With a focus on 'Grundvorstellungen' Part 1: a theoretical integration into current concepts

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Abstract: Current comparative studies such as PISA assess individual achievement in an attempt to grasp the concept of competence. Working with mathematics is then put into concrete terms in the area of application. Thereby, mathematical work is understood as a process of modelling: At first, mathematical models are taken from a real problem; then the mathematical model is solved; finally the mathematical solution is interpreted with a view to reality and the original problem is validated by the solution. During this cycle the main focus is on the transition between reality and the mathematical level. Mental objects are necessary for this transition. These mental objects are described in the German didactic with the concept of 'Grundvorstellungen'. In the delimitation to related educational constructs, 'Grundvorstellungen' can be described as mental models of a mathematical concept.

Kurzreferat: Gegenwärtige Vergleichsstudien in Mathematik schließen aufgrund der gemessenen Leistungen auf ein individuelles Merkmal, das durch das Konzept der Kompetenz operationalisiert wird. Diese spezielle Sichtweise wird durch das Verständnis einer mathematischen Grundbildung konkretisiert, bei dem Kompetenzen im Umgang mit Mathematik in zumeist anwendungsbezogenen Kontexten erfasst werden. Dabei wird mathematisches Arbeiten als ein Modellierungskreislauf aufgefasst, bei dem mathematische Modelle aus Umweltbezügen herausgelöst, innermathematisch verarbeitet und wieder im Hinblick auf den Umweltbezug interpretiert werden müssen. In diesem Zyklus ist die Übersetzung zwischen realer und mathematischer Ebene eine zentrale Tätigkeit, die nur dann gelingen kann, wenn mentale Objekte vorhanden sind, die diese Übertragungen ermöglichen. Solche mentalen Objekte werden in der deutschen Mathematik-Didaktik als Grundvorstellungen bezeichnet, wobei sich in Abgrenzung zu verwandten pädagogischen Konstrukten Grundvorstellungen als mentale Modelle mathematischer Inhalte beschreiben lassen.

ZDM-Classification: C30, D60, M10

1. Achievement and competence

The starting point of mathematical comparative studies like PISA is the shown achievement in a test instrument. Under the term *achievement* the measurable *result* of an individual is described on a stimulus. To differentiate this concept of the underlying *cognitive structure*, Weinert (2001) suggests using the concept of *competence*. Competences here are understood by the individual's abilities and skills to successfully solve problems in different situations.

This general point of view led Klieme, Funke, Leutner, Reimann and Witt (2001) to develop a working definition of the concept of competences, which seems to be useful in comparative studies and which is restricted to a cognitive view. Competences are therefore systems from specific, in principle, learnable skills, abilities and knowledge which enable the individual to cope with a couple of requests in different environments.

The characteristics of this definition of competences are:

- a *functional appreciation* whereby the indicator of competences is the mastering of definite requests; i.e. due to the results of a test the existence of a corresponding cognitive characteristic is defined.
- a *local specific appreciation*. Competences are obtained within a restricted area of contexts and situations; i.e. due to the mastering of specific situations, the competences for processing these situations can be identified.
- a *generalised appreciation* in which competences are understood as common arrangements which go beyond the description of a single achievement.

Competences are considered learnable. Research into the area of educational expertise offers the opinion that it is possible for every person to become an expert in all domains of learning, as long as he deals sufficiently with each domain (cf. Gruber & Rehrl, 2003).

2. Mathematical literacy

Starting with the idea that competences are learnable, the question of the different types of learning environments and the understanding of education at school is more interesting than mathematical precision. Klafki (1991) describes education as the ability of self-determination, where all-round education consists of three determining factors: (a) an education *for all*, (b) a *general* education and (c) an education with a *focus more towards general aspects*.

