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ORIGINAL ARTICLE

# "Rising to the challenge": using generalization in pattern problems to unearth the algebraic skills of talented pre-algebra students

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**Abstract** This study focuses on the generalization methods used by talented pre-algebra students in solving linear and non-linear pattern problems. A qualitative analysis of the solutions of three problems revealed two approaches to generalization: recursive–local and functional–global. The students showed mental flexibility, shifting smoothly between pictorial, verbal and numerical representations and abandoning additive solution approaches in favor of more effective multiplicative strategies. Three forms of reflection aided generalization: reflection on commonalities in the pattern sequence's structure, reflection on the generalization method, and reflection on the "tentative generalization" through verification of the pattern sequence. The latter indicates an intuitive grasp of the mathematical power of generalization. The students' generalizations evinced algebraic thinking in the discovery of variables, constants and their mutual relations, and in the communication of these discoveries using invented algebraic notation. This study confirms the centrality of generalizations in mathematics and their potential as gateways to the world of algebra.

## 1 Introduction

The role of generalization in mathematical achievement and learning cannot be overestimated. Mason termed it "the heartbeat of mathematics" (Mason 1996, p. 65), and the NCTM standards (2000) call for generalization as one of the main goals of mathematical instruction. Krutetskii (1976) classifies generalization as one of the higher cognitive abilities demonstrated by mathematics learners, and Polya (1957) attributes many scientific discoveries and mathematical results to what he calls "lucky generalization," claiming that generalization is essential in the development of mathematical knowledge (p. 117). Because of the higher ordered thinking involved in generalization, such as abstraction, holistic thinking, visualization, flexibility and reasoning, the ability to generalize is a feature that characterizes capable students and differentiates them from others (Greenes 1981; Sriraman 2003; Sternberg 1979).

As a process and a product of mathematics education, generalization has merits and importance as an instructional goal in itself. However, it can also serve as a means of constructing new knowledge, acting as an initiator for further learning in algebra (Amit and Klass-Tsirulnikov 2005) and probability (Sriraman 2003).

In the following paper, we will illustrate how capable students (aged 11–13) generalize pattern problems, and describe what kind of generalization strategies they use, how they communicate and justify their generalizations and what algebraic thinking notations they demonstrate. We will focus on linear and non-linear (quadratic) pattern problems, since these problems are accessible to young students on the one hand, while being loaded with mathematical generalization potential on the other.

After a short literature review, we will describe the three tasks the students were confronted with. Then, for each task, we will thoroughly analyze the generalization experiences the students underwent and discuss the results for each task, emphasizing their unique outcomes. We will conclude with an overall discussion and recommendations.

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## 2 Literature review

### 2.1 Generalization

Due to its centrality in mathematics education and to the different purposes of generalization, different researchers and theoreticians define generalization in different ways. Polya (1957) defines generalization as *"passing from the consideration of one object to the consideration of a set containing that object; or passing from the consideration of a restricted set to that of a more comprehensive set containing the restricted one"* (p. 108).

Generalization, according to Polya, is a gradual process. This process starts with "tentative generalization"—an effort to understand the observed facts, to make analogies and to further test special cases. The initial tentative generalization (or generalizations) leads to a finer one. No generalization is final, however, without a solid mathematical proof. At this point, Polya distinguishes between induction that is related to generalization and mathematical induction that is related to a rigorous proof. He reconciles between the two as follows: *"Mathematics presented with rigor is systematic deductive science, but mathematics in making is an experimental inductive science"* (p. 117).

Skemp (1986) perceives mathematical generalization as a sophisticated and powerful process, which involves reflection and a skillful reconstruction of one's existing schemes. Davis (1986) claims that generalization is a genuine need in mathematics education, achievable in at least two ways, either *"by the creation of abbreviated 'summary' versions of knowledge items, or, alternatively, by creating an isomorphism (a mapping) between two items"* (p. 360).

Harel and Tall (1991) view generalization as the process of "applying a given argument in a broader context" (p. 38), and distinguish between expansive generalization, reconstructive generalization and disjunctive generalization.

They recommend encouraging expansive generalization when possible by providing students with "experiences which lead to a meaningful understanding" of a situation.

Doerfler (1991) defines two types of generalizations: empirical generalization, which refers to the recognition of common features (in line with the identification of commonalities by Dreyfus 1991); and theoretical generalization, which has to do with a "system of action" in which invariants are identified, abstracted and inserted into prototypes comprising relations between objects. Mitchelmore (2002) labels two terms related to the process of generalization: abstract apart and abstract general. The first refers to finding similarities in specific given instances by which instrumental understanding is exhibited. The second refers to extending to a large number of instances, far beyond the given one, by which relational understanding is developed.

Ellis (2007) constructs a generalization taxonomy in which there are two main generalization levels: actions and reflection. The generalization action level includes forming an association between two or more mathematical objects, searching for similarities and relationships, and extending a pattern, relationship or rule into a more general structure. The reflection level refers to the ability to identify or use an existing generalization.

Krutetskii (1976) offers the following traditional, but operational, definition of generalization. He claims that

> "Any effective generalization in the realm of numerical and letter symbolism can be regarded from at least two aspects: one must be able to see a similar situation (where to apply it), and one must master the generalized type of solution, the generalized scheme of a proof or an argument (what to apply). In either case one must abstract oneself from specific content and single out what is similar, general, and essential in the structures of objects, relationships, or operations." (p. 237).

Kruteskii sees the ability to generalize mathematical objects, relations and operations as one of the building blocks of mathematical structure and as a special component for the development of mathematical capacity in gifted students. Sriraman's (2003) findings on the advantage that gifted students have in solving problems that lead to generalization are in accord with Kruteskii's views. The ability to use generalization is not an exclusively inherent attribute. On the contrary, it can be developed through "... *experiences that enable students to monitor and reflect on their work*, [which] *will in turn enable them to develop a capacity for generalization. This implies that the ability to generalize is the result of certain mathematical experiences*" Sriraman (2004, p. 205). The developmental approach to enhance the generalization capacity of young students, especially high achievers, was a well known phenomenon in the former Soviet Union (Amit and Burda 2005; Davydov 1990), and parts of this approach were adapted by one of the authors of this paper in the Kidumatica program (Amit et al. 2007).

### 2.2 Algebra, patterns, and generalization

The connection between algebra, patterns and generalization has been noted by numerous researchers. Mentioned below are a few such studies that are directly related to the study presented in this paper.

Usiskin (1988) offers five different concepts for algebra. The first is "Algebra as generalized arithmetic" (p. 11). In line with this concept, variables are thought of as pattern generalization. This notion of a variable as a pattern generalizer is "fundamental" in mathematical modeling. Moreover, Usiskin sums up the instruction of algebra in two words "translate and generalize" (p. 12).

Kaput (1999) defines algebra as "the generalization and formation of patterns and constraints" (p. 136). He suggests a route to algebra with understanding, which involves generalization from a very early age and in different domains, such as arithmetic, geometry and modeling situations:

> "Generalization involves deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situations themselves but rather on the patterns, procedures, structures, and the relations across and among them (which, in turn, become new, higher level objects of reasoning or communication). But expressing generalizations means rendering them into some language, whether in a formal language, or, for young children, in intonation and gesture" (p. 136).

In conclusion, Kaput joins others in recommending that teaching be conducted through meaningful experiences that will lead to algebraic understanding.

Experiments with pattern problems have been shown to be very efficient in revealing the ability to generalize and symbolize and in promoting the development of algebraic thinking in particular. These experiments vary in the types of pattern—numerical, pictorial, computational procedures or repeating patterns—and differ in different population, from children to pre-service schoolteachers (Amit and Neria 2007; Becker and Rivera 2004; English and Warren 1998; Ishida 1997; Radford 2006; Rivera 2007; Rivera and Becker 2005; Sriraman 2003; Stacy 1989; Zazkis and Lijendak 2002).

Concerning linear patterns, Stacey (1989) distinguishes between "near generalization" tasks, which include finding the next pattern or elements that can be reached by counting, drawing or forming a table, and "far generalization" tasks, in which finding a pattern requires an understanding of the general rule. Three main generalization strategies are defined in her study: (1) Recursive approach (adding strategy)—drawing a figure and counting or making a table. (2) Searching for the functional relationship—developing a mathematical expression from a figure. (3) Making incorrect proportional reasoning, using the ratio $f(x) = nx$, when the relation is $f(x) = ax + b$ ($b \neq 0$).

One major result of studies utilizing this division is that the more competent students attempt to search for a functional relationship, while the less proficient ones turn to the recursive approach or incorrect proportional reasoning (English and Warren 1998; Ishida 1997; Lee 1996). The latter methods may prevent students from identifying the general structure of the pattern (Orton and Orton 1999).

Radford (2006) suggests that in the process of generalization, students go through an experience of decision making, deciding on such things as what stays the same and what is changed, what should be emphasized and deemphasized, and what should be ignored. He stresses that for a patterning activity to be an algebraic one, students cannot rely on guess and test strategies; resourceful algebraic activities must be based on looking for commonalities and forming general concepts followed by the formation of generalizing expressions.

