

STRONG ERGODICITY AROUND COUNTABLE PRODUCTS OF COUNTABLE EQUIVALENCE RELATIONS

BY

ASSAF SHANI

*Department of Mathematics & Statistics, Concordia University
1455 De Maisonneuve Blvd. West, Montreal, QC H3G 1M8, Canada
e-mail: assaf.shani@concordia.ca*

ABSTRACT

This paper deals with countable products of countable Borel equivalence relations and equivalence relations “just above” those in the Borel reducibility hierarchy. We show that if E is strongly ergodic with respect to μ then $E^{\mathbb{N}}$ is strongly ergodic with respect to $\mu^{\mathbb{N}}$. We answer questions of Clemens and Coskey regarding their recently defined Γ -jump operations, in particular showing that the \mathbb{Z}^{k+1} -jump of E_{∞} is strictly above the \mathbb{Z}^k -jump of E_{∞} . We study a notion of equivalence relations which can be classified by infinite sequences of “definably countable sets”. In particular, we define an interesting example of such an equivalence relation which is strictly above $E_{\infty}^{\mathbb{N}}$, strictly below $=^+$, and is incomparable with the Γ -jumps of countable equivalence relations.

We establish a characterization of strong ergodicity between Borel equivalence relations in terms of symmetric models, using results from [Sha21]. The proofs then rely on a fine analysis of the very weak choice principles “every sequence of E -classes admits a choice sequence”, for various countable Borel equivalence relations E .

1. Introduction

Let E be an equivalence relation on a Polish space X . Say that E is **Borel** if it is a Borel subset of $X \times X$, and E is **countable** if each E -class is countable. Given a countable group Γ and a Borel action $a: \Gamma \curvearrowright X$, the corresponding **orbit equivalence relation** E_a is defined by $x E_a y$ if there is some $\gamma \in \Gamma$

Received November 14, 2019 and in revised form November 20, 2022

sending x to y . The Feldman–Moore theorem states that any countable Borel equivalence relation is the orbit equivalence relation of some Borel action of a countable group.

Given Borel equivalence relations E and F on Polish spaces X and Y respectively, a Borel map $f: X \rightarrow Y$ is a **homomorphism from E to F** , denoted $f: E \rightarrow_B F$, if for any $x, x' \in X$, if $x E x'$ then $f(x) F f(x')$. A Borel map $f: X \rightarrow Y$ is a **reduction of E to F** if for all $x, x' \in X$, $x E x' \iff f(x) F f(x')$. A Borel homomorphism $f: E \rightarrow_B F$ corresponds to a “Borel definable” map between E -classes and F -classes. This map is injective if and only if f is a reduction. Say that E is **Borel reducible to F** , denoted $E \leq_B F$, if there is a Borel reduction from E to F . This pre-order is used to measure the complexity of various equivalence relations. Say that $E <_B F$ if $E \leq_B F$ and $F \not\leq_B E$.

Let a be a Borel action of a countable group Γ on a probability measure space (X, μ) . Recall that the action is said to be **ergodic (with respect to μ)** if every a -invariant Borel set has either measure zero or measure one. An equivalent condition is: any Borel a -invariant function from X to $[0, 1]$ is constant on a measure one set. Similarly, the action is said to be **generically ergodic** if any a -invariant Borel set is either meager or comeager, equivalently, if any Borel a -invariant function from X to $[0, 1]$ is constant on a comeager set. Note that a Borel a -invariant function is precisely a Borel homomorphism from E_a to $=_{[0,1]}$, where $=_Y$ is the equality relation on Y .

Definition 1.1 (See [Kec ∞ , Definition 7.5]): Let E and F be Borel equivalence relations on Polish spaces X and Y respectively and let μ be a probability measure on X . Say that **E is (μ, \mathbf{F}) -ergodic** if for any Borel homomorphism $f: E \rightarrow_B F$ there is a Borel E -invariant measure one set $A \subseteq X$ such that f maps A into a single F -class.

We sometime say that E is F -ergodic with respect to μ . Note that E is ergodic with respect to μ if and only if E is $(\mu, =_{[0,1]})$ -ergodic. The notion of E_0 -ergodicity is also known as “strong ergodicity”, as first defined by Jones and Schmidt [JS87]. Let \mathbb{F}_2 be the free group on two generators, ν the $(\frac{1}{2}, \frac{1}{2})$ measure on $\{0, 1\}$. Define E_0 on $\{0, 1\}^{\mathbb{Z}}$ to be the orbit equivalence relation of the shift action $\mathbb{Z} \curvearrowright \{0, 1\}^{\mathbb{Z}}$, and E_∞ as the orbit equivalence relation of the shift action $\mathbb{F}_2 \curvearrowright \{0, 1\}^{\mathbb{F}_2}$. Then E_∞ is $(\nu^{\mathbb{F}_2}, E_0)$ -ergodic (see [HK05]). The notion of strong ergodicity is prevalent in the study of countable Borel

equivalence relations. In order to show that E is not Borel reducible to F , one often shows that E is F -ergodic with respect to some measure (see [AK00]).

When studying non-countable Borel equivalence relations, Baire category arguments are often used rather than measure theoretic ones. In this case the notion of generic strong ergodicity is often useful. Let E and F be Borel equivalence relations on Polish spaces X and Y respectively. Say that \mathbf{E} is **generically F-ergodic** if for any Borel homomorphism $f: E \rightarrow_B F$ there is a Borel E -invariant comeager set $A \subseteq X$ such that f maps A into a single F -class. Say that E is **generically ergodic** if it is generically $=_{[0,1]}$ -ergodic. If E is induced by an action of some countable group, this is equivalent to the action being generically ergodic.

1.1. STRONG ERGODICITY FOR INFINITE PRODUCTS. Given equivalence relations E_n on Polish spaces X_n respectively, let their **(full support) product** $\prod_n E_n$ be the equivalence relation on $\prod_n X_n$ defined by $x (\prod_n E_n) y$ if $x(n) (E_n) y(n)$ for all n . Let $E^{\mathbb{N}}$ be the product $\prod_n E$. These equivalence relations arise naturally in the study of the Borel reducibility hierarchy; see for example [HK97].

Let the **finite support product** $\prod_n^{\text{fin}} E_n$ be the equivalence relation on $\prod_n X_n$ defined by $x (\prod_n^{\text{fin}} E_n) y$ if $x (\prod_n E_n) y$ and $x(n) = y(n)$ for all but finitely many n . While the full support product of countable equivalence relation is no longer a countable equivalence relation, the finite support product is. The finite support product operation was studied by Kechris where he showed the following [Kec20, Lemma 4.2]: suppose E_n is a countable Borel equivalence relation and μ_n is a probability measure on X_n . Then $\prod_n^{\text{fin}} E_n$ is ergodic with respect to $\prod_n \mu_n$ if and only if E_n is ergodic with respect to μ_n for every n . Note that if the finite support product is ergodic, then so is the full support product. While strong ergodicity is not preserved under finite support products, we show that it is preserved under full support products.

LEMMA 1.2 (Corollary 3.9 below): *Suppose E_n is a countable Borel equivalence relation and μ_n is a probability measure on X_n . Let F be a countable Borel equivalence relation. Then*

$$\prod_n E_n \text{ is } \left(\prod_n \mu_n, F \right)\text{-ergodic if and only if } E_n \text{ is } (\mu_n, F)\text{-ergodic for all } n.$$

The non-trivial direction is right to left, that is, showing that the infinite product is F -ergodic with respect to the product measure. Let us note that full support products are necessary, even for E_0 -ergodicity. Let $X = \{0, 1\}$, $\mu = (\frac{1}{2}, \frac{1}{2})$ and $E = X^2$ be the equivalence relation with a single equivalence class. Then E is (μ, E_0) -ergodic but $\prod_n^{\text{fin}} E$ is not $(\mu^{\mathbb{N}}, E_0)$ -ergodic (as it is Borel bireducible with E_0). More generally, if μ does not concentrate on a single element and E is any equivalence relation, then $\prod_n^{\text{fin}} E$ is not $(\mu^{\mathbb{N}}, E_0)$ -ergodic. Thus a different approach is necessary for the lemma above, the proof of which appeals to set-theoretic definability in symmetric models. The proof shows in general that for ergodic E_n , homomorphisms $\prod_n E_n \rightarrow_B F$ are determined by homomorphisms defined on finite products $\prod_{n < m} E_n$, on a measure 1 set.

1.2. THE Γ -JUMPS OF CLEMENS AND COSKEY. Recently Clemens and Coskey [CC22] defined new “gentle” jump operators. In particular, by applying these operations to countable equivalence relations, these yield new interesting equivalence relations which are strictly above the countable products of countable equivalence relations, yet lower in the Borel reducibility hierarchy than previously known examples. For a countable group Γ , Clemens and Coskey define the Γ -**jump** of E , $E^{[\Gamma]}$ on X^Γ , by

$$x E^{[\Gamma]} y \iff (\exists \gamma \in \Gamma)(\forall \alpha \in \Gamma)x(\gamma^{-1}\alpha) E y(\alpha).$$

The Γ -jumps generalize the usual shift actions of countable groups. For example, the \mathbb{Z} -jump of $=_{\{0,1\}}$ is E_0 and the \mathbb{F}_2 -jump of $=_{\{0,1\}}$ is E_∞ , where $=_{\{0,1\}}$ is the equality relation on $\{0, 1\}$.

Clemens and Coskey show that for any countable infinite group Γ , if E is a generically ergodic countable Borel equivalence relation (for example, E may be E_0 or E_∞) then $E^{[\Gamma]}$ is E_∞ -generically ergodic. It follows that $E^{[\Gamma]}$ is not Borel reducible to a countable product of countable equivalence relations. In particular, they conclude that $E_0^{[\mathbb{Z}]}$ is strictly above $E_0^{\mathbb{N}}$ in the Borel reducibility hierarchy (it is open whether in fact $E_0^{[\mathbb{Z}]}$ is an immediate successor of $E_0^{\mathbb{N}}$). Whether distinct groups Γ produce non Borel bireducible jumps $E^{[\Gamma]}$ was left open. In particular, they asked if the equivalence relations $E_\infty^{[\mathbb{F}_2]}$ and $E_\infty^{\mathbb{N}} \times E_0^{[\mathbb{Z}]}$ are different than $E_\infty^{[\mathbb{Z}]}$. In Section 5 we show that $E_\infty^{\mathbb{N}} \times E_0^{[\mathbb{Z}]}$ and $E_\infty^{[\mathbb{Z}]}$ are \leq_B -incomparable (see Corollary 5.6 and Proposition 5.14), and that $E_\infty^{[\mathbb{Z}]}$ is strictly below $E_\infty^{[\mathbb{F}_2]}$ (and even $E_\infty^{[\mathbb{Z}^2]}$) with respect to Borel reducibility (see Corollary 1.4). Moreover, we completely characterize strong ergodicity between

Γ -jumps of countable Borel equivalence relations in terms of group theoretic properties.

THEOREM 1.3: *Let Γ and Δ be countable groups and E a generically ergodic countable Borel equivalence relation. The following are equivalent:*

- (1) *There is a subgroup $\tilde{\Delta}$ of Δ , a normal subgroup H of $\tilde{\Delta}$ and a group homomorphism from Γ to $\tilde{\Delta}/H$ with finite kernel;*
- (2) *$E^{[\Gamma]}$ is not generically $E_{\infty}^{[\Delta]}$ -ergodic.*

The same is also true for measures. That is, if E is ergodic with respect to some measure μ , then we may replace (2) with “ $E^{[\Gamma]}$ is not $(\mu^{\Gamma}, E_{\infty}^{[\Delta]})$ -ergodic”.

COROLLARY 1.4: *Let E be a generically ergodic countable Borel equivalence relation.*

- $E^{[\mathbb{Z}]} <_B E^{[\mathbb{Z}^2]} <_B E^{[\mathbb{Z}^3]} <_B \cdots <_B E^{[\mathbb{Z}^{<\omega}]} <_B E^{[\mathbb{F}_2]}$;
- $E^{\mathbb{Z}}$ and $E^{[\mathbb{Z}_2^{<\mathbb{N}}]}$ are \leq_B -incomparable.

Remark 1.5: This is in stark contrast to the situation with countable equivalence relations, in which case any action of a countable abelian group induces an equivalence relation which is Borel reducible to E_0 (see [GJ15]), and any countable Borel equivalence relation is Borel reducible to E_0 on a comeager set (see [KM04, Theorem 12.1]). A central tool in these and other arguments showing hyperfiniteness of countable Borel equivalence relations is the use of the “Marker Lemma” (see [KM04, Lemma 6.7]). The main tool in the proof of Theorem 1.3 is Lemma 5.1, which shows a strong failure of the Marker Lemma in the context of a countable group acting on a set of E_0 -classes (see Remark 5.2). This shows that some kind of appeal to the Marker Lemma is necessary to show the hyperfiniteness of, for example, \mathbb{Z}^2 -actions.

1.3. CLASSIFICATION BY COUNTABLE SEQUENCES OF DEFINABLY COUNTABLE SETS. Recall that $=^+$ is the equivalence relation on $\mathbb{R}^{\mathbb{N}}$ defined so that the map $x \in \mathbb{R}^{\mathbb{N}} \mapsto \{x(n) : n \in \mathbb{N}\}$ is a complete classification. That is, $x =^+ y$ if and only if $\{x(n) : n \in \mathbb{N}\} = \{y(n) : n \in \mathbb{N}\}$. The classifying invariants are countable sets of reals. However, given such arbitrary countable set of reals A , there is no enumeration of A which is definable from A . On the other hand, suppose E is a countable Borel equivalence relation, induced by a Borel action of a countable group Γ . The map $x \mapsto \{\gamma \cdot x : \gamma \in \Gamma\}$, sending x to its orbit, is a complete classification. In this case the classifying invariants are countable

sets of reals which can be enumerated in a simple way: given any y in the orbit of x , $\langle \gamma \cdot y : \gamma \in \Gamma \rangle$ provides an enumeration of the orbit.

For a countable Borel equivalence relation E , $E^{\mathbb{N}}$ can be classified by sequences of countable sets of reals $A = \langle A_n : n < \omega \rangle$ such that for each n there is an enumeration of A_n definable from A and elements in the transitive closure of A . Next we consider equivalence relations which can be classified in such a way, by countable sequences of *definably* countable sets of reals.

