gestures of students but accompanying them with a more specific and precise language in relation to the language used by students. The semiotic game, if it is used with awareness, may be a very good tool to introduce students to institutional knowledge.

Even if AiC or IDS lenses were not aware of the semiotic game, they did realize the important role of the teacher and his influence on the students' knowledge constructs for AiC and on the students' increased interest for IDS.

It might be interesting to point out that the analyzed video comes from a group of Italian researchers who usually rely in their approaches on semiotic games.

Gestures can very well attest to epistemic actions. Nevertheless, neither the AiC or IDS lenses paid attention to such gestures. This observation might demonstrate that for addressing certain semiotic concerns the TDS, AiC and IDS frameworks should need to be networked with other frameworks.

5 The process of networking between theoretical approaches: difficulties and methodological reflections

In the preceding sections, we demonstrated that the three theoretical frameworks potentially complement and thus enrich each other if links between them can be established. We pointed out potential benefits but we should also point to the problems that will necessarily arise in the process of linking between theoretical approaches. Therefore, the crucial question is not only *whether* the theoretical approaches can complement each other but *how* this can be achieved.

There is no doubt that the history of the domain of mathematics education is rich in constructions combining the affordances of several theoretical approaches for benefiting from their complementarities. And even in this article, when presenting how the TDS approaches social interactions, we have mentioned productive links made between concepts from the TDS and the ATD. But these two theoretical approaches were born in the same educational and didactic culture, have been developed jointly, and building links between them has been a constant effort of the didactic community since the 1980s. The problem we address in this article is a much more difficult one.

In the previous section, in which we described the three analyses of the video learning–teaching experience, we observed the researchers' common reaction that the video did not provide the data required in order to do the appropriate analysis from the point of view of each specific theoretical framework. This common reaction of insufficient data reflects in fact deeper questions about the minimal units of reality which are considered as pertinent in a given research paradigm in order to make sense of this reality and to permit the analysis of the observed facts. The video leads us as researchers to think about the interesting question what constitutes a significant unit for our didactic analysis. The answer might be different for different theoretical lenses.

5.1 Difficulties

Our efforts at answering the question *how* the three theoretical approaches can complement each other force us to make very clear the assumptions underlying each theoretical framework, some of which may be hidden. This is rewarding in itself but let us consider the difficulties that may arise in the process. Indeed, considering the three frameworks described in this paper, there might be possible contradictions between the underlying assumptions of the theoretical approaches.

Specifically, we have observed how each theoretical framework has its own way of considering the role of social interactions in the learning process: the social interactions are an important part of the context in AiC; but in relation to IDS, social interactions are not viewed as a part of the context: they are the basis that constitutes learning mathematics; and in the TDS, social interactions are part of the situation, the system of relationships between teacher, students and mathematics. Given these differences, the question arises *how* it is possible to establish links between the theoretical approaches without getting embroiled in contradictions between the basic assumptions underlying each theory.

To be more specific about the problems that may arise, let us limit our considerations temporarily to two theoretical approaches: As a consequence of the above differences, the categories of analysis of the AiC framework are different from those of the TDS. As stressed in the previous sections, the two frameworks use different categories in relation to the analysis of the development of mathematical knowledge. AiC approaches abstraction through three types of epistemic actions: recognizing, building-with, constructing; TDS distinguishes between three functionalities of mathematical knowledge: for acting, for communicating, for proving, which serve to organize the development of students' conceptualizations through appropriate situations. Should we use both categories of analysis? Should we try to find a smallest common denominator between the categories (which might turn out to be empty)?

Similar difficulties arise while using the lenses offered by interest-dense situations and AiC: although epistemic actions are used by both frameworks, not only are they different actions but they are viewed in different ways. Investigating whether there are (other?) epistemic actions that might be appropriate for considering both, interest-dense situations and knowledge construction is a complex issue.