Most studies of patterns focus on the linear. Only a few address non-linear patterns such as exponential patterns (De Bock et al. 2002; Ebersbach and Wilkening 2007; Kerbs 2003). Most common strategies for such problems were additive, and there was an evident tendency toward linearity, even when the patterns were non-linear.

A review of the literature reveals the importance and prominence of the subject of generalization, as well as its complexity, and the various theories surrounding the ability to generalize. Moreover, it illustrates the connection between algebra and generalization, particularly the generalization of patterns.

In this study, we examined the generalization methods of capable pre-algebra students when solving linear and non-linear pattern problems. The analysis included discovering and defining various stages of generalization, justification methods (presentation and argumentation), use of representations and the students' algebraic thinking methods.

The aim of this paper is to present an in-depth analysis of students' strategies in solving pattern problems that lead to generalizations. To accomplish this, we elaborate on the solution strategies that led to successful problem solving, as well as those that prevented students from obtaining generalizations or led them to generalize incorrectly. For each task, we will present examples of students' solutions, followed by a discussion relevant to the specific task, with emphasis on strategies for near and far generalizations, the transition from local-near generalizations to global ones, and the representation modes and mathematical symbols pre-algebra students use in solving generalization evoking tasks. The paper will conclude with an overall discussion of generalization tasks for pre-algebra junior high school students and the study's implications for teaching and research.

## 3 Methodology

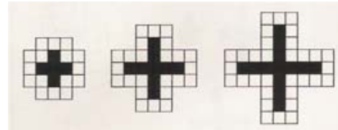
### 3.1 Participants

The participants in this research were capable students aged 11–13 (beginning of grades 6–7), who were members of "Kidumatica", an after-school math club. These students were club novices, having only just been accepted to Kidumatica and were in the midst of their first 2 weeks of club attendance. Like all the other club members (numbering a total of 400), these students come to Kidumatica on their own free will, as club attendance is in no way part of their regular school program. Students are either recommended for club membership by their teachers, or discover and approach Kidumatica on their own. All prospective members go through a series of examinations devised by the club staff to determine mathematical promise with as little reliance as possible on previous mathematical knowledge. The entrance exams are designed mainly to assess students' thinking, approach to problems, perseverance and motivation to succeed (i.e., not their IQ; these are unofficial intelligence tests). The students accepted to Kidumatica are highly motivated and certainly amongst the top students in their class, but not formally classified as "gifted." These students, who were beginning grades 6–7, had no prior extra-curricular studies, just their school curriculum, which did not include any formal algebra at this stage.[1,2]

### 3.2 Data sources

The main data sources were the students' solutions to three pattern tasks leading to generalizations (Figs. 1, 5, 10). Each task appeared in a different questionnaire that contained six non-routine problems, distributed to three equivalent groups of students. The questionnaires served as a pre-test aimed at investigating the abilities of the club's new participants, prior to any mathematical activities in the club, in order to assign the appropriate level of instruction. No grades were given and the results did not determine acceptance to the club—all of the participants had already been accepted. The students were asked to solve tasks and

The following illustration presents the three first patterns in a sequence:



A. How many white tiles are needed to make the next pattern?

B. How many white tiles are needed to make pattern 10?

C. Suggest a method to calculate the number of white tiles needed to make any pattern in this sequence.

D. Suggest a method to calculate the number of white tiles needed to make the $n^{th}$ pattern in this sequence

**Fig. 1** The linear pattern task

elaborate and explain their solutions. The questionnaire was designed accordingly. Each page had three parts. The tasks appeared in the upper part of the page and the rest of the page was divided into two columns: the right one was for the solution and the left one for explanations and elaborations of the solution (see Figs. 3, 4, 8). To ensure full justification of the students' solutions, we asked them very specific questions:

- What did I do?
- Why did I do it?
- If I changed my mind, why?
- If I did not answer, why?

The questionnaire format was carefully and specifically designed to ease the tasks of both students and teachers—first helping the students to clearly and fully explain their solutions, which later enabled those evaluating the exams to clearly see the students' solution paths and thought processes.[3]

The students were asked to write down *everything* in their questionnaires, but the pilot showed that they tended to write only the final "clean" solution (a well known phenomenon, Fried and Amit 2003). Therefore, in addition to the questionnaires, the researchers provided sketch pads for drafts to students who requested them. All the sketch pads were gathered and included in the analysis and proved an invaluable source of information regarding the students' solution methods and the stages of their generalization.

1 The club, which includes ~400 students aged 10–16, was founded nine years ago by one of the authors of this paper. The project aimed at addressing the special needs of students (most of whom come from underdeveloped or struggling areas), who possess mathematical ability but are not necessarily considered gifted, and who are interested in and desirous of learning more about mathematics. The club and its activities are designed to help these children by developing their mathematical and creative thinking skills.

2 The students participate in weekly mathematics workshops in specific topics (such as logic, problem-solving, number theory, etc.), and attend a full day devoted to science and social activity every five weeks. Classes are taught by teachers rich in mathematical knowledge and very experienced in working with talented students—most from the former Soviet Union. The students come from 50 schools in 14 different cities and villages. Some leave the club after one year, but most remain for several years. In its 9 years of existence, Kidumatica has won a string of awards in national and international Olympiads, and its graduates have moved on to prestigious university faculties.

3 In a pilot study, the students were asked to provide "justifications" for their solution paths, a term that proved misleading and problematic. Students were not familiar with the request to justify in mathematics and associated the instruction only to verbal tasks such as in history or social studies. Some students were confused by the origin of the word "justification," were unsure what it had to do with mathematics, and did not proceed. As a result, the wording was altered to *"What did I do; why did I do it; if I changed my mind—why; if I did not answer—why."* These instructions guided the students in providing justifications without using explicit instructions.

The students were given ample time to complete the questionnaires, which were distributed in the afternoon during club activity hours. The club's regular instructors were present throughout the entire time, as were the researchers, who answered questions for comprehension, and mainly observed the students and took notes.

### 3.3 The tool: three pattern tasks

Three generalization evoking tasks were chosen: a pictorial linear task; a pictorial non-linear task; and a verbally presented, non-linear task drawing on everyday life (Figs. 1, 5, 10).

The three tasks provided the researchers with a spectrum of patterns (linear, non-linear, etc.) and with a detailed and thorough view of the students' generalization capabilities at the beginning stage. As mentioned above, the questionnaires were distributed to three separate groups at the same time, in accordance with the cross-section approach. This method does not seek to monitor development over time as in previous studies, such as Krebs (2003), Lanin et al. (2006), Steele (2005), Warren et al. (2006).

The three chosen tasks were non-routine, requiring students to develop a strategy since they lacked a prefabricated solution strategy. In addition, of course, no similar problems were found in school textbooks.

A non-routine problem must demand that students use their preexisting knowledge in an unfamiliar way, thereby effectively reconstructing what they know. It must facilitate (i.e., provide an opportunity) the use of different representations. Lastly, the solution process must be fully documented and justified.

The problems selected were "mathematically rich" (Maher 2002), and held potential for the construction of new mathematical ideas and concepts—in this case, the potential for developing algebraic thinking. The three problems were generalization evoking, presenting different levels of cognitive demands and complexity (as will be elaborated).

The three pattern tasks had a similar structure: the "givens" consisted of a small finite set of example patterns. These were followed by 3–4 "items," which paced the required generalization process based on previous research on generalization (Stacey 1989; English and Warren 1998).

*Item A*: In accordance with the theoretical "near generalization" pattern, it served as a "warm up" item that enabled the solvers to examine and investigate the pattern.

*Item B*: In accordance with the theoretical "far generalization" pattern, a correct answer could be obtained by forming a "tentative generalization" (see Polya 1957 for the theory), or simply by extending the pattern using numbers or by drawing.

*Item C*: The "intuitive generalization" (informal generalization) pattern enabled the students to represent the generalization in any form they felt comfortable with. For the researchers, this item was an indicator of generalization abilities. It was based on prior research, which indicated that students found verbalizing generalizations easier than writing them symbolically (English and Warren 1998), and on the fact that the study participants were all pre-algebra students.

*Item D*: The "formal generalization" pattern contained an explicit requirement for representing a generalization in a formal mode, striving toward algebra. The aim of item D was to investigate how the students symbolize prior to formal studies in algebra.

Items C and D provided the distinction between those students who can "think algebraically" and those who can also "write algebraically" (Steele 2005) (note: in Task III, items C and D merged into one item). All of the tasks had been used in previous studies and been validated and deemed suitable by experts; in addition, they had been used in a pilot simulation to test the clarity of their translation and their suitability to the topic at hand.

### 3.4 Data analysis

Our aim in this paper is to present an in-depth analysis of a variety of students' strategies in solving pattern problems that lead to generalizations. Therefore, we chose a qualitative analysis (over a quantitative one) as the more effective means of revealing and portraying the "narrative of the generalization" and the development of algebraic skills.

All of the students' answers were analyzed qualitatively according to the following criteria: correctness, generalization strategy (including explanations and justifications), representational modes and symbolic use.

