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MAXIMUM PRINCIPLES AND DIRECT METHODS FOR TEMPERED FRACTIONAL OPERATORS

BY

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ABSTRACT

In this paper, we are concerned with the tempered fractional operator $-(\Delta+\lambda)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ and λ is a sufficiently small positive constant. We first establish various maximum principle principles and develop the direct moving planes and sliding methods for anti-symmetric functions involving tempered fractional operators. And then we consider tempered fractional problems. As applications, we extend the direct method of moving planes and sliding methods for the tempered fractional problem, and discuss how they can be used to establish symmetry, monotonicity, Liouville-type results and uniqueness results for solutions in various domains. We believe that our theory and methods can be conveniently applied to study other problems involving tempered fractional operators.

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1. Introduction

1.1. Background. In this paper, we are concerned with the tempered fractional Laplacian operator $-(\Delta+\lambda)^{\frac{\alpha}{2}}$, which is defined as

$$(\Delta+\lambda)^{\frac{\alpha}{2}}u(x):=-c_{n,\alpha}P.V.\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{e^{\lambda|x-y|}|x-y|^{n+\alpha}}dy, \tag{1.1}$$

where $P.V.$ stands for Cauchy principal value, and

$$c_{n,\alpha}=\begin{cases}\frac{\alpha\Gamma(\frac{n+\alpha}{2})}{2^{1-\alpha}\pi^{\frac{n}{2}}|\Gamma(1-\frac{\alpha}{2})|}, & \text{for } \lambda=0 \text{ or } \alpha=1,\\ \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}|\Gamma(-\alpha)|}, & \text{for } \lambda>0 \text{ and } \alpha\neq 1,\end{cases} \tag{1.2}$$

and Γ denotes the Gamma function.

To ensure that the right-hand side of the definition (1.2) is well-defined, we require that $u\in\mathscr{L}_\alpha(\mathbb{R}^n)\cap C^{1,1}_{\rm loc}(\mathbb{R}^n\setminus\{0\})$ with

$$\mathscr{L}_\alpha(\mathbb{R}^n):=\left\{u:\mathbb{R}^n\to\mathbb{R}\mid\int_{\mathbb{R}^n}\frac{e^{-\lambda|x|}|u(x)|}{1+|x|^{n+\alpha}}dx<+\infty\right\}. \tag{1.3}$$

It should be pointed out that the nonlocal operator $-(\Delta+\lambda)^{\frac{\alpha}{2}}$ can also be defined equivalently through Caffarelli and Silvestre's extension method; we refer to [5] and references therein for more details.

When $\lambda\to 0+$, the tempered fractional operator $-(\Delta+\lambda)^{\frac{\alpha}{2}}$ degenerates into the familiar fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, which is also a nonlocal integro-differential operator given by

$$(-\Delta)^{\frac{\alpha}{2}}u(x)=C_{N,\alpha}\,P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+\alpha}}dy, \tag{1.4}$$

where $0<\alpha<2$,

$$C_{N,\alpha}=\left(\int_{\mathbb{R}^N}\frac{1-\cos(2\pi y_1)}{|y|^{N+\alpha}}dy\right)^{-1}.$$

The fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is well-defined for any $u\in C^{1,1}_{\rm loc}(\mathbb{R}^N)\cap\dot{L}_\alpha(\mathbb{R}^N)$ with the function spaces

$$\dot{L}_\alpha(\mathbb{R}^N):=\left\{u:\mathbb{R}^N\to\mathbb{R}\mid\int_{\mathbb{R}^N}\frac{|u(x)|}{1+|x|^{N+\alpha}}dx<+\infty\right\}.$$

It can also be defined equivalently through Caffarelli and Silvestre's extension method (see [5]).

In recent years, problems involving fractional operators have attracted more attention due to their various applications in mathematical modeling, such as fluid mechanics, molecular dynamics, relativistic quantum mechanics of stars (see, e.g., [6, 21]), in conformal geometry (see, e.g., [15]) and in probability, and also finance (see [3, 4]) etc. In 1996, Bertoin [3] interpreted the fractional Laplacian as an infinitesimal generator for a stable Lévy diffusion process. The scaling limit of Lévy flight is the α-stable Lévy process, generated by the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$. In order to make the Lévy flight a more suitable physical model, the concept of the tempered Lévy flight was introduced. The scaling limit of the tempered Lévy flight is called the tempered Lévy process, which is generated by the tempered fractional Laplacian.

From the viewpoint of mathematics, the nonlocal nature of the fractional operator makes the nonlocal problem more challenging than the local problem. The pioneering work in the literature can be traced back to Caffarelli and Silvestre (see, e.g., [5]). They define the nonlocal operators via the extension method and reduce the nonlocal problem to a local problem in higher dimensions. Then, variational theory can be applied to study the existence and other related properties of the solutions to the nonlocal problem; we refer to [4,5,9–11,17,27,29,30,37,39,40,42] and references therein for a series of fruitful results in elliptic equations and systems. Later on, Chen and Li (see [15,17]) give another approach, which considers the equivalent IEs instead of PDEs by deriving the integral representation formulae of solutions to the equations involving fractional operators (see e.g. [16, 22–24, 29, 35, 42, 43]). However, these two approaches can not work directly for the problem with tempered fractional Laplacian operator $(\Delta + \lambda)^{\frac{\alpha}{2}}$.

Many results have been achieved for fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, but very few results for tempered fractional Laplacian $(\Delta + \lambda)^{\frac{\alpha}{2}}$. Now let us recall the work achieved on tempered fractional Laplacian. For instance, Zhang, Deng and Karniadakis [45] developed numerical methods for the tempered fractional Laplacian in the Riesz basis Galerkin framework. Zhang, Deng and Fan [44] designed the finite difference schemes for the tempered fractional Laplacian equation with the generalized Dirichlet type boundary condition. Duo and Zhang [28] proposed a finite difference method to discretize the n-dimensional (for $n \geq 1$) tempered integral fractional Laplacian and applied it to study the tempered effects on the solution of problems arising in various applications. Shiri, Wu and Baleanu [41] proposed a collocation methods for terminal value problems of tempered fractional differential equations. For more works on tempered fractional Laplacian, refer to [26, 36, 46] and the references therein.

1.2. Main results. In this paper, the main purpose is to establish various maximum principles for the tempered fractional Laplacian operator $(\Delta+\lambda)^{\frac{\alpha}{2}}$. Motivated by the direct methods for $(-\Delta)^s$ and $(-\Delta)_p^s$ established in [12–14, 18, 20, 38], we will develop direct moving planes and sliding methods for various tempered fractional problems, and then we illustrate how the maximum principles we obtained can be used in the method of moving planes and sliding to establish symmetry, monotonicity and uniqueness results for solutions in various domains. We believe that our method can be used to deal with other problems involving tempered fractional Laplacian operator $(\Delta+\lambda)^{\frac{\alpha}{2}}$.

First, we give some useful notation. Let T be any given hyper-plane in $\mathbb{R}^n$ and Σ be the half-space on one side of the plane T. Denote the reflection of a point x with respect to T by $\tilde{x}$ and

$$w(x)=u(\tilde{x})-u(x). \tag{1.5}$$

Our first result is to establish various maximal principles for the fractional tempered Laplacian operator $(\Delta+\lambda)^{\frac{\alpha}{2}}$.

Theorem 1.1 (Narrow region principle): *Assume that Ω is a bounded narrow region in Σ, such that it is contained in the region between T and T_Ω, where T_Ω is a hyper-plane that is parallel to T. Denote $d(\Omega):=\operatorname{dist}(T,T_\Omega)$. Suppose that $w\in\mathscr{L}_\alpha(\mathbb{R}^n)\cap C^{1,1}_{\rm loc}(\Omega)$ and is lower semi-continuous on $\overline{\Omega}$, and satisfies*

$$\begin{cases}(\Delta+\lambda)^{\frac{\alpha}{2}}w(x)-c(x)w(x)\le 0, & \text{at points } x\in\Omega \text{ where } w(x)<0,\\ w(x)\ge 0, & \text{in } \Sigma\backslash\Omega.\\ w(\tilde{x})=-w(x), & \text{in } \Sigma,\end{cases}$$

where $c(x)$ is uniformly bounded from below in $\{x\in\Omega\mid w(x)<0\}$. Then we have $w(x)\ge 0$ in Ω.

Furthermore, assume that

$$(\Delta+\lambda)^{\frac{\alpha}{2}}w(x)-c(x)w(x)\le 0, \quad \text{at points } x\in\Omega \text{ where } w(x)=0.$$

Then if $w=0$ at some point in Ω, we have $w=0$ almost everywhere in $\mathbb{R}^n$.

These conclusions hold for an unbounded open set Ω if we further assume that

$$\liminf_{|x|\to\infty} w(x)\ge 0.$$

Theorem 1.2 (Decay at infinity (I)): *Suppose* $0 \notin \Sigma$. *Let* Ω *be an unbounded open set in* Σ. *Assume* $w \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\rm loc}(\Omega)$ *and satisfies*

$$\begin{cases} (\Delta+\lambda)^{\frac{\alpha}{2}} w(x) - c(x)w(x) \le 0, & \textit{at points } x \in \Omega \textit{ where } w(x) < 0, \\ w(x) \ge 0, & \textit{in } \Sigma\backslash\Omega, \\ w(\tilde{x}) = -w(x), & \textit{in } \Sigma, \end{cases}$$

with

$$\liminf_{\substack{x\in\Sigma,\, w(x)<0 \\ |x|\to+\infty}} |x|^\alpha e^{\lambda|x|} c(x) \ge -\frac{c_{n,\alpha,\lambda}}{6},$$

where $c_{n,\alpha,\lambda}$ *is defined in* (2.11). *Then there exists a constant* $R_0 > 0$ *(depending only on* $c(x)$, μ, n *and* α, *but independent of* w *and* Σ*) such that, if* $\hat{x} \in \Omega$ *satisfying* $w(\hat{x}) = \min_{\overline{\Omega}} w(x) < 0$, *then* $|\hat{x}| \le R_0$.

Theorem 1.3 (maximum principles for anti-symmetric functions in unbounded domains): *Assume that* $w \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\rm loc}(\Sigma)$ *is bounded from below and* $w(x)$ *is an anti-symmetric function that is* $w(\tilde{x}) = -w(x)$ *in* Σ, *where* $\tilde{x}$ *is the reflection of* x *with respect to* T. *Suppose that, at any point* $x \in \Sigma$ *such that* $w(x) < 0$, w *satisfies*

$$(\Delta+\lambda)^{\frac{\alpha}{2}} w(x) - c(x)w(x) \le 0,$$

where $c(x)$ *satisfies* $\inf_{\{x\in\Sigma | w(x)<0\}} c(x) \ge 0$. *Then*

$$w(x) \ge 0, \quad \forall x \in \Sigma.$$

Furthermore, assume that

$$(\Delta+\lambda)^{\frac{\alpha}{2}} w(x) - c(x)w(x) \le 0, \quad \textit{at points } x \in \Sigma \textit{ where } w(x) = 0.$$

Then if $w = 0$ *at some point in* Σ, *we have* $w = 0$ *almost everywhere in* $\mathbb{R}^n$.

Remark 1.4: One can observe that we allow the function $c(x)$ in Theorem 1.1 and Theorem 1.2 to be negative; our assumptions differ from the corresponding assumptions for fractional Laplacian $(-\Delta)^s$ on Chen and Li [14]. From the proof of Theorem 1.3, we can derive the unbounded narrow region principle and decay at infinity II, which improved Theorem 1.1 and Theorem 1.2. We will not describe the content of the theorem here; refer to Section 2 for details.

The literature on maximum principles and the consequential qualitative properties of solutions (such as symmetry, monotonicity) is currently extensive. Different techniques have been developed to overcome technical difficulties arising from the particular nature of the operators under study. The most commonly used techniques are the moving plane method and the sliding method. Furthermore, the moving plane method can be traced back to the early 1950s. It was invented by Alexandroff to study surfaces with constant average curvature. An in-depth understanding of this method has become a potent tool for studying other fields, such as geometrical analysis, geometrical inequalities, conformal geometry, and PDEs. For more literature on moving plane (sphere) methods, refer to [19, 24, 25, 31–33] and the references therein. The sliding method developed by Berestycki and Nirenberg (see [1, 2]) provides a flexible alternative to approach symmetry and related issues. It has been adapted to the nonlocal setting in the earlier cited papers.

Applications of the direct method of moving planes. In Section 3, we will use several examples to illustrate how the key ingredients in the above can be used in the method of moving planes to establish symmetry, monotonicity of solutions to tempered fractional problems.

We first consider the **static nonlinear Schrödinger equations** involving the fractional tempered Laplacian operator:

$$-(\Delta+\lambda)^{\frac{\alpha}{2}}u(x)+V(x)u(x)=u^p(x),\quad \forall\, x\in\mathbb{R}^n, \tag{1.6}$$

where $V\in C^1(\mathbb{R}^n,\mathbb{R})$.

We will prove the following symmetry and monotonicity results for nonnegative solutions to (1.6) via the method of moving planes.

Theorem 1.5: *Assume that $u\in\mathscr{L}_\alpha(\mathbb{R}^n)\cap C^{1,1}_{\rm loc}(\mathbb{R}^n)$ is a nonnegative solution of (1.6) with $1<p<+\infty$. Suppose that potential $V(x)$ enjoys the following conditions:*

(i) *there exists $V_0>0$ such that $V(x)\geq V_0$ for all $x\in\mathbb{R}^n$;*
(ii) *$V(x)$ is a nonincreasing function, that is, for any $x_1<y_1$,*

$$V(x_1,\bar{x})\geq V(y_1,\bar{x}).$$

Moreover, if

$$\limsup_{|x|\to+\infty}u(x)=l<\Big(\frac{V_0}{p}\Big)^{\frac{1}{p-1}}, \tag{1.7}$$

then u must be radially symmetric and monotone decreasing about some point in $\mathbb{R}^n$.