Definition 1.6: Let

$$D = \{f \in (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}} : \forall n, i, j (f(n)(i) \text{ is computable from } f(n+1)(j))\}.$$

Define the equivalence relation E_{Π} on D by $x (E_{\Pi}) y$ if for each n ,

$$\{x(n)(i) : i \in \mathbb{N}\} = \{y(n)(i) : i \in \mathbb{N}\}.$$

That is, E_{Π} is $(=^+)^{\mathbb{N}}$ restricted to the domain D .

Given $x \in D$ let $A_n^x = \{x(n)(j) : j \in \mathbb{N}\}$. The map $x \mapsto A^x = \langle A_n^x : n \in \mathbb{N} \rangle$ is a complete classification of E_{Π} . For every n and any $z \in A_{n+1}^x$, all the reals in A_n^x are computable from z , so there is a definable enumeration of A_n^x using z . That is, given the sequence A^x we can definably witness that each A_n^x is a countable set. We show that E_{Π} is not a product of countable Borel equivalence relations, and is also different than the Γ -jumps.

THEOREM 1.7: (1) $E_{\infty}^{\mathbb{N}} <_B E_{\Pi} <_{B=^+}$ and E_{Π} is pinned;
 (2) $E_{\Pi} \not\leq_B E_{\infty}^{[\Gamma]}$ and $E_0^{[\Gamma]} \not\leq_B E_{\Pi}$ for any infinite countable group Γ .

See Definition 2.6 for the definition of pinned. Part (1) is proved in Section 4 and part (2) is proved in Section 5. In Section 4 we give a more general definition attempting to capture those equivalence relations which can be classified by countable sequences of definably countable sets of reals (Definition 4.1), and show that E_{Π} is maximal among those (Theorem 4.15).

Remark 1.8: The only previously known examples of equivalence relations between $E_{\infty}^{\mathbb{N}}$ and $=^+$ were the non-pinned equivalence relations constructed by Zapletal in [Zap11]. Clemens and Coskey [CC22] note that $E_{\infty}^{[\mathbb{Z}]}$ is strictly above $E_{\infty}^{\mathbb{N}}$, strictly below $=^+$ and is pinned, thus is much closer to products of countable Borel equivalence relations. By the results above the equivalence relation E_{Π} also sits in this gap, and is incomparable with the Γ -jumps of countable Borel equivalence relations.

1.4. **VERY WEAK CHOICE PRINCIPLES.** Using the developments in [Sha21], the results above are proved by first reformulating the questions in terms of symmetric models and choice principles. In particular, we isolate an equivalent condition for strong ergodicity between equivalence relations (Lemma 2.5). The results then rely on studying the following weak choice principles.

Definition 1.9: Let E be a countable equivalence relation on a Polish space X . Then **choice for countable sequences of E classes**, abbreviated $\text{CC}[E^{\mathbb{N}}]$, stands for the following statement: Suppose $A = \langle A_n : n \in \mathbb{N} \rangle$ is a countable sequence of sets $A_n \subseteq X$ such that each A_n is an E -class. Then $\prod_n A_n$ is not empty.

THEOREM 1.10 (Theorem 3.2 below): *Suppose E and F are countable Borel equivalence relations on Polish spaces X and Y respectively, and there is a Borel probability measure μ on X such that E is (μ, F) -ergodic. Then there is a model in which $\text{CC}[F^{\mathbb{N}}]$ holds yet $\text{CC}[E^{\mathbb{N}}]$ fails.*

In particular, there are many pairs of countable Borel equivalence relations E and F such that $\text{CC}[E^{\mathbb{N}}]$ and $\text{CC}[F^{\mathbb{N}}]$ are independent. A curious point here is that these models are constructed as intermediate extensions of a random real generic extension (using the measure $\mu^{\mathbb{N}}$). Furthermore, these arguments will not work using a Cohen real, due to the fact that all countable Borel equivalence relations are hyperfinite on a comeager set.

Recall that **countable choice for countable sets of reals**, abbreviated here as $\text{CC}[\mathbb{R}]^{\aleph_0}$, states that any countable sequence $A = \langle A_n : n \in \mathbb{N} \rangle$ of countable sets of reals $A_n \subseteq \mathbb{R}$ admits a choice function. This is a very weak choice principle, commonly studied in the literature (see [HR98]). Over ZF, for a countable equivalence relation E , $\text{CC}[E^{\mathbb{N}}]$ follows from $\text{CC}[\mathbb{R}]^{\aleph_0}$ (since any Polish space is Borel isomorphic to \mathbb{R}).

The proof that E_{Π} is not Borel reducible to $E_{\infty}^{\mathbb{N}}$ and to the Γ -jumps relies on finding a model in which $\text{CC}[\mathbb{R}]^{\aleph_0}$ fails yet for any countable Borel equivalence relation E , $\text{CC}[E^{\mathbb{N}}]$ holds (Theorem 4.10).

ACKNOWLEDGEMENTS. The results in this paper are partially from my PhD thesis. I would like to express my sincere gratitude to my advisor, Andrew Marks, for his guidance and encouragement, and for numerous informative discussions.

Furthermore, I wish to thank Clinton Conley, Yair Hayut, Alexander Kechris, Menachem Magidor and Jindrich Zapletal for helpful conversations. I am also indebted to John Clemens and Sam Coskey for sharing with me an early version of their paper.

Finally, I thank the anonymous referee for carefully reading this paper and providing many corrections, comments, and suggestions.

2. Preliminaries

The reader is referred to any standard text, such as [Jec03] or [Kun83] for the set theoretic notions used below.

From now on V always denotes some ground model over which we force. Most often V will be assumed to be a model of ZFC, and sometimes a model of ZF alone. We will often consider extensions $V \subseteq M$ where M is a model of ZF or ZFC. It will always be the case that M and V have the same ordinals. Shoenfield absoluteness will be used repeatedly, often for statements of the form “ f is a Borel reduction of E to F ”. The most common instance of this situation is when M is a generic extension of V . This is done sometimes with V being replaced with some sufficiently elementary transitive countable model. Another situation is when V is L^M , the constructible universe as calculated in M .

By “generic extension” we always mean using a set forcing. When no poset is involved, we say that a set X is generic over V if X is in some generic extension of V . If x is a real (which we identify with a subset of ω) in some generic extension of V , then x is in fact P -generic over V for some poset P . In this case we write $V[x]$ for $V(x)$. For a formula ψ and a model M , we denote the relativization of ψ to M by ψ^M .

Fact 2.1 (Folklore): Let V be a ZF model, and A a set in some extension of V . Then there exists a minimal transitive model of ZF containing V and A , denoted $V(A)$.

Even if V satisfies ZFC, the axiom of choice may fail in $V(A)$. This does not happen if A is a real, or more generally a set of ordinals.

Fact 2.2 (Folklore): Assume V satisfies ZFC. Suppose A is a set of ordinals in some extension. Then $V(A)$ satisfies the axiom of choice.

For the results in this paper it suffices to consider $V = L[r]$, for some real r (so that the Borel equivalence relations we discuss are in $L[r]$). In this case $V(A)$ is the usual Hajnal relativized L -construction, $L(r, A)$. Generally, $V(A)$ can be constructed as the union of $L(V_\alpha, A)$ over all ordinals α .

Working in some extension of V , let $\text{HOD}_{V,A}$ be the collection of all sets which are hereditarily definable using A , parameters from V , and parameters from the transitive closure of A . Then $\text{HOD}_{V,A}$ is a transitive model of ZF, extending V and containing A . In the examples considered in this paper it will not matter if one takes the minimal model $V(A)$ or the model $\text{HOD}_{V,A}$.

More important for the developments below, note that inside $V(A)$, the model $\text{HOD}_{V,A}$ must be everything, by minimality. We will use this below repeatedly, in the following way:

For any $X \in V(A)$, there is some formula ψ , parameters \bar{a} from the transitive closure of A and $v \in V$ such that X is the unique set satisfying $\psi(X, A, \bar{a}, v)$ in $V(A)$. Equivalently, there is a formula φ such that $X = \{x : \varphi(x, A, \bar{a}, v)\}$ in $V(A)$. In this case say that \bar{a} is a **support** for X . We will be particularly interested in sets with empty support. That is, those definable from A and parameters in V alone.

LEMMA 2.3 (Folklore): *Let V be a ZF model. Suppose P, Q are posets in V and $G \times H$ is generic for $P \times Q$ over V . (That is, G and H are mutually generic over V .) Then $V[G] \cap V[H] = V$.*

The reader is referred to [Gao09] or [Kano08] for a discussion on equivalence relations which are classifiable by countable structures. The main consequence of being classification by countable structures which we use in this paper is the existence of a complete classification which is absolute in the following way (see [Sha21, Fact 2.5]).

If E is classifiable by countable structures, there is a map $x \mapsto A_x$ (definable by a set theoretic formula) which is a complete classification of E and remains so in any generic extension. Furthermore, the assignment is absolute, in the sense that the statement $A_x = A$ cannot change in a forcing extension. We will call such $x \mapsto A_x$ an **absolute classification**. For example, the map $x \mapsto A_x = \{x(i) : i \in \mathbb{N}\}$ is an absolute classification of $=^+$.

Fix an absolute classification $x \mapsto A_x$ of E . The sets A_x are the classifying invariants. More generally, we will say that a set A is an **invariant** (with this fixed classification) if it is of the form A_x in some generic extension. (This

is sometimes called a potential invariant.) For example, any set of reals is an invariant for $=^+$, as it can be made countable by collapsing.

LEMMA 2.4 (see [Sha21, Lemma 3.6]): *Let V be a ZF model. Suppose E and F are Borel equivalence relations on X and Y respectively and $x \mapsto A_x$ and $y \mapsto B_y$ are absolute classifications of E and F respectively. Assume that A is a set in some extension of V so that A is an invariant for E in $V(A)$ (meaning there is some x in a generic extension of $V(A)$ so that $A = A_x$). Assume further that E is Borel reducible to F . Then there is an invariant B for F , definable from A and parameters in V alone, such that $V(A) = V(B)$.*

Proof sketch. Fix a Borel reduction f of E to F . The set B is defined as the unique set such that in any generic extension of $V(A)$, if there is an $x \in X$ with $A = A_x$, then $B = B_{f(x)}$. Since f is a reduction, A can be defined from B as the unique set such that for any x in a generic extension, if $B = B_{f(x)}$ then $A = A_x$. ■

Given an ideal I of Borel subsets of X let P_I be the poset of all Borel I -positive sets, ordered by inclusion, p extends q if $p \subseteq q$. The reader is referred to [Zap08] or [KSZ13] for the definition and a discussion on proper ideals. For the results in this paper we only need to consider the meager ideal, in which case P_I is Cohen forcing, or the null ideal in which case P_I is Random forcing.

The following lemma characterizes strong ergodicity between Borel equivalence relations, which are classifiable by countable structures, in terms of symmetric models. The proof follows from [Sha21, Section 3], where it is shown that a Borel homomorphism corresponds to a definable set in the relevant symmetric model.

LEMMA 2.5: *Suppose E and F are Borel equivalence relations on X and Y respectively and $x \mapsto A_x$ and $y \mapsto B_y$ are absolute classifications of E and F respectively. Let I be a proper ideal over X . The following are equivalent.*

- (1) *For every partial Borel homomorphism $f: E \rightarrow_B F$, defined on some I -positive set, f maps an I -positive set into a single F -class.*
- (2) *If $x \in X$ is P_I -generic over V and B is an invariant for F in $V(A_x)$ which is definable only from A_x and parameters in V , then $B \in V$.*

Proof sketch. (1) \implies (2). If x is P_I -generic and $B \in V(A_x)$ is definable using only A_x and a parameter in V then by [Sha21, Proposition 3.7] $B = B_{f(x)}$ for

some partial Borel homomorphism $f: E \rightarrow_B F$ defined on an I -positive set. By (1) f maps an I -positive set C into a single F -class. Taking two mutually generic $x, y \in C$, we see that B is in $V[x] \cap V[y] = V$.

(2) \implies (1). Given a partial Borel homomorphism f , let x be P_I -generic in the domain of f and let $B = B_{f(x)}$. By [Sha21, Lemma 3.6] $B \in V(A_x)$ is definable from A_x and parameters in V alone, and so $B \in V$ by (2). Let C be some I -positive set forcing that $B_{f(x)} = \check{B}$. Fix a sufficiently large countable transitive model M . Then all P_I -generics over M in C are mapped into a single F -class, since their images under f have the same invariant B . ■

We briefly recall the definition of pinned equivalence relations. See [Kano08, Definition 7.1.2] and [Zap11].

Definition 2.6: Let E be an analytic equivalence relation on a Polish space X . Let P be a poset and τ a P -name. The pair $\langle P, \tau \rangle$ is a **virtual E -class** if $P \times P$ forces that τ_l is E -equivalent to τ_r , where τ_l and τ_r are the interpretation of τ using the left and right generics respectively. A virtual E -class $\langle P, \tau \rangle$ is **pinned** if there is some $x \in X$ from the ground model such that P forces that τ is E -equivalent to \check{x} . Finally, E is **pinned** if any virtual E -class is pinned.

Assume E is a Borel equivalence relation and $x \mapsto A_x$ is a complete classification using hereditarily countable structures. Then a virtual E -class simply corresponds to a set A in the ground model and a pair $\langle P, \tau \rangle$ where P forces that $A_\tau = A$. To see this, let $G_l \times G_r$ be $P \times P$ -generic and let x_l, x_r be the interpretations of τ according to G_l, G_r respectively. Then $A = A_{x_l} = A_{x_r}$ is in $V[G_l] \cap V[G_r] = V$. So a virtual E -class is a set A (in the ground model) which is forced to be the invariant of some real. This virtual E -class is pinned if and only if A is A_x for some x in the ground model. Thus E is pinned if and only if “being an E -invariant” is absolute for forcing extensions. For example, the equivalence relation $=^+$ is not pinned, as the set of all reals \mathbb{R} is not the image of any $x \in \mathbb{R}^{\mathbb{N}}$ in the ground model, but it is after collapsing the continuum to be countable. On the other hand, suppose E is a countable Borel equivalence relation induced by some action $a: \Gamma \curvearrowright X$ of a countable group Γ , and consider the classification $x \mapsto \Gamma \cdot x$. Then “being an orbit of the action a ” is absolute, and E is pinned. Similarly, if E is countable then $E^{\mathbb{N}}$ is pinned: if $A = \langle A_n : n < \omega \rangle$ is a sequence of E -classes, then any $x \in \prod_n A_n$ satisfies that A is the invariant of x .