Having become aware of the substantial difficulties involved in any attempt to connect theoretical approaches, we raise the question what can (and what cannot) be possible aims of such an effort. Clearly, any attempt at unifying the three theoretical approaches, or even two of them, into an encompassing theory is doomed to failure before it even starts. Such an attempt would necessarily destroy the basic assumptions of all theoretical approaches involved, or at least of all but one. What, then, can we aim at? We propose to aim at establishing a network of links between the theoretical approaches. In networking, we want to retain the specificity of each theoretical framework with its basic assumptions, and at the same time profit from combining the different theoretical lenses. What we aim at is to develop meta-theoretical tools able to support the communication between different theoretical languages, which enable researchers to benefit from their complementarities.

5.2 Methodological reflections

In the example described in the previous section, the authors were comparing, combining, and contrasting theoretical approaches while attempting to apply them to the common dataset provided by the video. One might say that we were beginning to "network with theoretical approaches". Here, the term "networking with theoretical approaches" is used in a sensitizing way in order to find out how theoretical approaches can be combined, compared and contrasted. One of our aims is to develop heuristics about how networking with theoretical approaches takes place and what it could potentially lead to. Through negotiations and methodological and methodical reflections meta-theoretical tools might be developed.

We assumed that researchers networking with each other as theorists produce implicit knowledge about how "networking with theoretical approaches" could proceed. We further assumed that this implicit knowledge can be uncovered through reflections about the process.

How did we proceed? We chose an aspect of the learning process, which has some relevance in all three theoretical approaches, namely social interactions. We did not specify this aspect very precisely in order to leave it relevant for all three frameworks. We presented different views on this aspect and its roles in the different theoretical approaches. We compared and contrasted each pair of theoretical approaches in more detail focusing on benefits, additional insights, and tools which one theory can offer to the other and vice versa.

Our analysis of the complexity of the process of linking between theoretical approaches led us to the conclusion that the following heuristics might support networking:

- Use a common, but not precisely defined aspect that all the theoretical approaches share and produce an overview of the theoretical approaches according to this aspect;
- Find out what ideas each pair of theoretical approaches share;
- Compare and contrast each pair of theoretical approaches according to the common aspect; consider the benefit, additional insight, limitations and tools each of the approaches can offer for working with the others;
- Connect the results into a set of complementary views taking into account all three theoretical approaches, and describe how this might be able to assist our understanding of learning processes.

As mentioned in the introductory chapter our paper is an example for coordinating theoretical approaches. This might be a first step towards integrating approaches.

Comparing and contrasting the three theoretical approaches we began to understand the other theories and even our own theoretical approaches better than before. As we started to look for how combining theories might be possible while experiencing what the theories can mutually learn from each other we did not only understand the theories better, we found commonalities and differences. As we made progress in this networking process we discovered complementary aspects, which could be regarded as aspects to coordinate or integrate.

These insights lead to the impression that the presented networking strategies in the introductory chapter are nested in nature: a process of integrating or even synthesizing theories would include processes of combining and coordinating in order to find out in what way the theories are complementary; combining and coordinating includes investigating commonalities and differences through processes of comparing and contrasting; processes of comparing and contrasting involves understanding the theories in question and making them understandable.

6 Conclusion

In this paper, starting from the diversity of existing theoretical frameworks in mathematics education, and the impossibility of any one of these to give a full account of the complexity of learning processes in mathematics, we presented the idea of looking for fruitful combinations or networking between theoretical approaches. For exploring this idea, relying on discussions initiated at CERME4 and continued since then, among others at CERME5, we decided to select theoretical frameworks we were familiar with, and to investigate how these could be compared, contrasted and combined in a coherent way in order to increase our understanding of learning processes in mathematics. For this purpose, we selected three theoretical frames: the Theory of Didactical Situations, the nested epistemic actions model for abstraction in context, and the approach in terms of interest-dense situations; as an example, we discussed in some detail how each of these is taking into account social interactions. We observed in a concrete example how a combined use of the three frameworks permits to deepen the analysis of a given situation but we should also make clear that behind this important practical use there is the drive to make mathematics education as a discipline progress by achieving more internal cohesion.