Remark 1.6: In [14], Chen and Li obtained the radial symmetry and uniqueness of the solution for the equation $(-\Delta)^{\frac{\alpha}{2}}u(x)+u(x)=u^p(x)$; they required

$$\lim_{|x|\to+\infty} u(x)=l<\left(\frac{1}{p}\right)^{\frac{1}{p-1}}.$$

Similar to our proof process, their assumptions can be reduced to

$$\limsup_{|x|\to+\infty} u(x)=l<\left(\frac{1}{p}\right)^{\frac{1}{p-1}}.$$

Next, we consider the following tempered fractional Choquard equations:

$$-(\Delta+\lambda)^{\frac{\alpha}{2}}u(x)=\left(\frac{1}{|x|^\gamma}*u^{p_1}\right)u^{q_1},\quad \text{in } \mathbb{R}^n, \tag{1.8}$$

where $p_1, q_1\geq 1$. We prove

THEOREM 1.7: *Assume that $u\in\mathscr{L}_\alpha(\mathbb{R}^n)\cap C^{1,1}_{\text{loc}}(\mathbb{R}^n)$ is a nonnegative solution of problem* (1.8) *with the following property:*

$$\int_{\mathbb{R}^n}\frac{u^{p_1-1}(x)}{|x|^{2\gamma}}dx<+\infty\quad \text{and}\quad u(x)=o(\frac{1}{|x|^\gamma})\quad \text{as } |x|\to\infty. \tag{1.9}$$

Then u must be radially symmetric and monotone decreasing around some point in $\mathbb{R}^n$.

APPLICATIONS OF THE DIRECT SLIDING METHODS. In Section 4, we will extend the direct sliding methods to tempered fractional equations with gradient term:

$$-(\Delta+\lambda)^{\frac{\alpha}{2}}u(x)=f(x,u,\nabla u).$$

We shall establish monotonicity and uniqueness of solutions to the above equation in $\mathbb{R}^n$.

The sliding method was used to establish qualitative properties of solutions for partial differential equations (mainly involving the regular Laplacian) such as monotonicity and uniqueness. The main idea of sliding lies in comparing values of the solution for the equation at two different points, between which one point is obtained from the other by sliding the domain in a given direction, and then the domain is slid back to the limiting position. Unlike the previous work, our nonlinear term f in this section contains a gradient term. This brings us new challenges, and we need some new techniques and more careful analysis to deal with this problem. To this end, we give the following notations.

For any $x = (x', x_n)$ with $x' := (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $\tau \in \mathbb{R}$, let

$$x^\tau(x) := (x', x_n + \tau), \quad u^\tau(x) := u(x', x_n + \tau), \quad w^\tau(x) := u(x) - u^\tau(x).$$

For a bounded domain, we put $\Omega^\tau := \Omega - \tau e_n$ with $e_n = (0, \dots, 0, 1)$, which is obtained by sliding Ω downward τ units.

Our main result is as follows:

THEOREM 1.8: *Assume that $u \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^+_n) \cap C(\overline{\mathbb{R}^+_n})$ is a solution of*

$$\begin{cases} -(\Delta + \lambda)^{\frac{\alpha}{2}} u(x) = f(x, u, \nabla u), & \text{in } \mathbb{R}^n_+, \\ 0 < u(x) \le A, & \text{in } \mathbb{R}^n_+, \\ u(x) = 0, & \text{in } \mathbb{R}_n \setminus \mathbb{R}^n_+, \end{cases} \tag{1.10}$$

with $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) | x_n > 0\}$, $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}) = (p_1, \dots, p_n)$. Suppose that

$$\lim_{x \to +\infty} u(x', x_n) = A, \quad \text{uniformly for all } x' \in \mathbb{R}^{n-1}, \tag{1.11}$$

and the function $f(x, u, \nabla u)$ is bounded, Lipschitz continuous in all variables and satisfies

$$\begin{cases} f(x', x_n, u, p_1, p_2, \dots, p_n) \le f(x', x_n + \tau, u, p_1, p_2, \dots, p_n), \\ f \text{ is nonincreasing in } u \in [A - \delta, A] \text{ for some } \delta > 0. \end{cases} \tag{1.12}$$

Then u is strictly monotone increasing with respect to x_n. Moreover, u depends on x_n and uniqueness.

Remark 1.9: In [13], Chen and Weth develop a direct sliding method for problems involving $\triangle^s_p$ with $s \in (0, 1)$ and $p \ge 2$. However, their nonlinear term is $f(u)$. To the best of our knowledge, it is the first time the direct sliding methods have been applied to nonlocal tempered fractional problems.

Remark 1.10: Similar to our manuscript, one can also develop the direct moving planes and sliding methods for the following general fully nonlinear nonlocal operators:

$$F_{s,\lambda_f}(u)(x) := c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{G(u(x) - u(y))}{e^{\lambda f|x-y|}|x - y|^{n+2s}} dy, \tag{1.13}$$

where f is nondecreasing with respect to $|x-y|$, G is a local Lipschitz continuous function satisfying $G(0) = 0$ and u belongs to some appropriate function space.

This kind of operators were first introduced by Caffarelli and Silvestre in [5]. If $G(t) = |t|^{p-2}t$ and $\lambda_f \to 0$,

$$F_{s,\lambda_f} \to F_{s,0} = (-\Delta)^s_p.$$

If $G(t) = |t|^{p-2}t$ and $\lambda_f > 0$, we denote

$$F_{s,\lambda_f} := (-\Delta - \lambda_f)^s_p.$$

As a special case, when $G(t) = t$ and f is an identity map, F_{s,λ_f} degenerates into the general tempered fractional operators $-(\Delta+\lambda)^{\frac{\alpha}{2}}$. We leave these open problems to interested readers.

The paper is organized as follows. In Section 2, we establish various maximum principles for anti-symmetric functions involving tempered fractional operators in bounded or unbounded domains. As applications, in Section 3, after extending the direct method of moving planes, we show the symmetry and monotonicity of solutions to the tempered fractional problem, which are Theorem 1.5 and Theorem 1.7. Next, we devote ourselves to deriving Liouville-type results for solutions to the tempered fractional problem. In Section 4, we prove Theorem 1.8 via the direct sliding methods.

From now on and subsequently in the paper, we always use the same C to denote a constant whose value may be different from line to line, and only the relevant dependence is specified.

2. Various maximum principles

In this section, we shall establish various maximum principles in bounded and unbounded domains for anti-symmetric functions and for the tempered fractional operator, respectively. It is well-known that maximum principles are key ingredients in applying the method of moving planes. For the reader's convenience, we restate these theorems before their proofs.

We begin with the following:

Lemma 2.1: *Suppose that $w \in \mathscr{L}_\alpha(\mathbb{R}^n \setminus \{0\})$, $w(\tilde{x}) = -w(x)$ and $w \geq 0$ in Σ. If there exists $x_0 \in \Sigma$ such that $w(x_0)=0$, w is $C^{1,1}$ near x_0 and $(\Delta+\lambda)^{\frac{\alpha}{2}} w(x_0) \leq 0$, then $w = 0$ a.e. in $\mathbb{R}^n$.*

Proof. Since there exists $x_0 \in \Sigma$ such that $w(x_0) = \min_{x\in\Sigma} w(x) = 0$, one can infer from (1.2) that

$$\begin{aligned}
0 &\geq (\Delta+\lambda)^{\frac{\alpha}{2}} w(x_0)\\
&= -c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{w(x_0)-w(y)}{e^{\lambda|x_0-y|}|x_0-y|^{n+\alpha}} dy\\
&= c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{w(y)}{e^{\lambda|x_0-y|}|x_0-y|^{n+\alpha}} dy\\
&= -c_{n,\alpha} P.V. \int_{\Sigma} \Big(\frac{1}{e^{\lambda|x_0-\tilde{y}|}|x_0-\tilde{y}|^{n+\alpha}} - \frac{1}{e^{\lambda|x_0-y|}|x_0-y|^{n+\alpha}}\Big) w(y) dy\\
&\geq 0;
\end{aligned}$$

the last inequality holds because

$$|x_0-\tilde{y}| \geq |x_0-y|.$$

Thus we must have $w = 0$ a.e. in Σ and hence $w(x) = 0$ a.e. in $\mathbb{R}^n$. ∎

2.1. Maximum principles for anti-symmetric functions in bounded domain.

Theorem 2.2 (A maximum principle for anti-symmetric functions): *Let Ω be a bounded open set in Σ. Assume that the function $w \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{loc}(\Omega)$ in* (1.5) *and is lower semi-continuous on $\overline{\Omega}$. If*

$$\begin{cases} (\Delta+\lambda)^{\frac{\alpha}{2}} w(x) - c(x)w(x) \leq 0 & \text{at points } x \in \Omega \text{ where } w(x) < 0,\\ w(x) \geq 0, & \text{in } \Sigma\setminus\Omega,\\ w(\tilde{x}) = -w(x), & \text{in } \Sigma, \end{cases} \tag{2.1}$$

where $c(x) \geq 0$ for any $x \in \{x \in \Omega \mid w(x) < 0\}$, then $w(x) \geq 0$ in Ω.

Furthermore, assume that

$$(\Delta+\lambda)^{\frac{\alpha}{2}} w(x) - c(x)w(x) \leq 0, \quad \text{at points } x \in \Omega \text{ where } w(x) = 0. \tag{2.2}$$

Then if $w = 0$ at some point in Ω, we have $w = 0$ almost everywhere in $\mathbb{R}^n$.

These conclusions hold for an unbounded open set Ω if we further assume that $\liminf_{|x|\to\infty} w(x) \geq 0$.

Proof. If w is not nonnegative, then the lower semi-continuity of w on $\overline{\Omega}$ indicates that there exists a $\hat{x} \in \overline{\Omega}$ such that $w(\hat{x}) = \min_{\overline{\Omega}} w < 0$. One can further deduce from (2.1) that $\hat{x}$ is in the interior of Ω. It follows that

$$\begin{aligned}
&(\Delta+\lambda)^{\frac{\alpha}{2}} w(\hat{x}) - c(\hat{x})w(\hat{x}) \\
&\quad = -c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{w(\hat{x}) - w(y)}{e^{\lambda|\hat{x}-y|}|\hat{x} - y|^{n+\alpha}} dy - c(\hat{x})w(\hat{x}) \\
&\quad = -c_{n,\alpha} P.V. \int_{\Sigma} \frac{w(\hat{x}) - w(y)}{e^{\lambda|\hat{x}-y|}|\hat{x}-y|^{n+\alpha}} + \frac{w(\hat{x})+w(y)}{e^{\lambda|\hat{x}-y|}|\hat{x} - \tilde{y}|^{n+\alpha}} dy - c(\hat{x})w(\hat{x}) \\
&\quad \geq -c_{n,\alpha} \int_{\Sigma} \left\{ \frac{w(\hat{x}) - w(y)}{e^{\lambda|\hat{x}-y|}|\hat{x} - \tilde{y}|^{n+\alpha}} + \frac{w(\hat{x}) + w(y)}{e^{\lambda|\hat{x}-y|}|\hat{x} - \tilde{y}|^{n+\alpha}} \right\} dy \\
&\quad = -c_{n,\alpha} \int_{\Sigma} \frac{2w(\hat{x})}{e^{\lambda|\hat{x}-y|}|\hat{x} - \tilde{y}|^{n+\alpha}} dy > 0,
\end{aligned} \tag{2.3}$$

which contradicts (2.1). Hence $w(x) \geq 0$ in Ω.

Now we have proved that $w(x) \geq 0$ in Σ. If there is some point $\bar{x} \in \Omega$ such that $w(\bar{x}) = 0$, then from (2.2) and Lemma 2.1 we derive $w = 0$ almost everywhere in $\mathbb{R}^n$. This completes the proof of Theorem 2.2. ∎

THEOREM 2.3 (Narrow region principle): *Assume that Ω is a bounded narrow region in Σ, such that it is contained in the region between T and T_Ω, where T_Ω is a hyper-plane that is parallel to T. Denote $d(\Omega) := \operatorname{dist}(T, T_\Omega)$. Suppose that $w \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(\Omega)$ and is lower semi-continuous on $\overline{\Omega}$, Ω is narrow in the sense that $d(\Omega) < \min\{\frac{1}{2}, \lambda\}$, and satisfies*

$$\begin{cases} (\Delta + \lambda)^{\frac{\alpha}{2}} w(x) - c(x)w(x) \leq 0, & \text{at points } x \in \Omega \text{ where } w(x) < 0, \\ w(x) \geq 0, & \text{in } \Sigma \backslash \Omega, \\ w(\tilde{x}) = -w(x), & \text{in } \Sigma, \end{cases} \tag{2.4}$$

where $c(x)$ is uniformly bounded from below in $\{x \in \Omega \mid w(x) < 0\}$. Then we have $w(x) \geq 0$ in Ω.

Furthermore, assume that

$$(\Delta + \lambda)^{\frac{\alpha}{2}} w(x) - c(x)w(x) \leq 0, \quad \text{at points } x \in \Omega \text{ where } w(x) = 0. \tag{2.5}$$

Then if $w = 0$ at some point in Ω, we have $w = 0$ almost everywhere in $\mathbb{R}^n$.