Remark 2.7: While ZF proves that any countable Borel equivalence relation is pinned, and more generally that any F_σ equivalence relation is pinned, $\text{CC}[E^{\mathbb{N}}]$ is used to show that $E^{\mathbb{N}}$ is pinned. Furthermore this is necessary, as $E^{\mathbb{N}}$ is not pinned in the models considered in Section 3. Larson and Zapletal [LZ20] also noticed the consistency of ZF with “ $E_0^{\mathbb{N}}$ is not pinned”. They further study pinned equivalence relations in choiceless models, with a focus on models of DC.

2.1. NOTATION. We use ω to denote the set of natural numbers $\mathbb{N} = 0, 1, 2, \dots$. From now on we will use E^ω instead of $E^{\mathbb{N}}$. For an equivalence relation E on X and a subset $A \subseteq X$, its E -saturation is defined by

$$[A]_E = \{x \in X : (\exists y \in A)xEy\}.$$

Say that E and F are **Borel bireducible**, denoted $E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$. We write $=_X^+$ for the equivalence relation on X^ω identifying x, y if they enumerate the same subset of X . For any Polish space X , $=_X^+ \sim_B =_B^+$.

3. Countable products of countable equivalence relations

In this section we consider countable powers of countable Borel equivalence relations. That is, equivalence relations of the form E^ω where E is a countable Borel equivalence relation. For notational simplicity we give a proof of Lemma 1.2 for powers only, the general proof is similar. To each such equivalence relation E^ω we associated a choice principle $\text{CC}[E^\omega]$ (Definition 1.9), which states that any countable sequence of E -classes admits a choice function. First we note that if E is Borel reducible to F , then $\text{CC}[F^\omega]$ implies $\text{CC}[E^\omega]$, over ZF. More generally:

PROPOSITION 3.1: *Let E and F be countable Borel equivalence relations on Polish spaces X and Y respectively. If E^ω is Borel reducible to F^ω then $\text{CC}[F^\omega]$ implies $\text{CC}[E^\omega]$, over ZF.*

Proof. Assume that $\text{CC}[F^\omega]$ holds and fix a sequence $A = \langle A_n : n < \omega \rangle$ such that each A_n is an E -class. It remains to show that $\prod_n A_n \neq \emptyset$. By Lemma 2.4 there is an F^ω -invariant B such that $L(A) = L(B)$, $B = \langle B_n : n < \omega \rangle$ where each B_n is an F -class. Applying $\text{CC}[F^\omega]$, there is some $y \in \prod_n B_n$. Now

$$L(A) = L(B) \subseteq L[y]$$

and the latter is a model of ZFC, so there is some $x \in \prod_n A_n$ in $L(y)$. ■

Next we separate these choice principles.

THEOREM 3.2: *Let E and F be countable Borel equivalence relations on X and Y respectively, μ a Borel probability measure on X and suppose that E is (μ, F) -ergodic. Then there is a model in which $\text{CC}[F^\omega]$ holds yet $\text{CC}[E^\omega]$ fails.*

COROLLARY 3.3: *$\text{CC}[E_\infty^\omega]$ is strictly stronger than $\text{CC}[E_0^\omega]$.*

Adams and Kechris [AK00] showed that there is a continuum size family \mathcal{F} of countable Borel equivalence relations such that for any distinct $E, F \in \mathcal{F}$, E is F -ergodic and F is E -ergodic, with respect to some measures on their domains. By Theorem 3.2 we conclude:

COROLLARY 3.4: *There is a continuum size family \mathcal{F} of countable Borel equivalence relations such that for any distinct $E, F \in \mathcal{F}$, $\text{CC}[E^\omega]$ and $\text{CC}[F^\omega]$ are independent.*

Towards proving Theorem 3.2 fix E, F and μ as in the theorem. By the Feldman–Moore theorem we may fix a countable group Γ and a Borel action $a: \Gamma \curvearrowright X$ such that $E = E_a$. For a Borel probability measure ν on a Polish space Y , let $R(\nu)$ be the poset $P_{I(\nu)}$ where $I(\nu)$ is the ideal of ν -measure zero sets. That is, all ν -positive measure Borel subsets of Y ordered by inclusion. Let $x = \langle x_n : n < \omega \rangle \in X^\omega$ be a $R(\mu^\omega)$ generic over V , $A_n = [x_n]_E$ and $A = \langle A_n : n < \omega \rangle$ the corresponding E^ω -invariant.

We work now in $V(A)$; recall the definitions from Section 2. Given $Z \in V(A)$, there is some formula φ , a parameter $v \in V$ and finitely many parameters \bar{a} from the transitive closure of A (the support of Z) such that $Z = \{z : \varphi(z, A, \bar{a}, v)\}$. Since each A_i (which is in the transitive closure of A) is definable from A , we may assume that \bar{a} is contained in $\bigcup_i A_i$. If $a \in A_i$ then a is definable from $x(i)$. Thus the support of X can be taken to be of the form $\bar{a} = \langle x(i) : i \in s \rangle$ where $s \subseteq \omega$ is finite.

The following proposition establishes the basic symmetric-model analysis of $V(A)$ that will be used. The proof follows a similar outline to that of an analogous property of the “basic Cohen model” (see [Bla81, Proposition 2.1]). One difference is the required permutations, which are here the ones preserving E^ω . Furthermore, we are working with a Random real and not a Cohen real. We note that the proposition holds for a Cohen real x as well, with the proof slightly simpler.

PROPOSITION 3.5: *Let X be a Polish space, μ a probability measure on X , Γ a countable group, $a: \Gamma \curvearrowright X$ a μ -ergodic action, and $E = E_a$. Let $x \in X^\omega$ be $R(\mu^\omega)$ -generic and define A as above. Suppose $Z \in V(A)$ and $Z \subseteq V$, $s \subseteq \omega$ is finite and $\bar{x} = \langle x(i) : i \in s \rangle$ is a support for Z . Then $Z \in V[\bar{x}]$.*

In particular, any real $b \in V(A)$ is in $V[x \upharpoonright n]$ for some $n < \omega$.

Proof. Let $\Gamma^{<\omega}$ be the group of all infinite sequences $\langle \gamma_i : i < \omega \rangle$ such that $\gamma_i \in \Gamma$ and $\gamma_i = 1$ for all but finitely many i . Fix φ and $v \in V$ such that

$$Z = \{z : \varphi(z, v, \bar{x}, A)\}$$

in $V(A)$. Let $\Omega = \omega \setminus s$. Given $y \in X^\Omega$ denote by $\bar{x} \hat{\ } y$ the element of X^ω whose restriction to Ω is y and its restriction to s is \bar{x} . For $p \in R(\mu^\omega)$, say that p **agrees** with \bar{x} if $\{y \in X^\Omega : \bar{x} \hat{\ } y \in p\}$ has positive μ^Ω -measure. That is, it is a condition in $R(\mu^\Omega)$. Note that if p is a condition so that $x \in p$, then p agrees with \bar{x} .

We work now in the $R(\mu^\omega)$ -generic extension $V[x]$. Hence, if $z \in Z$ then $\varphi^{V(A)}(z, v, \bar{x}, A)$ holds in $V[x]$, so there is a condition p which agrees with \bar{x} forcing $\varphi^{V(\dot{A})}(\check{z}, \check{v}, \check{x}, \dot{A})$. Similarly, if $z \notin Z$ there is some p which agrees with \bar{x} forcing $\neg\varphi^{V(\dot{A})}(\check{z}, \check{v}, \check{x}, \dot{A})$. We will show that Z is defined in $V[\bar{x}]$ as the set of all $z \in V$ such that there is some condition $p \in R(\mu^\omega)$ which agrees with \bar{x} and forces that $\varphi^{V(\dot{A})}(\check{z}, \check{v}, \check{x}, \dot{A})$. It suffices to show the following: for any $z \in V$ there are no p_0, p_1 which agree with \bar{x} such that $p_1 \Vdash \varphi^{V(\dot{A})}(\check{z}, \check{v}, \check{x}, \dot{A})$ and $p_0 \Vdash \neg\varphi^{V(\dot{A})}(\check{z}, \check{v}, \check{x}, \dot{A})$.

For contradiction, assume we have p_0, p_1 as above. Let

$$q_i = \{y \in X^\Omega : \bar{x} \hat{\ } y \in p_i\}.$$

Fix a large enough countable model M and let \tilde{q}_i be the set of all $y \in q_i$ which are $R(\mu^\Omega)$ -generic over $M[\bar{x}]$. Note that \tilde{q}_i has positive measure. Since E is (μ, F) -ergodic, E is in particular ergodic with respect to μ , hence Γ acts ergodically. By [Kec20, Lemma 4.2] the countable group $\Gamma^{<\omega}$ acts ergodically on X^ω (which we identify here with X^Ω). It follows that there is some $g \in \Gamma^{<\omega}$ such that $(g^{-1} \cdot \tilde{q}_1) \cap \tilde{q}_0$ has positive measure. In particular there is some $x' \in (g^{-1} \cdot \tilde{q}_1) \cap \tilde{q}_0$. That is, both $x_0 = \bar{x} \hat{\ } x'$ and $x_1 = \bar{x} \hat{\ } g \cdot x'$ are $R(\mu^\omega)$ -generics over M , and they agree on \dot{A} and \check{x} . Furthermore, x_0 extends p_0 , thus $\varphi(z, v, \bar{x}, A)$ fails in $M(A)$, but x_1 extends p_1 , so $\varphi(z, v, \bar{x}, A)$ holds in $M(A)$, a contradiction. ■

COROLLARY 3.6: *In $V(\langle A_n : n < \omega \rangle)$ there is no choice function for $\langle A_n : n < \omega \rangle$. In particular, $\text{CC}[E^\omega]$ fails.*

Proof. Otherwise, there is a choice function

$$r \in \prod_n A_n$$

which is in $V[x \upharpoonright n]$ for some n , by the lemma. However, $r(n)$ is generic over $V[x \upharpoonright n]$, a contradiction. ■

COROLLARY 3.7: *If $B \in V(A)$ is a set of subsets of V , and B is countable in $V(A)$, then $V(B) \neq V(A)$.*

Proof. Using an enumeration of B , B can be coded as a subset of V and therefore B is in $V[x \upharpoonright n]$ for some $n < \omega$. ■

In particular, if F is a countable Borel equivalence relation and B is an F -invariant in $V(A)$, then B is a countable set of reals (an F -class), and therefore $V(A) \neq V(B)$. By Lemma 2.4 it follows that E^ω is not essentially countable. This recovers the well known fact that when E is as in Proposition 3.5 then E^ω is not essentially countable (as in this case $E_0^\omega \leq_B E^\omega$).

We will see that the choice separation in Theorem 3.2 corresponds to strong ergodicity between E^ω and F^ω , rather than E and F . First we show that the first follows from the latter.

LEMMA 3.8: *Suppose E, F are countable Borel equivalence relations on X and Y respectively. Let μ be a Borel probability measure on X and assume that E is (μ, F) -ergodic. Then E^ω is (μ^ω, F) -ergodic.*

COROLLARY 3.9: *Suppose E, F , and μ are as above. Then E is (μ, F) -ergodic if and only if E^ω is (μ^ω, F) -ergodic if and only if E^ω is (μ^ω, F^ω) -ergodic.*

Proof. Assume first that E is (μ, F) -ergodic, then E^ω is (μ^ω, F) -ergodic by the lemma above. Given a homomorphism $f: E^\omega \rightarrow F^\omega$ the projections

$$f_n(x) = f(x)(n)$$

are homomorphisms from E^ω to F . Thus each f_n is constant on a μ^ω -conull set A_n , and so f is constant on the conull set $A = \bigcap_n A_n$.

Conversely, assume that E^ω is (μ^ω, F^ω) -ergodic and fix $f: E \rightarrow F$. Then $f^\omega: X^\omega \rightarrow Y^\omega$, defined by $f^\omega(x)(n) = f(x(n))$, is a homomorphism from E^ω to F^ω . By assumption, there is a conull set $A \subseteq X^\omega$ which f^ω sends to a single F^ω -class, $\langle [y_i]_F : i < \omega \rangle$. Let A_0 be the projection of A to the first coordinate. Then A_0 is conull and for any $x \in A_0$, $f(x) \in [y_0]_F$. ■

The proof of Lemma 3.8 will appeal to Proposition 3.5 to reduce the problem to that of finite powers, in which case a direct measure theoretic argument works.

PROPOSITION 3.10: *Let E, F, μ be as in Lemma 3.8. Then E^n is (μ^n, F) -ergodic.*

Proof. Suppose that E and E' are countable Borel equivalence relations on X and X' respectively and are F -ergodic with respect to μ and μ' respectively. We show that $E \times E'$ is $(\mu \times \mu', F)$ -ergodic. The proposition is then established inductively. Fix a Borel homomorphism $f: E \times E' \rightarrow F$. For $x \in X$ define $f_x: X' \rightarrow Y$ by $f_x(x') = f(x, x')$. For each x , f_x is a homomorphism from E' to F . By assumption there is a μ' -measure 1 set $C_x \subseteq X'$ and $y_x \in Y$ such that $f_x(x') F y_x$ for any $x' \in C_x$. If $x_1 E x_2$ then $y_{x_1} F y_{x_2}$, since for any $x' \in C_{x_1} \cap C_{x_2}$, $y_{x_1} F f(x_1, x') F f(x_2, x') F y_{x_2}$.