Heymann (1996) follows Klafki's concept, whereby he suggests that an all-round education is a prerequisite for the individual's participation in society. This aim of an all-round education, which was established by Winter (1976 a, 1976b, 1996) in the area of mathematics conveys mathematics in three *basic experiences*. These experiences can be characterised as

(E1) application orientation,
(E2) structure orientation and
(E3) problem orientation

(cf. Blum & Henn, 2003; Winter, 1996).

Thereby, *application orientation* does not directly mean the preparation for specific situations in life, but rather, the possibility of a basic insight into nature, society and culture. *Structure orientation* looks more at the analysis of mathematical objects in relation to a deductive view of the world. The *problem orientation* on the other hand, emphasises the acquisition of heuristic abilities to recognise and use samples in problem solving processes. However, these three aspects are connected with each other.

Mathematics at school should make the prerequisites impart such basic mathematical knowledge in order to gain insights into various contexts of life in a reflective und understandable way.

This view supports the ideas of Freudenthal (1973, 1981, 1983) whereby the arrangement of coherences is an essential aim of mathematical teaching at school. From these ideas, an understanding for mathematical competence as an individual characteristic has been developed according to *mathematical literacy*, which studies such as PISA based on (see also Prediger's article in this journal).

Mathematical literacy is defined as:

"[…] *an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements and to use and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen.*"
(OECD 2003, p. 24)

Therefore the focus is on the application of mathematical concepts and models. These are based on the understanding of Winter's basic experiences as a general insight into mathematics and the structural knowledge of an individual within his own environment.

The combination of social abilities and attitudes can be seen in this definition of mathematical competences. Furthermore, it is important to constantly improve mathematical ability in order to understand and to be able to use mathematics in daily life and in the workplace. The National Council of Teachers of Mathematics (NCTM) gives the following examples:

" • *Mathematics for life*. Knowing mathematics can be personally satisfying and empowering. The underpinnings of everyday life are increasingly mathematical and technological. For instance, making purchasing decisions, choosing insurance or health plans, and voting knowledgeably all call for quantitative sophistication.
- *Mathematics as a part of cultural heritage*. Mathematics is one of the greatest cultural and intellectual achievements of human-kind, and citizens should develop an appreciation and understanding of that achievement, including its aesthetic and even recreational aspects.
- *Mathematics for the workplace*. Just as the level of mathematics needed for intelligent citizenship has increased dramatically, so too has the level of mathematical thinking and problem solving needed in the workplace, in professional areas ranging from health care to graphic design.
- *Mathematics for the scientific and technical community*. Although all careers require a foundation of mathematical knowledge, some are mathematics intensive. More students must pursue an educational path that will prepare them for lifelong work as mathematicians, statisticians, engineers, and scientists."
(NCTM 2000, p. 4)

This general concept of mathematical education is difficult to convey and to test in regard to this understanding of mathematical literacy. In view of mathematical work, it is the aim of the following section to transpose concretely this concept into a test instrument.

3. The mathematical process of modelling

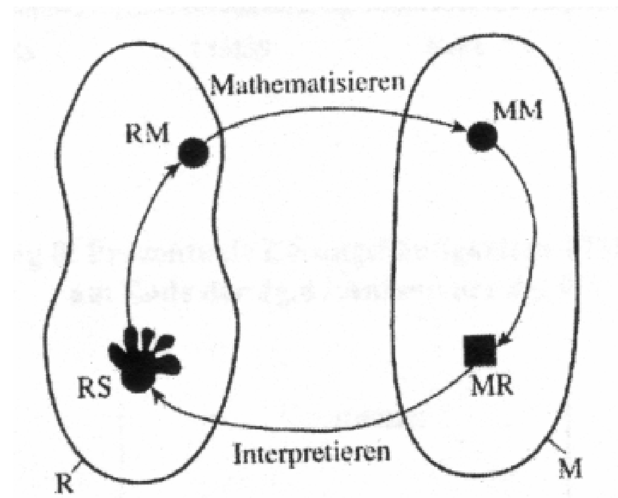
The explained opinion of mathematical literacy stands out due to an explicit application orientation. Grasping mathematics should be taught in the connection with real problems and how to use acquired knowledge in real situations later in life (cf. Griesel, 1976).