These conclusions hold for an unbounded open set Ω if we further assume that

$$\liminf_{|x|\to\infty} w(x) \geq 0.$$

Proof. Without loss of generalities, we may assume that

$$T = \{x \in \mathbb{R}^n \mid x_1 = 0\} \quad \text{and} \quad \Sigma = \{x \in \mathbb{R}^n \mid x_1 < 0\},$$

and hence $\Omega \subseteq \{x \in \mathbb{R}^n \mid -d(\Omega) < x_1 < 0\}$.

If w is not nonnegative in Ω, then the lower semi-continuity of w on $\overline{\Omega}$ indicates that there exists a $\bar{x} \in \overline{\Omega}$ such that $w(\bar{x}) = \min_{\overline{\Omega}} w < 0$. One can further deduce from (2.4) that $\bar{x}$ is in the interior of Ω. A direct computation shows that

$$\begin{aligned}(\Delta+\lambda)^{\frac{\alpha}{2}} w(\bar{x}) &= -c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{w(\bar{x}) - w(y)}{e^{\lambda|\bar{x}-y|}|\bar{x} - y|^{n+\alpha}} dy \\ &= -c_{n,\alpha} P.V. \int_{\Sigma} \frac{w(\bar{x}) - w(y)}{e^{\lambda|\bar{x}-y|}|\bar{x} - y|^{n+\alpha}} + \frac{w(\bar{x}) + w(y)}{e^{\lambda|\bar{x}-\tilde{y}|}|\bar{x} - \tilde{y}|^{n+\alpha}} dy \\ &\geq -c_{n,\alpha} P.V. \int_{\Sigma} \left\{ \frac{w(\bar{x}) - w(y)}{e^{\lambda|\bar{x}-\tilde{y}|}|\bar{x} - \tilde{y}|^{n+\alpha}} + \frac{w(\bar{x}) + w(y)}{e^{\lambda|\bar{x}-\tilde{y}|}|\bar{x} - \tilde{y}|^{n+\alpha}} \right\} dy \\ &= -c_{n,\alpha} \int_{\Sigma} \frac{2w(\bar{x})}{e^{\lambda|\bar{x}-\tilde{y}|}|\bar{x} - \tilde{y}|^{n+\alpha}} dy. \end{aligned} \tag{2.6}$$

Let

$$D := \{y = (y_1, y') \in \mathbb{R}^N \mid d(\Omega) < y_1 - (\bar{x})_1 < 2d(\Omega), |y' - (\bar{x})'| < 1\}.$$

Denote

$$t := y_1 - (\bar{x})_1, \quad \tau := |y' - (\bar{x})'|.$$

Then we find $|\bar{x} - \tilde{y}| < 2$. Through direct calculations, we have

$$\begin{aligned}\int_{\Sigma} \frac{1}{e^{\lambda|\bar{x}-\tilde{y}|}|\bar{x} - \tilde{y}|^{n+\alpha}} dy &\geq \int_D \frac{1}{e^{\lambda|\bar{x}-y|}|\bar{x} - y|^{n+\alpha}} dy \\ &\geq \int_D \frac{1}{e^{2\lambda}|\bar{x} - y|^{n+\alpha}} dy \\ &= \int_{d(\Omega)}^{2d(\Omega)} \int_0^1 \frac{\omega_{n-1}\tau^{n-2} d\tau}{e^{2\lambda}(t^2 + \tau^2)^{\frac{n+\alpha}{2}}} dt \\ &= \int_{d(\Omega)}^{2d(\Omega)} \frac{1}{e^{2\lambda} t^{1+\alpha}} \int_0^{\frac{1}{t}} \frac{\omega_{n-1}\rho^{n-2} d\rho}{(1 + \rho^2)^{\frac{n+\alpha+1}{2}}} dt \\ &\geq \int_{d(\Omega)}^{2d(\Omega)} \frac{1}{e^{2\lambda} t^{1+\alpha}} \int_0^1 \frac{\omega_{n-1}\rho^{n-2} d\rho}{(1 + \rho^2)^{\frac{n+\alpha}{2}}} dt \\ &\geq C_{n,\alpha,\lambda} \int_{d(\Omega)}^{2d(\Omega)} \frac{1}{t^{1+\alpha}} dt = \frac{C_{n,\alpha,\lambda}}{d(\Omega)^{\alpha}}, \end{aligned} \tag{2.7}$$

where we have used the substitution $\rho := \tau/t$ and $\omega_{n-1} = |B_1(0)|$ in $\mathbb{R}^{n-1}$.

Since $c(x)$ is uniformly bounded from below (with respect to $d(\Omega)$) in $\{x \in \Omega \mid w(x) < 0\}$, we combine (2.6) and (2.7) and can infer that

$$(\Delta+\lambda)^{\frac{\alpha}{2}} w(\bar{x}) - c(\bar{x})w(\bar{x}) \geq \Big[\frac{-C_{n,\alpha,\lambda}}{d(\Omega)^{\alpha}} - \inf_{\{x\in\Omega \mid w(x)<0\}} c(x)\Big] w(\bar{x}) > 0,$$

as $d(\Omega)$ is sufficiently small, which contradicts (2.4).

Now we have proved that $w(x) \geq 0$ in Σ. If there is some point $\bar{x} \in \Omega$ such that $w(\bar{x}) = 0$, then from (2.5) and Lemma 2.1 we derive $w = 0$ almost everywhere in $\mathbb{R}^n$. This finishes the proof of Theorem 2.3. ∎

2.2. Maximum principles for anti-symmetric functions in unbounded domain.

Theorem 2.4 (Decay at infinity (I)): *Suppose* $0 \notin \Sigma$. *Let* Ω *be an unbounded open set in* Σ. *Assume* $w \in \mathscr{L}_{\alpha}(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(\Omega)$ *and satisfies*

$$\begin{cases} (\Delta+\lambda)^{\frac{\alpha}{2}} w(x) - c(x)w(x) \leq 0, & \text{at points } x \in \Omega \text{ where } w(x) < 0, \\ w(x) \geq 0, & \text{in } \Sigma\backslash\Omega, \\ w(\tilde{x}) = -w(x), & \text{in } \Sigma, \end{cases} \tag{2.8}$$

with

$$\liminf_{\substack{x\in\Sigma,\, w(x)<0 \\ |x|\to+\infty}} |x|^{\alpha} e^{\lambda|x|} c(x) \geq -\frac{c_{n,\alpha,\lambda}}{6}, \tag{2.9}$$

where $c_{n,\alpha,\lambda}$ *is defined in* (2.11). *Then there exists a constant* $R_0 > 0$ *(depending only on* $c(x)$, μ, n *and* α, *but independent of* w *and* Σ*) such that, if* $\hat{x} \in \Omega$ *satisfying* $w(\hat{x}) = \min_{\overline{\Omega}} w(x) < 0$, *then* $|\hat{x}| \leq R_0$.

Proof. Without loss of generalities, we may assume that, for some $\lambda \leq 0$,

$$T = \{x \in \mathbb{R}^n \mid x_1 = \lambda\} \quad \text{and} \quad \Sigma = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}.$$

Since $w \in \mathscr{L}_{\alpha} \cap C^{1,1}_{\text{loc}}(\Omega)$ and $\hat{x} \in \Omega$ satisfying $w(\hat{x}) = \min_{\overline{\Omega}} w(x) < 0$, using similar calculations as (2.3), we get

$$(\Delta+\lambda)^{\frac{\alpha}{2}} w(\hat{x}) \geq -c_{n,\alpha,\lambda} \int_{\Sigma} \frac{2w(\bar{x})}{e^{\lambda|\bar{x}-\tilde{y}|}|\bar{x}-\tilde{y}|^{n+\alpha}} dy. \tag{2.10}$$

Note that $\lambda \leq 0$ and $\hat{x} \in \Omega$; it follows that

$$B_{|\hat{x}|}(\bar{x}) \subset \{x \in \mathbb{R}^n | x_1 > \lambda\},$$

where $\bar{x} := (2|\hat{x}| + (\hat{x})_1, (\hat{x})')$. Thus we derive that

$$\begin{aligned}\int_\Sigma \frac{1}{e^{\lambda|\bar{x}-\tilde{y}|}|\hat{x}-\tilde{y}|^{n+\alpha}}dy &\geq \int_{B_{|\hat{x}|}(\bar{x})} \frac{1}{e^{\lambda|\bar{x}-y|}|\hat{x}-y|^{n+\alpha}}dy \\ &\geq \int_{B_{|\hat{x}|}(\bar{x})} \frac{1}{e^{\lambda|\hat{x}|}|\hat{x}-y|^{n+\alpha}}dy \\ &\geq \int_{B_{|\hat{x}|}(\bar{x})} \frac{1}{3^{n+\alpha+1}e^{\lambda|\hat{x}|}|\hat{x}|^{n+\alpha}}dy \\ &\geq \frac{\omega_n}{3^{n+\alpha+1}e^{\lambda|\hat{x}|}|\hat{x}|^{\alpha}} =: \frac{c_{n,\alpha,\lambda}}{e^{\lambda|\hat{x}|}|\hat{x}|^{\alpha}},\end{aligned} \tag{2.11}$$

where $\omega_n := |B_1(0)|$ denotes the volume of the unit ball in $\mathbb{R}^n$. Then we can deduce from (2.8), (2.10) and (2.11) that

$$0 \geq (\Delta+\lambda)^{\frac{\alpha}{2}}w(\hat{x}) - c(\hat{x})w(\hat{x}) \geq \Big[-\frac{c_{n,\alpha,\lambda}}{e^{\lambda|\hat{x}|}|\hat{x}|^{\alpha}} - c(\hat{x})\Big]w(\hat{x}). \tag{2.12}$$

It follows from $w(\hat{x}) < 0$ and (2.12) that

$$|\hat{x}|^{\alpha}e^{\lambda|\hat{x}|}c(\hat{x}) \leq -c_{n,\alpha,\lambda}. \tag{2.13}$$

From (2.9), we infer that there exists an R_0 sufficiently large such that, for any $|x| > R_0$,

$$e^{\lambda|\hat{x}|}|\hat{x}|^{\alpha}c(\hat{x}) > -\frac{c_{n,\alpha,\lambda}}{3}. \tag{2.14}$$

Combining (2.13) and (2.14), we arrive at $|\hat{x}| \leq R_0$. Therefore, we must have $|\hat{x}| \leq R_0$. ∎

THEOREM 2.5 (Maximum principles for anti-symmetric functions in unbounded domains): *Assume that $w \in \mathscr{L}_\alpha(\mathbb{R}^n)\cap C^{1,1}_{\mathrm{loc}}(\Sigma)$ is bounded from below and $w(x)$ is an anti-symmetric function that is $w(\tilde{x}) = -w(x)$ in Σ, where $\tilde{x}$ is the reflection of x with respect to T. Suppose that, at any points $x \in \Sigma$ such that $w(x) > 0$, w satisfies*

$$(\Delta+\lambda)^{\frac{\alpha}{2}}w(x) - c(x)w(x) \leq 0, \tag{2.15}$$

where $c(x)$ satisfies

$$\inf_{\{x\in\Sigma | w(x)<0\}} c(x) > 0. \tag{2.16}$$

Then

$$w(x) \geq 0, \quad \forall x \in \Sigma. \tag{2.17}$$

Furthermore, assume that

$$(\Delta+\lambda)^{\frac{\alpha}{2}}w(x)-c(x)w(x)\leq 0, \quad \text{at points } x\in\Sigma \text{ where } w(x)=0. \tag{2.18}$$

Then if $w=0$ at some point in Σ, we have $w=0$ almost everywhere in $\mathbb{R}^n$.

Proof. Suppose that (2.17) is false. Since w is bounded from below, we have

$$M:=\inf_{\Sigma} w(x)<0.$$

Hence, there exist sequences $x^k\in\Sigma$ and $0<\beta_k<1$ with $\beta_k\to 1$ as $k\to\infty$ such that

$$w(x^k)\leq \beta_k M. \tag{2.19}$$

We may assume that

$$T=\{x\in\mathbb{R}^n|x_1=0\}, \quad \Sigma=\{x\in\mathbb{R}^n|x_1<0\}.$$

Then $\tilde{x}=(-x_1,x_2,\ldots,x_n)$. We denote $d_k:=\frac{1}{3}\text{dist}(x^k,T)$. Let

$$\psi(x)=\begin{cases} e^{\frac{|x|^2}{|x|^2-1}}, & |x|<1,\\ 0, & |x|\geq 1.\end{cases}$$

It is well known that $\psi\in C_0^\infty(\mathbb{R}^n)$, thus $|(\Delta+\lambda)^{\frac{\alpha}{2}}\psi(x)|\leq C$ for all $x\in\mathbb{R}^n$. Moreover, $(\Delta+\lambda)^{\frac{\alpha}{2}}\psi(x)$ is monotone decreasing with respect to $|x|$.

Set

$$\psi_k(x):=\psi\Big(\frac{x-\widetilde{(x^k)}}{d_k}\Big) \quad \text{and} \quad \tilde{\psi_k}(x)=\psi_k(\tilde{x})=\psi\Big(\frac{x-x^k}{d_k}\Big).$$

Then $\tilde{\psi_k}-\psi_k$ is anti-symmetric with respect to T, which means that

$$(\tilde{\psi_k}-\psi_k)(\tilde{x})=-(\tilde{\psi_k}-\psi_k)(x).$$

Now pick $\varepsilon_k=-(1-\beta_k)M$, then $w(x^k)-\varepsilon_k[\tilde{\psi_k}-\psi_k](x^k)\leq M$. We denote

$$w_k(x):=w(x)-\varepsilon_k[\tilde{\psi_k}-\psi_k](x).$$

Then w_k is also anti-symmetric with respect to T.