*Correctness of answers* refers to the final answers in items A and B only. Given that the students who participated in this research were pre-algebra students, the categorization of items C and D (the generalization items) into correct or incorrect answers by strict mathematical rules may be misleading, since they mostly comprised "invented" symbolism systems or verbalizations mixed with symbols. In concentrating exclusively on the correct final results, important information about mathematical thinking may be overlooked. We have noted that "wrong" answers are sometimes "good" answers because they demonstrate mathematical creativity and sophistication (Neria and Amit 2006).

#### *3.4.1 Generalization (solution) strategy*

Based on previous studies (English and Warren 1998; Ishida 1997; Lee 1996), this category included local and global generalizations, additive or recursive strategies versus functional generalizations. Strategies for generalizing functional rules were divided into numerical, such as the use of finite differences in a table, trial-and-error (random or systematic), visual, or a combination of numerical and visual strategies (Becker and Rivera 2004; Kerbs 2003; Rivera 2007).

### 3.4.2 *Mode of representation*

Generalizations were sorted according to students' representation modes: verbal, numerical or algebraic. Although in many cases students "made up" algebraic symbols since they lacked the knowledge of correct and precise algebraic symbolism, the analysis was focused on the "personal construction" of the symbolism.

The findings were validated through a method of revised and refined analysis with the aid of three experts. The two researchers went over the students' questionnaires individually, following the pre-assigned criteria. Afterwards, they met to refine the data and reach a consensus. The results were then given (in part) to a teacher in the Kidumatica Club for additional validation where necessary. The problems in this study met with a high level of consensus as early as the second stage of analysis.

In the following sections, some examples of solutions to each task will be presented, accompanied by an elaborate analysis and interpretation. The analysis will not necessarily be separated into different categories and sections, since these are closely interwoven with each other, but rather presented in a more holistic manner.

## 4 The linear pictorial pattern task: task 1

The task (Fig. 1) was adapted from Rivera and Becker (2005), and contained four graduated questions dealing with a pictorial linear sequence. In their research, Rivera and Becker investigated the figural versus numerical generalization modes of prospective schoolteachers. In the current study, we used the same task to investigate the generalization process of junior high school students participating in the Kidumatica Club.

### 4.1 Findings and interpretation: task I

A total of 50 students executed this task. The local and concrete applications in this linear pattern, such as finding the next pattern and the tenth pattern, proved to be an uncomplicated task for most of these mathematically capable students. As in previous studies (e.g., English and Warren 1998; Lee 1996), the constant difference property was usually recognized, enabling most students to find the fourth element of the pattern by adding 8 to the third one. The majority of the students who answered items A and B used additive strategies that usually entailed preparing a table or list. As for the other items, two types of generalizations were found, local recursive and global functional. We will demonstrate three examples of solutions to this task.

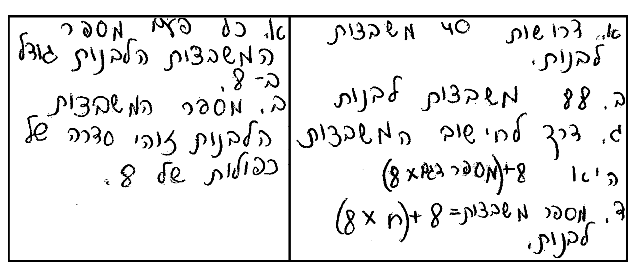
### 4.1.1 *Example 1, task I*

The student in Fig. 2 began by identifying the invariant, meaning the constant difference. In the answer to item A, he wrote (upper left side of the answer) *"each time the number of white tiles increases by 8"*, demonstrating an additive strategy. The answer to item B (lines 4–6 on the left side) *"the number of white tiles is a series of multiplications of 8"* indicates the transfer from an additive strategy (*increases by*) to a global multiplicative strategy in finding a commonality in the number of white tiles, all are multiplications of 8.

The flexibility in switching from an additive to a global multiplicative strategy enabled the student to distinguish, in item C, between the constants and variants (i.e., eight tiles between any two consecutive patterns is a constant, and the number of the pattern varies). Having made this decisive distinction, he is able to describe verbally a method to calculate *any* pattern in the sequence (fifth line on the right side): *"the method to calculate the squares is (8× the pattern number) +8"* and then to express (in item D) a correct method to calculate the squares for $n$:

$(8 \times n) + 8 =$ number of white tiles.

In fact, this student used an additive strategy only in item A for finding the next pattern. The solution and explanations indicate that after answering item A, he "jumped" to item C, found the generalization rule and then



a. each time the number of white tiles increases by 8
b. the number of white tiles is a series of multiplications of 8

a. 40 white tiles are required
b. 88 white tiles
c. The method to calculate the squares is: (8 x the pattern number) + 8
d. (8 X n ) + 8 = number of white tiles

**Fig. 2** The solution by a student for example 1, task I returned to answering item B, calculating the tenth pattern. His generalization process was: *near generalization, global generalization* and then *far generalization.*

This student was *flexible* enough to shift from an additive local approach to a global one. He *invented a symbolic system* describing his generalization method. In item C he wrote an arithmetic expression using numbers (*8*) and words (*the pattern number*) including unnecessary parentheses, as if he was not confident that arithmetic rules apply also when using symbols. In the next item (D), he converted the method of description into an "almost algebraic" equation. Instead of the pattern number, he wrote *n*, but he wrote the product and the number of tiles in words. In fact, this student *constructed a function* that describes the number of tiles in any chosen pattern as a function of its placement in the sequence. *There is no doubt that this student has revealed a solid foundation of algebraic thinking.*

### 4.1.2 Example 2, task I

The student in Fig. 3 began her answer by converting the figurative form into a numerical one (the bottom part of the answer). She wrote the index number on the upper line and continued the chart. She linked each index number with the number of white tiles. She calculated and checked if the difference of 8 was constant (the sketches and calculations in the right margins), proceeded to add 8 to the third pattern and got 40 (circled). She continued the process of adding 8 until she reached the tenth pattern and found the correct answer, 88.

At this stage, she left the additive strategy for a functional (multiplicative) one and presented a verbal generalization in item C: *"you can multiply 8 by the pattern number and add 8."* To verify her conclusion, she inserted the number 11 into her method (item C) and calculated the number of tiles in the 11th pattern and got 96. She then checked if the number 96 fitted into the rule of constant difference. She found that the difference between the tenth and eleventh patterns was also 8 and happily added a "smiley" and wrote *"I did it."*

In answering item D, the student translated her verbal rule into mathematical symbols. She ignored the instruction to use *n* as the pattern number, and defined three variables, A for the pattern number, B for the number of tiles and C for the number of tiles in the previous pattern (lower left side). She wrote the rule "$(8 \times A) + 8 = C$" using C instead of B. As in the first example, the student added unnecessary parentheses, just to be on "the safe side," possibly revealing an understandable insecurity in using algebraic symbols. She also added a method for verifying the solution, "*If* $B - C = 8$ *then the solution is correct.*"

This answer has some unique attributes associated with generalizations. The detailed solution path demonstrates transitions between representation modes: from the figurative givens to a numerical representation, from the numerical to the symbolic, and—in the reflection and verification—from the symbolic representation to the numerical and verbal. The flexibility is also demonstrated in the smooth transition from local recursive approaches to a global functional approach, and going back to the recursive one for verification purposes.

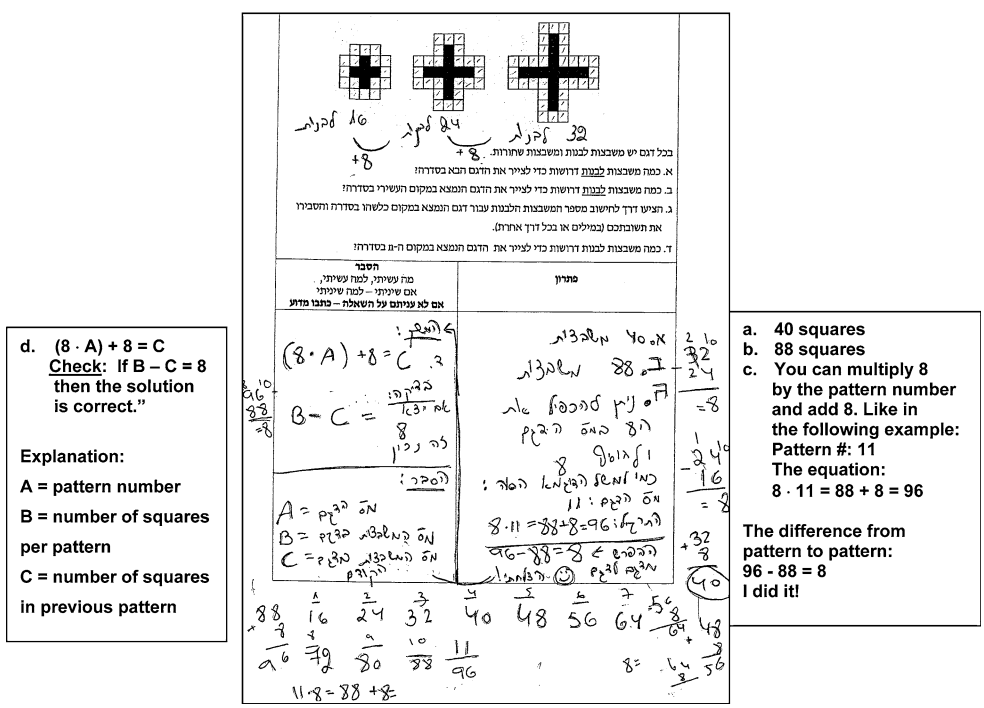



Fig. 3 The solution by a student for example 2, task I

As in example 1, the student showed algebraic thinking in defining variables and finding the functional relationship between them; she also initiated a verification method, without any explicit or implicit instruction to do so.