Let $D \subseteq X \times Y$ be the set of all pairs (x, y) such that for any measure 1 set $C \subseteq X'$ there is some $x' \in C$ with $f_x(x') F y$; D is Borel and has countable Y -sections. By Lusin–Novikov uniformization (see [Kec95]) there is a Borel function $g: X \rightarrow Y$ such that $(x, g(x))$ is in D for all $x \in X$. Note that $g(x) F y_x$ for any $x \in X$ and so g is a homomorphism from E to F . Since E is (μ, F) -ergodic, there is a measure 1 set $C \subseteq X$ and $y \in Y$ such that $g(x) F y$ for all $x \in C$. Let $A = \bigcup_{x \in C} \{x\} \times C_x$, a $\mu \times \mu'$ -measure 1 subset of $X \times X$. We claim that $f(x, x') F y$ for any $(x, x') \in A$, which concludes the proof. Indeed, $f(x, x') = f_x(x') F y_x F g(x) F y$. The first F -relation is since $x' \in C_x$, the second by the property of g , and the third since $x \in C$. ■

Proof of Lemma 3.8. Fix a Borel homomorphism $f: E^\omega \rightarrow_B F$. Fix a large enough countable transitive model M containing a code for f . Let x be $R(\mu^\omega)$ -generic over M , $A_n = [x(n)]_E$ and $A = \langle A_n : n < \omega \rangle$. By Lemma 2.4 the F -class of $f(x)$, $B = \{y : f(x) F y\}$, is in $M(A)$ and has empty support. In particular, $f(x) \in M(A)$. Note that B is a set of reals, so not necessarily a subset of the ground model M . However, each member of B , such as $f(x)$, is a

real and may be coded as a subset of ω . By Proposition 3.5, $f(x) \in M[x \upharpoonright m]$ for some $m < \omega$, and so $B \in M[x \upharpoonright m]$. Note that

$$M[x \upharpoonright m] = M(\langle A_i : i < m \rangle).$$

Fix φ and $v \in M$ such that $B = \{y : \varphi^{M(A)}(y, v, A)\}$. We claim that B is definable in $M(\langle A_i : i < m \rangle)$ using only $\langle A_i : i < m \rangle$ and v . This follows from Proposition 3.5. Viewing $M(\langle A_i : i < m \rangle)$ as the ground model, then B is a subset of the ground model and is definable in $M(\langle A_i : i < m \rangle)(\langle A_i : i \geq m \rangle)$ using only $\langle A_i : i \geq m \rangle$ and the ground model parameters $v, \langle A_i : i < m \rangle$.

By [Sha21, Lemma 3.7] there is a Borel homomorphism

$$g: E^m \longrightarrow_B F$$

defined on a μ^m -positive measure set C containing $x \upharpoonright m$ such that $B = B_{g(x \upharpoonright m)}$. By assumption and by Proposition 3.10, g sends a full measure subset of C to a single F -class. W.l.o.g. assume the above statements are forced by the maximal condition. Now the set D of all μ^ω -generics $x \in X^\omega$ such that $x \upharpoonright m \in C$ has positive μ^ω -measure and for any $x \in D$, $f(x) \in F_{g(x \upharpoonright m)}$ lies in a single F -class. Finally, since D is a set of generics its saturation $[D]_{E^\omega}$ is Borel (see [KSZ13, Theorem 2.29]), invariant and measure 1 (since E^ω is ergodic with respect to μ^ω). ■

By Corollary 3.6, Lemma 3.8 and Corollary 3.9, the following proposition will finish the proof of Theorem 3.2.

PROPOSITION 3.11: *Let E, F and μ be as above such that E^ω is (μ^ω, F^ω) -ergodic. Let $x \in X^\omega$ be μ^ω -Random real generic over V , $A_n = [x(n)]_E$ and $A = \langle A_n : n < \omega \rangle$. Then $V(A) \models \text{CC}[F^\omega]$.*

Proof. Suppose $B = \langle B_n : n < \omega \rangle \in V(A)$ is a sequence of F -classes. B is definable using A , parameters in V and some parameters \bar{a} from $\bigcup_n A_n$. If $\bar{a} = \emptyset$ then by Lemma 2.5, B is in V , and therefore admits a choice function in V . Generally, fix some m such that \bar{a} is in $V[x \upharpoonright m]$. Work now with $V[x \upharpoonright m]$ as the base model, forcing $\langle A_n : n \geq m \rangle$ over it. A similar argument shows that $B \in V[x \upharpoonright m]$, which is a model of ZFC, and so B admits a choice function. ■

We end this section with a simple remark about the choice principles $\text{CC}[E^\omega]$. Fix a countable Borel equivalence relation E on X and let $a: \Gamma \curvearrowright X$ be an action of a countable group Γ on X generating E . Note that if $\langle A_n : n < \omega \rangle$ is

a sequence of E classes, then any choice function $x \in \prod_n A_n$ codes a countable enumeration of $\bigcup_n A_n$ (using some fixed enumeration of Γ). It follows that $\text{CC}[E^\omega]$ is equivalent to the formally stronger statement, that the union of countably many E -classes is countable. In particular, it follows that if $F \subseteq E$, then $\text{CC}[E^\omega]$ implies $\text{CC}[F^\omega]$: given a sequence of F classes $\langle B_n : n < \omega \rangle$, let $A_n = [B_n]_E$ be the corresponding E -class. Now a wellordering of $\bigcup_n A_n$ gives a wellordering of $\bigcup_n B_n$.

4. Equivalence relations which can be classified by sequences of countable sets of reals

The following definition attempts to capture those equivalence relations which can be classified using classifying invariants of the form $\langle A_n : n < \omega \rangle$ where each A_n is a subset of a Polish space and A_n is definably enumerated using some elements in $\bigcup_n A_n$ as a parameter.

Definition 4.1: Let E be an equivalence relation such that the domain of E is a subset of some product space $X = \prod_n X_n$. Then E is said to be a **Pointwise Countable Product (PCP) relation** if there are Borel equivalence relations F_n on X_n such that

$$E = \prod_n F_n \upharpoonright \text{dom } E,$$

and for every n , for any $x E y$, $y(n)$ is Δ_1^1 in $x(n+1)$.

In this case, define $A_n^{E,x} \equiv \{y(n) : y E x\}$, the projection to X_n of the equivalence class of x (which will be denoted as A_n^x when E is unambiguous). The map sending x to $\langle A_n^x : n < \omega \rangle$ is a complete classification of E , using classifying invariants which are countable sequences of countable subsets of a Polish space. Furthermore, given an invariant $\langle A_n^x : n < \omega \rangle$, one can definably enumerate each A_n^x . That is, fix any element $z \in A_{n+1}^x$, then all elements in A_n^x are Δ_1^1 in z .

LEMMA 4.2: *Suppose E satisfies the following weakening of PCP: There is a function $\varphi : \omega \rightarrow \omega$ such that for all n and $f E g$ in $\text{dom } E$, $g(n)$ is Δ_1^1 in $f(0), \dots, f(\varphi(n))$. Then E is Borel reducible to a PCP relation.*

Proof. Define $\psi: \omega \rightarrow \omega$ inductively by

$$\psi(0) = 0 \quad \text{and} \quad \psi(n+1) = \sup_{k \leq \psi(n)} \varphi(k).$$

Let $Y_n = \prod_{m \leq \psi(n)} X_n$ and consider the map $\theta: \prod_n X_n \rightarrow \prod_n Y_n$ defined by

$$\theta(f)(n) = \langle f(m); m \leq \psi(n) \rangle.$$

Let $G_n = \prod_{m \leq \psi(n)} F_n$ and $\tilde{E} = \prod_n G_n$. It can be verified that θ is a reduction of E to \tilde{E} and $\tilde{E} \upharpoonright \theta[\text{dom } E]$ is a PCP relation. Note also θ is one-to-one, so if $\text{dom } E$ is Borel, so is $\theta[\text{dom } E]$. ■

Example 4.3: For any countable Borel equivalence relation E on X , E^ω is Borel reducible to a PCP equivalence relation. In this case, if $x, y \in X^\omega$ are E^ω -related, $y(n)$ is $\Delta_1^1(x(n))$.

Definition 4.4: Given a PCP relation E as above, let

$$A_n^E = \{(y, z) : \exists x \in \text{dom } E (y = x(n+1) \wedge z \in A_n^{E,x})\}.$$

If $z \in A_n^{E,x}$ then $(x(n+1), z) \in A_n^E$, but not necessarily the other way. In general both A_n^E and $A_n^{E,x}$ are analytic. By modifying the domain a little, we can get a PCP relation for which both $A_n^{E,x}$ and A_n^E are Borel, and in fact

$$A_n^{E,x} = \{z : (x(n+1), z) \in A_n^E \wedge z \in F_n x(n)\}.$$

For Proposition 4.6 below we will only need $A_n^{E,x}$. The result about A_n^E will be used in Theorem 4.15.

LEMMA 4.5: *Suppose E is a PCP Borel equivalence relation as in Definition 4.1 above. So $E = \prod_n F_n \upharpoonright \text{dom } E$ where $\text{dom } E$ is a Borel subset of $\prod_n X_n$. There is a Borel set D containing $\text{dom } E$ such that, for $F = \prod_n F_n \upharpoonright D$, F is a PCP relation and for any $x \in D$ and any n , the sets $A_n^{F,x}$ and A_n^F are Borel. Moreover, for any z , $z \in A_n^{F,x} \iff z \in F_n x(n)$ and $(x(n+1), z) \in A_n^E$.*

Proof. Given $x \in \prod_n X_n$ and $z \in X_n$, let $x[n, z]$ be the result of replacing the n 'th coordinate of x with z . The idea is to determine whether $z \in A_n^{E,x}$ by asking if $x[n, z]$ is in $\text{dom } E$. For natural examples, this is in fact the case. In general, we will have to add members to $\text{dom } E$. To do this in a controlled manner, preserving the PCP conditions, we will use a reflection argument.

Define first D_0 to be all $x \in \prod_n X_n$ such that there is some $y \in \text{dom } E$ for which $x \in \prod_n A_n^{E,y}$; D_0 is Σ_1^1 and $\prod_n F_n \upharpoonright D_0$ is still PCP. That is, if $x \in \prod_n F_n$ and $y \in \text{dom } E$ such that $x \in \prod_n A_n^{E,y}$, then $x \in D_0$.

then $x(n)$ is $\Delta_1^1(y(n+1))$ for all n . Define a property of subsets of X , $\Phi(A, B)$ as follows (we think of A as D_0 and B as its complement). $\Phi(A, B)$ holds if

- (1) For any x, y if $x, y \in A$ and $x(\prod_n F_n)y$ then $x(n)$ is $\Delta_1^1(y(n+1))$ for all n .
- (2) For any x, y , if $x \in A$ and for all m there is some $x' \in A$ such that $x'(\prod_n F_n)x$ and $x'(m) = y(m)$, then $y \notin B$.

Φ is hereditary, continuous upward in the second variable and is Π_1^1 on Σ_1^1 . Furthermore, $\Phi(D_0, \neg D_0)$ holds. By the second reflection theorem (Theorem 35.16 in [Kec95]), there is some Δ_1^1 set $D \supseteq D_0$ such that $\Phi(D, \neg D)$ holds. Let $F = \prod_n F_n \upharpoonright D$; F is PCP by condition (1) above.

Fix $x \in D$. By condition (2), for any $z \in A_n^{F,x}$, $x[n, z] \in D$. Also, if $x[n, z] \in D$ and $z F_n x(n)$, then $x F x[n, z]$ and so $z \in A_n^{F,x}$. We conclude that $z \in A_n^{F,x}$ if and only if $z F_n x(n)$ and $x[n, z] \in D$, which is Borel.

The relation $(y, z) \in A_n^F$ holds if and only if $\exists x$ s.t. $x(n+1) = y$ and $x[n, z] \in D$, if and only if $\forall x$ if $x(n+1) = y$ then $x[n, z] \in D$. Thus A_n^F is Σ_1^1 and Π_1^1 , and therefore is Borel.

Note also that if $z F_n x(n)$ and $(x(n+1), z) \in A_n^F$, then by (2) $x[n, z] \in D$, and so $z \in A_n^{F,x}$. ■

PROPOSITION 4.6: *Assume E is PCP and Borel, then*

- (1) $E \leq_B =^+$, and
- (2) E is pinned, hence is strictly below $=^+$.

Proof. By Lemma 4.5 we may assume that the sets $A_n^{E,x}$ are Borel. For each x and n the set $A_n^{E,x}$ is countable, so by Lusin-Novikov uniformization we may find Borel functions h_i^n so that for any $x \in \text{dom } E$, $A_n^{E,x} = \{h_i^n(x) : i \in \omega\}$.

- (1) Define a map $f: X \rightarrow (\mathbb{R}^\omega)^\omega$ by

$$f(x) = \langle \langle h_i^n(x); i < \omega \rangle; n < \omega \rangle,$$

sending x to the sequence $\langle A_n^x : n < \omega \rangle$. We show that f is a Borel reduction to $(=^+)^\omega$ which is Borel bireducible with $=^+$. If $x E y$ then $A_n^x = A_n^y$ for all n , and so $f(x)(n) =^+ f(y)(n)$ for each n . Conversely, assume

$$f(x)(n) =^+ f(y)(n)$$

for every n . For each n , there is some i such that $x_n = h_i^n(x)$, and then some j such that $h_i^n(x) = h_j^n(y)$, which is F_n -related to $y(n)$. It follows that $x(n) F_n y(n)$ for each n , thus $x E y$.

(2) Note that the statement

$$\forall x \forall y (y E x \implies \exists i (y(n) = h_i^n(x)))$$

is Π_1^1 , and therefore is absolute. Suppose P is some poset, τ a P -name, such that $P \times P \Vdash \tau_l E \tau_r$. Let G be P -generic over V and $x = \tau[G]$. We need to find some $x' \in V$ such that $x' E x$. Take H such that (G, H) is $P \times P$ generic over V and let $y = \tau[H]$.

Using absoluteness between the models $V[G]$, $V[H]$ and $V[G, H]$, and that $x E y$, it follows that $A_n^x = A_n^y$ for every n . Thus

$$A^x = \langle A_n^x : n < \omega \rangle \in V[G] \cap V[H] = V.$$

Applying countable choice in V , there is some z such that $z(n) \in A_n^x$ for every n . By absoluteness, there is some $x' \in V$ such that $x' \in \text{dom } E$ and $x'(n) F_n z(n)$ for every n . Thus $x'(n) F_n x(n)$ for each n , and $x' \in \text{dom } E$, hence $x' E x$ as required. ■

4.1. AN INTERESTING PCP EQUIVALENCE RELATION. We now define an interesting PCP relation which we denote E_Π . The definition below, which is the one used in all the proofs, is different than the one mentioned in the introduction (Definition 1.6). We will show in Proposition 4.16 below that the two definitions are Borel bireducible.