Note that for any $x\in\Sigma\setminus B_{d_k}(x^k)$, $w(x)\geq M$ and $\tilde{\psi_k}(x)=\psi_k(x)=0$. From the definition of $w_k(x)$, we also have

$$w_k(x^k)\leq M\leq w_k(x), \quad \forall\, x\in\Sigma\setminus B_{d_k}(x^k).$$

Hence the infimum of $w_k(x)$ in Σ is achieved in $B_{d_k}(x^k)$. Consequently, there exists a point $\overline{x}^k \in B_{d_k}(x^k)$ such that

$$w_k(\overline{x}^k) = \inf_{x\in\Sigma} w_k(x) \leq M. \tag{2.20}$$

By the choice of ε_k, it is easy to verify that $w(\bar{x}^k) \leq \beta_k M < 0$.

Next, we will evaluate the upper and lower bounds of $(\Delta+\lambda)^{\frac{\alpha}{2}} w_k(\bar{x}^k)$.

Since $\overline{x}^k \in B_{d_k}(x^k)$ and by the definition of d_k, we have $2d_k \leq |\overline{x}^k|$. As a consequence of (2.15), we obtain

$$\begin{aligned}
(\Delta+\lambda)^{\frac{\alpha}{2}} w_k(\bar{x}^k) &= (\Delta+\lambda)^{\frac{\alpha}{2}}\{w - \varepsilon_k[\tilde{\psi_k} - \psi_k]\}(\bar{x}^k)\\
&= (\Delta+\lambda)^{\frac{\alpha}{2}} w(\bar{x}^k) - (\Delta+\lambda)^{\frac{\alpha}{2}}\{\varepsilon_k[\tilde{\psi_k} - \psi_k]\}(\bar{x}^k)\\
&\leq -c(\bar{x}^k)w(\bar{x}^k) - (\Delta+\lambda)^{\frac{\alpha}{2}}\{\varepsilon_k[\tilde{\psi_k} - \psi_k]\}(\bar{x}^k)\\
&\leq -c(\bar{x}^k)w(\bar{x}^k) + \varepsilon_k|(\Delta+\lambda)^{\frac{\alpha}{2}}\{[\tilde{\psi_k} - \psi_k]\}(\bar{x}^k)|\\
&\leq -c(\bar{x}^k)w(\bar{x}^k) + \frac{2C\varepsilon_k}{d_k^\alpha}\\
&\leq \frac{2C\varepsilon_k}{d_k^\alpha},
\end{aligned} \tag{2.21}$$

where we have used the fact that

$$\begin{aligned}
|(\Delta+\lambda)^{\frac{\alpha}{2}}\psi_k(x)| =& \left| -c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{\psi_k(x)-\psi_k(y)}{e^{\lambda|x-y|}|x-y|^{n+\alpha}} dy \right|\\
=& \left| -c_{n,\alpha} P.V. \int_{B_{d_k}(x)} \frac{\psi_k(x)-\psi_k(y)}{e^{\lambda|x-y|}|x-y|^{n+\alpha}} dy \right|\\
&+ \left| -c_{n,\alpha} P.V. \int_{B^c_{d_k}(x)} \frac{\psi_k(x)-\psi_k(y)}{e^{\lambda|x-y|}|x-y|^{n+\alpha}} dy \right|\\
\leq& \left| \int_{B_{d_k}(x)} \frac{2c_{n,\alpha}||\psi||_{C^{1,1}(\mathbb{R}^n)}|\frac{x}{d_k} - \frac{y}{d_k}|^2}{|x-y|^{n+\alpha}} dy \right|\\
&+ \left| \int_{B^c_{d_k}(x)} \frac{c_{n,\alpha}}{e^{\lambda d_k}|x-y|^{n+\alpha}} dy \right|\\
\leq& \frac{C}{d_k^\alpha} + \frac{C}{d_k^\alpha e^{\lambda d_k}} \leq \frac{C}{d_k^\alpha}.
\end{aligned} \tag{2.22}$$

On the other hand, we derive the following:

$$\begin{aligned}
&(\Delta+\lambda)^{\frac{\alpha}{2}} w_k(\bar{x}^k)\\
=&-c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{w_k(\bar{x}^k)-w_k(y)}{e^{\lambda|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}} dy\\
=&-c_{n,\alpha} P.V. \int_{\Sigma} \Big[\frac{w_k(\bar{x}^k)-w_k(y)}{e^{\lambda|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}+\frac{w_k(\bar{x}^k)+w_k(y)}{e^{\lambda|\bar{x}^k-y|}|\bar{x}^k-\tilde{y}|^{n+\alpha}}\Big] dy\\
=&-c_{n,\alpha}\int_{\Sigma}\Big(\frac{1}{e^{\lambda|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}-\frac{1}{e^{\lambda|\bar{x}^k-\tilde{y}|}|\bar{x}^k-\tilde{y}|^{n+\alpha}}\Big)(w_k(\bar{x}^k)-w_k(y))dy\\
&-2c_{n,\alpha} w_k(\bar{x}^k)\int_{\Sigma}\frac{1}{e^{\lambda|\bar{x}^k-\tilde{y}|}|\bar{x}^k-\tilde{y}|^{n+\alpha}}dy\\
\geq&-2c_{n,\alpha} w_k(\bar{x}^k)\int_{\Sigma}\frac{1}{e^{\lambda|\bar{x}^k-\tilde{y}|}|\bar{x}^k-\tilde{y}|^{n+\alpha}}dy\\
\geq&-C_{n,\alpha,\lambda}\int_{H_k}\frac{w_k(\bar{x}^k)}{e^{\lambda d_k}|\bar{x}^k-\tilde{y}|^{n+\alpha}}dy\\
\geq&-\frac{C_{n,\alpha,\lambda} w_k(\bar{x}^k)}{e^{\lambda d_k} d_k^{\alpha}},
\end{aligned} \tag{2.23}$$

where $H_k := \{x=(x_1,x')\in\mathbb{R}^n \mid -\frac{d_k}{2}<x_1<0, |x'-(\bar{x}^k)'|<\frac{\sqrt{3}d_k}{2}\}$. Here we also used the following facts: $w_k(\bar{x}^k)\leq w_k(y), \forall\, y\in\Sigma$, which can be seen directly.

Next, we will carry out our proof by discussing two different cases and derive contradictions in both of these two cases.

Case (i): There exists a k_1 such that $d_k \geq k_1$ for all k. Collecting (2.21) and (2.23), we have

$$0<(\Delta+\lambda)^{\frac{\alpha}{2}} w_k(\bar{x}^k)\leq -c(\bar{x}^k)w(\bar{x}^k)+C\varepsilon_k\leq -c(\bar{x}^k)w(\bar{x}^k),$$

which implies $w(\bar{x}^k)\to 0$ as $k\to+\infty$, that is impossible.

Case (ii): Up to a subsequence (still denote by d_k) such that $d_k<\lambda$ for all k. In particular, in this case, we need only assume that

$$\inf_{\{x\in\Sigma|w(x)<0\}} c(x)\geq 0.$$

Combining (2.21) and (2.23), we derive

$$-\frac{C_{n,\alpha,\lambda} w_k(\bar{x}^k)}{e^{\lambda d_k} d_k^{\alpha}}\leq \frac{2C\varepsilon_k}{d_k^{\alpha} e^{\lambda d_k}}. \tag{2.24}$$

Noticing that $\varepsilon_k = -(1-\beta_k)M$, the above equation is equivalent to

$$C_{n,\alpha,\lambda} \leq 2C(1-\beta_k)e^{\lambda^2} \leq C(1-\beta_k), \tag{2.25}$$

which also will lead to a contradiction as $k \to +\infty$.

Now we have proved that $w(x) \leq 0$ in Σ. If there is some point $\bar{x} \in \Omega$ such that $w(\bar{x}) = 0$, then from (2.18) we derive immediately that $w = 0$ almost everywhere in $\mathbb{R}^n$. This concludes our proof of Theorem 2.5. ∎

Similar to Theorem 2.5, we can deduce the following unbounded narrow region principle and decay at infinity II. It should be mentioned that in Subsection 2.1, we assumed that the solution tends to zero near infinity. But in the following theorem, we deal with the situation where the asymptotic decay of the solutions is not assumed.

Theorem 2.6 (Unbounded narrow region principle): *Let $\Omega \subseteq \Sigma$ be an open set (possibly unbounded and disconnected) such that it is contained in the region between T and T_Ω, where T_Ω is a hyper-plane that is parallel to T. Suppose that $w \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ is bounded from below and satisfies*

$$\begin{cases} (\Delta+\lambda)^{\frac{\alpha}{2}} w(x) - c(x)w(x) \leq 0, & \text{at points } x \in \Omega \text{ where } w(x) < 0, \\ w(x) \geq 0, & \text{in } \Sigma\backslash\Omega, \\ w(\tilde{x}) = -w(x), & \text{in } \Sigma, \end{cases} \tag{2.26}$$

where $c(x)$ is uniformly bounded from below (with respect to $d(\Omega)$) in $\{x \in \Omega \mid w(x) < 0\}$. If we assume that

$$\inf_{\{x\in\Omega \,|\, w(x)<0\}} c(x) > -\frac{3^\alpha C_{n,\alpha,\lambda}}{e^{\frac{\lambda d(\Omega)}{3}} d(\Omega)^\alpha}, \tag{2.27}$$

where $C_{n,\alpha,\lambda}$ is the same as in (2.23),

$$d(\Omega) := \operatorname{dist}(T, T_\Omega) < 3\lambda.$$

Then, we have $w(x) \geq 0$ in Ω. Furthermore, assume that

$$(\Delta+\lambda)^{\frac{\alpha}{2}} w(x) - c(x)w(x) \leq 0, \quad \text{at points } x \in \Omega \text{ where } w(x) = 0. \tag{2.28}$$

If $w = 0$ at some point in Ω, then we have $w = 0$ almost everywhere in $\mathbb{R}^n$.

Proof. Theorem 2.6 can be proved via similar arguments as in the proof of Case (ii) in Theorem 2.5, we only mention some key ingredients. Indeed, it is easy to see that $d_k < \frac{1}{3}d(\Omega) < \lambda$. Collecting (2.21) and (2.23) we get

$$(\inf_{\{x\in\Omega|w(x)<0\}} c(x))w_k(\bar{x}^k) + \frac{2C\varepsilon_k}{d_k^\alpha} \geq \frac{2C\varepsilon_k}{d_k^\alpha} + c(\bar{x}^k)w_k(\bar{x}^k) \geq -\frac{C_{n,\alpha,\lambda}w_k(\bar{x}^k)}{e^{\lambda d_k}d_k^\alpha}. \tag{2.29}$$

We can choose k sufficiently large such that $\beta_k > 1 - \frac{C_{n,\alpha,\lambda}}{4e^{\lambda^2}C}$. Recalling that $\varepsilon_k = -(1-\beta_k)M$ and $w_k(\bar{x}^k) \leq M$, we have

$$\frac{2C\varepsilon_k}{d_k^\alpha} \leq -\frac{C_{n,\alpha,\lambda}w_k(\bar{x}^k)}{2d_k^\alpha e^{\lambda^2}} \leq -\frac{C_{n,\alpha,\lambda}w_k(\bar{x}^k)}{2d_k^\alpha e^{\lambda d_k}}.$$

Next, one can infer from (2.29) and $3d_k \leq d(\Omega)$ that

$$\inf_{\{x\in\Omega|w(x)<0\}} c(x) \leq -\frac{3^\alpha C_{n,\alpha,\lambda}}{e^{\frac{\lambda d(\Omega)}{3}}d(\Omega)^\alpha},$$

which contradicts (2.27). This completes the proof of Theorem 2.6. ∎

Remark 2.7: In Theorem 2.6, we do not need the additional assumption $\liminf_{|x|\to+\infty} w(x) \geq 0$. Hence, when $\lambda \to 0^+$, Theorem 2.6 extends the narrow region principle of the fractional Laplace equation in Chen and Li [14] to unbounded regions under some weak assumptions.

THEOREM 2.8 (Decay at infinity (II)): *Let Ω be an unbounded open set in Σ. Suppose that $w \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\Omega\setminus\{0\})$ is bounded from below and satisfies*

$$\begin{cases} (\Delta+\lambda)^{\frac{\alpha}{2}}w(x) - c(x)w(x) \leq 0, & \text{at points } x\in\Omega \text{ where } w(x)<0,\\ w(x)\geq 0, & \text{in } \Sigma\backslash\Omega,\\ w(\tilde{x}) = -w(x), & \text{in } \Sigma, \end{cases} \tag{2.30}$$

with

$$\liminf_{\substack{x\in\Omega,\, w(x)<0\\ |x|\to\infty}} |x|^{2s}c(x) > -\frac{C_{n,\alpha,\lambda}}{2^{\alpha+2}|\bar{x}^k|^\alpha e^{\lambda|\bar{x}^k|}}, \tag{2.31}$$

where the constant $C_{n,\alpha,\lambda}$ is the same as in (2.23). Then there exists an $R_0>0$ large enough and $\beta_0\in(0,1)$ close enough to 1 (R_0 and β_0 are independent of w and Σ) such that, if $\hat{x}\in\Omega$ satisfying

$$w(\hat{x}) \leq \beta_0 \inf_\Omega w(x) < 0, \tag{2.32}$$

then $|\hat{x}| \leq R_0$.

