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RATIONAL MAPS AND K3 SURFACES

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ABSTRACT

For a very general complex projective K3 surface S and a smooth projective surface A with trivial canonical class, we prove that there is no dominant rational map $A \dashrightarrow S$, which is not an isomorphism.

1. Introduction

The purpose of this note is to contribute to the study of rational maps with target a K3 surface (we will mainly consider smooth complex projective surfaces). A particular question is this:

Does the existence of a dominant rational map, which is not an isomorphism, from a surface with trivial canonical class to a given K3 surface constrain this K3 surface from being "general enough"?

The answer is provided by the following

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Theorem 1.1: *Suppose there exists a dominant rational map $\phi\colon A \dashrightarrow S$ from a surface A with trivial canonical class to a* K3 *surface S. Suppose also that* $\deg \phi \geq 2$. *Then the Picard number $\rho(S)$ is strictly greater than* 1.

The study of dominant rational maps between K3 and Abelian surfaces goes back to the papers [S-I] and [Mo] (there the authors used the lattice-theoretic point of view). Also, a special case of Theorem 1.1, concerning general quartic surfaces, follows from the results obtained by C. Voisin in [Vo, Theorem 2] (she took the IVHS approach). We unify these trends in a way by employing some simple birational and projective geometry (cf. Remark 1.6 below).

Let us collect some examples supporting Theorem 1.1.

Example 1.2 (Kummer surfaces): Take an Abelian surface A and consider the involution $\tau\colon A \longrightarrow A$ acting as $x \mapsto -x$ for all $x \in A$. The minimal resolution $\operatorname{Km} A \longrightarrow A/\tau$ of the quotient A/τ (extracting sixteen (-2)-curves out of ordinary double points) gives a K3 surface admitting a rational map $A \dashrightarrow \operatorname{Km} A$ of degree 2. One has $\rho(\operatorname{Km} A) = 16 + \rho(A)$ in this case.

In view of Example 1.2 it is worth noting that already the case of K3 surfaces in positive characteristic (≥ 5) indicating the setup of Theorem 1.1 is not so trivial. Indeed, it is known that the supersingular Kummer surfaces form a 1-dimensional family, whereas there is a 9-dimensional family of all supersingular K3 (see [L, Section 5] for precise results and references). This, together with the "rational sandwich theorem" from [L1], implies that there exist non-Kummer (supersingular) K3 surfaces dominated by Abelian ones.

Here are more examples over $\mathbb{C}$:

Example 1.3 (Shioda surfaces): Let S be an elliptic K3 surface with a section and two singular fibers of type II^* (plus some other singular fibers). It was proved by T. Shioda in [Shi] that there exists a Kummer surface $\operatorname{Km} C_1 \times C_2$, where C_i are elliptic curves, and two rational maps

$$\operatorname{Km} C_1 \times C_2 \dashrightarrow S, \quad S \dashrightarrow \operatorname{Km} C_1 \times C_2$$

of degree 2. One can also compute that $\rho(S) \geq 18$.

The existence of such S follows from the results in [N]. Namely, all K3 surfaces Y, admitting a primitive embedding of lattices

$$U \oplus E_8(-1)^{\oplus 2} \oplus \mathbb{Z}\langle -2d \rangle \hookrightarrow \operatorname{NS}(Y)$$

for some $d \geq 1$, form a moduli space of dimension one. Any such Y possesses a jacobian elliptic fibration (that is it has a section) and two singular fibers of type II^*. Also for very general such Y the equality

$$U \oplus E_8(-1)^{\oplus 2} \oplus \mathbb{Z}\langle -2d\rangle = \mathrm{NS}(Y)$$

holds. Moreover, the lattice $U \oplus E_8(-1)^{\oplus 2} \oplus \mathbb{Z}\langle -2d\rangle$ has a unique, up to isometries, primitive embedding into the K3 lattice (see [N1, Theorem 1.14.4]). Consequently, the transcendental lattice $T(Y)$ coincides with $U \oplus \mathbb{Z}\langle 2d\rangle$, and it follows from [Mo, Corollary 4.4] that Y is not a Kummer surface.

Finally, let us indicate that for even d there is a geometric construction of Y due to K. Hulek and M. Schütt, see [H-S]. They proved that any such Y is obtained by a quadratic base change from a rational (jacobian) elliptic surface. It was also shown that Y represents a specific member of the so-called **Barth–Peters family** (cf. [H-S, Lemma 4.8]).