The theoretical frames we have chosen are quite different and thus constitute good examples for illustrating the existing diversity in the field. Two of them are situation centered while the third one is learner centered. One of them began to develop about 30 years ago; it has been used by scores of researchers who have contributed to its development. Understanding the complex object it has become along the years is not easy, and many researchers in mathematics education have only a superficial knowledge of it. The two other frames are more recent constructions, developed and used up to now by rather small communities. They do not have such a large scope, and at least at a first sight it seems easier to become reasonably familiar with their main constructs.

Working collaboratively, we have tried to understand our respective didactical cultures, to identify interesting similarities and complementarities between our perspectives, and boundary objects that could support connections. Even focusing on social interactions, an aspect that plays an important role in all three frames, this was far from being an easy task. It required from each of us a costly effort of decentration. The cost of this effort evidences the strength of the coherences underlying our respective didactical cultures, and the specificities of the educational and research experiences underlying these. Looking back at this emergent work, what seems important is the fact that in spite of the diversity of our experiences and cultures, we share common concerns, and that the theoretical constructs we develop or use are the tools we have for approaching these concerns in an efficient way. Comparing, contrasting, and trying to build connections, we certainly understand better today the functionalities each of us gives to the theoretical constructs she/he uses, how she/he uses them and what she/he is able to produce thanks to them; we also see better the limits of our respective tools and what could be offered by networking them in ways that would not destroy their internal coherence. But what we achieved is just a first step.

In the long run this work will hopefully lead to a clearer meta-theoretical concept, which we might call "networking between theoretical approaches" and which might enhance the development of the theoretical work in our community regarding the need to grasp the complexity of our research objects better than we are able to do now.

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References

Artigue, M., Bartolini Bussi, M., Dreyfus, T., Gray, E., & Prediger, S. (2006a). Different theoretical perspectives and approaches in research in mathematics education: Report from Working Group 11. In M. Bosch (Ed.), *Proceedings of CERME 4* (pp. 1239–1243). Spain: Saint Feliu de Guíxols.

Artigue, M., Lenfant, A., & Roditi, E. (2006b). Comparison of different theoretical frameworks in didactic analyses of videotaped classrooms observation. In M. Bosch (Ed.), *Proceedings of CERME 4* (pp. 1316–1326). Spain: Saint Feliu de Guíxols.

Bikner-Ahsbahs, A. (2003). A social extension of a psychological interest theory. In N. A. Peitman, B. J. Dougherty, J. T. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PME-NA (27th Conference of the International Group for the Psychology of Mathematics Education)*, Vol. *2* (pp. 97–104). Honolulu, HI: University of Hawai'i, College of Education.

Bikner-Ahsbahs, A. (2005). *Mathematikinteresse zwischen Subjekt und Situation (Interest in mathematics between subject and situation)*. Hildesheim, Berlin: Franzbecker.

Bikner-Ahsbahs, A., & Prediger, S. (2006). Diversity of theories in mathematics education: How can we deal with it? *Zentralblatt für Didaktik der Mathematik (ZDM)*, *38*, 52–57.

Bloch, I. (2002). Différents modèles de milieu dans la théorie des situations. In J. L. Dorier, et al. (Eds.), *Actes de la 11ème Ecole d'Eté de Didactique des Mathématiques* (pp. 125–139). Grenoble: La Pensée Sauvage.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Castagnola, E., Dané, C., Impedovo, M., Paola, D., & Tomasi, L. (2005). *Mathematics in the Web*. http://www.matematica.it/paola/Corso%20di%20matematica.htm. Retrieved 16 August, 2007.

Chevallard, Y. (1985). *La transposition didactique*. Grenoble: La Pensée Sauvage.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique. Perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques*, *12*, 73–112.