Proof. Theorem 2.8 can be proved via similar contradiction arguments as for Theorem 2.5; we only mention some key ingredients. Suppose on the contrary that there exist sequences $\{x^k\} \in \Omega$ and $\{\beta_k\} \in (0,1)$ such that

$$|x^k| \to +\infty, \quad \beta_k \to 1, \quad \text{and} \quad w(x^k) \leq \beta_k \inf_{\Omega} w(x) < 0. \tag{2.33}$$

One can infer from (2.21) and (2.23) that

$$\frac{2C\varepsilon_k}{d_k^{\alpha+1}} + c(\bar{x}^k) w_k(\bar{x}^k) \geq -\frac{C_{n,\alpha,\lambda} w_k(\bar{x}^k)}{d_k^{\alpha} e^{\lambda d_k}}. \tag{2.34}$$

Now we take k sufficiently large such that

$$\beta_k > 1 - \frac{C_{n,\alpha,\lambda}}{4C}.$$

Note that $d_k := \frac{1}{3}\mathrm{dist}(x^k, T)$ and $\bar{x}^k \in B_{d_k}(x^k)$. We can infer from (2.34) that, for k large enough,

$$c(\bar{x}^k) \leq -\frac{C_{n,\alpha,\lambda}}{2 d_k^{\alpha} e^{\lambda d_k}} \leq -\frac{C_{n,\alpha,\lambda}}{2^{\alpha+2} |\bar{x}^k|^{\alpha} e^{\lambda |\bar{x}^k|}},$$

which contradicts (2.31) if we let $k \to +\infty$. ∎

Remark 2.9: From the proof of Theorem 2.8, we can see that $\hat{x}$ does not need to be the minimum point of w, only that $\hat{x}$ satisfies (2.32). Therefore, we say decay at infinity (II) Theorem 2.8 improves decay at infinity (I) Theorem 2.4.

3. A Direct method of moving planes for a tempered fractional Laplacian

In this section, applying various maximum principles for anti-symmetric functions established in Section 2, we will extend the direct method of moving planes to investigate symmetry and monotonicity of solutions to various problems involving the tempered fractional Laplacian operator $-(\Delta + \lambda)^{\frac{\alpha}{2}}$.

3.1. Schrödinger equations in $\mathbb{R}^n$.

Proof of Theorem 1.5. Choose an arbitrary direction to be the x_1-direction. In order to apply the method of moving planes, we need some notations. For arbitrary $\lambda \in \mathbb{R}$, let

$$T_\lambda := \{x \in \mathbb{R}^n | x_1 = \lambda\}$$

be the moving planes,

$$\Sigma_\lambda := \{x \in \mathbb{R}^n | x_1 < \lambda\} \tag{3.1}$$

be the region to the left of the plane, and

$$x^\lambda := (2\lambda - x_1, x_2, \dots, x_n)$$

be the reflection of x about the plane T_λ.

Assume that u is a nonnegative solution of the Schrödinger equations (1.6). To compare the values of $u(x)$ with $u(x^\lambda)$, we define

$$w_\lambda(x) := u(x^\lambda) - u(x), \quad \forall\, x \in \Sigma_\lambda.$$

Then, for any $\lambda \in \mathbb{R}$, at points $x \in \Sigma_\lambda$ where $w_\lambda(x) < 0$, we have

$$\begin{aligned}(\Delta + \lambda)^{\frac{\alpha}{2}} w_\lambda(x) &= u^p(x) - u^p(x^\lambda) + V(x^\lambda)u(x^\lambda) - V(x)u(x) \\ &\leq -pu^{p-1} w_\lambda(x) + V(x) w_\lambda(x) \\ &\leq -pu^{p-1} w_\lambda(x) + V_0 w_\lambda(x) \\ &=: c(x) w_\lambda(x),\end{aligned} \tag{3.2}$$

where $c(x) := V_0 - pu^{p-1}(x)$. From the assumption (1.7), we infer that, for any $\lambda \in \mathbb{R}$,

$$\liminf_{\substack{x \in \Sigma_\lambda,\, w_\lambda(x) < 0 \\ |x| \to +\infty}} c(x) > 0. \tag{3.3}$$

We carry out the moving planes procedure in two steps.

Step 1. We use Theorem 2.8 (decay at infinity (II)) to show that, for sufficiently negative λ,

$$w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda. \tag{3.4}$$

In fact, from assumption (1.7), we know that u is bounded from above and hence w_λ is bounded from below for any $\lambda \in \mathbb{R}$. Suppose that $\inf_{\Sigma_\lambda} w_\lambda < 0$. By (3.2) and (3.3), we can deduce from Theorem 2.8 (decay at infinity (II)) that there exist $R_0 > 0$ large and $0 < \gamma_0 < 1$ close to 1 (independent of λ) such that, if $\hat{x} \in \Sigma_\lambda$ satisfying $w_\lambda(\hat{x}) \leq \gamma_0 \inf_{\Sigma_\lambda} w_\lambda < 0$, then $|\hat{x}| \leq R_0$. This will lead to a contradiction provided that $\lambda \leq -R_0$. Thus we have, for any $\lambda \leq -R_0$,

$$w_\lambda \geq 0 \quad \text{in } \Sigma_\lambda.$$

Step 2. Step 1 provides a starting point, from which we can now move the plane T_λ to the right, to its limiting position, as long as (3.4) holds.

To this end, let us define

$$\lambda_0 := \sup\{\lambda \in \mathbb{R} \mid w_\mu \geq 0 \text{ in } \Sigma_\mu,\ \forall\, \mu \leq \lambda\}. \tag{3.5}$$

It follows from Step 1 that $-R_0 \leq \lambda_0 < +\infty$. One can easily verify that

$$w_{\lambda_0}(x) \geq 0, \quad \forall\, x \in \Sigma_{\lambda_0}. \tag{3.6}$$

Next, we show via contradiction arguments that

$$w_{\lambda_0}(x) \equiv 0, \quad \forall\, x \in \Sigma_{\lambda_0}. \tag{3.7}$$

Suppose on the contrary that

$$w_{\lambda_0} \geq 0 \text{ but } w_{\lambda_0} \not\equiv 0 \quad \text{in } \Sigma_{\lambda_0}; \tag{3.8}$$

then we must have

$$w_{\lambda_0}(x) > 0, \quad \forall\, x \in \Sigma_{\lambda_0}. \tag{3.9}$$

In fact, if (3.9) is violated, then there exists a point $\hat{x} \in \Sigma_{\lambda_0}$ such that

$$w_{\lambda_0}(\hat{x}) = \min_{\Sigma_{\lambda_0}} w_{\lambda_0} = 0.$$

Then it follows from (1.6) that

$$(\Delta + \lambda)^{\frac{\alpha}{2}} w_{\lambda_0}(\hat{x}) \leq 0, \tag{3.10}$$

and hence Lemma 2.1 implies that $w_{\lambda_0} \equiv 0$ in Σ_{λ_0}, which contradicts (3.8). Thus $w_{\lambda_0}(x) > 0$ in Σ_{λ_0}.

Then we show that the plane T_λ can be moved a little bit further from T_{λ_0} to the right. More precisely, there exists an $\delta > 0$, such that for any $\lambda \in [\lambda_0, \lambda_0+\delta]$, we have

$$w_\lambda(x) \geq 0, \quad \forall\, x \in \Sigma_\lambda. \tag{3.11}$$

In fact, (3.11) can be achieved by using the narrow region principle (Theorem 2.3) and the decay at infinity (II) (Theorem 2.8). First, since

$$c(x) := V_0 - pu^{p-1}(x) \quad \text{is uniformly bounded,}$$

we can choose $\delta_1 > 0$ small enough such that $(\Sigma_{\lambda_0+\delta_1} \setminus \overline{\Sigma_{\lambda_0-\delta_1}}) \cap B_{R_*}(0)$ is a narrow region, where $R_* := R_0 + |\lambda_0| \geq R_0$ with R_0 given by decay at infinity (II) (Theorem 2.8). From (3.9), we deduce that there exists a $c_0 > 0$ such that

$$w_{\lambda_0}(x) \geq c_0 > 0, \quad \forall\, x \in \overline{\Sigma_{\lambda_0-\delta_1} \cap B_{R_*}(0)}. \tag{3.12}$$

As a consequence, due to the continuity of w_λ with respect to λ, there exists a $0 < \delta_2 < \delta_1$ sufficiently small such that, for any $\lambda \in [\lambda_0, \lambda_0 + \delta_2]$,

$$w_\lambda(x) > 0, \quad \forall\, x \in \overline{\Sigma_{\lambda_0 - \delta_1} \cap B_{R_*}(0)}. \tag{3.13}$$

For any $\lambda \in [\lambda_0, \lambda_0 + \delta_2]$, if we suppose that $\inf_{\Sigma_\lambda} w_\lambda(x) < 0$, then the decay at infinity (II) (Theorem 2.8) implies that

$$w_\lambda(x) > \gamma_0 \inf_{\Sigma_\lambda} w_\lambda(x), \quad \forall\, x \in \Sigma_\lambda \setminus \overline{B_{R_0}(0)},$$

and hence the negative minimum $\inf_{\Sigma_\lambda} w_\lambda(x)$ can be attained in $B_{R_0}(0) \cap \Sigma_\lambda$. Then, from (3.13), we infer that, if $\inf_{\Sigma_\lambda} w_\lambda(x) < 0$, then the negative minimum $\inf_{\Sigma_\lambda} w_\lambda(x)$ can be attained in the narrow region $(\Sigma_\lambda \setminus \overline{\Sigma_{\lambda_0 - \delta_1}}) \cap B_{R_*}(0)$. Therefore, from the narrow region principle (Theorem 2.3), we get, for any $\lambda \in [\lambda_0, \lambda_0 + \delta_2]$,

$$w_\lambda(x) > 0, \quad \forall\, x \in (\Sigma_\lambda \setminus \overline{\Sigma_{\lambda_0 - \delta_1}}) \cap B_{R_*}(0), \tag{3.14}$$

and hence

$$w_\lambda(x) \geq 0, \quad \forall\, x \in \Sigma_\lambda. \tag{3.15}$$

Thus (3.11) holds, which contradicts the definition (3.5) of λ_0. Hence (3.7) must be valid.

The arbitrariness of the x_1-direction leads to the radial symmetry and monotonicity of $u(x)$ about some point $x_0 \in \mathbb{R}^n$. This completes the proof of Theorem 1.5. ∎

3.2. Tempered fractional Choquard equations. In this subsection, by applying the direct method of moving planes and giving some new integral estimates, we prove the following theorem on symmetry and monotonicity of nonnegative solutions to problem (1.8), which is Theorem 1.7.

There is a large amount of literature on the qualitative properties of solutions to Choquard type equations involving fractional Laplacians, or other nonlocal operators; refer to [7, 22, 31] and references therein for more details.

Proof of Theorem 1.7. Choose an arbitrary direction to be the x_1-direction. In order to apply the direct method of moving planes along the x_1-axis, we need some notations. For any $\lambda \in \mathbb{R}$, let T_λ, Σ_λ, x^λ and w_λ be defined the same as in subsection 3.1. Define $u_\lambda(x) := u(x^\lambda)$. Set

$$\Sigma_\lambda^- := \{x \in \Sigma_\lambda \mid w_\lambda(x) < 0\}.$$

Since the assumption (1.9) implies that u is bounded, so w_λ is also bounded. For $x \in \Sigma_\lambda^-$, a direct computation shows that

$$
\begin{aligned}
-(\Delta+\lambda)^{\frac{\alpha}{2}} w_\lambda(x) =& \Big(\frac{1}{|x|^\gamma} * u^{p_1}\Big)(x^\lambda) u_\lambda^{q_1}(x) - \Big(\frac{1}{|x|^\gamma} * u^{p_1}\Big) u^{q_1}(x) \\
=& q_1 \Big(\frac{1}{|x|^\gamma} * u^{p_1}\Big)(x) \xi^{q_1-1} w_\lambda(x) \\
&+ \Big(\int_{\mathbb{R}^n} \frac{u^{p_1}(y)}{|x^\lambda - y|^\gamma} dy - \int_{\mathbb{R}^n} \frac{u^{p_1}(y)}{|x-y|^\gamma} dy\Big) u_\lambda^{q_1}(x) \\
\geq& q_1 u^{q_1-1}(x) \Big(\frac{1}{|x|^\gamma} * u^{p_1}\Big)(x) w_\lambda(x) \\
&+ u_\lambda^{q_1}(x) \int_{\Sigma_\lambda} \Big(\frac{1}{|x-y|^\gamma} - \frac{1}{|x^\lambda-y|^\gamma}\Big)(u_\lambda^{p_1}(y) - u^{p_1}(y)) dy \\
\geq& q_1 u^{q_1-1}(x) \Big(\frac{1}{|x|^\gamma} * u^{p_1}\Big)(x) w_\lambda(x) \\
&+ p_1 u^{q_1}(x) \int_{\Sigma_\lambda^-} \Big(\frac{1}{|x-y|^\gamma} - \frac{1}{|x^\lambda-y|^\gamma}\Big) u^{p_1-1}(y) w_\lambda(y) dy \\
\geq& q_1 u^{q_1-1}(x) \int_{\mathbb{R}^n} \frac{u^{p_1}(y)}{|x-y|^\gamma} dy\, w_\lambda(x) \\
&+ p_1 u^{q_1}(x) \inf_{\Sigma_\lambda^-} w_\lambda \int_{\Sigma_\lambda^-} \Big(\frac{1}{|x-y|^\gamma} - \frac{1}{|x^\lambda-y|^\gamma}\Big) u^{p_1-1}(y) dy,
\end{aligned}
\tag{3.16}
$$

where ξ is a value between $u(x)$ and $u_\lambda(x)$.

