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THE SUSPENSION OF A 4-MANIFOLD AND ITS APPLICATIONS

BY

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ABSTRACT

Let M be a smooth, orientable, closed, connected 4-manifold and suppose that $H_1(M;\mathbb{Z})$ is finitely generated and has no 2-torsion. We give a homotopy decomposition of the suspension of M in terms of spheres, Moore spaces and $\Sigma\mathbb{C}P^2$. This is used to calculate any reduced generalized cohomology theory of M as a group and to determine the homotopy types of certain current groups and gauge groups.

1. Introduction

Let M be a smooth, orientable, closed, connected 4-manifold. This implies by Morse theory that M has a CW-structure with one 4-cell. Suppose that $H_1(M;\mathbb{Z})$ is finitely generated and has no 2-torsion. Specifically, assume

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that:

(1)
- $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^m \oplus \bigoplus_{j=1}^{n} \mathbb{Z}/b_j\mathbb{Z}$;
- each b_j is a prime power, where the prime is odd.

From (1), by Poincaré Duality, the integral homology of M is:

(2)

i	$H_i(M;\mathbb{Z})$
0	$\mathbb{Z}$
1	$\mathbb{Z}^m \oplus \bigoplus_{j=1}^n \mathbb{Z}/b_j\mathbb{Z}$
2	$\mathbb{Z}^d \oplus \bigoplus_{j=1}^n \mathbb{Z}/b_j\mathbb{Z}$
3	$\mathbb{Z}^m$
4	$\mathbb{Z}$
≥ 5	0

where $d \geq 0$ can be any integer. Our main theorem identifies the homotopy type of ΣM.

Theorem 1.1: *Let M be a smooth, orientable, closed, connected 4-manifold and suppose that $H_1(M;\mathbb{Z})$ is finitely generated and has no 2-torsion. If M is Spin then there is a homotopy equivalence*

$$\Sigma M \simeq \left(\bigvee_{i=1}^{m}(S^2 \vee S^4)\right) \vee \left(\bigvee_{j=1}^{n}(P^3(b_j) \vee P^4(b_j))\right) \vee \left(\bigvee_{k=1}^{d} S^3\right) \vee S^5.$$

If M is non-Spin then there is a homotopy equivalence

$$\Sigma M \simeq \left(\bigvee_{i=1}^{m}(S^2 \vee S^4)\right) \vee \left(\bigvee_{j=1}^{n}(P^3(b_j) \vee P^4(b_j))\right) \vee \left(\bigvee_{k=1}^{d-1} S^3\right) \vee \Sigma\mathbb{C}P^2.$$

In fact, Theorem 1.1 is a special case of a more general result about the suspension of 4-dimensional CW-complexes whose cohomology satisfies Poincaré Duality and has no 2-torsion (see Theorem 5.9). Such a classification fits into a long history of classifying CW-complexes with cells occurring in a small number of consecutive dimensions, with contributions, for example, by Whitehead [32, 33], Chang [4], Baues and Hennes [3], Baues and Drozd [2] and Pan and Zhu [21]. Apart from [33], these classifications occur in the stable range; the classification in Theorem 5.9 notably occurs unstably.

A key aspect of Theorem 1.1 is that the suspension of M involves only three types of spaces: spheres, Moore spaces and $\Sigma\mathbb{C}P^2$. Each is simple and characterizes a cohomological property: a sphere corresponds to an isolated $\mathbb{Z}$ summand, a Moore space corresponds to a torsion summand, and a $\Sigma\mathbb{C}P^2$ corresponds to two $\mathbb{Z}$ summands connected by the Steenrod operation Sq^2. The hypothesis that only odd torsion in cohomology is allowed is necessary to achieve this. For example, the suspension of $S^1 \times \mathbb{RP}^3$ is homotopy equivalent to $S^2 \vee \Sigma\mathbb{RP}^3 \vee \Sigma^2\mathbb{RP}^3$ which does not split as in Theorem 1.1 since $\Sigma\mathbb{RP}^3$ is indecomposable. The list of indecomposable wedge summands at the prime 2 would therefore be much more complex.

The simple description of ΣM in Theorem 1.1 is advantageous. It implies that the homotopy type of ΣM is completely determined by only two properties: (i) whether M is Spin or not and (ii) $H_*(M;\mathbb{Z})$ (or equivalently, $H^*(M;\mathbb{Z})$).

Interestingly, while suspending a manifold loses all the geometry, it does give access to many other properties. Theorem 1.1 is applied in three different contexts: to determine any reduced generalized cohomology theory of M, to determine the homotopy type of certain current groups associated to M, and to determine the homotopy type of certain gauge groups associated to M. These applications are discussed in detail in Section 6.

To prove Theorem 1.1 new methods are developed that use homology and cohomology to detect whether certain maps are null homotopic. This generalizes Neisendorfer's work in defining and determining the mod-p^r Hopf invariant [20].

2. Preliminary information on Moore spaces

This section records some information on the homotopy groups of Moore spaces which will be needed later. For $m \geq 2$ and $k \geq 2$, the **mod-k Moore space** $P^m(k)$ of dimension m is the homotopy cofibre of the degree k map on S^{m-1}. Notice that

$$\Sigma P^m(k) \simeq P^{m+1}(k).$$

Lemma 2.1: *If p is an odd prime and $r \geq 1$ then*

$$\pi_3(P^3(p^r)) \cong \mathbb{Z}/p^r\mathbb{Z}.$$

Proof. Consider the homotopy fibration $F^3(p^r) \longrightarrow P^3(p^r) \xrightarrow{q} S^3$ where q is the pinch map to the top cell. This induces an exact sequence

$$[S^3, \Omega S^3] \longrightarrow [S^3, F^3(p^r)] \longrightarrow [S^3, P^3(p^r)] \xrightarrow{q_*} [S^3, S^3].$$

At odd primes,

$$\pi_3(\Omega S^3) \cong 0.$$

Since $P^3(p^r)$ is rationally trivial and $\pi_3(S^3) \to \pi_3(S^3) \otimes \mathbb{Q}$ is injective, any composite $S^3 \xrightarrow{f} P^3(p^r) \xrightarrow{q} S^3$ must have degree zero. Hence $q_* = 0$. Thus, by exactness, $\pi_3(F^3(p^r)) \cong \pi_3(P^3(p^r))$.

To complete the proof it is now equivalent to show that $\pi_2(\Omega F^3(p^r)) \cong \mathbb{Z}/p^r\mathbb{Z}$. For $m \geq 1$, let $S^{2m+1}\{p^r\}$ be the homotopy fibre of the degree p^r map on S^{2m+1}. In particular, $S^{2m+1}\{p^r\}$ is $(2m-1)$-connected. By [19, Proposition 14.2] there is a homotopy equivalence

$$\Omega F^3(p^r) \simeq S^1 \times \left(\prod_{j=1}^{\infty} S^{2p^j-1}\{p^{r+1}\} \right) \times \Omega R^3(p^r)$$

where $R^3(p^r)$ is a wedge of mod-p^r Moore spaces consisting of a single copy of $P^4(p^r)$ and all other wedge summands being at least 3-connected. In particular, for $R^3(p^r)$, by the Hilton–Milnor Theorem there is an isomorphism $\pi_3(R^3(p^r)) \cong \pi_3(P^4(p^r))$. Further, the Hurewicz homomorphism implies that

$$\pi_3(P^4(p^r)) \cong H_3(P^4(p^r)) \cong \mathbb{Z}/p^r\mathbb{Z}.$$

Returning to the decomposition of $\Omega F^3(p^r)$, since each space $S^{2p^j-1}\{p^{r+1}\}$ is at least 3-connected, we obtain $\pi_2(\Omega F^3(p^r)) \cong \pi_2(\Omega R^3(p^r))$ and we have just seen that $\pi_2(\Omega R^3(p^r)) \cong \mathbb{Z}/p^r\mathbb{Z}$. ∎

LEMMA 2.2 ([24, Lemma 3.3]): *If p is an odd prime and $r \geq 1$ then $\pi_4(P^3(p^r)) \cong 0$ and $\pi_4(P^4(p^r)) \cong 0$.* ∎

LEMMA 2.3 ([19, Corollary 6.6]): *Let p be an odd prime, $s, t \geq 1$ and $m, n \geq 2$. Then there is a homotopy equivalence*

$$P^m(p^s) \wedge P^n(p^t) \simeq P^{m+n-1}(p^{\min(s,t)}) \vee P^{m+n}(p^{\min(s,t)}). \qquad ∎$$

LEMMA 2.4: *Let p be an odd prime and $s, t \geq 1$. Then*

$$\pi_3(\Sigma P^2(p^s) \wedge P^2(p^t)) \cong \mathbb{Z}/p^{\min(s,t)}\mathbb{Z}.$$

Proof. By Lemma 2.3 and for dimensional reasons there are isomorphisms

$$\pi_3(\Sigma P^2(p^s) \wedge P^2(p^t)) \cong \pi_3(P^4(p^{\min(s,t)}) \vee P^5(p^{\min(s,t)}) \cong \pi_3(P^4(p^{\min(s,t)})).$$

Since $P^4(p^{\min(s,t)})$ is 2-connected, by the Hurewicz Theorem there are isomorphisms

$$\pi_3(P^4(p^{\min(s,t)})) \cong H_3(P^4(p^{\min(s,t)}); \mathbb{Z}) \cong \mathbb{Z}/p^{\min(s,t)}\mathbb{Z}. \qquad \blacksquare$$

3. A homological test for a null homotopy I

In the next two sections we give homological and cohomological criteria determining when certain maps are null homotopic. These maps are from S^3 or $P^3(p^r)$ into a wedge $\bigvee_{i=1}^m P^3(p^{r_i})$. So the material in this section and the next focus on 3-dimensional Moore spaces.

In what follows we will use the terms "homotopy fibration diagram" and "homotopy cofibration diagram". To explain these, recall that there is a standard construction that turns any continuous, pointed map $f\colon X \longrightarrow Y$ that is a surjection on path-components into a fibration, in the sense that f factors as $p \circ \phi$ where $\phi\colon X \longrightarrow X'$ is a homotopy equivalence and $p\colon X' \longrightarrow Y$ is a fibration (see, for example, [23, Theorem 7.1.14]). The homotopy fibre of f is the fibre of p. As in [23, Section 7.6], a homotopy commutative square

(3)
$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

is equivalent up to homotopy to a strictly commutative square in which the horizontal maps are fibrations. This induces a map between fibres, that is, a map between the homotopy fibres of g' and g. It is notable that while the homotopy types of the fibres are determined by the homotopy classes of g' and g, the homotopy class of the induced map is not determined by the homotopy classes of f and f'. However, the induced map γ can be chosen via the standard construction above so that there is a homotopy commutative diagram of fibration sequences

$$\begin{tikzcd} \Omega X \arrow[r, "\partial'"] \arrow[d, "\Omega f"] & F' \arrow[r] \arrow[d, "\gamma"] & W \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \Omega Y \arrow[r, "\partial"] & F \arrow[r] & Y \arrow[r, "g"] & Z. \end{tikzcd}$$

Further, this diagram could be extended vertically as well, as in [23, Thoerem 7.6.2], to produce a homotopy commutative diagram in which each consecutive pair of horizontal maps and each consecutive pair of vertical maps is a homotopy fibration. Any such diagram originating from the square (3) and extending via homotopy fibrations horizontally or vertically in this manner is called a **homotopy fibration diagram**. A **homotopy cofibration diagram** is defined dually.

In general, let $i_1\colon \Sigma X \longrightarrow \Sigma X \vee \Sigma Y$ and $i_2\colon \Sigma Y \longrightarrow \Sigma X \vee \Sigma Y$ be the inclusions of the left and right wedge summands respectively. Let

$$[i_1, i_2]\colon \Sigma X \wedge Y \longrightarrow \Sigma X \vee \Sigma Y$$

be the Whitehead product of i_1 and i_2.

Let r, s, t be positive integers such that $s, t \geq r$. Then

$$H^2(P^3(p^s); \mathbb{Z}/p^r\mathbb{Z}) \cong H^2(P^3(p^t); \mathbb{Z}/p^r\mathbb{Z}) \cong \mathbb{Z}/p^r\mathbb{Z}.$$

Let u_s and u_t be the generators of $H^2(P^3(p^s); \mathbb{Z}/p^r\mathbb{Z})$ and $H^2(P^3(p^t); \mathbb{Z}/p^r\mathbb{Z})$ respectively. Then $H^2(P^3(p^s) \times P^3(p^t); \mathbb{Z}/p^r\mathbb{Z})$ is generated by $u_s \otimes 1$ and $1 \otimes u_t$.

Lemma 3.1: *Let p be a prime and let s and t be integers such that $s, t \geq 1$. Then there is an isomorphism*

$$H^4(P^2(p^s) \times P^2(p^t); \mathbb{Z}/p^{\min(s,t)}\mathbb{Z}) \cong \mathbb{Z}/p^{\min(s,t)}\mathbb{Z}$$

and $u_s \cup u_t$ is a generator.