Example 1.4 (Symplectic automophisms): An automorphism σ of a K3 surface A is called **symplectic** if σ induces trivial action on $H^0(A, \Omega^2_A) \simeq \mathbb{C}$. Let $G \subset \mathrm{Aut}(A)$ be a finite group of symplectic automorphisms. The minimal resolution $S \longrightarrow A/G$ provides a K3 surface S together with a dominant rational map $A \dashrightarrow S$. It is easy to see that $\rho(S) \geq 9$. Indeed, the morphism $A \longrightarrow A/G$ can not be étale, hence the resolution $S \longrightarrow A/G$ has an exceptional locus. Note that with a detailed analysis of possible G one can describe the Picard group of S more precisely (see [Hu, Chapter 15]).

Example 1.5: Let $S \subset \mathbb{P}^1 \times \mathbb{P}^2$ be given by a general divisor of type $(2, 3)$. Projection on the first factor yields an elliptic fibration $\pi : S \longrightarrow \mathbb{P}^1$. Then there exists a rational map $S \dashrightarrow S$ of degree 16 which induces the morphism $[4] : E \longrightarrow E$ on every smooth fiber E of π (cf. [D]). Note that $\rho(S) = 2$ and so the estimate in Theorem 1.1 is sharp. Also observe that the quadratic form on $\mathrm{NS}(S)$ is $2x^2 + 6xy$ and hence S does not contain (-2)-curves—contrary to the claim in the first version of our paper. We are indebted to S. Galkin and E. Shinder for communicating this construction to us.

Remark 1.6: Suppose that A in Theorem 1.1 is an Abelian surface and take the resolution of indeterminacies of ϕ as in the diagram (2.1) below. We have

$$T(W) = T(A)$$

for the transcendental lattices of A and its blow-up W. It also follows from the projection formula for the resolved morphism $q : W \longrightarrow S$ that q^* gives an embedding $T(S) \hookrightarrow T(W)$. Thus

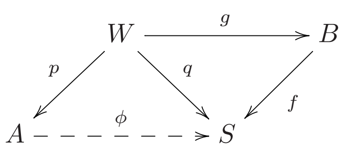
$$\mathrm{rk}\ T(S) \leq \mathrm{rk}\ T(A) \leq 5$$

and so $\rho(S) \geq 17$ (we would like to thank E. Shinder for pointing out this argument). Yet let us note that our result provides more impact on the geometry of S. In particular, when $A = S$ is general, Theorem 1.1 improves the main result of [Ch] (our proof is also considerably shorter). We refer to [B-S-V], [Ma], [N2] and [A-R-V], [B], [Ka] for relevant studies of dominant rational maps between different K3 surfaces and some other classes of algebraic varieties.

2. The proof of Theorem 1.1

Consider the following commutative diagram:

(2.1)



Here p blows up the indeterminacy locus of ϕ, g and f are a proper contraction and a finite morphism, respectively, provided by the Stein factorization of q, and B is a normal surface.

The idea behind the proof of Theorem 1.1 is that the exceptional set $\mathrm{Ex}(p)$ of p should constrain the geometry of S and ϕ.

The next lemma shows that ϕ does not contract any curves:

Lemma 2.1: *The inclusion* $\mathrm{Ex}(g) \subseteq \mathrm{Ex}(p)$ *holds.*

Proof. Restrict ϕ to appropriate $A' := A \setminus \{\text{a finite set of points}\}$ to obtain a morphism $\tilde{\phi}\colon A' \longrightarrow S$. Consider the exact sequence

$$0 \to \tilde{\phi}^*\Omega^1_S \xrightarrow{\gamma} \Omega^1_{A'} \to \Omega^1_{\tilde{\phi}} \to 0$$

of sheaves of differentials. It follows that

$$\mathrm{Supp}\,\Omega^1_{\tilde{\phi}} = V(\det\gamma), \text{the scheme of zeros},$$

for $\det\gamma$ being the morphism $\tilde{\phi}^*K_S \longrightarrow K_{A'}$ induced by γ. Further, since the canonical classes of A, A' and S are trivial, we get that $\det\gamma$ is a constant.