Now, at points $x \in \Sigma_\lambda^-$ where $w_\lambda(x) = \inf_{\Sigma_\lambda^-} w_\lambda$, we obtain

$$
(\Delta + \lambda)^{\frac{\alpha}{2}} w_\lambda(x) - c_\lambda(x) w_\lambda(x) \leq 0, \tag{3.17}
$$

with

$$
c_\lambda(x) := -q_1 u^{q_1-1}(x) \int_{\mathbb{R}^n} \frac{u^{p_1}(y)}{|x-y|^\gamma} dy - p_1 u^{q_1}(x) \int_{\Sigma_\lambda^-} \Big(\frac{1}{|x-y|^\gamma} - \frac{1}{|x^\lambda-y|^\gamma}\Big) u^{p_1-1}(y) dy.
$$

Next, we will prove that

$$
\lim_{x \in \Sigma_\lambda^-,\, |x| \to +\infty} c_\lambda(x) \geq 0, \tag{3.18}
$$

which implies, for any $\lambda \leq 0$,

$$
\begin{aligned}
\lim_{x \in \Sigma_\lambda^-,\, |x| \to +\infty} \Big\{ & q_1 u^{q_1-1}(x) \int_{\mathbb{R}^n} \frac{u^{p_1}(y)}{|x-y|^\gamma} dy \\
&+ p_1 u^{q_1}(x) \int_{\Sigma_\lambda^-} \Big(\frac{1}{|x-y|^\gamma} - \frac{1}{|x^\lambda-y|^\gamma}\Big) u^{p_1-1}(y) dy \Big\} = 0.
\end{aligned}
\tag{3.19}
$$

To this end, we first infer from the assumption (1.9) that

$$\int_{\mathbb{R}^n} \frac{u^{p_1}(x)}{|x|^\gamma} dx < +\infty,$$

and for any $\epsilon > 0$, there exists $R_\epsilon > 0$ large enough such that

$$u(x) \leq \frac{\epsilon}{|x|^\gamma}, \quad \forall\ |x| \geq R_\epsilon.$$

Split $\mathbb{R}^n$ into $A_1 \cup A_2 \cup A_3$, where

$$\begin{aligned}
A_1 &= \Big\{ y : |y - x| \geq \frac{|x|}{2} \text{ and } |y| < \sqrt{|x|} \Big\}, \\
A_2 &= \Big\{ y : |y - x| \geq \frac{|x|}{2} \text{ and } |y| \geq \sqrt{|x|} \Big\}, \\
A_3 &= \Big\{ y : |y - x| < \frac{|x|}{2} \Big\}.
\end{aligned}$$

For any $\lambda \leq 0$, taking $x \in \Sigma_\lambda^-$ with $|x| > 2R_\epsilon$ sufficiently large, by straightforward calculations we have

$$\begin{aligned}
&q_1 u^{q_1 - 1}(x) \int_{\mathbb{R}^n} \frac{u^{p_1}(y)}{|x - y|^\gamma} dy \\
&\leq \frac{q_1 2^\gamma \epsilon^{q_1 - 1}}{|x|^{\frac{\gamma}{2} + (q_1 - 1)\gamma}} \int_{A_1} \frac{u^{p_1}(y)}{|y|^\gamma} dy \\
&\quad + \frac{3^\gamma q_1 \epsilon^{q_1 - 1}}{|x|^{(q_1 - 1)\gamma}} \int_{A_2} \frac{u^{p_1}(y)}{|y|^\gamma} dy + \frac{2^{p_1} \epsilon^{p_1 + q_1 - 1}}{|x|^{p_1 + (q_1 - 1)\gamma}} \int_{A_3} \frac{1}{|x - y|^\gamma} dy \\
&\leq \frac{q_1 2^\gamma \epsilon^{q_1 - 1}}{|x|^{q_1 \gamma - \frac{\gamma}{2}}} \int_{\mathbb{R}^n} \frac{u^{p_1}(x)}{|x|^\gamma} dx \\
&\quad + \frac{3^\gamma q_1 \epsilon^{q_1 - 1}}{|x|^{(q_1 - 1)\gamma}} \int_{|y| \geq \sqrt{|x|}} \frac{u^{p_1}(y)}{|y|^\gamma} dy + C\epsilon^{p_1 + q_1 - 1} \\
&\leq C\epsilon^{q_1 - 1} + C\epsilon^{q_1 - 1} + C\epsilon^{p_1 + q_1 - 1} \\
&\leq C\epsilon^{q_1 - 1} + C\epsilon^{p_1 + q_1 - 1},
\end{aligned} \tag{3.20}$$

where we use the fact that of $|x-y|\geq\frac{|x|}{2}$ implies $|x-y|\geq\frac{|y|}{3}$. At the same time, we can also get

$$
\begin{aligned}
&p_1u^{q_1}(x)\int_{\Sigma_\lambda^-}\Big(\frac{1}{|x-y|^\gamma}-\frac{1}{|x^\lambda-y|^\gamma}\Big)u^{p_1-1}(y)dy\\
&\leq p_1u^{q_1}(x)\int_{\Sigma_\lambda^-}\frac{|x^\lambda-y|^\gamma-|x-y|^\gamma}{|x-y|^\gamma\cdot|x^\lambda-y|^\gamma}u^{p_1-1}(y)dy\\
&\leq p_12^\gamma|x|^\gamma u^{q_1}(x)\int_{\mathbb{R}^n}\frac{u^{p_1-1}(y)}{|x-y|^{2\gamma}}dy\\
&\leq\frac{C\epsilon^{q_1}}{|x|^{q_1\gamma}}\int_{A_1}\frac{u^{p_1-1}(y)}{|y|^{2\gamma}}dy \qquad (3.21)\\
&\quad+\frac{C\epsilon^{q_1}}{|x|^{q_1\gamma-\gamma}}\int_{A_2}\frac{u^{p_1-1}(y)}{|y|^{2\gamma}}dy+\frac{C\epsilon^{p_1+q_1-1}}{|x|^{(p_1+q_1-1)\gamma-\gamma}}\int_{A_3}\frac{1}{|x-y|^{2\gamma}}dy\\
&\leq\frac{C\epsilon^{q_1}}{|x|^{q_1\gamma}}\int_{\mathbb{R}^n}\frac{u^{p_1-1}(x)}{|x|^{2\gamma}}dx+C\epsilon^{q_1}+C\epsilon^{p_1+q_1-1}\\
&\leq C\epsilon^{q_1}+C\epsilon^{q_1}+C\epsilon^{p_1+q_1-1}\\
&\leq C\epsilon^{q_1}+C\epsilon^{p_1+q_1-1}.
\end{aligned}
$$

By letting $\epsilon\to0$, we obtain

$$
\begin{aligned}
\lim_{x\in\Sigma_\lambda^-,\,|x|\to+\infty}&\Big\{q_1u^{q_1-1}(x)\int_{\mathbb{R}^n}\frac{u^{p_1}(y)}{|x-y|^\gamma}dy\\
&+p_1u^{q_1}(x)\int_{\Sigma_\lambda^-}\Big(\frac{1}{|x-y|^\gamma}-\frac{1}{|x^\lambda-y|^\gamma}\Big)u^{p_1-1}(y)dy\Big\} \qquad (3.22)\\
&\leq C\epsilon^{q_1}+C\epsilon^{q_1-1}+C\epsilon^{p_1+q_1-1}.
\end{aligned}
$$

This indicates that (3.19) holds and hence (3.18) holds for any $\lambda\leq0$. Besides, from (3.20) and (3.21) one can also derive that $c_\lambda(x)$ is uniformly bounded from below (independent of λ).

We will carry out the direct method of moving planes in two steps.

STEP 1. Start moving the plane T_λ from λ sufficiently negative to the right along the x_1-axis.

Apparently, we only prove that for sufficiently negative λ,

$$w_\lambda(x)\geq0,\quad\forall\ x\in\Sigma_\lambda. \qquad (3.23)$$

In fact, one can infer from the assumption (1.9) that u is bounded from above and

$$\lim_{|x|\to+\infty} u(x) = 0,$$

and hence w_λ is bounded from below and

$$\lim_{|x|\to+\infty} w_\lambda(x) = 0.$$

Suppose that the conclusion (3.23) is false, then there exists $\hat{x} \in \Sigma_\lambda^-$ such that

$$w_\lambda(\hat{x}) = \inf_{\Sigma_\lambda} w_\lambda < 0.$$

Apply (3.17) and (3.18) and Theorem 2.4 to conclude that there exists $R_0 > 0$ large (independent of λ) such that

$$|\hat{x}| \leq R_0.$$

This will lead to a contradiction provided that $\lambda \leq -R_0$. Thus we have, for any $\lambda \leq -R_0$, $w_\lambda \geq 0$ in Σ_λ.

Step 2. Step 1 provides a starting point, from which we can now move the plane T_λ along the x_1-axis to its limiting position.

To this end, let us define

$$\lambda_0 := \sup\{\lambda \leq 0 \mid w_\mu \geq 0 \text{ in } \Sigma_\mu, \forall\, \mu \leq \lambda\}. \tag{3.24}$$

It follows from Step 1 that $-R_0 \leq \lambda_0 \leq 0$. One can easily verify that

$$w_{\lambda_0}(x) \geq 0, \quad \forall\, x \in \Sigma_{\lambda_0}. \tag{3.25}$$

We will carry out the proof by discussing two different cases.

Case (i): $\lambda_0 < 0$. Similar to case (ii) in the proof of Theorem 1.5. We can easily obtain

$$w_{\lambda_0}(x) \equiv 0, \quad \forall\, x \in \Sigma_{\lambda_0}. \tag{3.26}$$

Case (ii): $\lambda_0 = 0$. We can move the plane in the opposite direction along the x_1-direction until the limiting position $\bar{\lambda}_0 \geq \lambda_0 = 0$. Again, if $\bar{\lambda}_0 > 0$, by using the narrow region principle (Theorem 2.3) and decay at infinity theorem 2.4, we can deduce that $u_{\bar{\lambda}_0} \equiv u$ as in case (ii) in the proof of Theorem 1.5. If $\bar{\lambda}_0 = \lambda_0 = 0$, we immediately have $u_0 \equiv u$.

From the contradictions derived in Case (i) and Case (ii), we conclude that u is symmetric about the plane T_{λ_0} or $T_{\bar{\lambda}_0}$. Since the x_1-direction is chosen arbitrarily, we must have that u is radially symmetric and monotone decreasing around some point $x_0 \in \mathbb{R}^n$. This concludes our proof of Theorem 1.7. ∎

3.3. Liouville-type results for nonlocal double phase problem. In this subsection, as a direct application of Maximum Principles in unbounded domains Theorem 2.5, combined with the method of moving planes, we shall establish Liouville-type results for the nonlocal double phase problem in $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

As we know, Liouville theorems in conjunction with the blowing up and re-scaling arguments, are crucial in establishing a priori estimates and hence the existence of positive solutions to non-variational boundary value problems for a class of elliptic equations on bounded domains or Riemannian manifolds with boundaries. For more works on the Liouville-type results for elliptic equations or systems, refer to [14, 15, 23, 30, 34, 37, 39] and the references therein.

Our main results in this subsection are as follows.

Theorem 3.1 (Liouville Theorem in $\mathbb{R}^n$): *Assume that $u \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is a bounded solution to*

$$-(\Delta+\lambda)^{\frac{\alpha}{2}} u(x) = f(x, u(x)), \quad \text{in } \mathbb{R}^n, \tag{3.27}$$

where the function $f(x,u)$ satisfies

$$\sup_{\substack{t_1,\, t_2 \in [\inf u,\, \sup u] \\ t_1 > t_2}} \frac{f(x,t_1) - f(x,t_2)}{t_1 - t_2} < 0 \tag{3.28}$$

and $f(x_1, x', u) \le f(\bar{x}_1, x', u)$ with $x_1 \le \bar{x}_1$. Then

$$u \equiv C, \quad \text{in } \mathbb{R}^n.$$

Proof. Keep the definitions of T_λ, x^λ, Σ_λ, w_λ in Subsection 3.1. Since $u \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is bounded, it lead to

$$w_\lambda \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n) \text{ is bounded.}$$

At any points $x \in \Sigma$ where $w(x) > 0$, one has

$$\begin{aligned} (\Delta+\lambda)^{\frac{\alpha}{2}} w_\lambda(x) &= f(x, u(x)) - f(x^\lambda, u_\lambda(x)) \\ &\le f(x, u(x)) - f(x, u_\lambda(x)) \\ &=: c(x) w_\lambda(x), \end{aligned} \tag{3.29}$$

provided that

$$c(x) := \frac{f(x,u(x)) - f(x,u_\lambda(x))}{u_\lambda(x)}.$$

It readily follows from (3.28) that

$$\inf_{\{x\in\Sigma_\lambda | w_\lambda(x)>0\}} c(x) > 0.$$

Therefore, using Theorem 2.5, we arrive immediately at $w_\lambda \geq 0$ in Σ_λ.