### 4.1.3 Example 3 task I

A similar phenomenon, but acquired in a different way, can be seen in Fig. 4. Item A is solved using an additive approach. The student first counted the four white tiles located in the corners of the black ones, and found the number of the remaining white tiles by counting. When she represented this structure numerically, she found that the numbers *"jump by 8."*

Since the first pattern had 16 white tiles, she used it as a starting point to which one must go on adding. She then abandoned the pictorial givens and concentrated on the numerical representations looking for a commonality. She found an arithmetic expression to represent the number of tiles in the third (given) pattern: 16 + (2 × 8). She was confident enough in her generalization process to skip the calculation of the patterns between the fourth and the tenth, and directly calculated the tenth pattern using the arithmetic structure she found: 16 + (9 × 8). We cannot be sure about the sequence of her work, but can confidently say she found a generalization prior to calculating the tenth pattern. The global generalization she wrote: "i*n order to calculate the number of white tiles in a pattern situated in a certain place in the sequence, I have to use the following formula:* 16 + [(*x*−1)·8]," reflects the structure of her calculations.

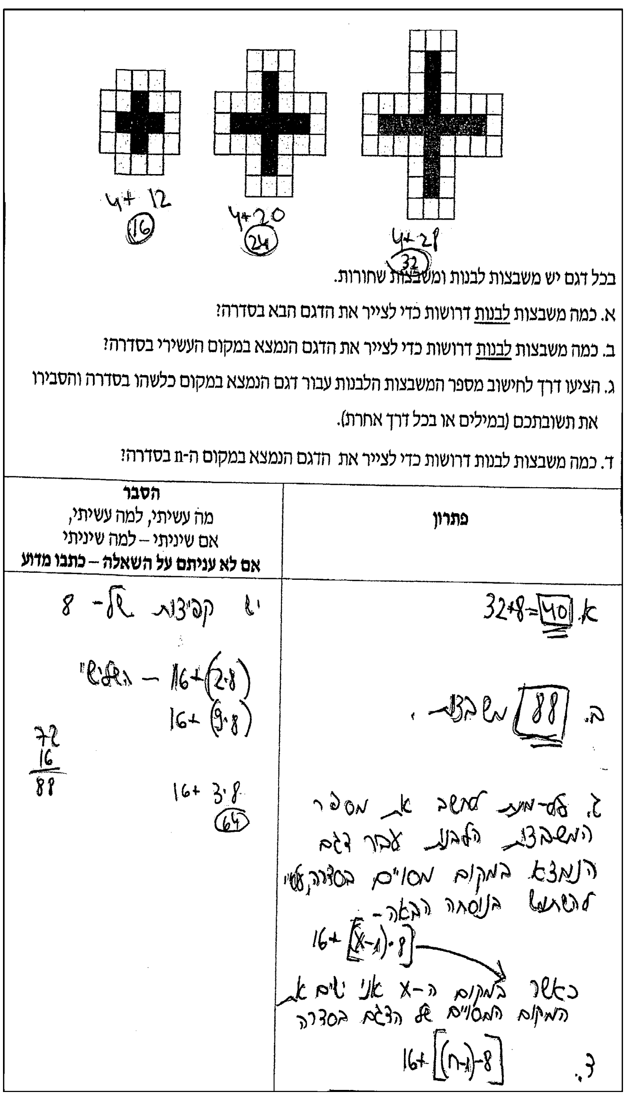



**Fig. 4** The solution by a student for example 3, task I

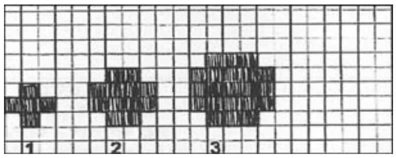
In her answer to item C, the student used *x*, and in answering item D, when the use of *n* was defined as the pattern number, she simply replaced the letters and wrote the algebraic expression "16 + [(*n*−1)·8]." Students' preference in using *x* and not *n* was detected in some of the solutions in this study, as well as by other researchers (Bardini et al. 2005), possibly reflecting prior knowledge that novice algebra students have about the discipline as "*something to do with x*."

Although the researchers provided verbal explanations regarding the meaning of *n*, many students found it difficult to write a symbolic representation. More than half of them did not answer item D at all; some wrote comments such as "*What is n?*" or "*I don't know what number n is*." Those who found the functional rule, but did not know how to represent it in an algebraic formal mode, wrote verbal generalizations such as: "*You do the pattern number and subtract one, and then multiply the result by 8 and add 16*."

## 4.2 Sub-discussion for task I

The analysis of the linear task revealed two major phenomena. In solving the task, all of the students began with the additive approach. Some (as illustrated by the examples above) moved on to a multiplicative approach and managed to achieve a functional or global generalization with which they were able to find the number of squares for each n, without relying on its precursor in the pattern. Others, however, did not switch from additive strategies to global ones. After finding the constant difference and answering the concrete items correctly (A, B), they attempted to generalize by continuing to apply additive strategies. These students either did not answer the generalizing items at all or generalized recursively (i.e., locally). In an example of such a local generalization, a student wrote for item C "*you have to add 8 to the existing number of white tiles and obtain the number in the next pattern*," and in representing

The following illustration presents the three first patterns in a sequence:




a. How many tiles are needed to make the next pattern?

b. How many tiles are needed to make pattern 10?

c. Suggest a method to calculate the number of tiles needed to make any pattern in this sequence.

d. Suggest a method to calculate the number of tiles needed to make the $n^{th}$ pattern in this sequence

**Fig. 5** The quadric pattern task

this generalization in an algebraic form (item D) wrote: "$n + 8 = n$."

The answers in Figs. 2–4 demonstrate several characteristics of mathematically capable students, such as generalization and abstraction abilities, flexibility in applying solution strategies, creativity and reflection, which are in accord with former research (Dahl 2004; Koichu and Berman 2005; Krutetskii 1976; Sriraman 2003). The solutions demonstrate a high level of generalization and abstraction abilities as well as good foundations of algebraic thinking. All of the students who found the generalizations had in fact constructed the function $f(x) = ax + b$.

## 5 The quadratic pictorial pattern task: task II

The quadratic pattern task (adapted from Zareba 2003) comprised the same four graduated items as the linear task (Fig. 5).

### 5.1 Findings and interpretation: task II

A vast majority of the 50 students who answered this task began solving it by drawing the next patterns. Those who turned to number sequences adopted a recursive approach and achieved local generalization (examples 4 and 5, Table 1, Figs. 6, 7), while those who adopted a figurative approach achieved global generalization (example 6, Fig. 8).

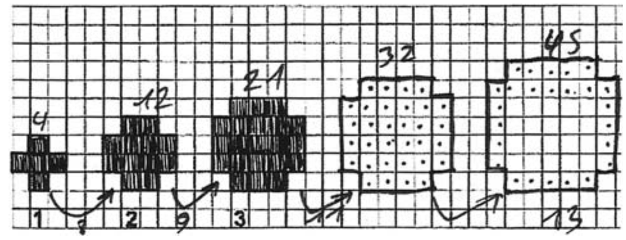



**Fig. 6** The solution by a student for example 4, task II

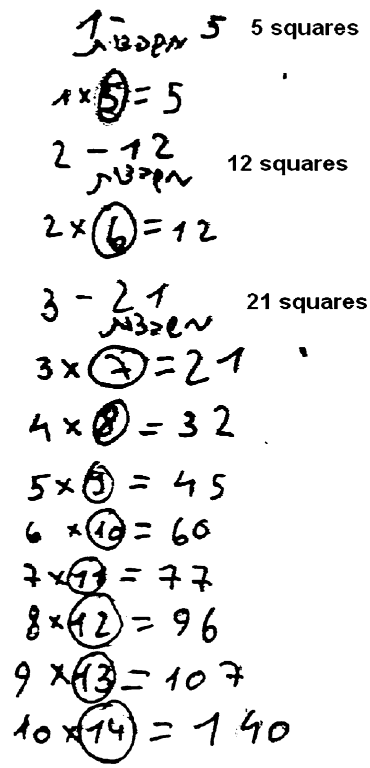



**Fig. 7** The solution by a student for example 5, task II

#### 5.1.1 *Example 4 task II: expansion by drawing*

In Fig. 6, the student began his answer by drawing the next two patterns and counting the number of squares (written above each pattern). He then calculated the difference between two consecutive patterns. Some students did not move beyond this point, as they were unable to find an immediate constancy in the numbers.