To solve a real-life mathematical problem, mathematical work is understood as a *process of modelling* (cf. Blum, 1996; Klieme, Neubrand & Luedtke, 2001). During this process there can be separate different phases:

(1) At first, the complexity of the real situation (RS) has to be focused to the specific problem in hand. You then get a model of reality. (2) This real model (RM) has to be transposed to a mathematical model on a mathematical level. (3) The mathematical model (MM) is solved and you get a mathematical result. (4) Finally the mathematical result (MR) is interpreted with a view to reality.

Figure 1 shows the different phases of such a modelling process. This, according to Blum (1996) emphasises a cyclical structure and the correlation between reality and mathematics at contrasting levels.

Figure 1. Blum's process of modelling (Blum 1996, p. 18)



Annotation:
R - reality, RS – real situation, RM – real model,
M - mathematical level, MM - mathematical model,
MR - mathematical result,
Mathematisieren – mathematical transition,
Interpretieren - interpretation

The circulation indicates that the process must perhaps be repeated due to the fact that models have to be modified or compared with other models (cf. Henn, 2002). The correspondence of the perceived world with the mathematical model indicates the understanding of an individual in regard to nature, society and culture (cf. Winter & Haas, 1997).

We can modify this illustration by taking a closer look at the solution. Then we are able to acquire the process of modelling according to Klieme, Neubrand and Luedtke (2001) and vom Hofe, Pekrun, Kleine and Goetz (2002, see figure 2 on the following page). In contrast to Blum's process, the mathematical transition is carried out directly from the real situation to the mathematical model. The processing of mathematical results is then carried out in the same way. The following step of 'interpretation' refers specifically to real consequences within these mathematical solutions. After that, these consequences are validated back to the initial problem. This phase particularly emphasises a necessary revision of the solution process.

All authors understand mathematical work according to the described literacy as a solution process between reality and mathematics. However, within the process there are different emphases: Blum outlines the steps of how to get to the mathematical model, whereas vom Hofe et al. point out the reversal, starting from the mathematical model back to the real situation.

A comparison of these didactic considerations, taking into account the educational points of view of Reusser (1996) and Holyoak (1984) shows similar ideas:

Reusser (1996) describes the different stages of mental processes which must be passed through the processing of mathematical problems. At first, we must work with the original problem which results in a real model ('episodic problem model') after various mental processes. This is then transposed into a mathematical model. After solving the mathematical model the numeric result is interpreted in regard to the original problem.

If we take a look Holyoak's process (1984) in solving a problem, we can see the mutual dependence and influencing of the real model ('problem model') and the mathematical model ('solution plan'), with a main emphasis on the interpretation of the solution.

If one compares this understanding of Reusser and Holyoak with the previous representations, then one establishes an obvious agreement about the different stages of processing mathematical problems. Reusser here, like Blum, emphasises the stages in the generation of a mathematical model. Holyoak and vom Hofe et al. point out the same stages of the solution process. There seems therefore, to be a general agreement about the necessity and the phases of mathematical work for solving application orientated problems. In any case, the components of this process are the validation of the results or the interpretation of the solution.

For the further exposition of mathematical competences it is important to look more closely at the transition between reality and mathematics which play a decisive role in the description of mathematical literacy in the following section.

4. 'Grundvorstellungen' as a mediator between reality and mathematics

In the process of modelling the transition between reality and mathematics represents a central task; a real situation is modelled on the one hand, to the mathematical level and, on the other hand the mathematical result is interpreted with respect to the real consequences. At first, we need *mental objects* of mathematical concepts for these transitions.