Proof. One case of the Künneth Theorem (see, for example, [10, Theorem 3.15]) is as follows. If X and Y are CW-complexes, R is a ring, and $H^k(Y; R)$ is a finitely generated R-module for all k, then the cross product

$$H^*(X; R) \otimes_R H^*(Y; R) \longrightarrow H^*(X \times Y; R)$$

is a ring isomorphism. In our case, if $r = \min(s, t)$ then both $H^*(P^2(p^s); \mathbb{Z}/p^r\mathbb{Z})$ and $H^*(P^2)(p^t); \mathbb{Z}/p^r\mathbb{Z})$ are finitely generated free $\mathbb{Z}/p^r\mathbb{Z}$-modules. Therefore, by the Künneth Theorem, there are isomorphisms

$$\begin{aligned} H^4(P^2(p^s) \times P^2(p^t); \mathbb{Z}/p^r\mathbb{Z}) &\cong H^2(P^2(p^s); \mathbb{Z}/p^r\mathbb{Z}) \otimes H^2(P^2(p^t); \mathbb{Z}/p^r\mathbb{Z}) \\ &\cong \mathbb{Z}/p^r\mathbb{Z} \otimes \mathbb{Z}/p^r\mathbb{Z} \cong \mathbb{Z}/p^r\mathbb{Z} \end{aligned}$$

and $u_s \cup u_t$ is a generator. ∎

Propositions 3.2 and 3.3 give useful tests for when a certain map is null homotopic.

PROPOSITION 3.2: *Let p be an odd prime and $s, t \geq 1$. Let*

$$f : S^3 \to \Sigma P^2(p^s) \wedge P^2(p^t)$$

be a map and let C be the homotopy cofibre of the composite

$$S^3 \xrightarrow{f} \Sigma P^2(p^s) \wedge P^2(p^t) \xrightarrow{[\imath_1, \imath_2]} P^3(p^s) \vee P^3(p^t).$$

The following are equivalent:

(a) *the map f is null homotopic;*
(b) $H^3(\Sigma P^2(p^s) \wedge P^2(p^t); \mathbb{Z}/p^{\min(s,t)}\mathbb{Z}) \xrightarrow{f^*} H^3(S^3; \mathbb{Z}/p^{\min(s,t)}\mathbb{Z})$ *is the zero map;*
(c) *all cup products in $\widetilde{H}^*(C; \mathbb{Z}/p^{\min(s,t)}\mathbb{Z})$ are zero.*

Proof. (a) $\Leftrightarrow$ (b). Let $u = \min(s, t)$ and consider the following string of isomorphisms:

$$\begin{aligned} \pi_3(\Sigma P^2(p^s) \wedge P^2(p^t)) &\cong H_3(\Sigma P^2(p^s) \wedge P^2(p^t); \mathbb{Z}) \\ &\cong H_3(P^4(p^u) \vee P^5(p^u); \mathbb{Z}) \\ &\cong H_3(P^4(p^u) \vee P^5(p^u); \mathbb{Z}/p^u\mathbb{Z}) \\ &\cong H^3(P^4(p^u) \vee P^5(p^u); \mathbb{Z}/p^u\mathbb{Z}) \\ &\cong H^3(\Sigma P^2(p^s) \wedge P^2(p^r); \mathbb{Z}/p^u\mathbb{Z}). \end{aligned}$$

The first isomorphism is due to the Hurewicz Theorem because $\Sigma P^2(p^s) \wedge P^2(p^t)$ is 2-connected. The second isomorphism holds by Lemma 2.3. The third isomorphism holds since

$$H_3(P^4(p^u) \vee P^5(p^u); \mathbb{Z}) \cong H_3(P^4(p^u); \mathbb{Z}) \cong \mathbb{Z}/p^u\mathbb{Z}$$

and changing homology coefficients from $\mathbb{Z}$ to $\mathbb{Z}/p^u\mathbb{Z}$ induces an isomorphism here. The fourth isomorphism holds by the Universal Coefficient Theorem. The fifth isomorphism holds by Lemma 2.3. Observe that under these isomorphisms the map $S^3 \xrightarrow{f} \Sigma P^2(p^s) \wedge P^2(p^t)$ is sent to

$$H^3(\Sigma P^2(p^s) \wedge P^2(p^t); \mathbb{Z}/p^u\mathbb{Z}) \xrightarrow{f^*} H^3(S^3; \mathbb{Z}/p^u\mathbb{Z}).$$

Thus f is null homotopic if and only if $f^* = 0$ in degree 3 mod-p^u cohomology.

(a)$\Rightarrow$(c). If f is null homotopic then $C \simeq P^3(p^s) \vee P^3(p^t) \vee S^4$ is a suspension, so all cup products in $\widetilde{H}^*(C; \mathbb{Z}/p^u\mathbb{Z})$ are zero.

(c)⇒(b). Consider the homotopy cofibration diagram

$$\begin{tikzcd} S^3 \arrow[r, "f"] \arrow[d, equal] & \Sigma P^2(p^s) \wedge P^2(p^t) \arrow[r] \arrow[d, "{[\imath_1,\imath_2]}"] & C_f \arrow[d] \\ S^3 \arrow[r, "{[\imath_1,\imath_2]\circ f}"] \arrow[d] & P^3(p^s) \vee P^3(p^t) \arrow[r] \arrow[d] & C \arrow[d, "d"] \\ * \arrow[r] & P^3(p^s) \times P^3(p^t) \arrow[r, equal] & P^3(p^s) \times P^3(p^t) \end{tikzcd} \tag{4}$$

where C_f is the homotopy cofibre of f and d is an induced map. As C_f is 2-connected, there is an isomorphism

$$d^* : H^2(P^3(p^s) \times P^3(p^t); \mathbb{Z}/p^u\mathbb{Z}) \to H^2(C; \mathbb{Z}/p^u\mathbb{Z}).$$

Therefore $H^2(C; \mathbb{Z}/p^u\mathbb{Z})$ is generated by $d^*(u_s \otimes 1)$ and $d^*(1 \otimes u_t)$.

The right column of (4) induces the exact sequence

$$\begin{aligned} &H^3(C; \mathbb{Z}/p^u\mathbb{Z}) \to H^3(C_f; \mathbb{Z}/p^u\mathbb{Z}) \\ &\qquad \xrightarrow{b} H^4(P^3(p^s) \times P^3(p^t); \mathbb{Z}/p^u\mathbb{Z}) \xrightarrow{d^*} H^4(C; \mathbb{Z}/p^u\mathbb{Z}). \end{aligned} \tag{5}$$

By Lemma 3.1, $H^4(P^3(p^s) \times P^3(p^t); \mathbb{Z}/p^u\mathbb{Z}) \cong \mathbb{Z}/p^u\mathbb{Z}$ is generated by the cup product $u_s \cup u_t$. The naturality of the cup product implies that

$$d^*(u_s \cup u_t) = d^*(u_s) \cup d^*(u_t).$$

But by assumption, cup products in $\widetilde{H}^*(C; \mathbb{Z}/p^u\mathbb{Z})$ are zero. Therefore $d^* = 0$ in (5), implying that b is onto. Hence the order of $H^3(C_f; \mathbb{Z}/p^u\mathbb{Z})$ is at least p^u.

On the other hand, the top row of (4) induces the exact sequence

$$\begin{aligned} &H^2(S^3; \mathbb{Z}/p^u\mathbb{Z}) \to H^3(C_f; \mathbb{Z}/p^u\mathbb{Z}) \\ &\qquad \xrightarrow{a} H^3(\Sigma P^2(p^s) \wedge P^2(p^t); \mathbb{Z}/p^u\mathbb{Z}) \xrightarrow{f^*} H^3(S^3; \mathbb{Z}/p^u\mathbb{Z}). \end{aligned} \tag{6}$$

Since $H^2(S^3; \mathbb{Z}/p^u\mathbb{Z}) = 0$, the map a is an injection, and by Lemma 2.3,

$$H^3(\Sigma P^2(p^s) \wedge P^2(p^t); \mathbb{Z}/p^u\mathbb{Z}) \cong \mathbb{Z}/p^u\mathbb{Z}.$$

Hence the order of $H^3(C_f; \mathbb{Z}/p^u\mathbb{Z})$ is at most p^u.

Thus $H^3(C_f; \mathbb{Z}/p^u\mathbb{Z})$ has order p^u. But this implies that a is a monomorphism between finite groups of the same order and so must be an isomorphism. Therefore f^* in (6) is the zero map. ∎

A similar argument to Proposition 3.2, but with variations, gives the following.

Proposition 3.3: *Let p be an odd prime and $r, s, t \geq 1$. Let*

$$f : P^3(p^r) \to \Sigma P^2(p^s) \wedge P^2(p^t)$$

be a map and let C be the homotopy cofibre of the composite

$$P^3(p^r) \xrightarrow{f} \Sigma P^2(p^s) \wedge P^2(p^t) \xrightarrow{[\imath_1,\imath_2]} P^3(p^s) \vee P^3(p^t).$$

Let $v = \min(r, s, t)$. Then the following are equivalent:

(a) *the map f is null homotopic;*
(b) *$H^3(\Sigma P^2(p^s) \wedge P^2(p^t); \mathbb{Z}/p^v\mathbb{Z}) \xrightarrow{f^*} H^3(P^3(p^r); \mathbb{Z}/p^v\mathbb{Z})$ is the zero map;*
(c) *all cup products in $\widetilde{H}^*(C; \mathbb{Z}/p^v\mathbb{Z})$ are zero.*

Proof. (a)⇔(b): Let $u = \min(s, t)$ and consider the following string of isomorphisms

$$\begin{aligned}
[P^3(p^r), \Sigma P^2(p^s) \wedge P^2(p^t)] &\cong H_3(\Sigma P^2(p^s) \wedge P^2(p^t); \mathbb{Z}/p^r\mathbb{Z}) \\
&\cong H^3(\Sigma P^2(p^s) \wedge P^2(p^t); \mathbb{Z}/p^r\mathbb{Z}) \\
&\cong H^3(P^4(p^u) \vee P^5(p^u); \mathbb{Z}/p^r\mathbb{Z}) \\
&\cong \begin{cases} \mathbb{Z}/p^r\mathbb{Z} & \text{if } r < u \\ \mathbb{Z}/p^u\mathbb{Z} & \text{if } r \geq u \end{cases} \\
&\cong \mathbb{Z}/p^v\mathbb{Z} \\
&\cong H^3(P^4(p^u) \vee P^5(p^u); \mathbb{Z}/p^v\mathbb{Z}) \\
&\cong H^3(\Sigma P^2(p^r \wedge P^2(p^s); \mathbb{Z}/p^v\mathbb{Z}).
\end{aligned}$$

The first isomorphism is due to the mod-p^r Hurewicz isomorphism since $\Sigma P^2(p^s) \wedge P^2(p^t)$ is 2-connected. The second isomorphism holds by the Universal Coefficient Theorem and the third holds by Lemma 2.3. The fourth isomorphism is the calculation of degree 3 cohomology, the fifth holds since $v = \min(r, s, t) = \min(r, u)$, the sixth is calculation again, and the seventh holds by Lemma 2.3. The transition from the second to the seventh is induced by the map of coefficient rings induced by the epimorphism $\mathbb{Z}/p^r\mathbb{Z} \longrightarrow \mathbb{Z}/p^v\mathbb{Z}$. Thus, under these isomorphisms, a map $f \colon P^3(p^r) \longrightarrow \Sigma P^2(p^s) \wedge P(p^t)$ is sent to the map it induces in mod-p^v cohomology. Thus f is null homotopic if and only if $f^* = 0$ in mod-p^v cohomology.