Definition 4.7: (E_Π) Let $X_n = \mathbb{R}^{\omega^{n+1}}$, where we think of X_{n+1} as X_n^ω . Let $X = \prod_n X_n$ and define:

$$D = \{f \in X : \forall n \forall j (f(n+1)(j) \text{ is an enumeration of } f(n)) \wedge \exists j (f(n) = f(n+1)(j))\}.$$

(By “ $f(n+1)(j)$ is an enumeration of $f(n)$ ” we mean that both have the same image, as elements in $X_n = X_{n-1}^\omega$.) We define the equivalence relation \mathbf{E}_Π (Π for product and permutations) on $D = \text{dom } E_\Pi$ as follows: $f E_\Pi g$ iff for every n , $\{f(n)(i) : i \in \omega\} = \{g(n)(i) : i \in \omega\}$. That is, E_Π is the equivalence relation $\prod_n (=_{\mathbb{R}^{\omega^n}}^+)$ restricted to the domain D .

The equivalence relation E_Π is PCP: if f, g are E_Π -related, then for any n there is some j such that $g(n) = g(n+1)(j)$, and therefore there is some i such that $g(n) = f(n+1)(i)$. In particular, $g(n)$ is $\Delta_1^1(f(n+1))$. The natural complete classification of E_Π is the map sending $f \in \text{dom } E_\Pi$ to the sequence $\langle A_n^{E_\Pi, f} : n < \omega \rangle$. The classifying invariants are sequences $\langle A_n : n < \omega \rangle$ such

that each A_n is a subset of the Polish space X_n and any element of A_{n+1} is a countable enumeration of A_n .

By Proposition 4.6, E_Π is pinned and strictly below $=^+$. We now turn to prove that E_Π is not Borel reducible to E_∞^ω . The proof relies on constructing a model in which $\text{CC}[\mathbb{R}]^{\aleph_0}$ fails yet $\text{CC}[E_\infty^\omega]$ holds.

Consider the poset

$$\{p: \text{dom } p \longrightarrow \omega : \text{dom } p \subseteq \omega \text{ is finite and } p \text{ is injective}\}.$$

This poset adds a generic permutation of ω , and is equivalent to the poset for adding a single Cohen real.

CONSTRUCTION 4.8: *Let P be the poset to add $\omega \times \omega$ mutually generic Cohen reals, indexed by $d_i, i < \omega$ where each d_i is an ω sequence of Cohen reals. We think of $d_0 \in \mathbb{R}^\omega$ and d_i for $i > 0$ as a sequence of permutations of ω . Define inductively $a_n \in \mathbb{R}^{\omega^{n+1}}$ as follows:*

- *Let $a_0 = d_0 \in \mathbb{R}^\omega$.*
- *Given a_n , let $a_{n+1} = \langle a_n \circ d_{n+1}(i); i < \omega \rangle$.*

Thinking of a_n as an element of $(\mathbb{R}^{\omega^n})^\omega$, let

$$A_n = \text{Im } a_n = \{a_n(i) : i \in \omega\}.$$

So $A_0 \subseteq \mathbb{R}$ is a set of mutually generic Cohen reals; $A_1 \subseteq \mathbb{R}^\omega$ is a set of mutually generic enumerations of A_0 ; and so forth, A_{n+1} is a set of generic enumerations of A_n . Let $A = \langle A_n; n < \omega \rangle$. Our model will be $V(A)$.

PROPOSITION 4.9: *In $V(A)$, $\prod_n A_n = \emptyset$. In fact: Suppose $h: \omega \longrightarrow \omega$, in $V(A)$, is unbounded. Then there is no function f such that for each n , $f(n)$ is a non-empty finite subset of $A_{h(n)}$.*

THEOREM 4.10: *In $V(A)$: For any countable Borel equivalence relation F , $\text{CC}[F^\omega]$ holds.*

It follows that $V(A)$ is not of the form $V(B)$ for any sequence of E_∞ -classes $B = \langle B_n : n < \omega \rangle$: Otherwise, by the theorem there is some $x \in \prod_n B_n$ in $V(A)$. As B is definable from x , it follows that

$$V(A) = V(B) = V(x),$$

which is a model of ZFC, by Fact 2.2, contradicting the proposition above.

Finally, note that A is an E_Π -invariant. Recall that this simply means that

$$A = \langle A_n^{E_\Pi, f} : n < \omega \rangle$$

for some $f \in D$ (possibly in some further generic extension, see the paragraph before Lemma 2.4). Indeed, define

$$f(n) = a_{n+1}(0) \quad \text{for all } n \in \omega.$$

Then $f \in X = \prod_n X_n$, $f(n)$ is an enumeration of A_n , and $f(n+1)(j)$ is an enumeration of $f(n)$ for any j . Moreover, as $f(n) = a_{n+1}(0)$ and the sequences $f(n+1)$, a_{n+1} enumerate the same set A_{n+1} , it follows that $f(n) = f(n+1)(j)$ for some j . So $f \in D$ is in the model $V[\langle d_i : i < \omega \rangle]$ and $\langle A_n^{E_\Pi, f} : n < \omega \rangle = A$, as required. (Such f can be found in any model in which $\prod_n A_n$ is not empty.)

By Lemma 2.4 we conclude:

COROLLARY 4.11: E_Π is not Borel reducible to E_∞^ω .

The following lemma provides the basic symmetric model properties of $V(A)$ (the existence of minimal supports). We first show how to prove Theorem 4.10 from the lemma and then sketch a proof of the lemma.

LEMMA 4.12: *There is map, definable in $V(A)$, sending a real $b \in V(A)$ to $n(b), E(b)$, where $n(b) < \omega$, $E(b)$ is the minimal finite subset of $A_{n(b)}$ such that $b \in V(E(b))$ and $n(b)$ is the minimal n for which such $E(b)$ exists.*

Proof of Proposition 4.9. Suppose h, f are as in the statement of the proposition. Using the linear ordering of the reals, define $g(n)$ to be the smallest member of $f(n)$. Now g is a real, so by Lemma 4.12 there is some m and a finite $E \subseteq A_m$ such that $g \in V(E)$. In particular, $h(n) \leq m$ for all n . ■

Proof of Theorem 4.10. Fix a countable Borel equivalence relation F on a Polish space X . Assume the parameters defining F are in V (otherwise, we need to add a fixed finite set to all the supports below). Assume $B = \langle B_n; n < \omega \rangle$ is a sequence of F -classes. For $x, y \in B_n$, y is Δ_1^1 in x (and a parameter for F). It follows that

$$x \in V(E) \iff y \in V(E)$$

for any E , therefore $n(x) = n(y)$ and $E(x) = E(y)$.

Consider the map $h: \omega \rightarrow \omega$ defined by $h(k) = n(x)$ for any $x \in B_k$, and $f: \omega \rightarrow \bigcup_n A_n$ defined by $f(k) = E(x)$ for any $x \in B_k$. These are well defined by the argument above. Note that, if $h(k) > 0$, then $f(k)$ is a non-empty subset of $A_{h(k)}$. By Proposition 4.9, h must be bounded. Fix such a bound $m \in \omega$, then $B_n \subseteq V(A_m)$ for all n . Fix an enumeration a of A_m in $V(A)$. Then $V(A_m) \subseteq V[a]$, and $V[a]$ satisfies the choice. In particular there is a well ordering in $V(A)$ of the reals in $V(A_m)$. Using this wellorder a choice function for $\langle B_n : n < \omega \rangle$ can be defined in $V(A)$. ■

For $b \in V(A)$ and a finite $E \subseteq \bigcup_n A_n$, say that E is a support for b if there is a formula ϕ and $v \in V$ such that $\phi(b, v, E, A)$ defines b in $V(A)$.

LEMMA 4.13: *For any real $b \in V(A)$, if E is a support for b then $b \in V(E)$.*

Proof. For notational simplicity consider the following special case: assume that $E = a_1(0) \in A_1$, we show that $b \in V(a_1(0))$. Let ϕ and $v \in V$ be such that

$$b = \{n \in \omega : V(A) \models \phi(n, v, a_1(0), A)\}.$$

First note that the sequence $\langle a_i : 1 \leq i < \omega \rangle$ can be added generically over $V(A_0)$. Define Q in $V(A_0)$ as the poset of all finite partial functions

$$p: \omega \times \omega \rightarrow A_0$$

such that $p(k, _)$ is injective. Then a_1 is Q -generic over $V(A_0)$. Let P_2 be the sub-poset of P to add the sequence $\langle d_2, d_3, \dots \rangle$. Then $\langle a_1, d_2, d_3, \dots \rangle$ is $Q \times P_2$ -generic over $V(A_0)$ and $\langle a_1, a_2, \dots \rangle$ can be defined in $V(A_0)[\langle a_1, d_2, d_3, \dots \rangle]$.

We show that b can be defined in $V(a_1(0)) = V(A_0)[a_1(0)]$ as the set of all $n \in \omega$ such that some condition in $Q \times P_2$ which agrees with $a_1(0)$ forces $\phi^{V(\dot{A})}(n, v, \dot{a}_1(0), \dot{A})$. The proof is similar to Proposition 3.5 (see also [Bla81, Proposition 2.1] or [Sha21, Lemma 2.4]). The main point is showing that two conditions which agree with $a_1(0)$ agree on $\phi^{V(\dot{A})}(n, v, \dot{a}_1(0), \dot{A})$.

Assume for contradiction that there are two conditions p, q in $Q \times P_2$ which agree on $a_1(0)$ yet force incompatible statements about $\phi^{V(\dot{A})}(n, v, \dot{a}_1(0), \dot{A})$ for some n . We may assume that $\langle a_1, d_2, d_3, \dots \rangle$ extends p . We will construct a generic $\langle a'_1, d'_2, d'_3, \dots \rangle$ which extends q , computes the same \dot{A} and satisfies $a'_1(0) = a_1(0)$, which leads to a contradiction.

First, let π_1 be a finite permutation preserving 0 such that $a'_1 = a_1 \circ \pi_1$ agrees with the restriction of q to Q ; a'_1 is Q -generic over $V(A_0)$. Note that $a'_1 \circ d_2(i)$ may no longer agree with $a_1 \circ d_2(i) = a_2(i)$. That is, $\langle a'_1, d_2, \dots \rangle$ calculates

the “wrong” A_2 . Let $d''_2(i) = \pi_1^{-1} \circ d_2(i)$, so $\langle a'_1, d''_2, \dots \rangle$ now calculates the correct a_2 . Note that d''_2 is a generic sequence permutation over $V(A_0)$. At this point q may not agree with d''_2 . Let π_2 be a finite permutation such that $d'_2 = d''_2 \circ \pi_2$ agrees with q . This is possible by genericity. Note that $\langle a'_1, d'_2, \dots \rangle$ still calculates the correct A_2 (but not a_2). Now set $d''_3(i) = \pi_2^{-1} \circ d_3(i)$, so that $\langle a'_1, d'_2, d''_3, \dots \rangle$ calculates the correct a_3 . Continue in this fashion, at each step making finite changes to the values of $d_{k+1}(i)$ to get “the correct a_{k+1} ” and then applying a finite permuting to the sequence $\langle d_{k+1}(i) : i < \omega \rangle$ to make it compatible with q . Since q has finite support, after finitely many steps we get $\langle a'_1, d'_2, \dots, d'_k, d_{k+1}, \dots \rangle$ which is compatible with q , which completes the proof. ■

If E is a support for b , then for all large enough n , and any $a \in A_n$, all the elements of E are definable from a , hence $\{a\}$ is a support for b . Note that if E is a support for b , n is maximal such that $E \cap A_n \neq \emptyset$, then $E \cap A_n$ is also a support for b . Let $n(b)$ be the minimal n such that there is some $E \subseteq A_n$ which is a support for b .

CLAIM 4.14: Fix a real $b \in V(A)$ and $n \in \omega$. If $E_1, E_2 \subseteq A_n$ are supports for b , then $E_1 \cap E_2$ is a support for b . By saying $\emptyset \subseteq A_n$ is a support for b , when $n > 0$, we mean that there is $E \subseteq A_{n-1}$ which is a support for b .

Proof. The members of A_n are enumerations of A_{n-1} , mutually generic over $V(A_{n-1})$. By mutual genericity, if b is in $V(A_{n-1})(E_1)$ and $V(A_{n-1})(E_2)$ then b is in $V(A_{n-1})(E_1 \cap E_2)$. If $E_1 \cap E_2 = \emptyset$, then $b \in V(A_{n-1})$ is definable from finitely many members of A_{n-1} , thus has a support in A_{n-1} . ■

Let $E(b)$ be the minimal $E \subseteq A_{n(b)}$ which is a support for b . The map $b \mapsto n(b)$, $E(b)$ satisfies the properties required by Lemma 4.12. This finishes the proof.

4.2. MORE ON E_Π . In this section we show that E_Π is maximal Borel PCP.

THEOREM 4.15: If E is a Borel PCP relation, then $E \leq_B E_\Pi$.

Proof. Given a Polish space Y , let $E_\Pi(Y)$ be defined as in Definition 4.7 above, replacing each \mathbb{R}^{ω^n} with Y^{ω^n} . Any Borel isomorphism between \mathbb{R} and Y gives a Borel isomorphism between $E_\Pi = E_\Pi(\mathbb{R})$ and $E_\Pi(Y)$. Let E be a PCP relation as in Definition 4.1 above, $\text{dom } E \subseteq \prod_n X_n$. The idea will be to construct a

reduction of E to $E_{\Pi}(X_0)$. Each $f \in \text{dom } E$ will be sent to an $E_{\Pi}(X_0)$ -invariant $\langle D_n : n < \omega \rangle$ where D_0 is equal to A_0^f , D_1 is a set of enumerations of D_0 , given by the elements of A_1^f , and so forth.