> "I have avoided the term *concept attainment* intentionally. Instead I speak of the constitution of mental objects, which in my view precedes concept attainment and which can be highly effective even if it is not followed by concept attainment."
> (Freudenthal 1983, p. 33)

With respect to the German didactic, one describes these cognitive objects traditionally as 'Grundvorstellungen' (cf. Oehl, 1962; vom Hofe, 1995). The roots of this concept were founded in the twentieth century under the influence of Piaget's work (1978). 'Grundvorstellungen' therefore, describe fundamental mathematical concepts or methods and its interpretation into real situations. They describe the relations between mathematical structures, individual psychological processes and real situations.

Vom Hofe (2003) points out three features of this concept:

a) No clear relationship exists between mathematical objects and specific 'Grundvorstellungen' because usually, mathematical objects are represented by several 'Grundvorstellungen' which stand in connection with each other. For example, the sum can be connected with either a *combination-'Grundvorstellung'* (Eva has € 3, Peter has € 4. How many € do they both have?), an *addition-'Grundvorstellung'* (Eva has € 3, her mother gives her € 4. How many € does Eva now have altogether?), or a *change-'Grundvorstellung'* (At first, Eva receives € 3 from her father, then she receives € 4 from her mother. How much does she receives from her parents altogether?) (cf. Kirsch, 1997).

b) One can distinguish 'Grundvorstellungen' in two ways: On the one hand, there exist *primary 'Grundvorstellungen'* which begin usually before mathematical instruction and these stand out due to concrete actions and concrete operations (e.g. the 'Grundvorstellungen' of the sum at a). On the other hand, *secondary 'Grundvorstellungen'* are developed during the time of mathematical instruction, which is indicated especially by mathematical representations (e.g. the 'Grundvorstellungen' of a function concept).

c) 'Grundvorstellungen' are neither fixed, nor used universally but are dynamic and develop within a networked mental system. The necessity for the development results in a varying range of validity: If 'Grundvorstellungen' are sustainable in one mathematical area, they must be extended in another area. For example, 'Grundvorstellung' of multiplication when using different numbers for the second factor has separate results. With natural numbers the product is always higher than the first factor; with fractional numbers however, the product can be higher (2nd factor >1) or lower (2nd factor <1) than the first factor.

The construction of such cognitive structures is described as the *formation of 'Grundvorstellungen'*. This formation is indicated (a) by recording the meaning of new concepts about known structures, (b) by the construction of mental objects which represent the concept, and (c) by the application of new situations.

The formation contains both the expansion and change of existing 'Grundvorstellungen' and the construction of new 'Grundvorstellungen'.

On this basis, 'Grundvorstellungen' can be characterised as:

- *specific mental objects* which show particularly structural and functional aspects of a mathematical subject;
- *dynamic objects* which can change and develop by new experiences in the course of time;
- *elements of a cognitive net* in which single 'Grundvorstellungen' are not isolated but are in correlation to others.

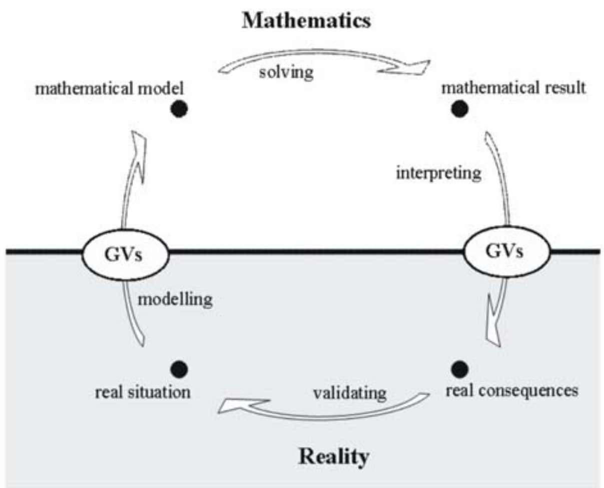
These characterisations point out, that 'Grundvorstellungen' can not be directly studied and require the need to be aware of the different types of behaviour. This point of view marks the *descriptive aspect* of the concept: Through the analysis of individual behaviour (e.g. at school, interviews, exams) the aim is to reconstruct the existing 'Grundvorstellungen' of mathematical objects. In contrast with this idea there is the *normative aspect*, whereupon 'Grundvorstellungen' are used as guidelines for the construction of mental objects of mathematical contents. The first aspect questions which 'Grundvorstellung' has been activated by a student; the second aspect questions which 'Grundvorstellung' has to be formatted by the student. If one compares the existing with the desired 'Grundvorstellungen', we have the idea outcome of an agreement. In many cases you can observe a deficit. This is a central topic of didactical research in the field of 'Grundvorstellungen'.