**Table 1** A student's method of using a table to understand pattern behavior

| Pattern number | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 | | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Number of squares | 5 | | 12 | | 21 | | 32 | | 45 | | 60 | | 77 | | 96 | | 117 | | 140 |
| Difference | | 7 | | 9 | | 11 | | 13 | | 15 | | 17 | | 19 | | 21 | | 23 | |

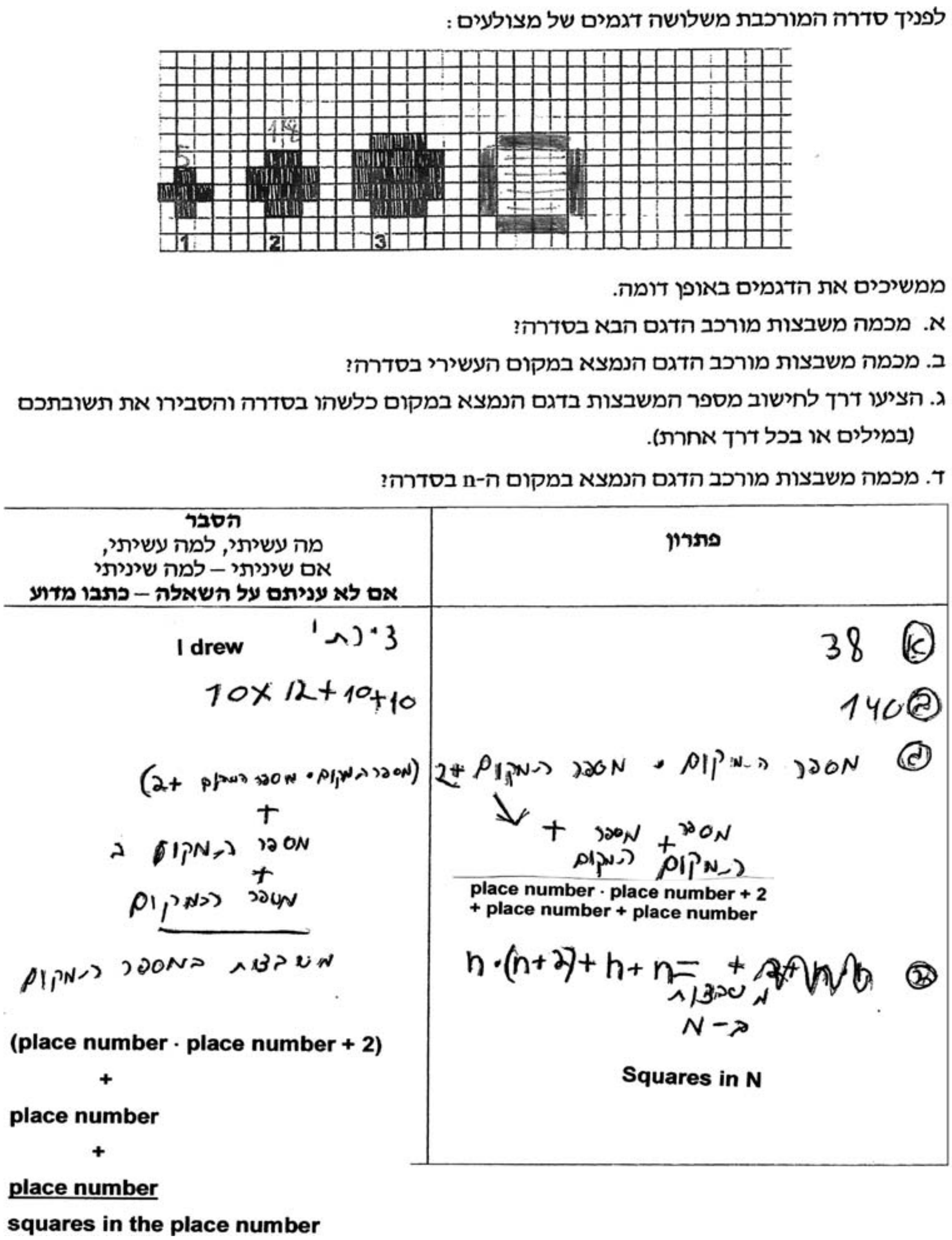



**Fig. 8** The solution by a student for example 6, task II

*5.1.2 Example 5 task II: a numerical approach to the non-linear problem*

Figure 7 illustrates a numerical approach. Once the student counted the squares in the givens, he abandoned the pictorial figures and concentrated on the numerical representation.

He wrote "*1–5 squares*," meaning pattern number 1 is comprised five squares and underneath "*1 x* ⑤ = *5*." He then repeated the process for the second and the third pattern. Grasping the regularity, he linked the number of a pattern (left column) and the number of tiles in this pattern (right column). The law for this link is: the numbers on the left column stand for the pattern numbers, which are multiplied by a sequence of ascending numbers starting with 5 (the circled numbers); the result of the products are the number of squares. The two lists, which are absolutely correct, have no figurative meaning. Extending the list enabled the student to achieve the local generalizations, but did not bring him closer to a global generalization. There was no attempt to generalize globally by giving a rule to calculate the number of squares for any pattern. These results are in line with Swaford and Langrall (2000), who found that although forming tables is useful in helping solvers make sense of a problem, it may also cause distraction from a more global view. This phenomenon seems to be more prominent when solving non-linear patterns, since the mathematical relationship between each pair of numbers is less obvious than in the case of linear patterns. Discarding the figural patterns and focusing on the numerical representations might be productive when dealing with linear patterns, where the constant difference can be recognized straight away, but in non-linear patterns this approach might be misleading.

Another solution path that led to local generalization was also based on a numerical representation and a recursive strategy. The student shifted from a figurative representation to a numerical one and found regularities that he tried to generalize.

On understanding the behavior of the pattern, he extended the sequence by one or two patterns, switched to a numerical representation (making a sequence of numbers) and found the following strategy: "*the difference between patterns 1 and 2 is 7, and between patterns 2 and 3 is 9; between patterns 3 and 4 it's 11 and so on. The difference increases by 2 (from 1 to 2, from 2 to 3, etc.), and then you add the number of squares to the difference between the next and the previous*." Implementing this strategy (making a list, see Table 1) enabled the student to answer items A and B, but not to generate a global generalization. He generalized the method, but not the pattern.

#### 5.1.3 *Example 6, task II: visualizing the situation and verbalizing generalization*

Approaches that led to global generalizations were based heavily on visualization. For example, in Fig. 8 we see that the student began by counting (he miscounted twice). He drew the next pattern and sought a global structure that would assist him in finding the tenth pattern. In his next drawing, he shifted toward a figurative solution. He dismantled the given figure into a central square surrounded by four rectangles. In his generalization process, he chose to use a slightly different structure. He dismantled the figure into a central rectangle whose sides were $n$ and $n + 2$, so that the area was the product of multiplying $n$ by

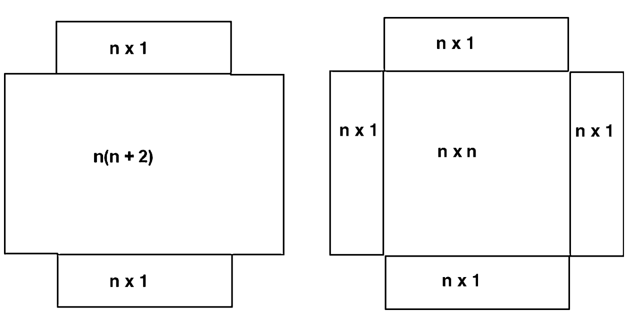



**Fig. 9** Examples of division into squares by the students to detect variables

$n + 2$, and then he added two additional rectangles whose sides were 1 and $n$ (Fig. 9). In his words: "(*place number × place number + 2) + place number + place number*." This approach was then implemented to find the tenth pattern.

The student found a functional generalization and then used it to find the tenth place. In this problem, it was very difficult to arrive at the tenth place by merely expanding the picture (though there were students who did this successfully), which necessitated moving to a generalization before returning to the specific case.

This student demonstrated a high level of algebraic thinking. He was able to find and translate the visual rule into an algebraic one, and to find a global functional relation between the squares of a pattern and the pattern location in the sequence. He used an algebraic notation for formal generalization (item D) in the form of:

$n \times (n + 2) + n + n.$

### 5.2 Sub-discussion for task II

Visualization had a central role in the effective solution of this non-linear pattern problem. This finding is in line with Kerbs (2003), who found that using a spatial approach when generalizing non-linear patterns leads to success, and Rivera (2007), who confirmed that visualizing both in linear and non-linear patterns promotes algebraic generalizations. The students who generalized productively were those who divided the pattern into parts, whose areas had a *constant relation* to the pattern place in the sequence. In this case, what remained constant throughout the generalization process was the manner of division and not the number of added squares, for example, the division into a square whose side is $n$ and four rectangles whose sides are 1 and $n$, or a rectangle whose sides are $n$ and $n + 2$ and two rectangles whose sides are 1 and n (Fig. 9). The students who were able to detect the variables (pattern number, dimensions) in the figural structure and differentiate them from the constants (shapes) achieved a correct global generalization.