Moreover, we have $\det\gamma \neq 0$, since γ induces an isomorphism of stalks over the generic point. Thus we obtain $\Omega^1_{\tilde{\phi}} = 0$. Then, since $q\big|_{W\setminus \mathrm{Ex}(p)} = \tilde{\phi}$ for a natural identification $A' = W \setminus \mathrm{Ex}(p)$, we obtain

$$\operatorname{Supp}\Omega^1_q \subseteq \mathrm{Ex}(p) \tag{2.2}$$

for the relative cotangent sheaf Ω^1_q of q.

Note that $\mathrm{Ex}(g) = \operatorname{Supp}\Omega^1_g$ because g is a proper contraction. Note also that $\operatorname{Supp}\Omega^1_g \subseteq \operatorname{Supp}\Omega^1_q$ by the exact sequence

$$g^*\Omega^1_f \to \Omega^1_q \to \Omega^1_g \to 0$$

(cf. (2.1)). The inclusion

$$\mathrm{Ex}(g) \subseteq \mathrm{Ex}(p)$$

of exceptional loci now follows from (2.2). ∎

Write $\mathrm{Ex}(p) = \bigcup_i E_i$ for smooth rational exceptional curves E_i. Let also $R := \sum_j Z_j$ be the ramification divisor of f with $Z_j \subset B$ some (not necessarily distinct) irreducible curves. This R is defined in terms of the canonical classes as follows:

$$R\big|_{B\setminus g(\mathrm{Ex}(g))} = K_{B\setminus g(\mathrm{Ex}(g))} \otimes (f^*K_S^{\vee})\big|_{B\setminus g(\mathrm{Ex}(g))} = K_{B\setminus g(\mathrm{Ex}(g))}. \tag{2.3}$$

We can also put $B \setminus g(\mathrm{Ex}(g)) = W \setminus \mathrm{Ex}(g)$ and identify R with its closure in W.

Lemma 2.2: *$R \subset W$ consists of exactly those E_i that are not contracted by g.*

Proof. We have $K_W = \sum_i n_i E_i$ for some $n_i \in \mathbb{N}$ by the ramification formula applied to p (recall that K_A is trivial). Now the claim follows from

$$R\big|_{B\setminus g(\mathrm{Ex}(g))} = K_{B\setminus g(\mathrm{Ex}(g))} = \sum_i n_i E_i\big|_{W\setminus \mathrm{Ex}(g)},$$

where the first identity is due to the Hurwitz formula applied to the finite morphism $f : B \longrightarrow S$, together with the fact that K_S is trivial. ∎

The following lemma proves a special case of Theorem 1.1:

Lemma 2.3: *Suppose that* $\operatorname{Supp} f^*(f(Z_j)) \subseteq R$ *for some j. Then $f(Z_j)$ is a (-2)-curve on S and hence $\rho(S) \geq 2$.*

Proof. We have $\operatorname{Supp} q^*f(Z_j) \subseteq \operatorname{Supp} g^*R$ by assumption. Note further that $\operatorname{Supp} g^*R \subseteq \mathrm{Ex}(p)$ by Lemmas 2.1 and 2.2. Consequently, we obtain

$$\operatorname{Supp} q^*f(Z_j) \subseteq \mathrm{Ex}(p)$$

and

$$(\deg q)(f(Z_j)^2) = (q^* f(Z_j)^2) < 0,$$

where the latter inequality is due to the fact that the matrix of $(E_i \cdot E_k)$ is negative definite (see [M]). The claim now follows because the arithmetic genus $p_a(f(Z_j)) = \frac{1}{2}(f(Z_j)^2) + 1 \geq 0$ and the equality holds iff $f(Z_j) \simeq \mathbb{P}^1$. ∎

Let us turn to the case not covered by Lemma 2.3:

LEMMA 2.4: *Suppose that either $R = \emptyset$ or for every j there exists an effective cycle $\widetilde{Z_j} \neq 0$ such that $f^*(f(Z_j)) \geq Z_j + \widetilde{Z_j}$ and $\operatorname{Supp} \widetilde{Z_j} \not\subseteq R$. Then ϕ is onto $S' := S \setminus \{$a finite set of points$\}$ and is unramified over S'.*[1]

Proof. If $R = \emptyset$, then Lemmas 2.1 and 2.2 imply that ϕ induces a finite surjective unramified morphism $A' \longrightarrow S'$, where $'$ indicates removing a finite set of points.