To fully understand this concept, we must take a short look back to mathematical literacy and the process of modelling: The normative aspect of 'Grundvorstellungen' shall be the focus in order to clarify the theoretical anchorage of the concept. If one looks at the modelling circulation of mathematical work, then 'Grundvorstellungen' hold the key position during the transition between mathematics and reality (figure 2). You are, so to speak, the *mediator* for individuals between the mathematical and real level.

5. 'Grundvorstellungen' as a mental models of mathematical objects

After we have taken a closer look at the concept of 'Grundvorstellungen' in connection with mathematical

Figure 2. 'Grundvorstellungen' within the process of modelling (vom Hofe, Kleine, Blum & Pekrun 2005, p. 3)



Annotation:
GVs - Grundvorstellungen
For an integration of figure 1 and 2: see Prenzel et al., 2004.

literacy, we want to place this concept in context with current educational and psychological concepts. In order to do this, we take a look at *mental models*, *schemes* and *conceptual change*. The structure of these different concepts can contribute to the further knowledge of 'Grundvorstellungen'.

Mental models

In psychology the concept of the *mental model* is used in different contexts with various meanings (cf. Seel, 2001). In the following section the meaning of mental models shall be outlined by Johnson-Laird (1983), whose ideas are used by numerous authors (e.g. Gruber & Ziegler, 1993; Mandl, Friedrich & Horn, 1988; Schnotz & Bannert, 1999). This concept firstly accepts that individuals build up internal models which represent a part of reality. Through this, individuals are able to organise their experiences and thinking, so that the mental model will then be an instrument to understand and explain reality. After that the following characteristics can be outlined:

- Mental models *concentrate on major qualities* in a given field.
- Mental models *illustrate phenomena* which are not immediately accessible.
- Mental models *steer the subjective perception* through their existing knowledge.
- Mental models *abstract* from reality by analogy.

Thereby, we can find two different kinds of analogies:
The mental object is determined by the surface structure of the real object (*surface analogy*).
Objects are represented mentally by processes of interpretation and conclusion (*structural and functional analogy*).

The existing mental model puts the "state of affairs" (Johnson-Laird 1983, p. 398) in the meaning of the *current* subjective understanding of an individual to the object. This should be emphasised that the development of the model is part of the constituting features of it. Beyond this, Norman (1983) stresses the incomplete and limited aspects of mental models.

Schnotz and Bannert (1999) assume that the surface features of objects have an essential influence on the structure of mental models. This makes it easier to recognise an analogy. According to the structure of knowledge, Gentner and Gentner (1983) and Holyoak (1984) consider the structural and functional analogy to be of greater importance.

Schemes

Schemes describe general and abstract knowledge which has been acquired by numerous experiences of single objects, persons, situations or actions (cf. Seel, 2001). According to Thorndike (1984), the structures of general knowledge represent typical models of these actions or objects. They are different from direct memories due to the abstraction of experiences. This feature of the concept indicates the economic character according to Wittmann (1997). Through this, individuals remove fortuities and unusual features of different objects and keep the cognitive effort to a minimum.