Lacking algebraic symbolic tools, some students verbalized the generalization and described the method for finding the number of squares in any place as follows: "*the number of squares in each of the four exterior rows is the pattern number, so you have to square the pattern number and add four times the pattern number*." In fact, they described a global functional generalization.

In their effort to generalize, students were engaged in "real algebra" (Doerfler 1991), and invented "verbal algebra" using any symbols or icons they already knew. All the students revealed mathematical thinking, and some could even materialize this thinking into sophisticated solutions.

## 6 The candle lighting task, a non-linear pattern task: task III

The candle lightning task has to do with a well-known tradition of the Jewish holiday Hanukah[4] (Fig. 10). This pattern problem differs from the previous two pattern problems in its presentation and context. The givens are presented verbally in contrast to the pictorial tasks described before, and the context is a real-life problem. Instead of the three figural examples given in the previous problems, in the Hanukah problem the students create their own examples based on prior experience. In retrospect, we found that this approach did not cause any obstacles; on the contrary, it was even helpful. Although its context is familiar to all of the students (item A, Fig. 10), the far generalization and the formal generalizing items (items B and C) were absolutely "non-routine" and had never before been shown in a textbook. From a mathematical perspective, there were two layers of pattern change. One was linear in nature (e.g., the growth in the number of candles from day to day, which was almost trivial), and the second was a quadratic growth (e.g., the sums of candles accumulated for each day).

On Hanukah we light candles on each day of the eight-day holiday. Each day we light one leading candle and additional candles according to the day of the holiday.
A. How many candles do we light altogether in all of the eight days of the holiday?
B. If the holiday was 30 days long, how many candles would we have to light?
C. If the holiday was $n$ days long, how many candles would we have to light?

**Fig. 10** The candle lighting task

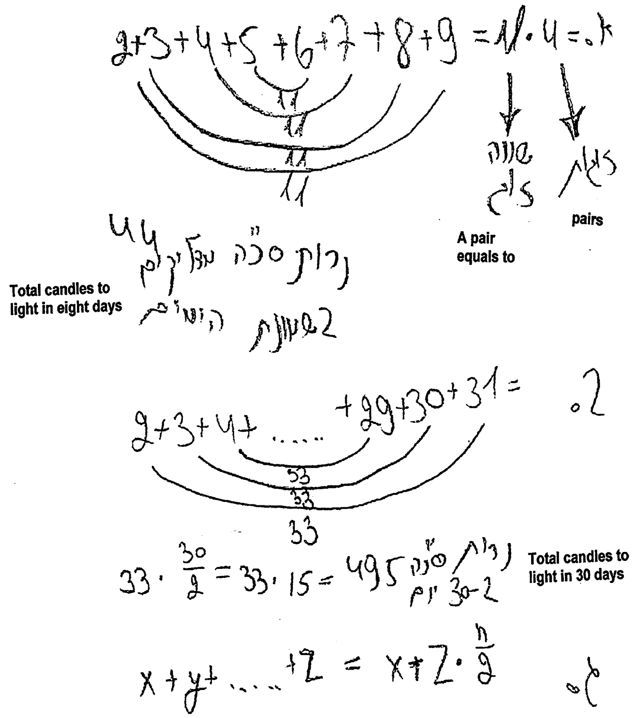



**Fig. 11** The solution by a student for example 7, task III

### 6.1 Findings and interpretation: task III

A total of 39 students received questionnaires containing this task. An additive strategy was adapted by the vast majority of students for solving item A. Many just added up the sequence of candles: 2 (first day) + 3 (second day) $+\cdots+$ 9 (eighth day) = 44.

This method worked well for a sequence of 8. Once there were 30 terms, this method turned out to be inefficient. Some students chose another track using the pairing strategy, which enabled them to extend the method to any number of patterns. This strategy is illustrated explicitly in Fig. 11 and elaborated below.

For the far generalization (item B), two different approaches appear: an additive approach for adding up the increasing numbers of candles, sometimes in a creative way; and a functional approach seeking a global rule, as seen below.

#### 6.1.1 *Example 7, task III: a generalized symbolized method of pairing*

In this example, the student drew a candelabrum (the "menorah") and the candles for each day, and discovered that by pairing the candles from both ends (first and eighth, second and seventh, etc.) she gets a constant number (11). She then multiplied the constant number by the number of pairs (4) and got the sum for 8 days, 44.

She applied the same method to find the sum of the sequence (item B) and even improved it: instead of counting the numbers of pairs, she halved the number of days. We can see that she found that for each "pair" (1st and 30th, 2nd and 29th, etc.) there is a constant sum of candles, 33. She multiplied this sum by half of the 30 days and arrived at the sum of 495 candles, which is the correct solution.

In short, the sophisticated method for summing up the number of candles lit in 8 days was generalized for 30 days and could be generalized for any number of days in item C.

[4] Hanukah is a Jewish holiday on which we light candles on each day of the 8-day holiday. On the first day, two candles are lit (the leading candle and one additional candle); on the second day, three candles are lit (the leading candle and two additional candles); on the third day, four candles are lit (the leading candle and three additional candles); and so on, until the eighth and last day of celebration.

However, lacking knowledge of algebraic notation, she invented her own (Fig. 11, lower part).

The student used $x$ for the number of candles lit on the first day (and not 2, as would be expected), $y$ stood for the number of candles lit on the second day, and $z$ for the number of candles on the last day; the sum is of the sequence of candles is $x + y + \cdots + z$. She then created the sum of a "pair" by adding the first and last numbers of the sequence, $x + z$. This sum was multiplied by half the number of days (*n: 2*). It is obvious that this student had generalized the arithmetic method that she had used before, and according to Usiskin, **algebra is generalizing arithmetic** (Usiskin 1988). There is no doubt that the student, though lacking in algebraic notation, definitely possessed **solid algebraic thinking**.

### 6.1.2 Example 8, task III: verbalizing a method by creative notation

The same pairing method was used by another student. This student wrote a correct verbal generalization: "The number of candles to light is the sum of candles to light on the first and the last days multiplied by half of the days in the holiday".[5] The generalizing process was applied immediately for a sequence of 30. For the global generalization (item C), she used her previous symbolic knowledge and formulated a surprising, creative "invention" of an algebraic representation (Fig. 12), writing: $x< + x> \times (n: 2)$ (note: the reader is encouraged to try and interpret the above expression before reading further). She took x as the number of candles on any day. However, to distinguish between the different x's, she added the mathematical symbols for smaller and bigger (<, >). Hence, in order to find the sum of candles for n days, she summed up the "small" $x$ (symbolized as $x<$, the number of candles on the first day) with the "big" $x$ ($x>$), the number of candles on the last day, and multiplied the sum by $n$ divided by 2 (half of the days). This is an extraordinary symbolization, which generalizes the method for finding the sum of the candles and indicates a perfect ability to use previous knowledge intelligently in a new situation (even if it is not exact). Again, although lacking instruction in formal algebra, this student is doing excellent algebra!

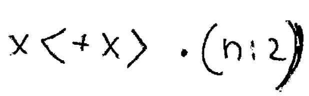



**Fig. 12** The solution by a student for example 8, task III

[5] None of the students were familiar with the formula for the sum of arithmetic progression.

### 6.1.3 Example 9, task III: a recursive approach for "partial sums"

An altogether different strategy that was found was forming tables of the sums of candles. For calculating $S_8$ and $S_{30}$, students simply added the numbers in the sequence (Table 2). In fact, by making the chart, they developed a recursive method to calculate the sums on the $n$th day. We can identify this method with the formula $S_n = s_{n-1} + A_n$, in which $S$ is the sum total and $A_n$ is the number of candles on the $n$th day. However, this local recursive method could not be generalized to a global one. For example, one of the students tried to adopt a functional approach expressed in an algebraic formula using only $n$ as a variable ($n \times 2$; $n \times 2 + 1$; $n \times 2 + 3$; $n \times 2 + 6$; $n \times 2 + 10$). Although he knew how to express the relationship between each individual day number and the total sum, he did not know how to proceed to generalize the list of algebraic expressions. In this solution, the student demonstrated a high level of mathematical thinking, although the recursive strategy led only to a local generalization. Having failed to find a functional relationship, he was neither able to calculate $S_{30}$ nor generalize globally. Nevertheless, these generalization experiences provide excellent foundations for algebraic development.

## 6.2 Sub-discussion for task III

Formal generalizations in Figs. 11 and 12 (although not absolutely correct) illustrate the students' abilities to generalize and think abstractly, while lacking the formal knowledge required for writing the correct answers. Two students identified the data pattern and figured out how to apply it to concrete situations, as required in items A and B. Although they failed to express an abstract generalized form that relates the sum of candles to the number of days, they demonstrated deep algebraic thinking and used creative notations that almost exhausted their previous mathematical knowledge.