In the second case, when $R \neq \emptyset$, Lemma 2.2 and the assumption imply that each $f(Z_j)$ is dominated by $p(g_*^{-1}(\widetilde{Z_j}))$ via ϕ because $g_*^{-1}(\widetilde{Z_j}) \not\subseteq \operatorname{Ex}(p)$. Then it follows from Lemma 2.1 and the equality $q(\operatorname{Ex}(p)) = f(\bigcup_j Z_j)$ that ϕ induces a finite surjective morphism $A' \longrightarrow S'$ for $A' = W \setminus \operatorname{Ex}(p)$ (cf. (2.1)). Finally, $\phi\big|_{A'}$ is unramified, since $R \subseteq \operatorname{Ex}(p)$ (cf. (2.3)). ∎

We will now assume that $\rho(S) = 1$ and complete the proof of Theorem 1.1 by establishing a contradiction (this should be contrasted with Example 1.5). Write $\operatorname{NS}(S) = \mathbb{Z} \cdot H_S$ for an ample generator H_S.

It follows from Lemmas 2.3 and 2.4 that ϕ induces a surjective unramified morphism $A' \longrightarrow S'$ as above. Then, since the complements $A \setminus A'$ and $S \setminus S'$ have codimension ≥ 2, the divisorial pull-back $\phi^* H_S$ of H_S to A is naturally defined. Namely, write H_S as a sum $\sum_i d_i D_i$ of prime Weil divisors D_i with $d_i \in \mathbb{Z}$, set $\phi^* D_i$ to be the Zariski closure of $\phi^{-1}(D_i \cap S')$ and extend by linearity. Note that ϕ^* preserves the linear equivalence.

So we have $\phi^* H_S = H_A$ for some divisor H_A on A. Let $n \in \mathbb{N}$ be such that nH_S is very ample. From the construction we get

$$n^2(H_A^2) = n^2(\deg \phi)(H_S^2) > 0$$

[1] We use interchangeably the terms "unramified" and "smooth" when applied to a finite morphism. Note also that in our case $\phi : A' \longrightarrow S'$ need not be a topological covering (the picture here is similar to , e.g., $f : \mathbb{C} \longrightarrow \mathbb{C}$, given by $f(z) = z^2(z-1)$, which becomes smooth on $\mathbb{C} \setminus \{0\}$).

and $(nH_A \cdot Z) = (nH_S \cdot \phi_*(Z)) > 0$ for every curve $Z \subset A$. Hence H_A is ample by the Nakai–Moishezon criterion. Then multiplying by a common factor one may assume that both H_A and H_S are very ample.

Consider a general smooth curve $C \in |H_A|$ with $C \subset A'$. Then the (scheme-theoretic) image $\phi(C) \subset S$ is defined. Furthermore, since actually $\phi(C) \subset S'$, the preimage $\phi^{-1}\phi(C)$ is defined as well.

Proposition 2.5: *$\phi(C)$ is a smooth curve.*

Proof. Suppose that $\phi(C)$ is singular at some point p. Then, since $\phi\big|_{A'}$ is a local isomorphism, there exist two distinct points $p_1, p_2 \in C$ and tangent vectors $t_i \in T_{p_i}C$ such that $\phi(p_i) = p$ and $\phi_*(t_1) \neq \phi_*(t_2)$ (cf. Figure 1).

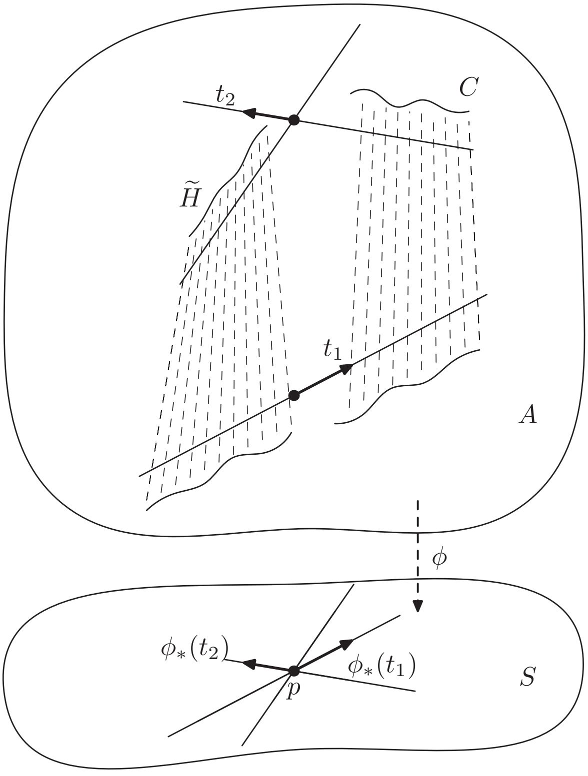


Figure 1.