By a similar argument we can prove that $w_\lambda \geq 0$ in $\mathbb{R}^n \setminus \Sigma$. Hence

$$w_\lambda \equiv 0, \quad \text{in } \mathbb{R}^n,$$

and hence u is symmetric with respect to T_λ. Since T_λ is arbitrary, we must have

$$u \equiv C, \quad \text{in } \mathbb{R}^n.$$

This finishes the proof of Theorem 3.1. ∎

THEOREM 3.2 (Liouville Theorem in $\mathbb{R}^n_+$): *Assume that* $u \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n)$ *is a nonnegative solution to*

$$\begin{cases} -(\Delta+\lambda)^{\frac{\alpha}{2}} u(x) = f(x,u(x)), & \text{in } \mathbb{R}^n_+, \\ u(x) = 0, & x \notin \mathbb{R}^n_+, \end{cases} \tag{3.30}$$

where the function $f(x,u)$ *satisfies,* $\frac{\partial f}{\partial u} \geq 0$ *and* $f(x,0) = 0$*. Suppose that*

$$\lim_{|x|\to+\infty} u(x) = 0, \tag{3.31}$$

then we have

$$u \equiv 0, \quad \text{in } \mathbb{R}^n.$$

Proof. We first claim that if there exists $x_0 \in \mathbb{R}^n_+$ such that $u(x_0) = 0$, then we have

$$u(x) \equiv 0, \quad \text{in } \mathbb{R}^n_+. \tag{3.32}$$

Note that u is a nonnegative solution of equation (3.30), which means that x_0 is the minimum point of the function $u(x)$, thus $\nabla u(x_0) = 0$.

On one hand, if $u(x) \not\equiv 0$, we have

$$\begin{aligned} -(\Delta+\lambda)^{\frac{\alpha}{2}} u(x_0) =& c_{n,\alpha,\lambda} P.V. \int_{\mathbb{R}^n} \frac{u(x_0)-u(y)}{e^{\lambda|x_0-y|}|x_0-y|^{n+\alpha}} dy \\ =& c_{n,\alpha,\lambda} P.V. \int_{\mathbb{R}^n} \frac{-u(y)}{e^{\lambda|x_0-y|}|x_0-y|^{n+\alpha}} dy < 0. \end{aligned} \tag{3.33}$$

On the other hand, since $\frac{\partial f}{\partial u} \geq 0$, we have

$$-\left(\Delta+\lambda\right)^{\frac{\alpha}{2}} u(x_0)=f(x_0,u(x_0))=f(x_0,0)=0. \tag{3.34}$$

This is a contradiction, which implies that $u(x) \equiv 0$ in $\mathbb{R}^n_+$.

In the following, we always assume that $u(x) > 0$. Then, we carry out the moving planes procedure in two steps.

Step 1. We shall show that, for $\lambda > 0$ sufficiently close to 0,

$$w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3.35}$$

Combining the assumption of Theorem 3.2 with $u > 0$, we have

$$\varliminf_{|x|\to\infty} w_\lambda(x) \geq 0.$$

Then similar to the proof of Theorem 1.5, applying narrow region principle 2.3 to functions w_λ, we get (3.35).

Step 2. We continue to move the plane T_λ along the x_n-axis to its limiting position as long as (3.35) holds. More precisely, let

$$\lambda_0 := \sup\{\lambda > 0 \mid w_\mu(x) > 0,\, x \in \Sigma_\mu, \mu \leq \lambda\}. \tag{3.36}$$

Now we show that

$$\lambda_0 = +\infty. \tag{3.37}$$

Suppose to the contrary that $\lambda_0 < +\infty$. Similar to the proof of Theorem 1.5, we can derive either

$$w_{\lambda_0}(x) = 0, \tag{3.38}$$

or

$$w_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0}. \tag{3.39}$$

If (3.39) is true, by a similar argument as in the proof of Theorem 1.5, we will show that the plane T_{λ_0} can be moved upward a little bit more, which contradicts the definition (3.37) of λ_0. Hence, (3.38) is true. It reveals that

$$u(x_1,x_2,\ldots,x_{n-1},2\lambda_0) = u(x_1,x_2,\ldots,x_{n-1},0) = 0,$$

which is a contradiction to $u > 0$. So (3.37) holds.

Therefore, u is increasing concerning the x_n-axis. In terms of the assumption (3.31) in Theorem 3.2, we know that is impossible. So $u(x) \equiv 0$ in $\mathbb{R}^n_+$.

We have completed the proof of Theorem 3.2. ∎

4. Direct sliding methods for a tempered fractional Laplacian

In this section, we will prove Theorem 1.8 via direct sliding methods. Similar to the direct method of moving planes, the maximum principles is a key ingredient in the sliding method. Hence, in this paper, we first establish the following maximum principles for the tempered fractional Laplacian operator $-(\Delta+\lambda)^{\frac{\alpha}{2}}$,

.

4.1. Maximum principles for the tempered fractional operator $-(\Delta+\lambda)^{\frac{\alpha}{2}}$. In this subsection, we shall establish various maximum principles for the tempered fractional Laplacian operator $-(\Delta+\lambda)^{\frac{\alpha}{2}}$ in unbounded domains. These maximum principles are key ingredients in applying the sliding method. We begin with the following elementary lemmas used in various places in the text.

Lemma 4.1 (Strong maximum principle): *Suppose that $u \in \mathscr{L}_\alpha(\mathbb{R}^n)$ and $u \geq 0$ in $\mathbb{R}^n$. If there exists $x_0 \in \mathbb{R}^n$ such that,*

$$\begin{cases} (\Delta+\lambda)^{\frac{\alpha}{2}} u(x_0) \leq 0, \\ u(x_0) = 0, \end{cases} \tag{4.1}$$

and u is $C^{1,1}$ near x_0, then $u = 0$ a.e. in $\mathbb{R}^n$.

Proof. Since there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = \min_{x\in\mathbb{R}^n} u(x) = 0$, it follows from the definition of $\mathcal{L}^s_\mu$ that

$$\begin{aligned} 0 &\geq (\Delta+\lambda)^{\frac{\alpha}{2}} u(x_0) \\ &= -c_{n,\alpha,\lambda} P.V. \int_{\mathbb{R}^n} \frac{u(x_0)-u(y)}{e^{\lambda|x_0-y|}|x_0-y|^{n+\alpha}} dy \\ &= c_{n,\alpha,\lambda} P.V. \int_{\mathbb{R}^n} \frac{u(y)}{e^{\lambda|x_0-y|}|x_0-y|^{n+\alpha}} dy \geq 0. \end{aligned}$$

Thus we must have $u = 0$ a.e. in $\mathbb{R}^n$. This finishes the proof of Lemma 4.1. ∎

Next, we will prove the maximum principles in unbounded open sets.

Theorem 4.2 (Maximum principles in unbounded open sets): *Assume that D is an open set in $\mathbb{R}^n$, possibly unbounded and disconnected. Let $\overline{D}$ be disjoint from an infinite open domain and set $\Gamma \subset \overline{D}^c$ such that*

$$\frac{|\Gamma \cap (B_{ar_x}(x)\backslash B_{r_x}(x))|}{|B_{ar_x}(x)\backslash B_{r_x}(x)|} \geq c_0 > 0, \quad \forall\, x \in D, \tag{4.2}$$

for some constants $a > 1$, $c_0 > 0$, *where* a, c_0 *are independent of* x *and* $r_x > 0$ *possibly depending on* x. *Suppose that* $u \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\rm loc}(D)$ *is bounded from below, and solves*

$$\begin{cases} (\Delta+\lambda)^{\frac{\alpha}{2}} u(x) - c(x)u(x) - \sum_{i=1}^n c_i(x)u_i(x) \leq 0, \\ \qquad\qquad\qquad\qquad\qquad \text{at points } x \in D \text{ where } u(x) < 0, \\ u(x) \geq 0, \qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^n \setminus D, \end{cases} \tag{4.3}$$

where $c(x) \geq 0$ *in the set* $\{x \in D \mid u(x) < 0\}$ *and* $u_i = \frac{\partial u}{\partial x_i}$. *Then* $u \geq 0 \quad \text{in} \, D$.

Furthermore, assume that

$$(\Delta+\lambda)^{\frac{\alpha}{2}} u(x) \leq 0, \quad \text{at points } x \in D \text{ where } u(x) = 0. \tag{4.4}$$

Then we have

$$\text{either} \quad u(x) > 0 \text{ in } D, \quad \text{or} \quad u(x) = 0 \text{ a.e. in } \mathbb{R}^n. \tag{4.5}$$

Proof. Suppose on the contrary that there exists some $x \in D$ such that $u(x) < 0$. Then we have

$$-\infty < M := \inf_{x\in\mathbb{R}^n} u(x) < 0. \tag{4.6}$$

There exist sequences $x^k \in D$ and $0 < \beta_k < 1$ with $\beta_k \to 1$ as $k \to \infty$ such that

$$u(x^k) \leq \beta_k M. \tag{4.7}$$

To this end, let

$$\psi(x) = \begin{cases} e^{\frac{|x|^2}{|x|^2-1}}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

It is well known that $\psi \in C_0^\infty(\mathbb{R}^n)$, therefore $|(\Delta+\lambda)^{\frac{\alpha}{2}}\psi(x)| \leq C_0$ for any $x \in \mathbb{R}^n$. Now we define

$$\Psi_k(x) := \psi\Big(\frac{x - x^k}{r_{x^k}}\Big),$$

where we choose $r_{x^k} < \frac{1}{\lambda}$. Take $\epsilon_k := -(1-\beta_k)M$. Since $u \geq M$ and $\Psi_k = 0$ in $\mathbb{R}^n \setminus B_{r_{x^k}}(x^k)$, we have

$$u(x^k) - \epsilon_k \Psi_k(x^k) \leq M \leq u(x) - \epsilon_k \Psi_k(x), \tag{4.8}$$

for any $x \in \mathbb{R}^n \setminus B_{r_{x^k}}(x^k)$. Consequently, there exists $\bar{x}^k \in B_{r_{x^k}}(x^k)$ such that

$$u(\bar{x}^k) - \epsilon_k \Psi_k(\bar{x}^k) = \inf_{x\in\mathbb{R}^n} [u(x) - \epsilon_k \Psi_k(x)] \leq M, \tag{4.9}$$

which implies that

$$u(\bar{x}^k) \leq u(x^k) - \epsilon_k \Psi_k(x^k) + \epsilon_k \Psi_k(\bar{x}^k) \leq u(x^k) \leq \beta_k M < 0. \tag{4.10}$$

Note that $\nabla(u(\bar{x}^k) - \epsilon_k \Psi_k(\bar{x}^k)) = 0$, and hence

$$\nabla u(\bar{x}^k) = 0, \quad \text{as } k \to \infty. \tag{4.11}$$

Therefore, through direct computations and recalling (4.9), we derive that

$$\begin{aligned}
&(\Delta+\lambda)^{\frac{\alpha}{2}}[u - \epsilon_k \Psi_k](\bar{x}^k) \\
&= - c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(\bar{x}^k) - \epsilon_k \Psi_k(\bar{x}^k) - u(y) + \epsilon_k \Psi_k(y)}{e^{\lambda|\bar{x}^k - y|}|\bar{x}^k - y|^{n+\alpha}} dy \\
&= - c_{n,\alpha} \Big[P.V. \int_{B_{r_{x^k}}(x^k)} \frac{u(\bar{x}^k) - \epsilon_k \Psi_k(\bar{x}^k) - u(y) + \epsilon_k \Psi_k(y)}{e^{\lambda|\bar{x}^k - y|}|\bar{x}^k - y|^{n+\alpha}} dy \\
&\quad + \int_{(B_{r_{x^k}}(x^k))^c} \frac{u(\bar{x}^k) - \epsilon_k \Psi_k(\bar{x}^k) - u(y) + \epsilon_k \Psi_k(y)}{e^{\lambda|\bar{x}^k - y|}|\bar{x}^k - y|^{n+\alpha}} dy \Big] \\
&\geq - c_{n,\alpha} \int_{(B_{r_{x^k}}(x^k))^c} \frac{u(\bar{x}^k) - \epsilon_k \Psi_k(\bar{x}^k) - u(y) + \epsilon_k \Psi_k(y)}{e^{\lambda|\bar{x}^k - y|}|\bar{x}^k - y|^{n+\alpha}} dy \\
&= - c_{n,\alpha} \int_{(B_{r_{x^k}}(x^k))^c} \frac{u(\bar{x}^k) - \epsilon_k \Psi_k(\bar{x}^k) - u(y)}{e^{\lambda|\bar{x}^k - y|}|\bar{x}^k - y|^{n+\alpha}} dy \\
&\geq - c_{n,\alpha} \int_{\Gamma \cap (B_{ar_{x^k}}(x^k) \setminus B_{r_{x^k}}(x^k))} \frac{u(\bar{x}^k) - \epsilon_k \Psi_k(\bar{x}^k)}{e^{\lambda|\bar{x}^k - y|}|\bar{x}^k - y|^{n+\alpha}} dy \\
&\geq - c_{n,\alpha} (u(\bar{x}^k) - \epsilon_k \Psi_k(\bar{x}^k)) \int_{\Gamma \cap (B_{ar_{x^k}}(x^k) \setminus B_{r_{x^k}}(x^k))} \frac{1}{e^{\lambda|\bar{x}^k - y|}|\bar{x}^k - y|^{n+\alpha}} dy \\
&\geq - C_{n,\alpha} (u(\bar{x}^k) - \epsilon_k \Psi_k(\bar{x}^k)) \frac{1}{e^{\lambda a r_{x^k}} r_{x^k}^{\alpha}} \\
&\geq \frac{-C_{n,a,\alpha,\lambda} M}{e^{\lambda r_{x^k}} r_{x^k}^{\alpha}},
\end{aligned} \tag{4.12}$$

where the last inequality holds due to $\mu \geq 0$.

On the other hand, we will evaluate the lower bound of $(\Delta+\lambda)^{\frac{\alpha}{2}}[u - \epsilon_k \Psi_k](\bar{x}^k)$.