The additive regression approach demonstrated by students revealed sound mathematical thinking and sophisticated ideas (e.g., "partial sums"), but was inefficient and did not lead to a global generalization. The functional approach that related the sum of lighted candles

**Table 2** A recursive approach for calculating the sum of candles in 30 days

| Day | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | … | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Number of candles | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | … | 31 |
| Sum | 2 | 5 | 9 | 14 | 20 | 27 | 35 | 44 | 54 | … | 495 |

to $n$, the number of days, enabled the students to generalize the problem for any $n$ using well-elaborated methods that could be translated into creative algebraic notations (Figs. 11, 12).

As in non-linear patterns, strategies with the starting point of forming numerical lists prevented students from finding the global structure required for generalizations. Although not all the students generalized correctly, the process of solving revealed a high level of mathematical thinking, even in the wrong answers.

## 7 Discussion

The results of the study are based on the performance of 139 capable pre-algebra students from grades 6–7, who carried out three pattern tasks: the first was a linear pattern task that was presented graphically and given to 50 students; the second was a non-linear pictorial pattern task, also given to 50 students; and the third was a non-linear pattern task that was presented verbally to 39 students, and which drew its format from everyday life.

The students, all of whom attended the Kidumatica Mathematics Club, solved the questionnaires individually (i.e., not in groups) and were required to provide full descriptions and justifications of their solution process. The format of the questionnaires allowed and required the students to answer in full.

In this study, the students exhibited abilities for generalization in linear and non-linear problems. They discovered constancies based on a few given examples from a series, generalized the situation and communicated their solution using various representations and sophisticated methods of argumentation. Concurrently, these students made intelligent use of their preexisting knowledge and created new knowledge, indicating solid algebraic thinking, which may serve them later as a basis for formal education.

Each task was divided into items in such a way as to allow the students to achieve full generalization gradually, in accordance with the theories of English and Warren (1998) and Stacey (1989). The items were arranged so as to require, in sequence, first near generalization then far generalization, followed by non-formal generalization and formal generalization. In retrospect, the study revealed that the students did not adopt the process suggested by the questionnaire format, creating a generalization process of their own, on which we will elaborate further.

Before continuing the discussion of the data, we wish to draw the reader's attention to a limitation imposed on this study by the nature of its participants. The students presented here have an affinity for mathematics. While they are not "gifted" in the formal sense and have not undergone any tests for giftedness, they have an active interest in mathematics and had shown enough mathematical aptitude to pass the entrance examinations for Kidumatica, a prestigious mathematics club for youth. As such, they are highly motivated to succeed and confident in their abilities to do so (passing the entrance exams having served as proof).

Notwithstanding these attributes, which set them apart from "the average students," the students on this study were all in their first 2 weeks of Kidumatica membership and had not yet had time to learn anything new in the club. Therefore, to a certain extent, they can be compared to the other top students in their classes, who may share their confidence and motivation, though not members of Kidumatica. This severely limits the ability to generalize from their performance. Therefore, in this study we make no assertions that our findings are applicable to the entire student population. Rather, we offer the study as proof of the existence of a phenomenon that can lead to further, more generalized research.

The purpose of this study was to explore the thinking processes of students and the methods in which they execute generalization processes. In accordance, the presentation of the study does not conduct quantitative statistical comparisons, but focuses on a narrative description of our analysis.

We chose to confront the students with generalization, because of its important position in mathematics and of its unique contribution to the mathematical development of talented students in particular (Davis 1986; Krutetskii 1976; Mason 1996; Polya 1957; Skemp 1986; Sriraman 2003). We chose to use pattern problems because of their great potential for revealing and creating algebraic knowledge (Amit and Neria 2007; English and Warren 1998; Kerbs 2003; Stacey 1989; Zazkis and Liljedahl 2002).

### 7.1 Characterizations of generalization methods

#### *7.1.1 Mental flexibility*

The processes of generalization revealed a high level of *"problem solving management"* and decision making (in line with Radford 2006). The students exhibited flexible thinking, which manifested itself at a number of levels. For instance, we found this flexibility in the transition from one form of representation to another. Students shifted from graphical representation to numerical representation, and later to verbal representation and symbolic ones. In every case, the transition was done intelligently and led to the continuation of the solution process.

Another aspect of the students' *mental flexibility* was found in switching from one solution method to another. Students who began solving a problem using a recursive–additive approach and reached a dead end, or decided that the method was ineffective, usually *did not remain fixated* on their first choice but showed flexibility in trying an alternate approach. The ability of the students in this study to make use of this mental flexibility can be attributed to the high motivation of these students (as in Sriraman 2003), and to their confidence in their ability, which allows them to make the necessary sophisticated "detours."

### *7.1.2 Manifold reflections*

Reflection is a cornerstone of the generalization process (Doerfler 1991; Ellis 2007; Harel and Tall 1991; Sriraman 2003). Skemp (1986) even claims that generalization is "sophisticated" because it "involves reflection on the form or method while temporarily ignoring the content" (p. 58).

Three forms of reflection were evident in this study. The first type of reflection occurred in the process of comprehending the rules that govern the given pattern. In this stage, the students "observed" the pattern, grasped its central attributes and usually performed a near generalization. Reflection of this kind led to several forms of results. In some cases, it led to one or two additional patterns (Example 4, Fig. 6). In others, the result of the first stage of reflection was a series of numbers (Example 2, Fig. 3; Example 5, Fig. 7).

The second form of reflection was the "observation" of the method (reflection on the method). Such reflection led students to the (intelligent) division into separate shapes (i.e., squares and rectangles) of the graphical pattern in Task II (Example 6, Fig. 8), and/or to the method of summing up symmetrical pairs of the lit candles in Task III (Example 7, Fig. 11). This second stage involved an abstraction of a method, which is cognitively demanding and demonstrates a high order of thinking skills. The third form is reflection on the generalization itself. The student has achieved a "generalization," either verbal or in a symbolic formula, and reflection manifests itself as "going backwards." This means implementing the generalization on a specific level, then checking and comparing it with the results obtained in an alternate way. For instance, in Example 2 (Fig. 3), a student found a generalization, used it to check the 11th pattern in a sequence and compared the result to a result she had found using an additive method. Another way of "going backwards" (particularly on the sketch pads), involved reconstructing the process either from beginning to end or (mostly) from the end backwards.

Multiple reflective forms are a prominent phenomenon in the generalization process in all of the tasks, linear and non-linear, used in this study. This highlights, once again, the inseparable connection between generalization and reflection.

Here, we wish to note that the sketch pads were an invaluable tool in the students' reflection. The questionnaires themselves were usually "clean," when handed over, with very little scribbling, exhibiting scant explicit evidence of reflection. Our sources for analyzing the students' thinking processes, including reflection, were the sketch pads, where the students felt free to express themselves—maybe because they did not know, while working on the problems, that we would be collecting their drafts as well. For them, the drafts were something private, where they were allowed to think and express anything, while the questionnaires were public, to be given to the teacher or researcher (exactly in line with the findings by Fried and Amit 2003). The importance of drafts should be taken into account when researchers wish to learn about the thinking processes of students preparing written answers.

### *7.1.3 Stages in the generalization process*

Broadly speaking, the generalization process that the students underwent is in agreement with the research literature. If we examine the examples presented in this paper, we can see that the first stage is always operational—the students found similarities in the patterns (Dreyfus 1991) and used them to define a basic set of rules, which they applied by continuing the pattern sequence. One could say that all the students, who were confronted with pattern problems of any kind went through this stage, which had various names in different sources, such as "action generalization" (Ellis 2007), "empirical generalization" (Doerfler 1991), "abstract apart" (Mitchelmore 2002), "abbreviated summary" (Davis 1986), "expansive generalization" (Harel and Tall 1991), "factual generalization" (Radford 2006), etc.

The second generalization stage is conceptual in nature. In this stage, the students define the underlying structure of the pattern sequence and the relations between objects, variants and constants. This stage is defined in the research literature under different names, such as "abstract general," "theoretical generalization," "reflection level," etc. (Ellis 2007; Doerfler 1991; Mitchelmore 2002).

This study showed that the first stage of generalization was accessible to a vast majority of the students, and they managed to obtain results to those parts of the tasks (namely items A and B) that required an operative level of generalization. However, the theoretical-conceptual level of generalization was achieved by only some of the students. Those who successfully completed the tasks are among those who mastered the theoretical-conceptual level of generalization.

The two generalization methods described above greatly resemble the following two generalization strategies:

a. The recursive–operational–local strategy.
b. The functional–conceptual–global strategy.

In the first stage, the students extend the sequence to the next member based on the one before it. This works both in linear and non-linear pattern problems. For instance, in a linear problem (Task I), the next member of the sequence is found by adding 8 white squares to the previous member (Examples 1, 2, and 3). In the non-linear graphical problem (Task II), the next member is found by enlarging the previous one according to the rules of the sequence (Example 4) or by continuing a series of numbers (Example 7). In the Hanukah candle problem, the recursion manifested itself twofold, first in finding the number of candles (simple), and secondly, and mainly, in finding the sum of candles that had been lit up to the given day by adding the number of candles for that day with the sum from the day before: $S_n = S_{n-1} + A_n$ (Example 9).