Consider a general curve $H \in \phi^*|H_S| \subset |H_A|$ which is tangent to C along both t_i (these H constitute a codimension 3 linear system $\mathcal{L}_A \subset \phi^*|H_S|$ by generality of C). Then the curve $\phi(H \cap A')$ is singular at p.[2] The linear system $\mathcal{L}_S \subset |H_S|$ of the closures of such $\phi(H \cap A')$ has codimension 3, since $\phi^* H_S = H_A$, and consists of all members from $|H_S|$ singular at p. Indeed, these members are cut out on S by hyperplanes containing the tangent plane T_pS, which is a codimension 3 condition.

On the other hand, take a general curve $\widetilde{H} \in \phi^*|H_S|$ through p_1, p_2, which has tangency along t_1, but not along t_2 (i.e., $\widetilde{H}$ is the preimage of a curve from $|H_S|$ tangent to only one branch of $\phi(C)$ at p). Then $\phi(\widetilde{H} \cap A')$ is singular at p for $\phi_*(t_1) \neq \phi_*(t_2)$ and $\phi\big|_{A'}$ being a local isomorphism, hence $\widetilde{H} \in \mathcal{L}_A$, a contradiction. ∎

Remark 2.6: Using [D, Proposition 2.1, (ii)] together with our Proposition 2.5 it is tempting to claim a contradiction and conclude the proof of Theorem 1.1. But this, however, disagrees with Example 1.5 because the argument in [D] applies to arbitrary A, S and ϕ of degree ≥ 2. The reason for this confusion is that [D, Proposition 2.1, (ii)] is formulated a bit imprecisely. A more accurate formulation should be that $\phi(C)$ is at most nodal. In fact, Dedieu considers the scheme J parameterizing various pairs $(C^*, x_1 + x_2)$, where $C^* \in |H_A|$, $x_i \in C^*$ are distinct points with $\phi(x_1) = \phi(x_2)$, and shows that $\dim J = \dim |H_A|$. Then the fiber of the natural projection $J \longrightarrow |H_A|$ over a general C is either finite (in which case $\phi(C)$ is nodal and $\phi\big|_C$ is its normalization) or empty (in which case $\phi\big|_C$ is an isomorphism).

Corollary 2.7: $\phi^{-1}\phi(C) = C$.

Proof. Firstly, the closure of the preimage $\phi^{-1}\phi(C)$ is connected, since $\phi^* H_S = H_A$ is ample. Now, if $\phi^{-1}\phi(C)$ has an irreducible component $C' \neq C$, then there is a point $p \in C \cap C'$, so that $\phi(C)$ is singular at $\phi(p)$ for $\phi\big|_{A'}$ being a local isomorphism (cf. Figure 1). But the latter contradicts Proposition 2.5. ∎

We have $\phi(C) \in |kH_S|$ for some $k \geq 1$ because $\rho(S) = 1$ by assumption. Now observe that $\phi_* C = \phi_* \phi^* H_S = (\deg \phi) H_S$ by the projection formula and also $\phi_* C = d\phi(C) = dkH_S$ for $d := \deg \phi\big|_C$. Hence $\deg \phi = dk$. On the other

[2] Note that a priori $H \not\subset A'$ and ϕ has indeterminacies on H.

hand, we have $d = \deg \phi$ by Corollary 2.7, and so $k = 1$. Thus $\phi(C) \in |H_S|$ and it follows that $|H_A| = \phi^*|H_S|$ for $C = \phi^{-1}\phi(C)$. In particular, ϕ must be regular because $C \subset A'$ by construction, hence it induces an étale cover, which is impossible for the K3 surface S.

Theorem 1.1 is completely proved.

We conclude the paper by asking the following question (suggested by the referee):

Is there an analog of Theorem 1.1 for non-projecitve compact complex surfaces?

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