(b)⇔(c): Consider the homotopy cofibration diagram

$$\begin{tikzcd}
P^3(p^r) \arrow[r, "f"] \arrow[d, equal] & \Sigma P^2(p^s)\wedge P^2(p^t) \arrow[r] \arrow[d, "{[\imath_1,\imath_2]}"] & C_f \arrow[d] \\
P^3(p^r) \arrow[r, "{[\imath_1,\imath_2]\circ f}"] \arrow[d] & P^3(p^s)\vee P^3(p^t) \arrow[r] \arrow[d] & C \arrow[d, "d"] \\
* \arrow[r] & P^3(p^s)\times P^3(p^t) \arrow[r, equal] & P^3(p^s)\times P^3(p^t)
\end{tikzcd}$$

where C_f is the homotopy cofibre of f and d is an induced map. As C_f is 2-connected,

$$d^* : H^2(P^3(p^s)\times P^3(p^t);\mathbb{Z}/p^v\mathbb{Z}) \to H^2(C;\mathbb{Z}/p^v\mathbb{Z})$$

is an isomorphism. Therefore $H^2(C;\mathbb{Z}/p^v\mathbb{Z})$ is generated by $d^*(u_s\otimes 1)$ and $d^*(1\otimes u_t)$. The diagram also induces a diagram of exact sequences

$$\begin{tikzcd}
H^3(C_f;\mathbb{Z}/p^v\mathbb{Z}) \arrow[r, "a"] \arrow[d, "b"] & H^3(\Sigma P^2(p^s)\wedge P^2(p^t);\mathbb{Z}/p^v\mathbb{Z}) \arrow[r, "f^*"] \arrow[d, "c"] & H^3(P^3(p^r);\mathbb{Z}/p^v\mathbb{Z}) \\
H^4(P^3(p^s)\times P^3(p^t);\mathbb{Z}/p^v\mathbb{Z}) \arrow[r, equal] \arrow[d, "d^*"] & H^4(P^3(p^s)\times P^3(p^t);\mathbb{Z}/p^v\mathbb{Z}) \arrow[d] & \\
H^4(C;\mathbb{Z}/p^v\mathbb{Z}) \arrow[r] & H^4(P^3(p^s)\vee P^3(p^s);\mathbb{Z}/p^v\mathbb{Z}) = 0 &
\end{tikzcd}$$

where a, b and c are names for the maps induced in cohomology. Observe that, in the middle column, $s,t\geq v$ so

$$H^3(\Sigma P^2(p^s)\wedge P^2(p^t);\mathbb{Z}/p^v\mathbb{Z}) \cong H^4(P^3(p^s)\times P^3(p^t);\mathbb{Z}/p^v\mathbb{Z}) \cong \mathbb{Z}/p^v\mathbb{Z},$$

implying that c is an isomorphism. Therefore, the commutativity of the top square implies that a is surjective if and only if b is. On the other hand, the top row implies that a is surjective if and only if f^* is the zero map, while the left column implies that b is surjective if and only if d^* is the zero map. Thus $f^*=0$ if and only if $d^*=0$. Since $H^4(P^3(p^s)\times P^3(p^t);\mathbb{Z}/p^v\mathbb{Z})$ is generated by $u_s\cup u_t$, $d^*=0$ if and only if $H^4(C;\mathbb{Z}/p^v\mathbb{Z})$ has no cup products. Hence $f^*=0$ if and only if $H^4(C;\mathbb{Z}/p^v\mathbb{Z})$ has no cup products. ∎

4. A homological test for a null homotopy II

In this section we aim towards Proposition 4.4, which gives homological and cohomological criteria for when certain maps are null homotopic, and which is applicable much more widely than Propositions 3.2 and 3.3. It also generalizes a result of Neisendorfer [20, Corollary 11.12] on the mod-p^r Hopf invariant. We rephrase that result in weaker form for a better comparison to Proposition 4.4.

LEMMA 4.1: *Let p be an odd prime and $r, s \geq 1$. Let $f\colon P^3(p^r) \longrightarrow P^3(p^s)$ be a map and let C_f be its cofibre. If*

- *$f_*\colon \tilde{H}_*(P^3(p^r);\mathbb{Z}) \longrightarrow \tilde{H}_*(P^3(p^s);\mathbb{Z})$ is the zero map, and*
- *all cup products in $\widetilde{H}^*(C_f;\mathbb{Z}/p^{\min(r,s)}\mathbb{Z})$ are zero,*

then f is null homotopic. ∎

Lemma 4.1 will be generalized to maps $f\colon X \longrightarrow \bigvee_{i=1}^m P^3(p^{r_i})$ for $X = S^3$ or $X = P^3(p^r)$. This requires some initial work, the first aspect of which is a general lemma concerning trivial cup products related to maps of wedges.

LEMMA 4.2: *Let $f\colon\bigvee_{i=1}^m A_i \longrightarrow \bigvee_{j=1}^n B_j$ be a map with homotopy cofibre C_f and suppose that $f^*=0$ for cohomology with coefficient group G and all cup products in $\widetilde{H}^*(C_f;G)$ are zero. For $1 \leq \imath \leq m$ and $1 \leq \jmath \leq n$, let $f_{\imath,\jmath}$ be the composite*

$$f_{\imath,\jmath}\colon A_\imath \hookrightarrow \bigvee_{i=1}^m A_i \xrightarrow{f} \bigvee_{j=1}^n B_j \longrightarrow B_\jmath$$

where the left map is the inclusion of the $\imath^{th}$ wedge summand and the right map is the pinch onto the $\jmath^{th}$ wedge summand. If $C_{f_{\imath,\jmath}}$ is the homotopy cofibre of $f_{\imath,\jmath}$ then all cup products in $\widetilde{H}^*(C_{f_{\imath,\jmath}};G)$ are zero.

Proof. We use an intermediate map. Let $f_\jmath$ be the composite

$$f_\jmath\colon \bigvee_{i=1}^m A_i \xrightarrow{f} \bigvee_{j=1}^n B_j \longrightarrow B_\jmath \tag{7}$$

and let $C_{f_\jmath}$ be the homotopy cofibre of $f_\jmath$. Consider the homotopy cofibration diagram

$$\begin{tikzcd}
\bigvee_{i=1}^m A_i \arrow[r, "f"] \arrow[d, equal] & \bigvee_{j=1}^n B_j \arrow[r] \arrow[d] & C_f \arrow[d, "d"] \\
\bigvee_{i=1}^m A_i \arrow[r, "f_\jmath"] & B_\jmath \arrow[r] & C_{f_\jmath}
\end{tikzcd}$$

where d is an induced map of cofibres. Take cohomology with coefficient group G. The homotopy cofibration diagram induces a map between long exact sequences in cohomology. By hypothesis, $f^* = 0$ so the definition of $f_\jmath$ implies that $f_\jmath^* = 0$ as well. Therefore, for every $k \geq 1$, there is a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & H^k(\bigvee_{i=1}^m \Sigma A; G) \arrow[r] \arrow[d, equal] & H^k(C_{f_\jmath} : G) \arrow[r] \arrow[d, "d^*"] & H^k(B_\jmath; G) \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
0 \arrow[r] & H^k(\bigvee_{i=1}^m \Sigma A; G) \arrow[r] & H^k(C_f; G) \arrow[r] & H^k(\bigvee_{i=1}^n B_i; G) \arrow[r] & 0.
\end{tikzcd}$$

A diagram chase shows that d^* is injective, and this is true for all $k \geq 1$. Thus, by the naturality of the cup product, the vanishing of cup products in $\widetilde{H}^*(C_f; G)$ implies their vanishing in $\widetilde{H}^*(C_{f_\jmath}; G)$.

Next, notice that the definition of $f_{\imath,\jmath}$ in the statement of the lemma and $f_\jmath$ in (7) imply that $f_{\imath,\jmath}$ is the composite $A_\imath \hookrightarrow \bigvee_{i=1}^m A_i \xrightarrow{f_\jmath} B_\jmath$. This factorization induces a homotopy cofibration diagram

$$\begin{tikzcd}
A_\imath \arrow[r] \arrow[d, equal] & \bigvee_{i=1}^m A_i \arrow[r, "h"] \arrow[d, "f_\jmath"] & \bigvee_{\substack{i=1 \\ i\neq\imath}}^m A_i \arrow[d, "g"] \\
A_\imath \arrow[r, "f_{\imath,\jmath}"] \arrow[d] & B_\jmath \arrow[r] \arrow[d] & C_{f_{\imath,\jmath}} \arrow[d, "d'"] \\
* \arrow[r] & C_{f_\jmath} \arrow[r, equal] & C_{f_\jmath}
\end{tikzcd}$$

where h is the pinch map, and g and d' are induced maps. Since $f_\jmath^* = 0$ and

$$h^* \colon H^*\Big(\bigvee_{\substack{i=1 \\ i\neq\imath}}^m A_i; G\Big) \to H^*\Big(\bigvee_{i=1}^m A_i; G\Big)$$

is an injection, the top right square implies that $g^* = 0$. Therefore, from the right vertical cofibration in the preceding diagram we obtain a surjection

$$(d')^* \colon H^*(C_{f_\jmath}; G) \to H^*(C_{f_{\imath,\jmath}}; G).$$

As cup products in $\widetilde{H}^*(C_{f_\jmath}; G)$ are zero and $(d')^*$ is a surjection, cup products in $\widetilde{H}^*(C_{f_{\imath,\jmath}}; G)$ are also zero. ∎

Next, we make a transition from a hypothesis that a map is zero in cohomology as in Lemma 4.2 to a map being zero in homology. In general, if the coefficient group G in Lemma 4.2 is a field, then the Universal Coefficient Theorem immediately implies that if $f_* = 0$ then $f^* = 0$. The coefficient ring we care about is $\mathbb{Z}/p^r\mathbb{Z}$, so we need to be more cautious. Perhaps overdoing it, we focus on the 3-dimensional Moore space case again.

Lemma 4.3: *Let p be an odd prime and let $r \geq 1$. Let $X = P^3(p^r)$ or S^3 and let $f : X \to \bigvee_{i=1}^m P^3(p^{r_i})$ be a map. If $f_* : \tilde{H}_*(X;\mathbb{Z}) \to \tilde{H}_*(\bigvee_{i=1}^m P^3(p^{r_i});\mathbb{Z})$ is trivial, then for any abelian group G the map*

$$f^* : \tilde{H}^*\left(\bigvee_{i=1}^m P^3(p^{r_i}); G\right) \to \tilde{H}^*(X;G)$$

is trivial.

Proof. It suffices to prove the lemma in the $m = 1$ case. For $X = P^3(p^r)$ it is obvious that $f^* : \tilde{H}^j(P^3(p^{r_1});G) \to \tilde{H}^j(P^3(p^r);G)$ is trivial except possibly for $j \in \{2,3\}$. By the Universal Coefficient Theorem, there are natural isomorphisms

$$H^2(P^3(p^r);G) \cong Hom(H_2(P^3(p^r);\mathbb{Z}),G))$$

and

$$H^3(P^3(p^r);G) \cong Ext(H_2(P^3(p^r);\mathbb{Z}),G)).$$

By hypothesis, $f_*\colon H_2(P^3(p^r);\mathbb{Z}) \to H_2(P^3(p^{r_1});\mathbb{Z})$ is the zero map, so the naturality of the Universal Coefficient Theorem implies that

$$f^*\colon H^j(P^3(p^{r_1});G) \to H^j(P^3(p^r);G)$$

is the zero map for $j \in \{2,3\}$.

For $X = S^3$, it suffices to show that $f^* : H^3(P^3(p^{r_1});G) \to H^3(S^3;G)$ is trivial. Let $\rho : P^3(p^{r_1}) \to S^3$ be the pinch map to the top cell and consider the composite

$$H^3(S^3;G) \xrightarrow{\rho^*} H^3(P^3(p^{r_1});G) \xrightarrow{f^*} H^3(S^3;G). \tag{8}$$

Observe that the long exact sequence in cohomology determined by the homotopy cofibration $S^2 \to P^3(p^{r_1}) \xrightarrow{\rho} S^3$ implies that ρ^* in (8) is an epimorphism. Therefore, in (8), $f^* = 0$ if and only if $f^* \circ \rho^* = 0$. But $\rho \circ f$ is a self-map of S^3 which factors through a rationally contractible space, implying that it is null homotopic. Hence $f^* \circ \rho^* = 0$, and so $f^* = 0$. ∎

In general, the Hilton–Milnor Theorem states that there is a homotopy equivalence

$$\Omega\Big(\bigvee_{i=1}^{m}\Sigma Y_i\Big)\simeq\prod_{\alpha\in\mathcal{I}}\Omega\Sigma(Y_1^{\wedge\alpha_1}\wedge\cdots\wedge Y_m^{\wedge\alpha_m}) \tag{9}$$

where $\mathcal{I}$ runs over a module basis for the free Lie algebra $L\langle v_1,\ldots,v_m\rangle$, and if $\alpha\in L\langle v_1,\ldots,v_m\rangle$ is a module basis element then for $1\le i\le m$ the integer α_i records the number of instances of v_i in α. Here, if $\alpha_i=0$ for some i then the smash product $Y_1^{\wedge\alpha_1}\wedge\cdots\wedge Y_m^{\wedge\alpha_m}$ is regarded as omitting Y_i rather than being a point; for example, $Y_1^{\wedge 2}\wedge Y_2^{\wedge 0}\wedge Y_3^{\wedge 3}$ is regarded as $Y_1^{\wedge 2}\wedge Y_3^{\wedge 3}$. Moreover, for $1\le k\le m$ let

$$\iota_k\colon\Sigma Y_k\longrightarrow\bigvee_{i=1}^{m}\Sigma Y_i$$

be the inclusion of the k^{th} wedge summand. For $\alpha\in\mathcal{I}$, let

$$w_\alpha\colon\Sigma(Y_1^{\wedge\alpha_1}\wedge\cdots\wedge Y_m^{\wedge\alpha_m})\longrightarrow\bigvee_{i=1}^{m}\Sigma Y_i$$

be the iterated Whitehead product formed from the maps ι_k where each instance of v_k in α is represented by the map ι_k. Then the homotopy equivalence (9) is realized by multiplying together the maps Ωw_α using the loop structure on $\Omega(\bigvee_{i=1}^{m}\Sigma Y_i)$.