In using the recursive strategy, students certainly show an ability to generalize. They demonstrate a method for expanding the sequence in an operational way and exhibit signs of the generalization of a method. However, this strategy is severely limited due to its locality. The detection of each member of the sequence requires finding the one before it. In order to find the 100th member of a sequence using this method, the student would first have to find the preceding 99. The recursive strategy was therefore found effective in this study for, at most, finding a near generalization or maybe a far one (going even as high as the 30th member of a sequence), but was found to be inefficient beyond this limited scope.

The functional strategy was found to be immeasurably more effective than the recursive one. According to this strategy, in the first stage, that of observing for commonalities, the student recognizes and defines two central elements in the pattern sequence—the variants (or variables) and the invariants (or constants). In the second stage, the student finds a connection between the variables and the constants, in effect defining dependence between them. This is the functional connection that is the very heart of generalization. For instance, in Task I, the students defined the number of squares as a variable, together with the pattern's place in the sequence. They defined the expansion rate in the number of white squares from pattern to pattern as a constant (a constant rate of change), thus arriving at a functional connection between the two. In Task II, the variables were also the total number of squares and the place in the sequence, but the rate of change was not constant, hence the functional connection was more complex. In Task III, several students found an unexpected but impressive variant in the form of the sum of candles on the first and last days (or the second and second to last, etc.), but not all of them were able to actually compute the number due to the complexity of the problem.

Students who used the functional strategy arrived at the correct results using far generalization, even when this meant finding the 30th member of a sequence. Moreover, the functional strategy led them to a method for finding a pattern in an *n*th position in the sequence.

One cannot compare the two strategies described above (the recursive–operational–local and the functional–conceptual–global) and the theories of conceptual understanding set forth by Dubinsky (1991). He saw the procedural, operational approach as a primary level of conceptual understanding, a more advanced level being achieved through reification and the transformation of the process into an object (a good example of this theory is the evolution of the concept of functions).

### 7.2 Algebraic thinking and notation

Pattern tasks have been found to be efficient in providing students with experiences that promote and reveal the development of algebraic thinking among students and preservice teachers alike (Amit and Neria 2007; Becker and Rivera 2004; English and Warren 1998; Radford 2006; Stacy 1989; Zazkis and Lijendak 2002). This idea is strongly supported by the findings of this study. The participating students exhibited algebraic thinking and found algebraic symbols with which to communicate their ideas (a note on algebraic symbols: most students, although being pre-algebra ones, are aware that algebra is something to do with "English letters" instead of numbers. Indeed, though their mother tongue is Hebrew, all of the students used Latin letters as symbols). As seen in the results (see previous section), most of the symbolism was not correct in the strict mathematical sense. It was, however, sufficient to clearly convey the students' ideas (see Examples 2, 3, 11, 12).

The students in this study (who were, once again, mathematically capable) conducted their solution processes with very little trial and error—what little there was appeared only on the sketchpads. They went very quickly in search of more systematic ways of moving from the specific patterns to generalizations. They did, therefore, make their way toward algebra, in line with the ideas of Radford (2006).

In discussions held with several mathematicians (personal communications), it emerged that they see algebra as mainly symbolic manipulation. According to this view, what the students in our study did was not algebra. We, however, embrace other definitions of algebra, under which our students did indeed "do algebra." Usiskin (1988) sees algebra as the generalization of arithmetic. The students in this study generalized the arithmetical processes with which they began using a combination of letters, words and numbers. For instance, in Examples 2 and 6, the students calculated first in numbers then expressed the process leading to the formula in words.

Kaput (1999) defines algebra as the "generalization and formation of patterns and constraints" (p. 136), and the results of this study synchronize exactly with this definition. We have seen that the students defined variables and constants, found a functional connection between various elements that are dependent on others and presented this connection in a verbal or semi-symbolic way, though certainly not in the accepted form for describing functions.

Doerfler (1991) gives algebraic legitimacy to what the students in this study did, stating explicitly that any symbolization is to be considered algebra so long as generalization processes are present. If we examine the results, we will see that the students, lacking formal algebraic tools, performed a "verbalization of algebra." For example, they wrote such equations as "*number of squares = place number x n*." In another technique, they translated the generalization into symbols, as in the following case, where "*x*< + >*x (n:2)*" means "add the smallest × (the first in the sequence in Task III) to the largest × (last in the sequence) and multiply by half the number of days."

## 7.3 Beyond linearity

The purpose of this study was to find how far talented students could be challenged by generalization problems. We wished to observe their generalization behavior and algebraic capacity when they had neither formal education in algebra nor any experience in completing generalization tasks. We therefore did not content ourselves with linear tasks (in themselves rich in generalization potential, we do not belittle their importance), but challenged the students with non-linear problems as well (Tasks II and III), which require a high cognitive level of complex solution paths in the generalization process.

Previous studies have not shown a uniform approach to non-linear problems. Some have found that while students are able to generalize arithmetic situations they are familiar with, such as proportional or linear relationships, they have difficulty generalizing less familiar arithmetic situations, such as non-linear relationships (De Bock et al. 2002; Swaford and Langrall 2000). Our study, however, agrees with other studies showing that pre-algebra students have the intuition to deal with non-linear patterns (Ebersbach and Wilkening 2007; Lanin et al. 2006; Rivera 2007). Moreover, the non-linear problems led the students to high level experiences and "pulled them" to the very limits of their generalization ability, forcing them to make intelligent use of their existing knowledge and reconstruct it to fit new situations. For instance, in Task II (Example 6, Fig. 8), the student performed a visualization of the given figures. He broke the complex shape down into simpler, more familiar shapes, the areas of which he knows how to calculate (squares and rectangles). He connected the required generalization, i.e., the number of squares in each pattern, to the concept of area, a connection in no way trivial for a student whose geometric and algebraic concepts are compartmentalized.

This phenomenon of students being pushed to their intellectual edge recurred in the candle lighting problem (see Example 7, Fig. 11). Here, students "discovered" Gauss's method, or found a recursive strategy for calculating sums in an arithmetic sequence, then communicated the results using the "verbal algebra" they invented. True, they did not reach the perfect and final mathematical conclusion, but demonstrated powerful mathematical thinking.

## 7.4 Inductive-deductive: mathematics in action or re-sequencing generalization

One unique phenomenon exhibited by the students participating in this study was the intuitive, intelligent and immediate use of generalization. According to the research literature, the consecutive stages in the generalization process of a pattern sequence are near and far generalization, followed by semi-formal and formal generalization. The first two generalization types are usually found inductively (scientific induction, not mathematical) through the systematic expansion of the sequence toward the desired final position. This is also possible for far generalization through arithmetical calculations, though these may be long and exhausting (as seen in several cases in items B and C of Tasks II and III). The semi-formal generalization is based on the arithmetic method used in the two previous ones. The tasks in this study were built around this theory, and the items (A, B, C, D) in each task correspond to the stages the theory lays out.

Our study, however, revealed a different generalization sequence. Some students reached the "near generalization" in item A (adding one or two members to the sequence) in accordance with the rules they had found. Then, they immediately "jumped" to the "semi-formal generalization" stage required in item C, where they gave verbal or symbolic explanations of their generalization methods. Following this, they "went back" to the "far generalization" in item B by using the generalization rule to find the pattern in the required place (in Task II it was the 10th, in Task III the 30th). These actions describe a process of induction to find the general case, followed by a deduction to find specific cases, pointing to a strong intuitive understanding of the importance and power of generalization. In this study, not only did the students disregard the order of the items in the given task (thus proving their strong psychological mettle), they also "invented" an intelligent use of mathematical deduction.

This procedural sequence for performing generalization is defined by Polya (1957) as "mathematics in making": "Mathematics presented with rigor is systematic deductive science, but mathematics in making is an experimental inductive science" (p. 117). To avoid wronging mathematical rigorousness, it must be made clear (to teachers and students) that a generalization is not valid until it has been proven mathematically. Until then, it is a "tentative generalization" (Ploya 1957) leading up to the final generalization. The students' generalization was indeed mathematics in action.

## 8 Conclusion

This study has shown that when capable students are faced with pattern tasks that are generalization evoking, they display high mathematical abilities. They appear competent in performing generalizations of complex patterns, and in finding recursive methods for local generalizations and functional methods for global generalizations.

The importance of pattern problems—linear, and particularly non-linear ones—lies in their extensive mathematical potential. They not only encourage generalization, they also require students to pool their existing knowledge resources and build upon them. Thus, they are a gateway to new knowledge, in this case, algebraic knowledge.

The results of this study suggest that generalization via pattern problems can be an invaluable means for developing and revealing the intuitive algebraic skills of pre-algebra students prior to their formal instruction on the topic. The cognitive demands of the pattern problems in the study led the students to "invent" for themselves the tools and strategies required to solve the tasks they were given and to communicate their solutions to others. The students were able to appreciate the importance and utility of algebra that paved their road to the algebraic culture.

In conclusion, this study confirms the existing idea that generalizations bear mathematical potential and are effective in mathematical empowerment.

Furthermore, just as students are exposed to various problem-solving strategies and provided with appropriate experience, they must also be exposed to generalization strategies and gain experience in solving problems that promote generalization. The importance of such an experience regarding students' mathematical empowerment has been proven beyond all doubt.

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