First, we extend the domain of E to assume the conclusion of Lemma 4.5, so A_n^E is Borel for any n . By definition of A_n^E , if $(y, z) \in A_n^E$ then z is $\Delta_1^1(y)$. So A_n^E has countable sections. Fix Borel functions h_i^n so that

$$\{h_i^n(y) : i \in \omega\} = \{z : (y, z) \in A_n^E\},$$

for $y \in X_{n+1}$. Recall also from Lemma 4.5 that for $f \in \text{dom } E$,

$$A_n^{E,f} = \{z \in X_n : z F_n f(n) \wedge (f(n+1), z) \in A_n^E\}.$$

In other words, given $f \in \text{dom } E$, $\{h_i^n(f(n+1)) : i \in \omega\}$ enumerates a set whose intersection with $[f(n)]_{F_n}$ is precisely $A_n^{E,f}$.

Add two new and distinct elements, which we will call 0 and 1, to the space X_0 . Let $Y = X_0 \cup \{0, 1\}$, where 0, 1 are both isolated. Fix Borel injections

$$\theta_n : X_{n+1} \longrightarrow \{0, 1\}^\omega$$

such that the constant sequences $\langle 0, 0, \dots \rangle$ and $\langle 1, 1, \dots \rangle$ are not in the images.

Fix $f \in \text{dom } E$. First we define some auxiliary functions (depending on f): given $x \in X_1$ define $\varphi_0^f(x) \in Y^\omega$ by:

- $\varphi_0^f(x)(2i) = h_i^0(x)$ if $h_i^0(x) F_0 f(0)$;
- $\varphi_0^f(x)(2i) = 0$ otherwise;
- $\varphi_0^f(x)(2i + 1) = \theta_0(x)(i)$.

So $\varphi_0^f(x)$ enumerates A_0^f (in the even entries) and codes x (in the odd entries).

Inductively, given $x \in X_{n+2}$ define $\varphi_{n+1}^f(x) \in (Y^{\omega^{n+1}})^\omega$ by

- $\varphi_{n+1}^f(x)(2i) = \varphi_n^f(h_i^{n+1}(x))$ if $h_i^{n+1}(x) F_{n+1} f(n+1)$;
- $\varphi_{n+1}^f(x)(2i) = 0$ otherwise;
- $\varphi_{n+1}^f(x)(2i + 1) = \theta_{n+1}(x)(i)$.

(Here, the elements are in the space Y^{ω^n} . By 0, 1 we mean the constant functions of such, in the corresponding space.) Note that if $f E g$ then φ_n^f and φ_n^g are equal. This is because the definition of φ^f depends only on the equivalence classes $[f(n)]_{F_n}$.

Finally, define a map $\Psi : \text{dom } E \longrightarrow \prod_n Y^{\omega^{n+1}}$ by

$$\Psi(f)(n) = \varphi_n^f(f(n+1)).$$

It remains to show that Ψ is a reduction of E to $E_{\Pi}(Y)$. First we show that the range of Ψ is included in the domain of $E_{\Pi}(Y)$. For $f \in \prod_n X_n$ and $n \in \omega$, we need to show that $\Psi(f)(n) = \Psi(f)(n+1)(j)$ for some j . By the choice of the functions h_i^n there is some j such that $h_j^{n+1}(f(n+2)) = f(n+1)$. Thus

$$\begin{aligned} \Psi(f)(n+1)(2j) &= \varphi_{n+1}^f(f(n+2))(2j) = \varphi_n^f(h_j^{n+1}(f(n+2))) \\ &= \varphi_n^f(f(n+1)) = \Psi(f)(n). \end{aligned}$$

Next we show that

$$f E g \implies \Psi(f) E_{\Pi} \Psi(g).$$

For $n = 0$, note that for any $f E g$:

$$\begin{aligned} \text{Im}\Psi(f)(0) &= \text{Im}\varphi_0^f(f(1)) = A_0^f \cup \{0, 1\} = A_0^g \cup \{0, 1\} \\ &= \text{Im}\varphi_0^g(g(1)) = \text{Im}\Psi(g)(0). \end{aligned}$$

Given $f E g$, n and i , we need to find some j such that

$$\Psi(g)(n+1)(i) = \Psi(f)(n+1)(j).$$

For odd i , it follows by the choice of θ that such j exists. Consider now an even number of the form $2i$. If $h_i^{n+1}(g(n+2)) F_{n+1} g(n+1)$ then $\Psi(g)(n+1)(2i) = 0$ and again we are done. Otherwise, $h_i^{n+1}(g(n+2)) \in A_{n+1}^g = A_{n+1}^f$, so there is some j such that $h_i^{n+1}(g(n+2)) = h_j^{n+1}(f(n+2))$. Thus

$$\begin{aligned} \Psi(g)(n+1)(2i) &= \varphi_{n+1}^g(g(n+2))(2i) = \varphi_n^g(h_i^{n+1}(g(n+2))) \\ &= \varphi_n^f(h_j^{n+1}(f(n+2))) = \varphi_{n+1}^f(f(n+1))(2j) \\ &= \Psi(f)(n+1)(2j). \end{aligned}$$

Finally, it remains to show that if $\Psi(f) E_{\Pi} \Psi(g)$ then $f E g$. Recall that, since E is a PCP relation, it is a product of relations F_n on X_n . It suffices to show that $f(n) F_n g(n)$ for all n . Assume $\Psi(f) E_{\Pi} \Psi(g)$ and fix $n \in \omega$. Fix i such that $\Psi(g)(n) = \Psi(g)(n+1)(i)$, and find a j such that

$$\Psi(g)(n+1)(i) = \Psi(f)(n+1)(j).$$

It follows that both i, j are even. Let $i = 2l, j = 2k$. Then it also follows that $h_l^{n+1}(g(n+2))$ is F_{n+1} -related to $g(n+1)$, and $h_k^{n+1}(f(n+2))$ is F_{n+1} -related to $f(n+1)$. Now

$$\begin{aligned} \varphi_n^g(g(n+1)) &= \Psi(g)(n) = \Psi(f)(n+1)(j) = \varphi_{n+1}^f(f(n+2))(j) \\ &= \varphi_n^f(h_k^{n+1}(f(n+2))). \end{aligned}$$

Note that x is coded in the odd entries of $\varphi_n(x)$ using the functions θ_{n+1} . Thus from the equation above it follows that $g(n + 1) = h_k^{n+1}(f(n + 2)) \in A_{n+1}^f$, hence $g(n + 1) F_{n+1} f(n + 1)$. Finally, for the 0'th entry,

$$g(0) \in \text{Im}\Psi(g)(0) = \text{Im}\Psi(f)(0),$$

so there is some j such that

$$g(0) = \Psi(f)(0)(j) = \varphi_0(f(1))(j).$$

It follows from the definition of φ_0 that $g(0) \in A_0^f$, thus $g(0) F_0 f(0)$. We established that $g(n) F_n f(n)$ for all n , thus $g E f$ and the proof is done. ■

We will use the theorem above to establish a few basic properties of the equivalence relation E_Π .

PROPOSITION 4.16: *Let*

$$D = \{f \in (\mathbb{R}^\omega)^\omega : \forall n, i, j (f(n)(i) \text{ is computable in } f(n+1)(j))\}.$$

Then $(=^+)^\omega \upharpoonright D \sim_B E_\Pi$. *That is, Definitions 4.7 and 1.6 agree.*

Proof. Fix recursive bijections $\phi_n: \mathbb{R}^{\omega^n} \rightarrow \mathbb{R}$ where $\phi_0 = \text{id}$. Define a map $\prod_n (\mathbb{R}^{\omega^n})^\omega \rightarrow (\mathbb{R}^\omega)^\omega$ by $f \mapsto \langle \phi_n \circ f(n); n < \omega \rangle$. This map is a reduction of $\prod_n (=_{\mathbb{R}^{\omega^n}})^+$ to $(=^+)^\omega$ and its image, when restricted to the domain of E_Π , is contained in D . Thus $E_\Pi \leq_B (=^+)^\omega \upharpoonright D$. Note that $(=^+)^\omega \upharpoonright D$ is Borel PCP, hence is Borel reducible to E_Π by Theorem 4.15. ■

COROLLARY 4.17: $(E_\Pi)^\omega \sim_B E_\Pi$.

Proof. $(E_\Pi)^\omega$ can be represented as an equivalence relation with domain contained in a space $\prod_{m,n} X_{m,n}$, satisfying that: if $f(E_\Pi)^\omega g$ then $g(m, n)$ is Borel in $f(m, n + 1)$. Let $s: \omega \rightarrow \omega \times \omega$ be the snake enumeration of $\omega \times \omega$, and let $X_n = X_{s(n)}$. This gives an isomorphism of $\prod_n X_n$ to $\prod_{m,n} X_{m,n}$. The pullback of $(E_\Pi)^\omega$ produces a relation which is Borel isomorphic to $(E_\Pi)^\omega$ and satisfies the conditions of Lemma 4.2. ■

We conclude by noting the following generalization of PCP equivalence relations.

Definition 4.18: Let E be an equivalence relation such that the domain of E is a Borel subset of some product space $X = \prod_n X_n$. Then E is said to be PCP* if there are Borel equivalence relations F_n on X_n such that $E = \prod_n F_n \upharpoonright \text{dom } E$, and for every x and every n , the set $A_n^{E,x} \equiv \{y(n) : yEx\}$ is countable.

Any PCP equivalence relation is PCP*. The proof of Proposition 4.6 can be established for PCP* equivalence relations by similar arguments. That is, any PCP* equivalence relation is pinned and strictly below $=^+$. We do not know whether there is any PCP* equivalence relation which is not reducible to E_{Π} .

5. Applications to the Clemens–Coskey jumps

Recall the definition of the Γ -jumps from Section 1.2. Let E be a countable Borel equivalence relation on a Polish space X and Γ a countable group. Given $x \in X^\Gamma$, for each $\gamma \in \Gamma$ let $A_\gamma^x = [x(\gamma)]_E$, the E -class of $x(\gamma)$. For a Γ -indexed sequence $A' = \langle A'_\alpha : \alpha \in \Gamma \rangle$, define $\gamma \cdot A' = \langle A'_{\gamma^{-1}\alpha} : \alpha \in \Gamma \rangle$. Define

$$A^x = \{\gamma \cdot \langle A_\alpha^x : \alpha \in \Gamma \rangle : \gamma \in \Gamma\}.$$

The map $x \mapsto A^x$ is a complete classification of $E^{[\Gamma]}$.

Fix a countable infinite group Γ . Let E be either ergodic with respect to a measure μ on X or generically ergodic. Let $x \in X^\Gamma$ be a μ^Γ -Random generic, or Cohen generic, respectively. Consider $A = A^x$, its $E^{[\Gamma]}$ -invariant. We will study the model $V(A)$. Note that this model is equal to $V(\langle A_\gamma : \gamma \in \Gamma \rangle)$. The latter is generated by a countable sequence of E -classes, which is the model studied in Section 3. By genericity the elements of $\{A_\alpha : \alpha \in \Gamma\}$ are distinct. In particular there is a well defined action of Γ on $\{A_\alpha : \alpha \in \Gamma\}$ defined by

$$\gamma \cdot A_\alpha = A_{\gamma^{-1}\alpha}.$$

LEMMA 5.1: *In $V(A)$, the elements $\{A_\alpha : \alpha \in \Gamma\}$ are indiscernibles over A and parameters in V .*

Remark 5.2: In our context the Marker Lemma (see [KM04, Lemma 6.7]) is manifested in the following way. Consider the shift action of Γ on $\{0, 1\}^\Gamma$ and let $x \in \{0, 1\}^\Gamma$. The natural invariant will be the unordered set $\{\gamma \cdot x : \gamma \in \Gamma\}$ together with the action of Γ on this set. Then the Marker Lemma provides arbitrarily sparse subsets of $\{\gamma \cdot x : \gamma \in \Gamma\}$, definable using only the invariant

and parameters in V . On the other hand, when Γ is acting on a collection of E_0 -classes rather than on reals, Lemma 5.1 shows that the Marker Lemma fails in a strong way.

Proof. We present the case where x is Random generic. The case for a Cohen generic is similar and slightly easier. Work in the big generic extension $V[x]$. Let 1 denote the identity of the group Γ . Fix a formula ϕ , $v \in V$ such that $\phi^{V(A)}(A_1, A, v)$ holds. We will show that $\phi^{V(A)}(A_{\gamma^{-1}}, A, v)$ holds for arbitrary $\gamma \in \Gamma$. Assume towards a contradiction that $\phi^{V(A)}(A_{\gamma^{-1}}, A, v)$ fails, and let p be a condition forcing the above. Since the shift action preserve the product measure, $\gamma \cdot p$ is a condition. Furthermore, since p forces $\phi^{V(\dot{A})}(\dot{A}_1, \dot{A}, \dot{v})$ then $\gamma \cdot p$ forces $\phi^{V(\dot{A})}(\dot{A}_{\gamma^{-1}}, \dot{A}, \dot{v})$.