In our case, we have

$$\Omega\Big(\bigvee_{i=1}^{m}P^3(p^{r_i})\Big)\simeq\prod_{\alpha\in\mathcal{I}}\Omega\Sigma P^2(p^{r_1})^{\wedge\alpha_1}\wedge\cdots\wedge P^2(p^{r_m})^{\wedge\alpha_m}.$$

Observe that $P^2(p^{r_1})^{\wedge\alpha_1}\wedge\cdots\wedge P^2(p^{r_m})^{\wedge\alpha_m}$ is $((\alpha_1+\cdots+\alpha_m)-1)$-connected. Suppose that X' is 2-dimensional. Then

$$[X',\Omega\Sigma P^2(p^{r_1})^{\wedge\alpha_1}\wedge\cdots\wedge P^2(p^{r_m})^{\wedge\alpha_m}]\cong 0\quad\text{if }(\alpha_1+\cdots+\alpha_m)\ge 3.$$

Observe also that there are m cases for which $(\alpha_1+\cdots+\alpha_m)=1$ and $\binom{m}{2}$ cases for which $(\alpha_1+\cdots+\alpha_m)=2$. So if $X=\Sigma X'$ then

$$\begin{aligned}\Big[X,\bigvee_{i=1}^{m}P^3(p^{r_i})\Big]&\cong\Big[X',\Omega\Big(\bigvee_{i=1}^{m}P^3(p^{r_i})\Big)\Big]\\&\cong\Big[X',\prod_{j=1}^{m}\Omega P^3(p^{r_j})\times\prod_{k\neq l}\Omega\Sigma P^2(p^{r_k})\wedge P^2(p^{r_l})\Big]\\&\cong\prod_{j=1}^{m}[X,P^3(p^{r_j})]\times\prod_{k\neq l}[X,\Sigma P^2(p^{r_k})\wedge P^2(p^{r_l})].\end{aligned}$$

Further, the j^{th} factor $[X, P^3(p^{r_j})]$ is mapped to $[X, \bigvee_{i=1}^{m} P^3(p^{r_i})]$ by the inclusion ι_j and the $\binom{m}{2}$ factors $[X, \Sigma P^2(p^{r_k}) \wedge P^2(p^{r_l})]$ may be arranged so that they map to $[X, \bigvee_{i=1}^{m} P^3(p^{r_i})]$ by the Whitehead products

$$\Sigma P^2(p^{r_k}) \wedge P^2(p^{r_l}) \xrightarrow{[\iota_k,\iota_l]} P^3(p^{r_k}) \vee P^3(p^{r_l}) \hookrightarrow \bigvee_{i=1}^{m} P^3(p^{r_i})$$

where $1 \leq k < l \leq m$. Hence if $f\colon X \longrightarrow \bigvee_{i=1}^{m} P^3(p^{r_i})$ then we may write

$$f \simeq \sum_{j=1}^{m} \iota_j \circ g_j + \sum_{1\leq k<l\leq m} [\iota_k, \iota_l] \circ h_{k,l} \tag{10}$$

for maps $X \xrightarrow{g_j} P^3(p^{r_j})$ and $X \xrightarrow{h_{k,l}} \Sigma P^2(p^{r_k}) \wedge P^2(p^{r_l})$.

Proposition 4.4: *Let $X = P^3(p^r)$ where p is an odd prime and $r \geq 1$ or let $X = S^3$ and set $r = \infty$. Let $f\colon X \to \bigvee_{i=1}^{m} P^3(p^{r_i})$ be a map and let C_f be its cofibre. If*

- *$f_* : \tilde{H}_*(X; \mathbb{Z}) \to \tilde{H}_*(\bigvee_{i=1}^{m} P^3(p^{r_i}); \mathbb{Z})$ is the zero map and*
- *all cup products in $\widetilde{H}^*(C_f; \mathbb{Z}/p^{\min(r,r_i)}\mathbb{Z})$ are zero for all $1 \leq i \leq m$,*

then f is null homotopic.

Proof. Since X is S^3 or $P^3(p^r)$ we have $X \simeq \Sigma X'$ where X' is 2-dimensional. Therefore, by (10), we have

$$f \simeq \sum_{j=1}^{m} \iota_j \circ g_j + \sum_{1\leq k<l\leq m} [\iota_k, \iota_l] \circ h_{k,l}$$

for maps $X \xrightarrow{g_j} P^3(p^{r_j})$ and $X \xrightarrow{h_{k,l}} \Sigma P^2(p^{r_k}) \wedge P^2(p^{r_l})$. To show that f is null homotopic it suffices to show that each g_j and $h_{k,l}$ is null homotopic.

First consider the map g_j when $X = P^3(p^r)$. Notice that g_j is the composite

$$g_j : P^3(p^r) \xrightarrow{f} \bigvee_{i=1}^{m} P^3(p^{r_i}) \xrightarrow{q} P^3(p^{r_j})$$

where q is the pinch map onto the j^{th} wedge summand. Since f induces the zero map in integral homology, so does g_j. by Lemma 4.3, g_j therefore induces the zero map in mod-$p^{\min(r,r_j)}$ cohomology. By hypothesis, all cup products in $\widetilde{H}^*(C_f; \mathbb{Z}/p^{\min(r,r_j)}\mathbb{Z})$ are zero, so by Lemma 4.2, all cup products in $\widetilde{H}^*(C_{g_j}; \mathbb{Z}/p^{\min(r,r_j)}\mathbb{Z})$ are also zero. Thus, by Lemma 4.1, g_j is null homotopic.

Next, consider the map g_j when $X = S^3$. Now g_j is the composite

$$S^3 \xrightarrow{f} \bigvee_{i=1}^{m} P^3(p^{r_i}) \xrightarrow{q} P^3(p^{r_j}).$$

Consider the composite

$$\overline{g}_j\colon P^3(p^{r_j}) \xrightarrow{\pi} S^3 \xrightarrow{g_j} P^3(p^{r_j})$$

where π is the pinch map to the top cell. The argument in the previous paragraph implies that $\overline{g}_j$ is null homotopic. Therefore g_k extends across the cofibre of π, implying that g_k factors as a composite $S^3 \xrightarrow{p^{r_j}} S^3 \xrightarrow{\gamma_j} P^3(p^{r_j})$ for some map γ_j. By Lemma 2.1,

$$\pi_3(P^3(p^{r_j})) \cong \mathbb{Z}/p^{r_j}\mathbb{Z},$$

so $g_j \simeq p^{r_j} \cdot \gamma_j$ is null homotopic.

At this point, we have shown that for either $X = S^3$ or $P^3(p^r)$ we have g_j null homotopic for $1 \leq j \leq m$. Thus (10) implies that $f \simeq \sum_{1\leq k<l\leq m}[\iota_k, \iota_l] \circ h_{k,l}$. Let

$$q_{k,l}\colon \bigvee_{i=1}^{m} P^3(p^{r_i}) \longrightarrow P^3(p^{r_k}) \vee P^3(p^{r_l})$$

be the pinch map onto the k^{th} and l^{th} wedge summands. Observe that every Whitehead product $[\iota_s, \iota_t]$ for $1 \leq s < t \leq m$ composes trivially with $q_{k,l}$ except $[\iota_k, \iota_l]$. Therefore $q_{k,l} \circ f \simeq q_{k,l} \circ (\sum_{1\leq s<t\leq m}[\iota_s, \iota_t] \circ h_{s,t}) \simeq [\iota_k, \iota_l] \circ h_{k,l}$. That is, $q_{k,l} \circ f$ is homotopic to the composite

$$\overline{h}_{k,l}\colon X \xrightarrow{h_{k,l}} \Sigma P^2(p^{r_k}) \wedge P^2(p^{r_l}) \xrightarrow{[\iota_k,\iota_l]} P^3(p^{r_k}) \vee P^3(p^{r_l}).$$

Since f induces the zero map in integral homology, so does $\overline{h}_{k,l}$. Let $C_{\overline{h}_{k,l}}$ be the homotopy cofibre of $\overline{h}_{k,l}$. By hypothesis, cup products in $\widetilde{H}^*(C_f; \mathbb{Z}/p^{\min(r,r_i)}\mathbb{Z})$ are zero for $1 \leq i \leq m$ so cup products in

$$\widetilde{H}^*(C_f; \mathbb{Z}/p^{\min(r,r_k,r_l)}\mathbb{Z})$$

are zero. By Lemma 4.2 (with $B_j = P^3(p^{r_k}) \vee P^3(p^{r_l})$), cup products in

$$\widetilde{H}^*(C_{\overline{h}_{k,l}}; \mathbb{Z}/p^{\min(r,r_k,r_l)}\mathbb{Z})$$

are also zero. Therefore, by Proposition 3.2 in the case $X = S^3$ and Proposition 3.3 in the case $X = P^3(p^r)$, the map $h_{k,l}$ is null homotopic. As this is true for all $1 \leq k < l \leq m$ we obtain $f \simeq *$. ∎

5. The homotopy type of the suspension of certain CW-complexes

In this section we assume M to be a 4-dimensional finite CW-complex that has one 4-cell and homology as follows:

$$\begin{array}{c|c} i & H_i(M;\mathbb{Z}) \\ \hline 0 & \mathbb{Z} \\ 1 & \mathbb{Z}^\ell \oplus \bigoplus_{j=1}^{n} \mathbb{Z}/b_j\mathbb{Z} \\ 2 & \mathbb{Z}^d \oplus \bigoplus_{\bar{j}=1}^{\bar{n}} \mathbb{Z}/\bar{b}_{\bar{j}}\mathbb{Z} \\ 3 & \mathbb{Z}^m \\ 4 & \mathbb{Z} \\ \geq 5 & 0 \end{array} \tag{11}$$

Here each b_j and $\bar{b}_{\bar{j}}$ is a power of an odd prime.

First consider the integer summands of $H_1(M;\mathbb{Z})$. Since the Hurewicz homomorphism $\pi_1(M) \to H_1(M;\mathbb{Z})$ is an epimorphism, each direct summand $\mathbb{Z}$ of $H_1(M;\mathbb{Z})$ is generated by the Hurewicz image of some map $\alpha_i\colon S^1 \longrightarrow M$. Let

$$a\colon \bigvee_{i=1}^{\ell} S^1 \longrightarrow M$$

be the wedge sum of the maps α_i and let W be the homotopy cofibre of a.

Lemma 5.1: *The map Σa has a left homotopy inverse and there is a homotopy equivalence*

$$\Sigma M \simeq \left(\bigvee_{i=1}^{\ell} S^2\right) \vee \Sigma W.$$

Proof. The Hurewicz Theorem implies that the image of a_* is

$$H_1(M;\mathbb{Z})_{\text{free}} \cong \mathbb{Z}^\ell.$$

The Universal Coefficient Theorem implies that $H^1(M;\mathbb{Z})_{\text{free}} \cong H_1(M;\mathbb{Z})_{\text{free}}$. Let $a_i \in H_1(M;\mathbb{Z})$ be the image of $(\alpha_i)_*$ and $\bar{a}_i \in H^1(M;\mathbb{Z})$ be the dual of a_i. Then $\bar{a}_i$ is represented by a map $\epsilon_i\colon M \longrightarrow K(\mathbb{Z},1) \simeq S^1$ and the composite $S^1 \xrightarrow{\alpha_i} M \xrightarrow{\epsilon_i} S^1$ is the identity map. After suspending, one may use the co-H structure to give a map

$$\epsilon\colon \Sigma M \longrightarrow \bigvee_{i=1}^{\ell} S^2$$

which is a left homotopy inverse for Σa. Therefore, with respect to the homotopy cofibration, $\bigvee_{i=1}^{\ell} S^2 \xrightarrow{\Sigma a} \Sigma M \xrightarrow{\Sigma w} \Sigma W$ where $w: M \to W$ is the quotient map, if σ is the comultiplication on ΣM, the composite

$$e\colon \Sigma M \xrightarrow{\sigma} \Sigma M \vee \Sigma M \xrightarrow{\epsilon \vee \Sigma w} \left(\bigvee_{i=1}^{\ell} S^2\right) \vee \Sigma W$$

induces an isomorphism in homology. As the domain and range of e are simply-connected, Whitehead's Theorem implies that e is a homotopy equivalence. ∎

The description of $H_*(M;\mathbb{Z})$ in (11) implies that the homology of W is as follows:

i	$H_i(W;\mathbb{Z})$
0	$\mathbb{Z}$
1	$\bigoplus_{j=1}^{n} \mathbb{Z}/b_j\mathbb{Z}$
2	$\mathbb{Z}^d \oplus \bigoplus_{\bar{j}=1}^{\bar{n}} \mathbb{Z}/\bar{b}_{\bar{j}}\mathbb{Z}$
3	$\mathbb{Z}^m$
4	$\mathbb{Z}$
≥ 5	0

We wish to give a homotopy decomposition of ΣW as a wedge of spheres and Moore spaces. To do so we analyze the homology decomposition of ΣW.

Define $M(\mathbb{Z}/k\mathbb{Z}, n) = P^{n+1}(k)$ and $M(\mathbb{Z}, n) = S^n$, and for any finitely generated abelian groups A and B define

$$M(A \oplus B, n) = M(A, n) \vee M(B, n).$$

Then $\tilde{H}_i(M(A,n);\mathbb{Z})$ is A for $i = n$ and zero otherwise. The following lemma describes the homology decomposition of a simply-connected CW-complex.