> "The term “schema” itself dates al least from Kant (1787/1963), who developed the idea that a person’s experiences are collected together in memory and that these collections are defined by certain common elements. Since these common elements identify categories of experiences, they permit the synthesis of abstract knowledge that represents the category."
> (Thorndike 1984, p. 170)

Piaget's idea of *accommodation* and *assimilation* has an essential influence on the current understanding. Accommodation here means the adaptation of a scheme to new stimuli, while assimilation expresses the selective use of existing schemes on new stimuli. It is due to this idea that we can characterise schemes which are explained below:

- Schemes *steer the attention* to assimilate new information more effectively (Seel, 2001).
- Depending on the dominance of assimilation or accommodation, existing schemes can be extended (*generalisation*) or limited (*discrimination*) (Wittmann, 1997).
- '*Slots*' can be components of a scheme. This is understood by the fact that either, numerous values can be part of a scheme (e.g. the type of the engine in the scheme 'car'), or they are able to further exist as a 'black box' (e.g. a black box 'drive' in the scheme 'car'). This consideration implicates a hierarchical structure of the construct in which sub-schemes can exist (e.g. 'double' as a sub-scheme of 'extend') (cf. Mandl et al., 1988).
- There is a *cognitive network* into which schemes are tied.
- By the processing of new experiences and impressions schemes are the subject of *permanent change* (Brewer, 1987).

Conceptual Change

According to Schnotz (1998), Duit (1999) and Stark (2003) the construct *conceptual change* aims to alter existing mental structures which students acquired in the past and which subsequently has proved inadequate in the course of time. This concept also has roots in Piaget's idea of accommodation (cf. Duit & Treagust, 2003).

At this point, it does not suffice just to enlarge the existing structure but a change must take place. Conceptual change describes this transformation of 'intuitive' knowledge to a scientific appreciation.

Shuell (1996) points out that most research of conceptual change was carried out in the natural sciences. The reason for research in this area was due to the fact that 'students' preinstrumentional beliefs' (Chinn & Brewer, 1993, p. 1) contradict scientific knowledge particularly in the natural sciences. For example, Vosniadou (1992) reports a common child concept where the earth is a flat plate which reflects immediate experiences.

> "When children enter school, they are, therefore, not 'empty vessels', but have already acquired a common-sense understanding of their nature and social environment based on experiences in everyday life. Unfortunately, this common-sense understanding frequently does not correspond to or is incompatible with the knowledge taught in school. Obviously, learning requires not only enrichment of knowledge and integration of new information, but also reorganization of existing knowledge, usually referred to as conceptual change.“
> (Schnotz, Vosniadou & Carretero 1999, p. VIII)

Vosniadou, Ioannides, Dimitrakopoulou and Papademetriou (2001) emphasise that conceptual change holds potential in cognitive conflicts because everyday experiences and scientific principles often contradict themselves. These conflicts are frequently avoided due to new (scientific) information which is interwoven with old ideas so that at least there is a mixture of intuitive and desired knowledge. Schnotz (1998) describes this mixture as a *synthetic model.* He sees in these kinds of missing concepts an active and creative test of the formation of a mental coherence.

In reference to the previous example of the child concept, we can find the following synthetic models about the earth: (a) separate earths (flat plate for living and a sphere as the celestial body) or (b) earth as a hollow sphere (persons live inside a hollow sphere on flat ground) (see Vosniadou, 1992 for more examples).

Hereby, the difficulties of this concept become obvious: The focusing on scientific contexts to initiate restructuring processes (Stark, 2003). Missing concepts are not problematic because they are incorrect from a scientific point of view; the problems with scientific contexts result in the application where they are not functional for daily life.

This means that missing concepts seem to be 'correct' from the child's point of view for daily life. A scientific view frequently plays a subordinate role in the current experienced world of children. Therefore, according to

Stark it is critical to use scientific models as the single and only aim in learning. In the understanding of conceptual change however, the scientific knowledge is the purpose of change (cf. Duit, 1999).