Consider p and $\gamma \cdot p$ as positive measure subsets of X^Γ . By [Kec20, Lemma 4.2] the finite support power $\prod_{\gamma \in \Gamma}^{\text{fin}} E$ is μ^Γ -ergodic. It follows that we may find generics x_1, x_2 such that x_1 extends p , x_2 extends $\gamma \cdot p$, and furthermore x_1 and x_2 are E^Γ -equivalent and they differ in only finitely many coordinates. (See the arguments in the proof of Proposition 3.5.) Note that $A_\gamma^{x_1} = A_\gamma^{x_2}$ for every γ . Since x_1 extends p , working in $V[x_1]$ we conclude that $\phi^{V(A^{x_1})}(A_{\gamma^{-1}}^{x_1}, A^{x_1}, v)$ fails. However, since x_2 extends $\gamma \cdot p$, working in $V[x_2]$ we conclude that $\phi^{V(A^{x_2})}(A_{\gamma^{-1}}^{x_2}, A^{x_2}, v)$ holds, a contradiction. ■

COROLLARY 5.3: *No non-empty finite subset $\bar{A} \subseteq \{A_\alpha : \alpha \in \Gamma\}$ is definable in $V(A)$ from A and parameters in V alone.*

Proof. Assume for contradiction there is such \bar{A} , defined as the unique solution to $\phi(\bar{A}, A, v)$. Fix γ such that

$$\gamma \cdot \bar{A} \equiv \{\gamma \cdot A_\alpha : A_\alpha \in \bar{A}\}$$

is different than \bar{A} (possible since Γ is infinite.) Then ϕ also uniquely defines $\gamma \cdot \bar{A}$, a contradiction. ■

CLAIM 5.4: *If $B \in V(A)$ is a countable set of subsets of V , definable from A and parameters in V alone, then $B \in V$.*

Proof. Fix some enumeration f of B . Note that $\text{supp}(b) \subseteq \text{supp}(f)$ for any $b \in B$. Let \bar{A} be the minimal finite subset of $\{A_\alpha : \alpha \in \Gamma\}$ such that $\text{supp}(b) \subseteq \bar{A}$ for any $b \in B$. Since \bar{A} is definable from A , \bar{A} is empty by Corollary 5.3. It follows that B is a subset of V . Then $B \in V$ by Proposition 3.5. ■

By Lemma 2.5 we conclude:

COROLLARY 5.5: *Let Γ be any countable infinite group. Suppose E is a countable Borel equivalence relation and either (1) E is ergodic with respect to μ or (2) E is generically ergodic. Then for any countable Borel equivalence relation F*

- (1) $E^{[\Gamma]}$ is (μ^Γ, F) -ergodic;
- (2) $E^{[\Gamma]}$ is generically F -ergodic.

Part (2) was proved by Clemens and Coskey [CC22]. From part (1) we conclude:

COROLLARY 5.6: $E_\infty^{[\mathbb{Z}]}$ is not Borel reducible to $E_\infty^{\mathbb{N}} \times E_0^{[\mathbb{Z}]}$.

Proof. Fix a measure μ for which E_∞ is (μ, E_0) -ergodic. Then $E_\infty^{[\mathbb{Z}]}$ is $(\mu^\mathbb{Z}, E_\infty)$ -ergodic by the discussion above, and is therefore $(\mu^\mathbb{Z}, E_\infty^{\mathbb{N}})$ -ergodic. Any reduction from $E_\infty^{[\mathbb{Z}]}$ to $E_\infty^{\mathbb{N}} \times E_0^{[\mathbb{Z}]}$ would give a reduction to $E_0^{[\mathbb{Z}]}$ on a $\mu^\mathbb{Z}$ -measure 1 set. Since E_∞ is not reducible to $E_0^{[\mathbb{Z}]}$ on any μ -measure 1 set (see [CC22]), this would give a contradiction. ■

PROPOSITION 5.7: $E_\Pi \not\preceq E_\infty^{[\Gamma]}$ and $E_0^{[\Gamma]} \not\preceq E_\Pi$ for any infinite countable group Γ .

Proof. Let A be an $E_0^{[\Gamma]}$ -invariant of a Cohen generic real as above. Suppose $B = \langle B_0, B_1, \dots \rangle$ is an E_Π -invariant definable from A alone. By Claim 5.4 $B_k \in V$ for every k , thus B is in fact a subset of V , and so $B \in V$ by Proposition 3.5. It follows that $E_0^{[\Gamma]}$ is generically E_Π -ergodic.

Let now A be the E_Π -invariant from Construction 4.8. Let B be a $E_\infty^{[\Gamma]}$ -invariant in $V(A)$. Then $V(B) = V(\langle B_\gamma : \gamma \in \Gamma \rangle)$. Since $V(A)$ is not generated by any countable sequence of E_∞ -classes (by Theorem 4.10), it follows that $V(A) \neq V(B)$, so $E_\Pi \not\preceq E_\infty^{[\Gamma]}$ by Lemma 2.4. ■

We now turn to prove Theorem 1.3, establishing strong ergodicity between the Γ -jumps for different values of Γ . The central tools used in the proof are Proposition 3.5 and Lemma 5.1, which hold for both measure and category. We focus on Baire category arguments, though the analogous results for measures also hold, as mentioned after Theorem 1.3. Note that when working mod meager sets, all the countable Borel equivalence relations below may be replaced with E_0 , by generic hyperfiniteness [KM04, Theorem 12.1].

LEMMA 5.8: *Let Γ be a countable group and $\tilde{\Gamma}$ a finite normal subgroup of Γ . Let E be countable Borel equivalence relation on X which is generically ergodic. Then $E^{[\Gamma]}$ is not generically $E_{\infty}^{[\Gamma/\tilde{\Gamma}]}$ -ergodic.*

Proof. Fix a Borel linear ordering $<$ of X and let m be the size of $\tilde{\Gamma}$. Define an equivalence relation \tilde{E} on X^m , relating two finite sequences \bar{x}, \bar{y} if there is a permutation π of m such that $x \circ \pi$ and y are E^m -related. Note that \tilde{E} is a countable Borel equivalence relation, thus it suffices to show that $E^{[\Gamma]}$ is not generically $\tilde{E}^{[\Gamma/\tilde{\Gamma}]}$ -ergodic.

Let $x \in X^{\Gamma}$ be Cohen generic and A^x its $E^{[\Gamma]}$ -invariant. For $\gamma \in \Gamma$ define $B_{\gamma\tilde{\Gamma}}$ to be all choice functions in $\{A_{\gamma\gamma'} : \gamma' \in \tilde{\Gamma}\}$. Note that $B_{\gamma\tilde{\Gamma}}$ can be viewed as an \tilde{E} -class. Define $B = \{\langle B_{g^{-1}\gamma\tilde{\Gamma}} : \gamma\tilde{\Gamma} \in \Gamma/\tilde{\Gamma} \rangle : g \in \Gamma/\tilde{\Gamma}\}$. Then B is an $\tilde{E}^{[\Gamma/\tilde{\Gamma}]}$ -invariant definable in $V(A)$ from A and parameters in V alone, and $B \notin V$. By Lemma 2.5 it follows that $E^{[\Gamma]}$ is not generically $\tilde{E}^{[\Gamma/\tilde{\Gamma}]}$ -ergodic. ■

COROLLARY 5.9: *Suppose there is a finite normal subgroup $\tilde{\Gamma}$ of Γ , a subgroup $\tilde{\Delta}$ of Δ and a normal subgroup H of $\tilde{\Delta}$ such that $\Gamma/\tilde{\Gamma}$ embeds into $\tilde{\Delta}/H$. Then $E^{[\Gamma]}$ is not generically $E_{\infty}^{[\Delta]}$ -ergodic.*

Proof. Clemens and Coskey [CC22] show that for any equivalence relation F , if Λ is either a subgroup or a quotient of Γ , then $E^{[\Lambda]}$ is Borel reducible to $E^{[\Gamma]}$. It follows that $E_{\infty}^{[\Gamma/\tilde{\Gamma}]}$ is Borel reducible to $E_{\infty}^{[\Delta]}$, so by the lemma above $E^{[\Gamma]}$ is not generically $E_{\infty}^{[\Delta]}$ -ergodic. ■

The corollary is the implication (1) \implies (2) of Theorem 1.3. The following proposition gives the reverse implication and will establish the theorem.

PROPOSITION 5.10: *Let Γ and Δ be countable groups, E a generically ergodic countable Borel equivalence relation and F a countable Borel equivalence relation. Suppose $E^{[\Gamma]}$ is not generically $F^{[\Delta]}$ -ergodic. Then there is a subgroup $\tilde{\Delta}$ of Δ , a normal subgroup H of $\tilde{\Delta}$ and a group homomorphism from Γ to $\tilde{\Delta}/H$ with finite kernel.*

Proof. By Lemma 2.5 there is a Cohen generic $x \in X^{\Gamma}$, $A = A^x$ its $E^{[\Gamma]}$ -invariant, such that in $V(A)$ there is an $F^{[\Delta]}$ -invariant B , definable from A and parameters in V alone, such that $B \notin V$. Let $B = \{\langle B_{\delta^{-1}\xi} : \xi \in \Delta \rangle : \delta \in \Delta\}$, where each B_{ξ} is an F -class.

CLAIM 5.11: *There is some ξ and $b \in B_{\xi}$ which is not in V .*

Proof. Otherwise, it follows that each B_ξ is in V and so $\langle B_{\delta^{-1}\xi} : \xi \in \Delta \rangle$ is a subset of V for each δ . B is countable in $V(A)$: fix some $B' = \langle B'_\xi : \xi \in \Delta \rangle \in B$. Let $\delta \cdot B' = \langle B'_{\delta^{-1}\xi} : \xi \in \Delta \rangle$, then $\langle \delta \cdot B' : \delta \in \Delta \rangle$ is an enumeration of B . Thus B is a countable set of subsets of V , definable from A alone, so B is in V by Claim 5.4. ■

Fix some $b \in B_\xi$ not in V . Since b is a real there is a minimal finite $s \subseteq \Gamma$ such that for $\bar{x} = \langle x(\gamma) : \gamma \in s \rangle$, $b \in V[\bar{x}]$. Let $\bar{A} = \langle A_\gamma : \gamma \in s \rangle$. Note that $V(\bar{A}) = V[\bar{x}]$ and $B_\xi \subseteq V(\bar{A})$. Say that \bar{A} is the support for B_ξ . For a fixed \bar{A} there could be many B_ξ whose support is \bar{A} . We utilize the following coding functions to ensure that the set of ξ for which \bar{A} is a support for B_ξ forms a subgroup. These are variations of the coding functions used by Clemens and Coskey in [CC22] to show that $E^{[\mathbb{Z}]}$ is Borel reducible to $=^+$.

For $B' \in B$ define $p_{B'}: \Delta^2 \times \Gamma \rightarrow \{0, 1\}$ by $p_{B'}(\delta_1, \delta_2, \gamma) = 1$ if and only if the support of $(\delta_1 \cdot B')_1$ is the γ -shift of the support of $(\delta_2 \cdot B')_1$. Let $P = \{p_{B'} : B' \in B\}$. (Note that $(\delta_1 \cdot B')_1$ is $B'_{\delta_1^{-1}}$, which has a well defined support according to the previous paragraph.) P is a set of reals definable from A alone (since B is definable from A alone). Furthermore P is countable: fix any $p_{B'} \in P$ and define $p_\delta(\delta_1, \delta_2, \gamma) = 1$ if and only if $p_{B'}(\delta_1 \delta, \delta_2 \delta, \gamma) = 1$. Given any other $p_{B''} \in P$, fix δ such that $B'' = \delta \cdot B'$, then $p_{B''}(\delta_1, \delta_2, \gamma) = 1$ if and only if the support of $(\delta_1 \cdot B'')_1$ is a γ -shift of the support of $(\delta_2 \cdot B'')_1$, if and only if the support of $(\delta_1 \delta \cdot B')_1$ is a γ -shift of the support of $(\delta_1 \delta \cdot B')_1$, if and only if $p_\delta(\delta_1, \delta_2, \gamma) = 1$. It follows that $\langle p_\delta : \delta \in \Delta \rangle$ enumerates P . By Claim 5.4 we conclude that P is in V , and so each $p \in P$ is in V .

Fix $p \in P$ and $B^* \in B$ such that $p_{B^*} = p$ and \bar{A} is the support of B^*_1 . Let $\tilde{\Delta}$ be the set of all $\delta \in \Delta$ such that $p_{\delta \cdot B^*} = p$ and the support of $(\delta \cdot B^*)_1$ is $\gamma \cdot \bar{A}$ for some $\gamma \in \Gamma$. Note that for any γ there is a δ as above, by indiscernibility. Define $H \subseteq \tilde{\Delta}$ as the set of all $\delta \in \tilde{\Delta}$ such that B^*_1 and $(\delta \cdot B^*)_1$ have the same support \bar{A} . From now on \bar{A} , B^* and p are fixed. For $\delta \in \tilde{\Delta}$ say that γ is **the support of δ** if $\gamma \cdot \bar{A}$ is the support of $(\delta \cdot B^*)_1$.

LEMMA 5.12: *Suppose $\delta, \delta' \in \tilde{\Delta}$ with supports γ, γ' respectively. Then $\delta^{-1} \delta' \in \tilde{\Delta}$ with supports $\gamma^{-1}, \gamma \gamma'$ respectively.*

Proof. Suppose $\delta \in \tilde{\Delta}$ and $\gamma \cdot \bar{A}$ is the support for $(\delta \cdot B^*)_1$. Then $p_{\delta \cdot B^*}(1, \delta^{-1}, \gamma) = 1$. Since $p_{B^*} = p = p_{\delta \cdot B^*}$, $p_{B^*}(1, \delta^{-1}, \gamma) = 1$ as well. Thus $\gamma^{-1} \cdot \bar{A}$ is the support for $(\delta^{-1} \cdot B^*)_1$. Furthermore, $p_{\delta^{-1} \cdot B^*}(\delta_1, \delta_2, \gamma) = 1$ if

and only $p_{B^*}(\delta_1\delta^{-1}, \delta_2\delta^{-1}, \gamma) = 1$ if and only if $p_{\delta \cdot B^*}(\delta_1\delta^{-1}, \delta_2\delta^{-1}, \gamma) = 1$ if and only if $p_{B^*}(\delta_1\delta^{-1}\delta, \delta_2\delta^{-1}\delta, \gamma) = 1$ if and only if $p_{B^*}(\delta_1, \delta_2, \gamma) = 1$. Therefore δ^{-1} is in $\tilde{\Delta}$.

Assume now δ, δ' are both in $\tilde{\Delta}$, where $\gamma \cdot \bar{A}$ and $\gamma' \cdot \bar{A}$ are the supports for $(\delta \cdot B^*)_1$ and $(\delta' \cdot B^*)_1$ respectively. Then

$$p_{B^*}(\delta, 1, \gamma) = 1 = p_{\delta' \cdot B^*}(\delta, 1, \gamma),$$

therefore the support of $(\delta\delta' \cdot B^*)_1$ is the γ -shift of the support of $(\delta' \cdot B^*)_1$, that is, it is $\gamma\gamma' \cdot \bar{A}$. Furthermore,

$$\begin{aligned} p_{\delta\delta' \cdot B^*}(\delta_1, \delta_2, \zeta) &= p_{\delta' \cdot B^*}(\delta_1\delta, \delta_2\delta, \zeta) = p_{B^*}(\delta_1\delta, \delta_2\delta, \zeta) = p_{\delta \cdot B^*}(\delta_1, \delta_2, \zeta) \\ &= p_{B^*}(\delta_1, \delta_2, \zeta). \end{aligned}$$

Therefore $\delta\delta'$ is in $\tilde{\Delta}$. ■

COROLLARY 5.13: $\tilde{\Delta}$ is a subgroup of Δ and H is a normal subgroup of $\tilde{\Delta}$.