Lemma 5.2 ([10, Theorem 4H.3]): *Let X be an n-dimensional simply-connected CW-complex and let $H_i = H_i(X;\mathbb{Z})$. Then there is a sequence of subcomplexes $\{X_i\}_{i=1}^{n}$ such that*

(1) *$H_i(X_m;\mathbb{Z}) \cong H_i(X;\mathbb{Z})$ for $i \leq m$ and $H_i(X_m;\mathbb{Z}) = 0$ for $i > m$;*
(2) *$X_2 = M(H_2, 2)$ and $X \simeq X_n$;*
(3) *X_{m+1} is the mapping cone of a map $f_m\colon M(H_{m+1}, m) \to X_m$ that induces a trivial homomorphism $(f_m)_*\colon H_m(M(H_{m+1}, m);\mathbb{Z}) \to H_m(X_m;\mathbb{Z})$.*

In our case, to describe the homology decomposition of ΣW we need some notation. Let

$$P = \bigvee_{j=1}^{n} P^3(b_j) \quad \overline{P} = \bigvee_{\bar{j}=1}^{\bar{n}} P^3(\bar{b}_{\bar{j}}) \quad \text{and} \quad S = \bigvee_{k=1}^{d} S^2.$$

Starting with $W_2 = P$, Lemma 5.2 implies that there are homotopy cofibrations

$$\begin{aligned} S \vee \overline{P} &\xrightarrow{f_2} W_2 \longrightarrow W_3, \\ \bigvee_{i=1}^{m} S^3 &\xrightarrow{f_3} W_3 \longrightarrow W_4, \\ S^4 &\xrightarrow{f_4} W_4 \longrightarrow \Sigma W, \end{aligned} \tag{12}$$

where f_2, f_3 and f_4 induce the zero map in integral homology. In Lemmas 5.5 and 5.7 we will show that the maps f_2 and f_3 are null homotopic, and in Lemma 5.8 we will show that the map f_4 is either null homotopic or factors in an entirely controllable way. As this will involve analyzing maps between Moore spaces of different torsion orders, a preliminary lemma is required.

Lemma 5.3: *Let X be a finite CW-complex. If p and q are distinct primes and $m, n \geq 3$, then any map $f\colon P^m(p^r) \to \Sigma X \vee P^n(q^t)$ is homotopic to the composite*

$$P^m(p^r) \xrightarrow{f'} \Sigma X \hookrightarrow \Sigma X \vee P^n(q^t)$$

where f' is the composite $P^m(p^r) \xrightarrow{f} \Sigma C \vee P^n(q^t) \xrightarrow{\text{pinch}} \Sigma X$.

Proof. First we show that $[P^m(p^r), Z \wedge P^m(q^t)]$ is trivial for any finite path-connected CW-complex Z. By the Künneth Theorem there is an exact sequence

$$0 \to \bigoplus_{i=1}^{n} \tilde{H}_i(Z) \otimes \tilde{H}_{n-i}(P^n(q^t)) \to \tilde{H}_n(Z \wedge P^n(q^t))$$
$$\to \bigoplus_{i=1}^{n} \mathrm{Tor}(\tilde{H}_i(Z), \tilde{H}_{n-i-1}(P^n(q^t))) \to 0.$$

This implies that the groups $\tilde{H}_*(Z \wedge P^n(q^t))$ are finite abelian and consist only of q-torsion. Therefore, by Serre's Theorem, the homotopy groups $\pi_i(Z \wedge P^n(q^t))$ are also finite abelian and consist only of q-torsion. The homotopy cofibration

$$S^{m-1} \xrightarrow{p^r} S^{m-1} \longrightarrow P^m(p^r)$$

induces an exact sequence

$$\pi_m(Z \wedge P^n(q^t)) \xrightarrow{p^r} \pi_m(Z \wedge P^n(q^t)) \longrightarrow [P^m(p^r), Z \wedge P^n(q^t)] \\ \longrightarrow \pi_{m-1}(Z \wedge P^n(q^t)) \xrightarrow{p^r} \pi_{m-1}(Z \wedge P^n(q^t)).$$

Since multiplying $\pi_i(Z \wedge P^n(q^t))$ by p^r is an isomorphism for $i \geq 1$, by exactness we obtain $[P^m(p^r), Z \wedge P^n(q^t)] \cong 0$.

Next, the homotopy class of f is in $[P^m(p^r), \Sigma X \vee P^n(q^t)]$. Noting that both $P^m(p^r)$ and $P^n(q^t)$ are suspensions since $m, n \geq 3$, the Hilton–Milnor Theorem implies that

$$[P^m(p^r), \Sigma X \vee P^n(q^t)] \cong \prod_{\alpha \in \mathcal{I}} [P^m(p^r), \Sigma X^{\wedge \alpha_1} \wedge (P^{n-1}(q^t))^{\wedge \alpha_2}]$$

where $\mathcal{I}$ runs over a Hall basis for the free Lie algebra $L\langle u, v\rangle$ and α_1, α_2 count the number of instances of u, v respectively in the bracket corresponding to α. The argument in the first paragraph implies that if $\alpha_2 \geq 1$ then each factor $[P^m(p^r), \Sigma X^{\wedge \alpha_1} \wedge (P^{n-1}(q^t))^{\wedge \alpha_2}]$, which is isomorphic to $[P^m(p^r), Z \wedge P^n(q^t)]$ for $Z = X^{\wedge \alpha_1} \wedge (P^{n-1}(q^t))^{\wedge \alpha_2 - 1}$, equals zero. The Hall basis for $L\langle u, v\rangle$ only has one term with $\alpha_2 = 0$, and that is u (when $\alpha_1 = 1$). Thus

$$[P^m(p^r), \Sigma X \vee P^n(q^t)] \cong [P^m(p^r), \Sigma X].$$

Hence f factors through f' up to homotopy. ∎

We also need a lemma concerning cup products in W_3.

Lemma 5.4: *Cup products vanish in $\widetilde{H}^*(W_3; \mathbb{Z}/p^r\mathbb{Z})$.*

Proof. Recall that W is a 4-dimensional CW-complex with a single 4-cell. Let Y be the 3-skeleton of W. Then by cellular approximation and the definition of W_3 the inclusion $W_3 \hookrightarrow \Sigma W$ factors as a composite

$$W_3 \xrightarrow{g} \Sigma Y \hookrightarrow \Sigma W.$$

Suppose that there are elements $x, y \in \widetilde{H}^*(W_3; \mathbb{Z}/p^r\mathbb{Z})$ such that $x \cup y \neq 0$. Since W_3 is simply-connected and of dimension 4, it must be the case that $|x| = |y| = 2$. By Lemma 5.2

$$g^* : H^2(\Sigma Y; \mathbb{Z}/p^r\mathbb{Z}) \to H^2(W_3; \mathbb{Z}/p^r\mathbb{Z})$$

is an isomorphism. Let $\bar{x}, \bar{y} \in H^2(\Sigma Y; \mathbb{Z}/p^r\mathbb{Z})$ be elements such that $x = g^*(\bar{x})$ and $y = g^*(\bar{y})$. Since ΣY is a suspension, all cup products in $\widetilde{H}^*(\Sigma Y; \mathbb{Z}/p^r\mathbb{Z})$

are zero. In particular, we have $\bar{x} \cup \bar{y} = 0$. The naturality of the cup product therefore implies that

$$x \cup y = g^*(\bar{x}) \cup g^*(\bar{y}) = g^*(\bar{x} \cup \bar{y}) = 0,$$

a contradiction. Hence it must be the case that all cup products in

$$\widetilde{H}^*(W_3; \mathbb{Z}/p^r\mathbb{Z})$$

are zero. ∎

Lemma 5.5: *There is a homotopy equivalence $W_3 \simeq P \vee \Sigma S \vee \Sigma\overline{P}$.*

Proof. We will show that the map $S \vee \overline{P} \xrightarrow{f_2} W_2$ in (12) is null homotopic, implying the statement of the lemma. It will be helpful to partition the Moore spaces in $\overline{P}$ by primes. Recall that $\overline{P} = \bigvee_{\bar{j}=1}^{\bar{n}} P^3(\bar{b}_{\bar{j}})$ where each $\bar{b}_{\bar{j}}$ is an odd prime power. List the primes appearing as $\{p_1, \ldots, p_t\}$. Write

$$\overline{P} = \bigvee_{s=1}^{t} \overline{P}_s \quad \text{where } \overline{P}_s = \bigvee_{\ell=1}^{\bar{n}_s} P^3(p_s^{r_{s,\ell}}).$$

Note that $\bar{n} = \bar{n}_1 + \cdots + \bar{n}_t$. Isolating $\overline{P}_1$, let

$$\overline{Q} = \bigvee_{s=2}^{t} \overline{P}_s$$

so that $\overline{P} = \overline{P}_1 \vee \overline{Q}$. For convenience, write p_1 as p and $r_{1,\ell}$ as r_ℓ for $1 \leq \ell \leq n_1$ so that $\overline{P}_1 = \bigvee_{\ell=1}^{n_1} P^3(p^{r_\ell})$. Correspondingly, write $P = P_1 \vee Q$ where P_1 is the wedge of all the mod-p^t Moore spaces in P for some $t \geq 1$, and Q is the wedge of mod-q^s Moore spaces for all primes $q \neq p$. Note that as the torsion in $\overline{P}$ and P may be different, it is possible that for the given prime p the wedge P_1 is trivial. Taking $n_1 = 0$ in the trivial case, write $P_1 = \bigvee_{k=1}^{n_1} P^3(p^{r_k})$. The homotopy cofibration $S \vee \overline{P} \xrightarrow{f_2} W_2 = P \longrightarrow W_3$ may then be rewritten as

$$S \vee \overline{P}_1 \vee \overline{Q} \xrightarrow{f_2} P_1 \vee Q \longrightarrow W_3.$$

To show that f_2 is null homotopic it is equivalent to show that each of the composites

$$f_S : S \hookrightarrow S \vee \overline{P}_1 \vee \overline{Q} \xrightarrow{f_2} P_1 \vee Q,$$

$$f_P : \overline{P}_1 \hookrightarrow S \vee \overline{P}_1 \vee \overline{Q} \xrightarrow{f_2} P_1 \vee Q,$$

$$f_Q : \overline{Q} \hookrightarrow S \vee \overline{P}_1 \vee \overline{Q} \xrightarrow{f_2} P_1 \vee Q,$$

is null homotopic. Since f_2 induces the trivial map in integral homology, so do each of f_P, f_Q and f_S.

First, consider f_S. Since S is 2-dimensional, $P_1 \vee Q$ is 1-connected, and f_S induces the trivial map in degree two integral homology, the Hurewicz homomorphism implies that f_S is null homotopic.

Next, consider f_P. Since $\overline{P}_1 = \bigvee_{\ell=1}^{\bar{n}_1} P^3(p^{r_\ell})$, to show that f_P is null homotopic it suffices to show that the restriction

$$f_P^\ell \colon P^3(p^{r_\ell}) \hookrightarrow \overline{P}_1 \xrightarrow{f_P} P_1 \vee Q$$

of f_P to the ℓ^{th} wedge summand is null homotopic. Since Q consists of mod-q^s Moore spaces for primes $q \neq p$, Lemma 5.3 implies that f_P^ℓ factors as a composite

$$P^3(p^{r_\ell}) \xrightarrow{g_P^\ell} P_1 \hookrightarrow P_1 \vee Q$$

for some map g_P^ℓ. We will show that g_P^ℓ is null homotopic, thereby implying that f_P^ℓ is null homotopic.

Observe that as f_P induces the zero map in homology, so does f_P^ℓ and therefore so does g_P^ℓ. Let $C_{g_P^\ell}$ be the homotopy cofibre of g_P^ℓ and recall that $P_1 = \bigvee_{k=1}^{n_1} P^3(p^{r_k})$. If cup products vanish in $\widetilde{H}^*(C_{g_P}^\ell; \mathbb{Z}/p^{\min(r_\ell,r_k)}\mathbb{Z})$ for $1 \leq k \leq n_1$, then Proposition 4.4 implies that g_P^ℓ is null homotopic.

It remains to show that cup products vanish in $\widetilde{H}^*(C_{g_P}^\ell; \mathbb{Z}/p^{\min(r_\ell,r_k)}\mathbb{Z})$. First, as g_P^ℓ induces the zero map in integral homology, by Lemma 4.3 it also induces the zero map in mod-$p^{\min(r_\ell,r_k)}$ cohomology. Second, notice that g_P^ℓ is homotopic to the composite

$$P^3(p^{r_\ell}) \xrightarrow{f_P^\ell} P_1 \vee Q \xrightarrow{\text{pinch}} P_1.$$

The definitions of f_P^ℓ and f_P then imply that g_P^ℓ is homotopic to the composite

$$P^3(p^{r_\ell}) \longrightarrow \overline{P}_1 \longrightarrow S \vee \overline{P} \vee Q \xrightarrow{f_2} P_1 \vee Q \xrightarrow{\text{pinch}} P_1.$$

As W_3 is the homotopy cofibre of f_2 and cup products vanish in

$$\widetilde{H}^*(W_3; \mathbb{Z}/p^{\min(r_\ell,r_k)}\mathbb{Z})$$

by Lemma 5.4, the factorization of g_P^ℓ through f_2 and Lemma 4.2 imply that cup products vanish in $\widetilde{H}^*(C_{g_P}^\ell; \mathbb{Z}/p^{\min(r_\ell,r_k)}\mathbb{Z})$.