The connection of 'Grundvorstellungen' within these concepts

If one compares 'Grundvorstellungen' with the represented concepts, then (1) mental models and schemes describe knowledge structures in similar ways. (2) Conceptual change focuses on the reorganisation processes of stated to desired structures; it must be seen in relation to the formation of 'Grundvorstellungen'.

At first, we will take a look at the connection between 'Grundvorstellungen' and mental models and schemes. Norman, Gentner and Stevens (1976) as well as Hasemann (1986) connect schemes and mental models by understanding that schemes represent the structural frame in which information is put together within inferred unities. The activation of schemes is the prerequisite for the discovery of analogy relations. Mental models are created on the basis of these analogies. *Mental models are designed therefore, on the basis of schematic knowledge.* Both theories are in hierarchical relation to each other where schemes are prerequisites for mental models.

The concept scheme is frequently used for the description of mathematical problem solving processes as Wittmann (1997) has explained: It takes an individual to find a solution to a given problem by trying to adapt different schemes in relation to the problem definition (accommodation) or by trying to redraft the problem to make it accessible to a given scheme (assimilation).

These executions of a problem solving process apply particularly to the represented modelling circulation (figure 2): On the one hand, individuals try to adapt and use different existing schemes again and again; on the other hand, they try to remodel the problem situation to take it work with already known schemes. This repetition can be identified with numerous turns within the process of modelling. By this understanding, *schemes form a structural frame for mathematical work* to discover known connections. Finding such coherences within the framework requires the activation of mental models to mediate between the different stages of the modelling process. 'Grundvorstellungen' are a particular form of such cognitive objects which cross between reality and mathematics and they are obtained on a mathematical content. With respect to this limitation 'Grundvorstellungen' are a part of the concept of mental models: *'Grundvorstellungen' are mental models of mathematical content*.

Let us take a look at the changes in mental structures in the following section. The formation of 'Grundvorstellungen' describes the construction and the expansion of existing mental structures, which are connected with each other. In contrast to this, conceptual change describes the transformation of inadequate structures. Within the theory of 'Grundvorstellungen' missing concepts are described as such inadequate structures.

These *missing concepts* represent a kind of knowledge which repeatedly leads to faulty conclusions. Missing concepts primarily become problematical, if they hinder the further education: They have an influence on the further learning process as a kind of pre-disposition about a certain object. However, for the acquisition of competences a pre-disposition plays an important role (cf. Gruber, 2000; Mandl, Gruber & Renkl, 1993).

For the development of competences in mathematics missing concepts are an essential problem. Conceptual change describes a theoretical framework for the transportation of missing concepts in adequate 'Grundvorstellungen'. In this way *conceptual change is an aspect of the formation of 'Grundvorstellungen'*.

6. Summary

The aim of the measurement in comparative studies is to conclude on a cognitive characteristic on which the achievement is based on. With such a characteristic the concept of the competence is chosen. The achieved result of a test is the indicator for competence. A precise finished attempt of a test instrument is possible by using the concept of mathematical literacy. Whereby, mathematical competences are defined as the sensible use of mathematical modelling in real-life situations.

There is a special focus in this article on the transition between real situations and the mathematical level whilst working with problems. For these transitions mental objects are necessary which mediate between reality and mathematics. To be more precise we use the term 'Grundvorstellungen'.

'Grundvorstellungen' can be described as mental models for mathematical objects. They are constructed on the basis of schematic structures. The formation of 'Grundvorstellungen' is marked by the construction of adequate mental models which can be applied to various contexts. In these contexts the construction of corresponding cognitive structures can be seen as the defined mathematical competences. Missing concepts are problematical because they greatly hinder the further learning progress. The transportation of such missing concepts to adequate 'Grundvorstellungen' can be described as conceptual change.

Following this article's insight into the representation of the central position of 'Grundvorstellungen' in mathematical literacy and its relation to current educational concepts, the second part of this series will aim to deal with the question how 'Grundvorstellungen' can be used as empirical criterion in comparative studies.

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