Proof. It follows from the lemma that $\tilde{\Delta}$ and H are subgroups. For example, if $h \in H$ then its support is $1 \in \Gamma$. By the lemma $h^{-1} \in \tilde{\Delta}$ with support $1^{-1} = 1$, that is, $h^{-1} \in H$. We now show that H is a normal subgroup of $\tilde{\Delta}$. Suppose $h \in H$, $\delta \in \tilde{\Delta}$ with support γ . By the lemma $h\delta \in \tilde{\Delta}$ with support γ . Applying the lemma again it follows that $\delta^{-1}h\delta \in \tilde{\Delta}$ with support $\gamma^{-1}\gamma = 1$, thus $\delta^{-1}h\delta \in H$. ■

For $\gamma \in \Gamma$ define $H_\gamma \subseteq \tilde{\Delta}$ as the set of all $\delta \in \tilde{\Delta}$ whose support is γ . It follows from Lemma 5.12 that each H_γ is a coset of H and that the map $\gamma \mapsto H_\gamma$ is a group homomorphism. Furthermore, if γ is in the kernel then $H_\gamma = H$ and so $\gamma \cdot \bar{A} = \bar{A}$. There could be only finitely many such γ 's, so the kernel is finite. ■

We now show that $E_\infty^{[\mathbb{Z}]}$ and $E_0^{[\mathbb{Z}]} \times E_\infty^\omega$ are pairwise \leq_B -incomparable. By Corollary 5.6 it suffices to show the following irreducibility.

PROPOSITION 5.14: $E_0^{[\mathbb{Z}]} \times E_0^\omega$ is not Borel reducible to $E_\infty^{[\mathbb{Z}]}$.

Remark 5.15: In this case, we do not have strong ergodicity with respect to either measure or category. Since $E_0 \leq_B E_\infty$, then $E_0^{[\mathbb{Z}]} \leq_B E_\infty^{[\mathbb{Z}]}$. Fix a Borel reduction f of $E_0^{[\mathbb{Z}]}$ to $E_\infty^{[\mathbb{Z}]}$. Then the map $(x, y) \mapsto f(x)$ is a non-trivial Borel homomorphism from $E_0^{[\mathbb{Z}]} \times E_0^\omega$ to $E_\infty^{[\mathbb{Z}]}$. The pre-image of a single $E_\infty^{[\mathbb{Z}]}$ -class will

involve only one $E_0^{[\mathbb{Z}]}$ -class, and will be meager and measure zero, with respect to a natural product topology, or a natural product measure.

Proof of Proposition 5.14. Since we are dealing with an additive group we will write the action in an additive way: $A_\alpha + \gamma = A_{\alpha+\gamma}$. Towards a contradiction, assume that there is a Borel reduction of $E_0^{[\mathbb{Z}]} \times E_0^\omega$ to $E_\infty^{[\mathbb{Z}]}$. Let $(x, y) \in (2^\omega)^\mathbb{Z} \times (2^\omega)^\omega$ be Cohen generic. Let A, B be the $E_0^{[\mathbb{Z}]} \times E_0^\omega$ -invariant. That is, $A = A^x$ as above and $B = \langle [y(n)]_{E_0} : n < \omega \rangle$. By Lemma 2.4 there is an $E_\infty^{[\mathbb{Z}]}$ -invariant $C \in V(A, B)$ such that $V(A, B) = V(C)$ and C is definable from A, B and parameters in V . Let $C_k, k \in \mathbb{Z}$ be E_∞ -classes such that

$$C = \{ \langle C_{n+k} : n \in \mathbb{Z} \rangle : k \in \mathbb{Z} \}.$$

CLAIM 5.16: (1) *The members of $\{A_n : n \in \mathbb{Z}\}$ are indiscernible over A and parameters from $V(B)$.*

(2) *For any $X \subseteq V$ there is a minimal support (s, t) where s is a finite subset of \mathbb{Z}, t a finite subset of ω such that*

$$X \in V(\langle x(k) : k \in s \rangle, \langle y(n) : n \in t \rangle) = V(\langle A_k : k \in s \rangle, \langle B_n : n \in t \rangle).$$

Part (1) follows from Lemma 5.1, working over $V(B)$ as the ground model. Part (2) is proved similarly: working over $V(B)$ we get that X is in $V(B)[\bar{x}]$. Now working over $V[\bar{x}]$ we get that X is in $V[\bar{x}][\bar{y}]$. In particular, any $c \in C_k$ has a minimal support (s, t) . The support is the same for any other $c' \in C_k$, and so we say that (s, t) is the support of C_k as well. Given such s, t we will denote $\bar{x} = \langle x(k) : k \in s \rangle, \bar{A} = \langle A_k : k \in s \rangle, \bar{y} = \langle y(n) : n \in t \rangle$ and $\bar{B} = \langle B_n : n \in t \rangle$. We sometimes call the pair (\bar{x}, \bar{y}) or (\bar{A}, \bar{B}) the support of C_k as well.

For each $C' \in C$ ($C' = \langle C'_k : k \in \mathbb{Z} \rangle$) define $p_{C'} : \mathbb{Z}^2 \rightarrow \{0, 1\}$ by $p_{C'}(t, l) = 1$ if and only if C'_t and C'_l have the same support. Let $P = \{p_{C'} : C' \in C\}$. Then P is a countable set of reals, definable from (A, B) alone, and therefore $P \in V(B)$. It follows that $P \in V(\langle B_n : n \leq m \rangle)$ for some m . By replacing V with $V(\langle B_n : n \leq m \rangle)$ and B with $\langle B_n : n \geq m \rangle$, we may assume that $P \in V$.

Fix $p \in P$ in V for which there is some $C' \in C$ with $p = p_{C'}$ and the support of C'_0 is (\bar{A}, \bar{B}) . Suppose there is some $C'' \in C$ such that $p_{C''} = p$ and C''_0 has the same support (\bar{A}, \bar{B}) . Fix m such that $C'' = C' + m$, then for every $k \in \mathbb{Z}, C'_{mk}$ has the same support (\bar{A}, \bar{B}) . That is, we associate to (\bar{A}, \bar{B}) an arithmetic sequence in the \mathbb{Z} -ordering on $\{C_n : n \in \mathbb{Z}\}$. If not such C'' exists then this arithmetic sequence is a singleton ($m = 0$). Note that for the distinct pairs (\bar{A}, \bar{B}) and (\bar{A}', \bar{B}') the corresponding arithmetic sequences must be disjoint.

CLAIM 5.17: *If \bar{A} is not empty then for any \bar{B} the arithmetic sequence corresponding to (\bar{A}, \bar{B}) is a singleton (that is, $m = 0$).*

Proof. Otherwise, there is an arithmetic sequence with common difference m corresponding to (\bar{A}, \bar{B}) with $m > 0$. By indiscernibility, for any t there is an arithmetic sequence with common difference m corresponding to $(\bar{A} + t, \bar{B})$. Since there could be only finitely many disjoint arithmetic sequences with fixed common difference m , we arrive at a contradiction. ■

CASE 1: Suppose there is a support (\bar{A}, \bar{B}) for some C_t where \bar{A} is not empty. By the claim (\bar{A}, \bar{B}) defines uniquely some $C' \in C$ with $\text{supp } C'_0 = (\bar{A}, \bar{B})$. As before, there is some $k \in \mathbb{Z}$ such that $\{(\bar{A} + l, \bar{B}) : l \in \mathbb{Z}\}$ corresponds to the k -arithmetic sequence $\{C'_{kl} : l \in \mathbb{Z}\}$ with $\text{supp } C'_{kl} = (\bar{A} + l, \bar{B})$.

- CLAIM 5.18: (1) *Assume that (\emptyset, \bar{B}') is a support corresponding to an m -arithmetic sequence. Then $0 < m \leq k$.*
 (2) *Assume \bar{A}' is not empty. Any (\bar{A}', \bar{B}') is a support corresponding to a singleton, so $\{(\bar{A}' + l, \bar{B}') : k \in \mathbb{Z}\}$ corresponds to an m -arithmetic sequence for some m . Then $m \leq k$.*

Proof. We prove (1), the proof of (2) is similar. Fix l such that for the unique C' corresponding to $(\bar{A} + l, \bar{B})$, there is $0 < j < k$ with $\text{supp } C'_j = (\emptyset, \bar{B}')$. By indiscernibility, this is true for $(\bar{A} + l, \bar{B})$ for any l . Fixing C' , it follows that for any l there is some $0 < j < k$ with $\text{supp } C'_{kl+j} = (\emptyset, \bar{B}')$. We conclude that $m > 0$ and $m \leq k$. ■

There could be at most finitely many disjoint m -arithmetic sequences with $0 < m \leq k$. Therefore there are only finitely many \bar{B}' for which there is some \bar{A}' (possibly empty) so that (\bar{A}', \bar{B}') is a support of some C_t . It follows that there is some $m < \omega$ such that $C_t \in V(A, \langle B_n : n < m \rangle)$ for all t . Forcing now $\langle B_n : n \geq m \rangle$ over $V(A, \langle B_n : n < m \rangle)$, we get that C is a set of subsets of the ground model $V(A, \langle B_n : n < m \rangle)$, which is countable and definable from $\langle B_n : n \geq m \rangle$ and parameters in $V(A, \langle B_n : n < m \rangle)$. By Corollary 3.7 it follows that $V(A, \langle B_n : n < m \rangle)(\langle B_n : n \geq k \rangle) \neq V(A, \langle B_n : n < m \rangle)(C)$, contradicting the assumption that $V(C) = V(A, B)$.

CASE 2: For any C_k its support is of the form (\emptyset, \bar{B}) for some \bar{B} . It follows that $C \in V(B)$, a contradiction. ■

The proposition also implies that $E^{[\mathbb{Z}]} <_B (E^{[\mathbb{Z}]})^2$ for any generically ergodic countable Borel equivalence relation E . Similar arguments show that

$$E^{[\mathbb{Z}]} <_B (E^{[\mathbb{Z}]})^2 <_B (E^{[\mathbb{Z}]})^3 <_B \dots .$$

References

- [AK00] S. Adams and A. S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, Journal of the American Mathematical Society **13** (2000), 909–943.
- [Bla81] A. Blass, *The model of set theory generated by countably many generic reals*, The Journal of Symbolic Logic **46** (1981), 732–752.
- [CC22] J. D. Clemens and S. Coskey, *New jump operators on Borel equivalence relations*, Journal of Mathematical Logic **22** (2022), Article no. 2250015,
- [DJK94] R. Dougherty, S. Jackson and A. S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Transactions of the American Mathematical Society **341** (1994), 193–225.
- [Gao09] S. Gao, *Invariant Descriptive Set Theory*, Pure and Applied Mathematics (Boca Raton), Vol. 293, CRC Press, Boca Raton, FL, 2009.
- [GJ15] S. Gao and S. Jackson, *Countable abelian group actions and hyperfinite equivalence relations*, Inventiones Mathematicae **201** (2015), 309–383.
- [HK97] G. Hjorth and A. S. Kechris, *New dichotomies for Borel equivalence relations*, Bulletin of Symbolic Logic **3** (1997), 329–346.
- [HK05] G. Hjorth and A. S. Kechris, *Rigidity theorems for actions of product groups and countable Borel equivalence relations*, Memoirs of the American Mathematical Society **177** (2005).
- [HR98] P. Howard and J. E. Rubin, *Consequences of the Axiom of Choice*, Mathematical Surveys and Monographs, Vol. 59, American Mathematical Society, Providence, RI, 1998.
- [Jec73] T. J. Jech, *The axiom of choice*, Studies in Logic and the Foundations of Mathematics, Vol. 75, North-Holland, Amsterdam, 1973.
- [Jec03] T. Jech, *Set Theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.
- [JS87] V. F. R. Jones and K. Schmidt, *Asymptotically invariant sequences and approximate finiteness*, American Journal of Mathematics **109** (1987), 91–114.
- [Kano08] V. Kanovei, *Borel Equivalence Relations*, University Lecture Series, Vol. 44, American Mathematical Society, Providence, RI, 2008.
- [KSZ13] V. Kanovei, M. Sabok and J. Zapletal, *Canonical Ramsey Theory on Polish Spaces*, Cambridge Tracts in Mathematics, Vol. 202, Cambridge University Press, Cambridge, 2013.
- [Kec95] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, Vol. 156, Springer, New York, 1995
- [Kec20] A. S. Kechris, *Quasi-invariant measures for continuous group actions*, in *Trends in Set Theory*, Contemporary Mathematics, Vol. 752, American Mathematical Society, Providence, RI, pp. 113–119.

- [Kec ∞] A. S. Kechris, *The Theory of Countable Borel Equivalence Relations*, Cambridge Tracts in Mathematics, Vol. 234, Cambridge University Press, Cambridge, 2024.
- [KM04] A. S. Kechris and B. D. Miller, *Topics in Orbit Equivalence*, Springer, Berlin–Heidelberg, 2004.
- [Kun83] K. Kunen, *Set Theory*, Studies in Logic and the Foundations of Mathematics, Vol. 102, North-Holland, Amsterdam, 1983.
- [LZ20] P. B. Larson and J. Zapletal, *Geometric Set Theory*, Mathematical Surveys and Monographs, Vol. 248, American Mathematical Society, Providence, RI, 2020
- [Mos39] A. Mostowski, *Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip*, *Fundamenta Mathematicae* **32** (1939), 201–252.
- [Sha21] A. Shani, *Borel reducibility and symmetric models*, *Transactions of the American Mathematical Society* **374** (2021), 453–485.
- [Zap08] J. Zapletal, *Forcing Idealized*, Cambridge Tracts in Mathematics, Vol. 174, Cambridge University Press, Cambridge, 2008.
- [Zap11] J. Zapletal, *Pinned equivalence relations*, *Mathematical Research Letters* **18** (2011), 1149–1156.