Finally, consider f_Q. Separating out the mod-$p_s^{r_s}$ Moore spaces in Q one prime at a time as was done for p_1 and $\overline{P}_1$, the same argument as for f_P can be used iteratively. Thus f_Q is null homotopic and the proof is complete. ∎

Observe that the space W_4 in (12) is the same as the suspension of the 3-skeleton of W. That is, $W_4 \simeq \Sigma Y$ for Y the 3-skeleton of W. Our approach to dealing with the maps f_3 and f_4 in (12) will be to use the fact that both W_4 and ΣW are suspensions. This requires a general lemma.

LEMMA 5.6: *Let A_i be simply connected for $1 \leq i \leq m$. Suppose that there is a map $g\colon \bigvee_{i=1}^m A_i \longrightarrow \Sigma X$ and a sequence $\{i_1, \ldots, i_k\}$ with $1 \leq i_1 < \cdots < i_k \leq m$ such that, for $1 \leq j \leq k$, the pinch map $q_j\colon \bigvee_{i=1}^m A_i \longrightarrow A_{i_j}$ extends across g to a map $r_j\colon \Sigma X \longrightarrow A_{i_j}$. Then the composite $b\colon \bigvee_{j=1}^k A_{i_j} \hookrightarrow \bigvee_{i=1}^m A_i \xrightarrow{g} \Sigma X$ has a left homotopy inverse.*

Proof. Let r be the composite

$$r\colon \Sigma X \xrightarrow{\sigma} \bigvee_{j=1}^k \Sigma X \xrightarrow{\bigvee_{j=1}^k r_j} \bigvee_{j=1}^k A_{i_j}$$

where σ is defined using the comultiplication on ΣX. We claim that $r \circ b$ is homotopic to a homotopy equivalence. Observe that for $1 \leq j \leq k$ we have

$$\tilde{q}_j \circ r \simeq r_j$$

where $\tilde{q}_j : \bigvee_{j=1}^k A_{i_j} \to A_{i_j}$ is the pinch map. By hypothesis, $r_j \circ g \simeq q_j$, so by definition of b we also have $r_j \circ b \simeq \tilde{q}_j$. Therefore $\tilde{q}_j \circ r \circ b \simeq r_j \circ b \simeq \tilde{q}_j$. In homology, the direct sum of finitely many $\mathbb{Z}$-modules is the same as the direct product, so the map

$$\widetilde{H}_*\Big(\bigvee_{j=1}^k A_i; \mathbb{Z}\Big) \xrightarrow{r_* \circ b_*} \widetilde{H}_*\Big(\bigvee_{j=1}^k A_i; \mathbb{Z}\Big) \cong \bigoplus_{j=1}^k \widetilde{H}_*(A_j; \mathbb{Z})$$

is determined by the projection to each $\widetilde{H}_*(A_j; \mathbb{Z})$. This projection is given by $(\tilde{q}_j)_*$. Thus the fact that $(\tilde{q}_j)_* = (\tilde{q}_j)_* \circ r_* \circ b_*$ implies that $r_* \circ b_*$ is the identity map. Hence, by Whitehead's Theorem, $r \circ b$ is a homotopy equivalence. ∎

LEMMA 5.7: *There is a homotopy equivalence $W_4 \simeq P \vee \Sigma S \vee \Sigma \overline{P} \vee \bigvee_{i=1}^m S^4$.*

Proof. By (12) and Lemma 5.5 there is a homotopy cofibration

$$\bigvee_{i=1}^m S^3 \xrightarrow{f_3} P \vee \Sigma S \vee \Sigma \overline{P} \longrightarrow W_4$$

where f_3 induces the trivial map in integral homology. We will show that f_3 is null homotopic and then the statement of the lemma follows.

Consider the composites

$$\begin{aligned}
&S^3 \hookrightarrow \bigvee_{i=1}^{m} S^3 \xrightarrow{f_3} P \vee \Sigma S \vee \Sigma\overline{P} \longrightarrow P \longrightarrow P^3(b_j),\\
(13)\qquad &S^3 \hookrightarrow \bigvee_{i=1}^{m} S^3 \xrightarrow{f_3} P \vee \Sigma S \vee \Sigma\overline{P} \longrightarrow \Sigma S \longrightarrow S^3,\\
&S^3 \hookrightarrow \bigvee_{i=1}^{m} S^3 \xrightarrow{f_3} P \vee \Sigma S \vee \Sigma\overline{P} \longrightarrow \Sigma\overline{P} \longrightarrow P^4(\bar{b}_{\bar{j}}),
\end{aligned}$$

where the three right-hand maps pinch onto a single wedge summand. Let g be the first composite in (13) and let C_g be its cofibre. Since the cofibre of f_3 is W_4 which is the suspension of the 3-skeleton of W, all cup products in $\widetilde{H}^*(W_4;\mathbb{Z}/p_j^{r_j}\mathbb{Z})$ are zero. Therefore, by Lemma 4.2, all cup products in $\widetilde{H}^*(C_g;\mathbb{Z}/p_j^{r_j}\mathbb{Z})$ are zero. Hence, by Proposition 4.4, g is null homotopic.

Since f_3 induces the zero map in integral homology, the second and third composites in (13) are null homotopic by the Hurewicz Theorem. These null homotopies hold for the inclusion of each S^3 into $\bigvee_{i=1}^{m} S^3$, so f_3 composes trivially with each of the pinch maps $P \vee \Sigma S \vee \Sigma\overline{P} \longrightarrow X$ for $X = P^3(b_j)$, S^3 or $P^4(\bar{b}_{\bar{j}})$. Thus each of these pinch maps extends to a map $W_4 \longrightarrow X$. Since W_4 is a suspension, Lemma 5.6 implies that the map $P \vee \Sigma S \vee \Sigma\overline{P} \longrightarrow W_4$ has a left homotopy inverse. Hence f_3 is null homotopic. ∎

LEMMA 5.8: *Suppose that $H^*(W;\mathbb{Z})$ has no 2-torsion. If the Steenrod operation Sq^2 acts trivially on $H^*(W;\mathbb{Z}/2\mathbb{Z})$ then there is a homotopy equivalence*

$$\Sigma W \simeq P \vee \Sigma S \vee \Sigma\overline{P} \vee \left(\bigvee_{i=1}^{m} S^4\right) \vee S^5.$$

If Sq^2 acts nontrivially on $H^*(W;\mathbb{Z}/2\mathbb{Z})$ then there is a homotopy equivalence

$$\Sigma W \simeq P \vee \bigvee_{k=2}^{d} S^3 \vee \Sigma\overline{P} \vee \left(\bigvee_{i=1}^{m} S^4\right) \vee \Sigma\mathbb{C}P^2.$$

Proof. By (12) and Lemma 5.7 there is a homotopy cofibration

$$S^4 \xrightarrow{f_4} P \vee \Sigma S \vee \Sigma\overline{P} \vee \bigvee_{i=1}^{m} S^4 \longrightarrow \Sigma W$$

where f_4 induces the trivial map in integral homology. Consider the composites

$$
\begin{aligned}
&S^4 \xrightarrow{f_4} P \vee \Sigma S \vee \Sigma\overline{P} \vee \bigvee_{i=1}^{m} S^4 \longrightarrow P \longrightarrow P^3(b_j),\\
&S^4 \xrightarrow{f_4} P \vee \Sigma S \vee \Sigma\overline{P} \vee \bigvee_{i=1}^{m} S^4 \longrightarrow \Sigma S \longrightarrow S^3,\\
&S^4 \xrightarrow{f_4} P \vee \Sigma S \vee \Sigma\overline{P} \vee \bigvee_{i=1}^{m} S^4 \longrightarrow \Sigma\overline{P} \longrightarrow P^4(\bar{b}_{\bar{j}}),\\
&S^4 \xrightarrow{f_4} P \vee \Sigma S \vee \Sigma\overline{P} \vee \bigvee_{i=1}^{m} S^4 \longrightarrow \bigvee_{i=1}^{m} S^4 \longrightarrow S^4,
\end{aligned}
\tag{14}
$$

where the middle and right maps pinch onto a single wedge summand.

Suppose that Sq^2 acts trivially on $H^*(W;\mathbb{Z}/2\mathbb{Z})$. Since each b_j and $\bar{b}_{\bar{j}}$ is a power of an odd prime, by Lemma 2.2,

$$\pi_4(P^3(b_j)) \cong \pi_4(P^3(\bar{b}_{\bar{j}})) \cong 0 \quad \text{and} \quad \pi_4(P^4(b_j)) \cong \pi_4(P^4(\bar{b}_{\bar{j}})) \cong 0,$$

implying the first and third composites in (14) are null homotopic. Since $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by a map η which is detected by Sq^2, the assumption that Sq^2 acts trivially on $H^*(W;\mathbb{Z}/2\mathbb{Z})$ implies that the second composite in (14) is null homotopic. Since f_4 induces the zero map in homology, the Hurewicz homomorphism implies that the fourth composite in (14) is null homotopic. Thus each of the pinch maps $P \vee \Sigma S \vee \Sigma\overline{P} \vee \bigvee_{i=1}^{m} S^4 \longrightarrow X$ for $X = P^3(b_j)$, S^3, $P^4(\bar{b}_{\bar{j}})$ or S^4 extends to a map $\Sigma W \longrightarrow X$. Therefore, by Lemma 5.6, the map $P \vee \Sigma S \vee \Sigma\overline{P} \vee \bigvee_{i=1}^{m} S^4 \longrightarrow \Sigma W$ has a left homotopy inverse. Hence f_4 is null homotopic, implying that

$$\Sigma W \simeq P \vee \Sigma S \vee \Sigma\overline{P} \vee \left(\bigvee_{i=1}^{m} S^4\right) \vee S^5.$$

Next, suppose that Sq^2 acts nontrivially on $H^*(W;\mathbb{Z}/2\mathbb{Z})$. Arguing as before, the first, third and fourth composites in (14) are null homotopic. As Sq^2 detects the generator η of $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$, the nontrivial action of Sq^2 on $H^*(W;\mathbb{Z}/2\mathbb{Z})$ implies that the second composite in (14) is nontrivial for at least one of the pinch maps $\Sigma S = \bigvee_{k=1}^{d} S^3 \longrightarrow S^3$. Possibly the second composite in (14) could be nontrivial for several such pinch maps. However, by [24], any map

$$h\colon S^4 \xrightarrow{\bigvee_{k=1}^{d} \epsilon_k \eta} \bigvee_{k=1}^{d} S^3 \quad \text{with } \epsilon_k \in \{0,1\} \text{ for all } 1 \le k \le d,$$

and having at least one $\epsilon_k = 1$, can be composed with a self-equivalence e of $\bigvee_{k=1}^{d} S^3$ so that $e \circ h$ is homotopic to the composite $S^4 \xrightarrow{\eta} S^3 \hookrightarrow \bigvee_{k=1}^{d} S^3$ where the inclusion can be assumed to be the first wedge summand. Altering the copy of ΣS in $P \vee \Sigma S \vee \Sigma\overline{P} \vee \bigvee_{i=1}^{m} S^4$ by the same self-equivalence e, we obtain that each of the pinch maps $P \vee \bigvee_{k=2}^{d} S^3 \vee \Sigma\overline{P} \vee \bigvee_{i=1}^{m} S^4 \longrightarrow X$ for $X = P^3(b_j)$, S^3 for $2 \leq k \leq d$, $P^4(\bar{b}_{\bar{j}})$ or S^4 extends to a map $\Sigma W \longrightarrow X$. Therefore, by Lemma 5.6, the map $P \vee \bigvee_{k=2}^{d} S^3 \vee \Sigma\overline{P} \vee \bigvee_{i=1}^{m} S^4 \longrightarrow \Sigma W$ has a left homotopy inverse. Therefore f_4 factors as the composite

$$S^4 \xrightarrow{\eta} S^3 \hookrightarrow P \vee \bigvee_{k=1}^{d} S^3 \vee \Sigma\overline{P} \vee \bigvee_{i=1}^{m} S^4$$

implying that $\Sigma W \simeq P \vee \bigvee_{k=2}^{d} S^3 \vee \Sigma\overline{P} \vee (\bigvee_{i=1}^{m} S^4) \vee \Sigma\mathbb{C}P^2$ since $\Sigma\mathbb{C}P^2$ is the homotopy cofibre of η. ∎

Combining the homotopy decomposition $\Sigma M \simeq (\bigvee_{i=1}^{m} S^2) \vee \Sigma W$ in Lemma 5.1 with that of ΣW in Lemma 5.8, we obtain a homotopy decomposition for ΣM.