Chevallard, Y. (2002). Organiser l'étude. Cours 1—Structures & Fonctions. Cours 3 - Écologie & Régulation. In J. L. Dorier, et al. (Eds.), *Actes de la 11ème École d'Eté de Didactique des Mathématiques* (pp. 3–22, 41–56). Grenoble: La Pensée Sauvage.

Dreyfus, T., & Kidron, I. (2006). Interacting parallel constructions: A solitary learner and the bifurcation diagram. *Recherches en Didactique des Mathématiques*, *26*, 295–336.

Dreyfus, T., Hershkowitz, R., & Schwarz, B. B. (2001). Abstraction in context II: The case of peer interaction. *Cognitive Science Quarterly*, *1*, 307–368.

Hersant, M., & Perrin-Glorian, M. J. (2005). Characterization of an ordinary teaching practice with the help of the theory of didactical situations. *Educational Studies in Mathematics*, *59*, 113–151.

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, *32*, 195–222.

Hershkowitz, R., Hadas, N., & Dreyfus, T. (2006). Diversity in the construction of a group's shared knowledge. In J. Novotná, H. Moraová, M. Krátká, N. Stehlíková (Eds.), *Proceedings of the*

30th Annual Conference of the International Group for Psychology of Mathematics Education, Vol. *3* (pp. 297–304). Prague: Charles University.

Hershkowitz, R., Hadas, N., Dreyfus, T., & Schwarz, B. B. (2007). Abstracting processes, from individuals' constructing of knowledge to a group's "shared knowledge". *Mathematics Education Research Journal*, *19*(2), 41–68.

Kidron, I. (2006). Conceptualization of the limit by means of the discrete continuous interplay: different theoretical approaches. In M. Bosch (Ed.), *Proceedings of CERME 4* (pp. 1295–1304). Spain: Saint Feliu de Guíxols.

Kidron, I., & Dreyfus, T. (2007). *Justification and interacting parallel constructions of knowledge.* Technical report, available from the authors.

Laborde, C., & Perrin-Glorian, M. J. (Eds.) (2005). Teaching situations as objects of research: Empirical studies within theoretical perspectives. *Educational Studies in Mathematics*, *59* (complete special issue).

Margolinas, C. (2004). Points de vue de l'élève et du professeur. Essai de développement de la théorie des situations didactiques. Habilitation Dissertation, Université de Provence.

Perrin-Glorian, M. J. (1992). Théorie des situations didactiques: naissance, développement, perspectives. In M. Artigue, et al. (Eds.), *Vingt ans de didactique des mathématiques* (pp. 97–147). Grenoble: La Pensée Sauvage.

Schwarz, B. B., Hershkowitz, R., & Azmon, S. (2006). The role of the teacher in turning claims to arguments. In J. Novotná, H. Moraová, M. Krátká, N. Stehlíková (Eds.), *Proceedings of the 30th Annual Conference of the International Group for Psychology of Mathematics Education*, Vol. *5* (pp. 65–72). Prague: Charles University.

Schwarz, B. B., Dreyfus, T., & Hershkowitz, R. (2008). *The nested epistemic actions model of abstraction in context.* Technical Report, Hebrew University of Jerusalem, Israel.

Sensevy, G., Schubauer-Leoni, M. L., Mercier, A., Ligozat, F., & Perrot, G. (2005). An attempt to model the teacher's action in the mathematics class. *Educational Studies in Mathematics*, *59*(1–3), 153–181.

Treffers, A., & Goffree, F. (1985). Rational analysis of realistic mathematics education—The Wiskobas program. In L. Streefland (Ed.), *Proceedings of the 9th International Conference for the Psychology of Mathematics Education*, Vol. *2* (pp. 97–121). Utrecht: OW&OC.

Wagner, H.-J. (1999). *Rekonstruktive Methodologie (Reconstructive methodology)*. Opladen: Leske+Budrich.

Warfield, V. M. (2006). *Invitation to Didactique*. http://www.math.washington.edu/~warfield/Inv%20to%20Did66%207-22-06.pdf. Retrieved 16 August, 2007.