THEOREM 5.9: *Let M be a 4-dimensional CW-complex that has one 4-cell and has homology as in* (11). *If Sq^2 acts trivially on $H^*(M;\mathbb{Z}/2\mathbb{Z})$ then there is a homotopy equivalence*

$$\Sigma M \simeq \left(\bigvee_{i=1}^{\ell} S^2\right) \vee \left(\bigvee_{k=1}^{d} S^3\right) \vee \left(\bigvee_{l=1}^{m} S^4\right) \vee \left(\bigvee_{j=1}^{n} P^3(b_j)\right) \vee \left(\bigvee_{\bar{j}=1}^{\bar{n}} P^4(\bar{b}_{\bar{j}})\right) \vee S^5.$$

If Sq^2 acts non-trivially on $H^*(M;\mathbb{Z}/2\mathbb{Z})$ then there is a homotopy equivalence

$$\Sigma M \simeq \left(\bigvee_{i=1}^{\ell} S^2\right) \vee \left(\bigvee_{k=1}^{d-1} S^3\right) \vee \left(\bigvee_{l=1}^{m} S^4\right) \vee \left(\bigvee_{j=1}^{n} P^3(b_j)\right) \vee \left(\bigvee_{\bar{j}=1}^{\bar{n}} P^4(\bar{b}_{\bar{j}})\right) \vee \Sigma\mathbb{C}P^2.$$

As a special case we prove Theorem 1.1.

Proof of Theorem 1.1. By assumption M is a smooth, orientable, closed, compact 4-manifold. Then, by Morse Theory, M has a CW-structure with one 4-cell. Since $H_1(M;\mathbb{Z})$ is finitely generated and has no 2-torsion, (1) holds and so $H_*(M;\mathbb{Z})$ is as in (2). Since (2) is a special case of (11), Theorem 5.9 applies to decompose ΣM. Observe that if M is Spin then the Steenrod operation Sq^2 acts trivially on $H^*(M;\mathbb{Z}/2\mathbb{Z})$, so Theorem 5.9 implies that there is a homotopy equivalence

$$\Sigma M \simeq \left(\bigvee_{i=1}^{m}(S^2 \vee S^4)\right) \vee \left(\bigvee_{j=1}^{n}(P^3(b_j) \vee P^4(b_j))\right) \vee \left(\bigvee_{k=1}^{d} S^3\right) \vee S^5,$$

while if M is non-Spin then Sq^2 acts nontrivially, so Theorem 5.9 implies that there is a homotopy equivalence

$$\Sigma M \simeq \left(\bigvee_{i=1}^{m}(S^2 \vee S^4)\right) \vee \left(\bigvee_{j=1}^{n}(P^3(b_j) \vee P^4(b_j))\right) \vee \left(\bigvee_{k=1}^{d-1} S^3\right) \vee \Sigma\mathbb{C}P^2. \qquad \blacksquare$$

6. Applications

Suppose that M is a 4-dimensional manifold satisfying the hypotheses of Theorem 1.1. In this section we give three applications of the homotopy decomposition of ΣM.

The first application is to calculate $E^*(M)$ as a group for any reduced generalized cohomology theory E^*. Examples include complex and real K-theory and cobordism.

Proposition 6.1: *Let M be a smooth, orientable, closed, connected 4-manifold satisfying the hypotheses of Theorem 1.1 and let E^* be a reduced generalized cohomology theory. If M is Spin there is a group isomorphism*

$$\begin{aligned} E^n(M) \cong & \bigoplus_{i=1}^{m}(E^n(S^1) \oplus E^n(S^3)) \oplus \bigoplus_{j=1}^{n}(E^n(P^2(b_j)) \oplus E^n(P^3(b_j))) \\ & \oplus \bigoplus_{k=1}^{d} E^n(S^2) \oplus E^n(S^4). \end{aligned}$$

If M is non-Spin there is a group isomorphism

$$\begin{aligned} E^n(M) \cong & \bigoplus_{i=1}^{m}(E^n(S^1) \oplus E^n(S^3)) \oplus \bigoplus_{j=1}^{n}(E^n(P^2(b_j)) \oplus E^n(P^3(b_j))) \\ & \oplus \bigoplus_{k=2}^{d} E^n(S^2) \oplus E^n(\mathbb{C}P^2). \end{aligned}$$

Proof. Let X, A and B be CW-complexes such that $\Sigma X \simeq \Sigma A \vee \Sigma B$. Using the axioms of reduced generalized cohomology theories, we obtain a string of group isomorphisms

$$\begin{aligned} E^n(X) &\cong E^{n+1}(\Sigma X) \\ &\cong E^{n+1}(\Sigma A \vee \Sigma B) \\ &\cong E^{n+1}(\Sigma A) \oplus E^{n+1}(\Sigma B) \\ &\cong E^n(A) \oplus E^n(B). \end{aligned}$$

In our case, the asserted group isomorphisms for $E^n(M)$ follow immediately from the above group isomorphisms and the homotopy decomposition of ΣM in Theorem 1.1. ∎

The second application is to current groups. Let X be a smooth manifold and let G be a connected Lie group. The **current group** associated to X and G is the space of smooth maps from X to G, which is homotopy equivalent to $\mathrm{Map}(X, G)$. The most famous example is the loop group $\mathrm{Map}(S^1, G)$. Current groups have received considerable attention, notably in [5, 17, 22].

In our case, consider $\mathrm{Map}(M, G)$. There is a fibration

$$\mathrm{Map}^*(M, G) \longrightarrow \mathrm{Map}(M, G) \xrightarrow{\mathrm{ev}} G$$

where ev evaluates a map at the basepoint of M. The multiplication on G induces one on $\mathrm{Map}(M, G)$ so the right inverse of ev induced by projecting M to the constant map implies that there is a homotopy equivalence

$$\mathrm{Map}(M, G) \simeq G \times \mathrm{Map}^*(M, G). \tag{15}$$

Note that $\mathrm{Map}^*(S^n, G) = \Omega^n G$. For $k \in \mathbb{Z}$, let $G \xrightarrow{k} G$ be the k^{th}-power map and let $G\{k\}$ be its homotopy fibre. Applying $\mathrm{Map}^*(\ , G)$ to the homotopy cofibration

$$S^n \xrightarrow{k} S^n \longrightarrow P^{n+1}(k)$$

gives a homotopy fibration

$$\mathrm{Map}^*(P^{n+1}(k), G) \longrightarrow \Omega^n G \xrightarrow{k} \Omega^n G,$$

implying that

$$\mathrm{Map}^*(P^{n+1}(k), G) \simeq \Omega^n G\{k\}.$$

PROPOSITION 6.2: *Let M be a smooth, orientable, closed, connected 4-manifold satisfying the hypotheses of Theorem 1.1 and let G be a connected topological group. If M is Spin there is a homotopy equivalence*

$$\mathrm{Map}(M,G) \simeq G\times\prod_{i=1}^{m}(\Omega G\times\Omega^3 G)\times\prod_{j=1}^{n}(\Omega G\{b_j\}\times\Omega^2 G\{b_j\})\times\left(\prod_{k=1}^{d}\Omega^2 G\right)\times\Omega^4 G.$$

If M is non-Spin there is a homotopy equivalence

$$\begin{aligned}\mathrm{Map}(M,G) \simeq & G\times\prod_{i=1}^{m}(\Omega G\times\Omega^3 G)\times\prod_{j=1}^{n}(\Omega G\{b_j\}\times\Omega^2 G\{b_j\})\\ & \times\left(\prod_{k=2}^{d}\Omega^2 G\right)\times\mathrm{Map}^*(\mathbb{C}P^2,G).\end{aligned}$$

Proof. In general, if $\Sigma X\simeq\Sigma A\vee\Sigma B$ then

$$\begin{aligned}\mathrm{Map}^*(X,G) &\simeq \mathrm{Map}^*(\Sigma X,BG)\simeq\mathrm{Map}^*(\Sigma A,BG)\times\mathrm{Map}^*(\Sigma B,BG)\\ &\simeq \mathrm{Map}^*(A,G)\times\mathrm{Map}^*(B,BG).\end{aligned}$$

In our case, the homotopy decomposition of ΣM in Lemma 1.1 implies that if M is Spin there is a homotopy equivalence

$$\mathrm{Map}^*(M,G) \simeq \prod_{i=1}^{m}(\Omega G\times\Omega^3 G)\times\prod_{j=1}^{n}(\Omega G\{b_j\}\times\Omega^2 G\{b_j\})\times\left(\prod_{k=1}^{d}\Omega^2 G\right)\times\Omega^4 G$$

and if M is non-Spin there is a homotopy equivalence

$$\begin{aligned}\mathrm{Map}^*(M,G) \simeq & \prod_{i=1}^{m}(\Omega G\times\Omega^3 G)\times\prod_{j=1}^{n}(\Omega G\{b_j\}\times\Omega^2 G\{b_j\})\\ & \times\left(\prod_{k=2}^{d}\Omega^2 G\right)\times\mathrm{Map}^*(\mathbb{C}P^2,G).\end{aligned}$$

The asserted homotopy decompositions for $\mathrm{Map}(M,G)$ now follow from (15). ∎

The third application is to gauge groups. Let G be a simply-connected, simple compact Lie group and let M be an orientable, closed, compact 4-manifold. Then $[M,BG]\cong\mathbb{Z}$ so for each $k\in\mathbb{Z}$ there is a principal G-bundle P_k with second Chern class k. The **gauge group** $\mathcal{G}_k(M)$ of P_k is the group of G-equivariant automorphisms of P_k that fix M. Gauge groups are of paramount importance in mathematical physics and geoemetry, and recently their homotopy theory has received a great deal of attention [8, 9, 13, 14, 15, 16, 24, 25, 26, 27, 28, 29, 30, 31].

By [1, 7] there is a homotopy equivalence $B\mathcal{G}_k(M) \simeq \mathrm{Map}_k(M, BG)$ where the right side is the component of the space of continuous (not necessarily pointed) maps from M to BG containing the map inducing P_k. From the mapping space point of view there is an evaluation fibration sequence

$$G \xrightarrow{\partial_k} \mathrm{Map}^*_k(M, BG) \longrightarrow \mathrm{Map}_k(M, BG) \xrightarrow{\mathrm{ev}} BG$$

where ev evaluates a map at the basepoint of M and ∂_k is the fibration connecting map. Notice that the homotopy fibre of ∂_k is $\mathcal{G}_k(M)$.

In Propositions 6.3 and 6.4 the Spin and non-Spin cases of smooth, orientable, closed, connected 4-manifolds are considered separately due to some additional delicacy in the non-Spin case.

Proposition 6.3: *Let M be a smooth, orientable, closed, connected 4-manifold and let G be a simply-connected, compact, simple Lie group. If M is Spin and satisfies the hypotheses of Theorem 1.1 then there is a homotopy equivalence*

$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(S^4) \times \prod_{i=1}^{m}(\Omega G \times \Omega^3 G) \times \prod_{j=1}^{n}(\Omega G\{b_j\} \times \Omega^2 G\{b_j\}) \times \left(\prod_{l=1}^{d} \Omega^2 G\right).$$

Proof. The pinch map $q\colon M \longrightarrow S^4$ to the top cell induces an isomorphism $[S^4, BG] \longrightarrow [M, BG]$, so by the naturality of the evaluation fibration there is a homotopy fibration diagram

$$\begin{array}{ccccccc}
G & \longrightarrow & \mathrm{Map}^*_k(S^4, BG) & \longrightarrow & \mathrm{Map}_k(S^4, BG) & \xrightarrow{\mathrm{ev}} & BG \\
\| & & \downarrow & & \downarrow & & \| \\
G & \longrightarrow & \mathrm{Map}^*_k(M, BG) & \longrightarrow & \mathrm{Map}_k(M, BG) & \xrightarrow{\mathrm{ev}} & BG.
\end{array} \tag{16}$$

Consider the homotopy cofibration sequence $S^3 \xrightarrow{f} M_3 \longrightarrow M \xrightarrow{q} S^4$ where M_3 is the 3-skeleton of M and f is the attaching map for the top cell. This induces a homotopy fibration $\mathrm{Map}^*(S^4, BG) \longrightarrow \mathrm{Map}^*(M, BG) \longrightarrow \mathrm{Map}^*(M_3, BG)$. Since $\mathrm{Map}^*(M_3, BG)$ has one component, restricting to the k^{th} component of $\mathrm{Map}^*(M, BG)$ we obtain a homotopy fibration

$$\mathrm{Map}^*_k(S^4, BG) \longrightarrow \mathrm{Map}^*_k(M, BG) \longrightarrow \mathrm{Map}^*(M_3, BG).$$

Notice that the connecting map for this homotopy cofibration is Σf, which is null homotopic by Theorem 1.1 since it is assumed that M is Spin.

From the left square in (16) we therefore obtain a homotopy fibration diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \Omega\mathrm{Map}_k^*(M,BG) \arrow[r, equal] \arrow[d] & \Omega\mathrm{Map}_k^*(M,BG) \arrow[d, "b"] \\
\mathcal{G}_k(S^4) \arrow[r] \arrow[d, equal] & \mathcal{G}_k(M) \arrow[r, "a"] \arrow[d] & \mathrm{Map}^*(\Sigma M_3, BG) \arrow[d, "(\Sigma f)^*"] \\
\mathcal{G}_k(S^4) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & \mathrm{Map}_k^*(S^4, BG) \arrow[d] \\
* \arrow[r] & \mathrm{Map}_k^*(M,BG) \arrow[r, equal] & \mathrm{Map}_k^*(M,BG)
\end{tikzcd}$$

where a and b are induced maps. Since $(\Sigma f)^*$ is null homotopic, b has a right homotopy inverse. The homotopy commutativity of the top right square then implies that a has a right homotopy inverse. Therefore, using the multiplication on $\mathcal{G}_k(M)$ we obtain a homotopy equivalence

$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(S^4) \times \mathrm{Map}^*(\Sigma M_3, BG).$$

As M is Spin, the homotopy decomposition of ΣM in Theorem 1.1 implies that

$$\Sigma M_3 \simeq \left(\bigvee_{i=1}^{m} (S^2 \vee S^4)\right) \vee \left(\bigvee_{j=1}^{n} (P^3(b_j) \vee P^4(b_j))\right) \vee \left(\bigvee_{l=1}^{d} S^3\right).$$

Substituting this into $\mathrm{Map}^*(\Sigma M_3, BG)$ then gives the homotopy equivalence asserted in the statement of the Proposition. ∎

Next, consider the non-Spin case. We aim for an argument mirroring the Spin case, but using a map $M \longrightarrow \mathbb{CP}^2$ instead of the pinch map $M \longrightarrow S^4$. However, the existence of such a map is not obvious. We produce a near substitute using the approach in [24]. To do so an extra hypothesis is introduced on $\pi_1(M)$ involving the graph product of groups.

Let $\Gamma = (V, E)$ be a finite undirected graph with vertex set V and edge set E, and let $\hat{G} = \{G_v | v \in V\}$ be a collection of groups associated to the vertices of Γ. The **graph product** $\Gamma\hat{G}$ of $\hat{G}$ over Γ is the quotient group F/R, where $F = *_{v\in V} G_v$ is the free product of G_v's and R is the normal subgroup generated by commutator groups $[G_u, G_v]$ wherever (u, v) is in E. For example, if Γ is a complete graph then $\Gamma\hat{G} = \bigoplus_{v\in V} G_v$ or if Γ is a graph of discrete points then $\Gamma\hat{G} = *_{v\in V} G_v$.

If each G_v is cyclic then the abelianization of $\Gamma\hat{G}$ is $\bigoplus_{v\in V} G_v$. It is known that if a group H is finitely presented then there is a smooth, orientable, closed, connected 4-manifold whose fundamental group is H (see, for example, [6, Theorem 1.2]). For example, if $\Gamma\hat{G}$ is a graph product of cyclic groups $\{G_v\}_{v\in V}$ then there is a smooth, orientable, closed, connected 4-manifold with $\pi_1(M) \cong \Gamma\hat{G}$ and $H_1(M;\mathbb{Z}) \cong \bigoplus_{v\in V} G_v$. A specific interesting case is when $M = M' \times S^1$ where M' is a smooth, orientable, closed, connected 3-manifold with $\pi_1(M')$ the graph product of copies of $\mathbb{Z}$ (a right-angled Artin group) or copies of $\mathbb{Z}/2\mathbb{Z}$ (a right-angled Coxeter group).

Proposition 6.4: *Let M be a smooth, orientable, closed, connected 4-manifold and let G be a simply-connected, compact, simple Lie group. Let $\Gamma\hat{G}$ be a graph product of $\{G_i\}_{i=1}^{m+n}$ where $G_i = \mathbb{Z}$ for $1 \le i \le m$, $G_{j+m} = \mathbb{Z}/b_j\mathbb{Z}$ for $1 \le j \le n$, and each b_j is odd. If M is non-Spin and $\pi_1(M) \cong \Gamma\hat{G}$ then there is a homotopy equivalence*

$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(\mathbb{C}P^2) \times \prod_{i=1}^{m}(\Omega G \times \Omega^3 G) \times \prod_{j=1}^{n}(\Omega G\{b_j\} \times \Omega^2 G\{b_j\}) \times \left(\prod_{l=2}^{d} \Omega^2 G\right).$$

Proof. For $1 \le i \le m$, denote the generator of $G_i = \mathbb{Z}$ by α_i. For $1 \le j \le n$, denote the generator of $G_{j+m} = \mathbb{Z}/b_j\mathbb{Z}$ by β_j. Then each α_i has infinite order and each β_j has finite order b_j. Since the Hurewicz homomorphism $h : \pi_1(M) \to H_1(M;\mathbb{Z})$ is the abelianization, $h(\alpha_i)$ has infinite order and $h(\beta_j)$ has order b_j. They generate the direct summands of

$$H_1(M) \cong \bigoplus_{i=1}^{m} \mathbb{Z} \oplus \bigoplus_{j=1}^{n} \mathbb{Z}/b_j\mathbb{Z}.$$

In particular, M satisfies the hypotheses of Theorem 1.1.

For $1 \le i \le m$, each α_i is represented by a map $x_i\colon S_1 \longrightarrow M$ of infinite order and for $1 \le j \le n$, each β_j is represented by a map $y_j\colon S^1 \longrightarrow M$ of order b_j. Since β_j has order b_j, it extends to a map $\tilde{\beta}_j : P^2(b_j) \to M$. Let

$$\xi\colon \left(\bigvee_{i=1}^{m} S^1\right) \vee \left(\bigvee_{j=1}^{n} P^2(b_j)\right) \longrightarrow M$$

be the wedge sum of the maps α_i and $\tilde{\beta}_j$. The graph product hypothesis on $\pi_1(M)$ implies that ξ induces an epimorphism on π_1. By (1), ξ_* is an isomorphism in degree 1 integral homology, and the description of $H_*(M;\mathbb{Z})$

in (11) together with the homotopy decomposition of ΣM in Theorem 1.1 implies that $\Sigma\xi$ has a left homotopy inverse. Define the space C and the map g by the homotopy cofibration

$$\left(\bigvee_{i=1}^{m} S^1\right) \vee \left(\bigvee_{j=1}^{n} P^2(b_j)\right) \xrightarrow{\xi} M \xrightarrow{g} C.$$

Since ξ induces an epimorphism on π_1, C is simply-connected. This implies that C can be given a minimal CW-structure with one cell corresponding to each homology class, and $H_*(C;\mathbb{Z})$ is determined by $H_*(M;\mathbb{Z})$ since ζ_* has a left inverse. Since $\Sigma\xi$ has a left homotopy inverse, Σg has a right homotopy inverse. Explicitly, the homotopy equivalence for ΣM in Theorem 1.1 implies that

$$\Sigma C \simeq \left(\bigvee_{i=1}^{m} S^4\right) \vee \left(\bigvee_{j=1}^{n} P^4(b_j)\right) \vee \left(\bigvee_{l=1}^{d-1} S^3\right) \vee \Sigma\mathbb{C}P^2.$$

This homotopy equivalence may not desuspend but observe that if C_3 is the 3-skeleton of C then

$$\Sigma C_3 \simeq \left(\bigvee_{i=1}^{m} S^4\right) \vee \left(\bigvee_{j=1}^{n} P^4(b_j)\right) \vee \left(\bigvee_{l=1}^{d-1} S^3\right) \vee S^3.$$

Because C_3 has cells only in dimensions 2 and 3, the attaching maps for the 3-cells are in the stable range, so this homotopy equivalence desuspends and we have

$$C_3 \simeq \left(\bigvee_{i=1}^{m} S^3\right) \vee \left(\bigvee_{j=1}^{n} P^3(b_j)\right) \vee \left(\bigvee_{l=1}^{d-1} S^2\right) \vee S^2.$$

Let D be the subwedge of C_3 given by

$$D = \left(\bigvee_{i=1}^{m} S^3\right) \vee \left(\bigvee_{j=1}^{n} P^3(b_j)\right) \vee \left(\bigvee_{l=1}^{d-1} S^2\right).$$

Then the composite of inclusions $D \longrightarrow C_3 \longrightarrow C$ has homotopy cofibre X, where $\Sigma X \simeq \Sigma\mathbb{C}\mathbb{P}^2$.

Define the map q' by the composite $q' \colon M \xrightarrow{g} C \longrightarrow X$ and define the space Y and the maps f' and δ by the homotopy cofibration sequence

$$M \xrightarrow{q'} X \xrightarrow{f'} Y \xrightarrow{\delta} \Sigma M \xrightarrow{\Sigma q'} \Sigma X.$$

As $\Sigma q'$ has a right homotopy inverse $s \colon \Sigma X \longrightarrow \Sigma M$, the composite

$$Y \vee \Sigma X \xrightarrow{\delta \vee s} \Sigma M \vee \Sigma M \xrightarrow{\nabla} \Sigma M$$

is a homotopy equivalence, where ∇ is the fold map. This implies that δ has a left homotopy inverse and hence f' is null homotopic. Further, when combined with the homotopy equivalence for ΣM in Theorem 1.1, it implies that there is a homotopy equivalence

$$Y \simeq \Big(\bigvee_{i=1}^{m}(S^2 \vee S^4)\Big) \vee \Big(\bigvee_{j=1}^{n}(P^3(b_j) \vee P^4(b_j))\Big) \vee \Big(\bigvee_{l=1}^{d-1} S^3\Big). \tag{17}$$

Now replace the homotopy cofibration $M \xrightarrow{q} S^4 \xrightarrow{\Sigma f} \Sigma M_3$ and the null homotopy for Σf in the argument for the Spin case with the homotopy cofibration $M \longrightarrow X \xrightarrow{f'} Y$ and the null homotopy for f' to obtain a homotopy equivalence

$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(X) \times \mathrm{Map}^*(Y, BG).$$

Substituting the homotopy equivalence for Y in (17) into $\mathrm{Map}^*(Y, BG)$ then gives a homotopy equivalence

$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(X) \times \prod_{i=1}^{m}(\Omega G \times \Omega^3 G) \times \prod_{j=1}^{n}(\Omega G\{b_j\} \times \Omega^2 G\{b_j\}) \times \Big(\prod_{l=2}^{d} \Omega^2 G\Big). \tag{18}$$

Notice that X only contains one 2-cell and one 4-cell, so it is the cofibre of $a\eta$ for some odd number a. While X may not be homotopy equivalent to $\mathbb{CP}^2$, and while $\mathcal{G}_k(X)$ may not be homotopy equivalent to $\mathcal{G}_k(\mathbb{CP}^2)$, by [24, Lemma 2.12] there is a homotopy equivalence $\mathcal{G}_k(X) \times \Omega^2 G \simeq \mathcal{G}_k(\mathbb{CP}^2) \times \Omega^2 G$ for $d \geq 2$. If $d = 1$, by the construction of X, the map $M \to X$ induces isomorphisms

$$H^2_{\mathrm{free}}(M;\mathbb{Z}) \cong H^2_{\mathrm{free}}(X;\mathbb{Z}) \quad \text{and} \quad H^4(M : Z) \cong H^4(X;\mathbb{Z}).$$

Furthermore, the cup products of degree 2 free elements are preserved under these identifications. So X is a Poincaré complex and must be $\mathbb{CP}^2$. Consequently, $\mathcal{G}_k(X) \simeq \mathcal{G}_k(\mathbb{CP}^2)$. Thus, in all cases, from (18) we obtain the asserted homotopy decomposition of $\mathcal{G}_k(M)$. ∎

Propositions 6.3 and 6.4 greatly generalize the results in [24], which considered the special cases when $\pi_1(M)$ is: (i) free, (ii) isomorphic to $\mathbb{Z}/p^r\mathbb{Z}$, or (iii) a free product of groups in (i) and (ii). It is worth emphasizing that the decomposition of $\mathcal{G}_k(M)$ can be simply read off from $H_*(M;\mathbb{Z})$.

Further, Huang and Wu [11] proved a cancellation result in p-local homotopy theory. From this we obtain the following.

COROLLARY 6.5: *Let M be a manifold as in Propositions 6.3 or 6.4 and let p be a prime. If M is Spin, there is a p-local homotopy equivalence $\mathcal{G}_k(M) \simeq \mathcal{G}_l(M)$ if and only if there is a p-local homotopy equivalence $\mathcal{G}_k(S^4) \simeq \mathcal{G}_l(S^4)$. If M is non-Spin, there is a p-local homotopy equivalence $\mathcal{G}_k(M) \simeq \mathcal{G}_l(M)$ if and only if there is a p-local homotopy equivalence $\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_l(\mathbb{CP}^2)$.* ∎

A classification of when there is a p-local homotopy equivalence

$$\mathcal{G}_k(S^4) \simeq \mathcal{G}_l(S^4)$$

for any prime p has been determined for $G = SU(2)$ [15], $G = SU(3)$ [8], $G = SU(5)$ [30], and $G = Sp(2)$ [28]. For example, when $G = SU(3)$ there is a p-local homotopy equivalence $\mathcal{G}_k(S^4) \simeq \mathcal{G}_l(S^4)$ if and only if $(k, 12) = (l, 12)$, where (a, b) is the greatest common denominator of integers a and b. Partial classifications have been determined in many other cases [9, 13, 14, 16, 25, 29, 31].

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