Indeed, since (4.10) implies $u(\bar{x}^k) < 0$ and hence $\bar{x}^k \in D$, applying (4.3) and (4.11) we see that

$$\begin{aligned}(\Delta+\lambda)^{\frac{\alpha}{2}}[u-\epsilon_k\Psi_k](\bar{x}^k) &= (\Delta+\lambda)^{\frac{\alpha}{2}}u(\bar{x}^k) - (\Delta+\lambda)^{\frac{\alpha}{2}}(\epsilon_k\Psi_k(\bar{x}^k)) \\ &\leq c(\bar{x}^k)u(\bar{x}^k) + \sum_{i=1}^n c_i(\bar{x}^k)u_i(\bar{x}^k) + \frac{C_0\epsilon_k}{r_{x^k}^{\alpha}} \\ &\leq \frac{C_0\epsilon_k}{r_{x^k}^{\alpha}}.\end{aligned} \tag{4.13}$$

Putting (4.12) and (4.13) together, we deduce that

$$-\frac{C_{n,\alpha,a,\lambda}M}{r_{x^k}^{\alpha}e^{\lambda r_{x^k}}} \leq \frac{C_0\epsilon_k}{r_{x^k}^{\alpha}}, \tag{4.14}$$

which implies

$$C_{n,\alpha,a,\lambda} \leq C_0(1-\beta_k)e^{\lambda r_{x^k}} \leq C_0 e(1-\beta_k).$$

This will lead to a contradiction for k sufficiently large.

Furthermore, if there exists a point $\tilde{x} \in D$ such that $u(\tilde{x}) = 0$, then it follows from (4.4) and Lemma 4.1 that $u = 0$ a.e. in $\mathbb{R}^n$. Therefore, we have

$$\text{either} \quad u(x) > 0 \text{ in } D, \quad \text{or} \quad u(x) = 0 \text{ a.e. in } \mathbb{R}^n.$$

This completes our proof of Theorem 4.2. ∎

From the proof of Theorem 4.2, we can deduce the following:

Theorem 4.3 (Narrow region principle in unbounded open sets): *Let D be an open set in $\mathbb{R}^n$ (possibly unbounded and disconnected) and*

$$d(D) := \sup_{x\in D} \operatorname{dist}(x, D^c)$$

be the width of D, under the assumptions (4.2) *and*

$$r_x = \operatorname{dist}(x, D^c) \leq d(D).$$

Suppose that $u \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(D)$ is bounded from above and solves

$$\begin{cases}(\Delta+\lambda)^{\frac{\alpha}{2}}u(x) - c(x)u(x) - \sum_{i=1}^n c_i(x)u_i(x) \leq 0, \\ \qquad\qquad\qquad\qquad \text{at points } x \in D \text{ where } u(x) < 0, \\ u(x) \geq 0, \qquad\qquad\qquad\qquad \text{in } D^c,\end{cases} \tag{4.15}$$

where $c(x)$ is uniformly bounded from below in $\{x \in D \mid u(x) < 0\}$ (with respect to $d(D)$) and $u_i = \frac{\partial u}{\partial x_i}$. If we assume that

$$\inf_{\{x \in D \mid u(x) < 0\}} c(x) > -\frac{C_{n,\alpha,\lambda}}{2d(D)^{\alpha} e^{\lambda d(D)}}, \tag{4.16}$$

where $C_{n,\alpha,\lambda} > 0$ is the same constant as in (4.12), then

$$u(x) \leqslant 0, \quad \text{in } D. \tag{4.17}$$

Furthermore, assume that

$$(\Delta + \lambda)^{\frac{\alpha}{2}} u(x) \geq 0, \quad \text{at points } u \in D \text{ where } u(x) = 0; \tag{4.18}$$

then we have

$$\text{either} \quad u(x) > 0 \text{ in } D, \quad \text{or} \quad u(x) = 0 \text{ a.e. in } \mathbb{R}^n. \tag{4.19}$$

Proof. The proof is similar to that of Theorem 4.2. We only give a sketch.

One can infer from (4.12) and (4.13) that

$$\frac{-C_{n,\alpha,\lambda} M}{r_{x^k}^{\alpha} e^{\lambda r_{x^k}}} \leq c(\bar{x}^k) u(\bar{x}^k) + \frac{C_0 \epsilon_k}{r_{x^k}^{\alpha} e^{\lambda r_{x^k}}}, \tag{4.20}$$

which implies

$$-c(\bar{x}^k) u(\bar{x}^k) \leq \frac{C_{n,a,\alpha,\lambda} M}{r_{x^k}^{\alpha} e^{\lambda r_{x^k}}} + \frac{C_0 \epsilon_k}{r_{x^k}^{\alpha}} < \frac{C_{n,a,\alpha,\lambda} M}{2 r_{x^k}^{\alpha} e^{\lambda r_{x^k}}}.$$

With a suitable choice of k sufficiently large such that $\beta_k \geq 1 - \frac{C_{n,\alpha,\lambda}}{2C_0}$, we derive from (4.6) and (4.20) that

$$c(\bar{x}^k) \leq -\frac{C_{n,a,\alpha,\lambda} M}{2 r_{x^k}^{\alpha} e^{\lambda r_{x^k}} u(\bar{x}^k)} \leq -\frac{C_{n,\alpha,\lambda}}{2 r_{x^k}^{\alpha} e^{\lambda r_{x^k}}} \leq -\frac{C_{n,\alpha,\lambda}}{2d(D)^{\alpha} e^{\lambda d(D)}},$$

which contradicts (4.16).

Furthermore, if there exists a point $\tilde{x} \in D$ such that $u(\tilde{x}) = 0$, then it follows from (4.4) and Lemma 4.1 that $u = 0$ a.e. in $\mathbb{R}^n$. Therefore, we have

$$\text{either} \quad u(x) > 0 \text{ in } D, \quad \text{or} \quad u(x) = 0 \text{ a.e. in } \mathbb{R}^n.$$

This completes our proof of Theorem 4.3. ∎

4.2. Proof of Theorem 1.8. In this subsection, by employing maximum principles for a tempered fractional Laplacian operator $-(\Delta+\lambda)^{\frac{\alpha}{2}}$ and the direct sliding methods, we shall investigate the monotonicity and uniqueness of solutions in half-space $\mathbb{R}^+_n$, which is Theorem 1.8.

Proof. From the definition of w^τ we can see that

$$-(\Delta+\lambda)^{\frac{\alpha}{2}} w^\tau(x) = c^\tau(x) w^\tau(x) + \sum_{i=1}^{n} c_i^\tau(x) w_i^\tau(x), \quad \text{in } D^\tau,$$

where $c_i^\tau(x) = \frac{\partial f}{\partial p_i}$, $w_i^\tau(x) = \frac{\partial u^\tau}{\partial x_i} - \frac{\partial u}{\partial x_i}$,

$$c^\tau(x) := \begin{cases} \frac{f(x,u^\tau(x),\nabla u) - f(x,u(x),\nabla u)}{u^\tau(x)-u(x)}, & \text{if } u^\tau(x) \neq u(x) \\ 0, & \text{if } u^\tau(x) = u(x) \end{cases}$$

is an L^∞ function satisfying $|c^\tau(x)| \le C, \forall\, x \in D^\tau$, since $f(x,u,\nabla u)$ is Lipschitz continuous in u.

Our goal is to show that

$$w^\tau(x) > 0, \quad \text{in } D^\tau, \quad \forall\; 0<\tau<\tau_0, \tag{4.21}$$

which indicates that u is strictly increasing in the x_n direction.

Step 1. We show that, for τ sufficiently large, we have

$$w^\tau(x) \le 0, \quad \text{in } \mathbb{R}^n. \tag{4.22}$$

Indeed, one can infer from (1.11) that there exists a sufficiently large M_1 such that for any $x_n > M_1$, $u \in [A-\delta, A]$, where $\delta > 0$. Next we only prove that, for any $\tau \ge M_1$.

$$w^\tau(x) \le 0, \quad \text{in } \mathbb{R}^n. \tag{4.23}$$

Suppose that (4.23) is not true, then there exists a constant $K > 0$ such that

$$\sup_{x\in\mathbb{R}^n_+} w^\tau(x) = K > 0,$$

and hence for some $\tau_1 \ge M_1$ there exists a sequence $x_k \in \mathbb{R}^n_+$ such that

$$w^{\tau_1}(x^k) \to K, \quad \text{as } k \to \infty. \tag{4.24}$$

We denote $D = \{x \in \mathbb{R}^n | w^{\tau_1}(x) > \frac{K}{3}\}$. Since $\tau_1 \ge M$ and $x \in \mathbb{R}^n_+$, we deduce that, for any $x \in D$, we have $u(x) \ge u^{\tau_1}(x) \ge A - \delta$.

Through a simple computation, for any $x \in D$, we get

$$\begin{aligned}
&-(\Delta+\lambda)^{\frac{\alpha}{2}}\Big(w^{\tau_1}(x)-\frac{K}{3}\Big)\\
&\quad= -(\Delta+\lambda)^{\frac{\alpha}{2}} w^{\tau_1}(x)\\
&\quad= f(x,u(x),(\nabla_x u)(x)) - f(x', x_n+\tau_1 e_n, u^{\tau_1}(x), (\nabla_x u^{\tau_1})(x))\\
&\quad\le f(x,u(x),(\nabla_x u)(x)) - f(x, u^{\tau_1}(x), (\nabla_x u^{\tau_1})(x))\\
&\quad=: -c(x)w^{\tau_1}(x) - \sum_{i=1}^{n} c_i(x) w_i^{\tau_1}(x)\\
&\quad\le -c(x)\Big(w^{\tau_1}(x)-\frac{K}{3}\Big) - \sum_{i=1}^{n} c_i(x)\Big(w^{\tau_1}(x)-\frac{K}{3}\Big)_i(x),
\end{aligned} \tag{4.25}$$

and using also (1.12), we see that

$$c(x) = -\frac{f(x,u,\nabla_x u) - f(x,u^{\tau_1},\nabla_x u)}{u-u^{\tau_1}} \ge 0.$$

Similar to (4.25), we derive that $w^{\tau_1}(x) - \frac{K}{3}$ satisfies

$$\begin{cases} (\Delta+\lambda)^{\frac{\alpha}{2}}(w^{\tau_1}(x)-\frac{K}{3}) - c(x)(w^{\tau_1}(x)-\frac{K}{3}) & \\ \qquad\qquad - \sum_{i=1}^{n} c_i(x)(w^{\tau_1}(x)-\frac{K}{3})_i(x) \ge 0, & x \in D,\\ w^{\tau_1}(x) \le 0, & x \in \mathbb{R}^n_+ \setminus D. \end{cases}$$

Again with Theorem 4.2 (by choosing $-u$ instead of u), we derive that $w^{\tau_1}(x) \le 0$ in $\mathbb{R}^n_+$, which contradicts (4.24). Therefore, equation (4.23) must hold.

Step 2. Inequality (4.23) provides a starting point for us to carry out the sliding procedure. Next, we decrease τ as long as inequality (4.22) holds until its limiting position. Define

$$\tau_0 := \inf\{\tau \mid w^{\tau}(x) \le 0 \text{ in } \mathbb{R}^n,\ 0<\tau<M_1\}. \tag{4.26}$$

We aim to prove that $\tau_0 = 0$.

Otherwise, suppose that $\tau_0 > 0$. We will show that τ_0 can be upward a little bit more, that is, there exists an $\varepsilon > 0$ small enough such that

$$w^{\tau}(x) \le 0, \quad \text{in } \mathbb{R}^n_+,\ \forall\, \tau_0-\varepsilon < \tau \le \tau_0,$$

which contradicts the definition (4.26) of τ_0.

(I) Firstly, we prove that

$$\sup_{\mathbb{R}^{n-1}\times(0,M_1+1]} w^{\tau_0}(x) < 0. \tag{4.27}$$

If not, then

$$\sup_{\mathbb{R}^{n-1}\times(0,M_0+1]} w^{\tau_0}(x) = 0.$$

So there exists a sequence $\{x^k\} \subset \mathbb{R}^{n-1} \times (0, M_0+1]$ such that $w^{\tau_0}(x^k) \to 0$, as $k \to \infty$.

Next we claim that

$$x^k \notin \partial\mathbb{R}^n_+. \tag{4.28}$$

Without loss of generality, let x_n^k be the n-th component of x_k. The above equation is equivalent to $x_n^k \neq 0$. If (4.28) is not true, one can infer from (1.10) that $u(x^k) = 0$. Therefore, $u((x^k)', (x^k)_n) = u((x^k)', (x^k)_n + \tau_0) = 0$, which contradicts the fact that $u > 0$ in $\mathbb{R}^n_+$. Thus our claim (4.28) is true.

Since $x^k \notin \partial\mathbb{R}^n_+$, then there exists a $\delta > 0$ such that $B_\delta(x^k) \in \mathbb{R}^n_+$. Set

$$\psi(x) = \eta\Big(\frac{x - x^k}{\delta}\Big),$$

where η is defined by

$$\eta(x) = \begin{cases} e^{\frac{|x|^2}{|x|^2-1}}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Now we can pick $\varepsilon_k > 0$ ($\to 0$ as $k \to \infty$) such that, for any $\mathbb{R}^n_+ \backslash B_\delta(x^k)$,

$$w^{\tau_0}(x) + \varepsilon_k\psi_k(x) \leq 0. \tag{4.29}$$

Therefore, there exists $\bar{x}^k \in B_\delta(x^k)$ such that

$$w^{\tau_0}(\bar{x}^k) + \varepsilon_k\psi_k(\bar{x}^k) = \max_{\mathbb{R}^n}(w^{\tau_0}(x^k) + \varepsilon_k\psi_k(x)) > 0,$$

which implies $\nabla(w^{\tau_0}(\bar{x}^k) + \varepsilon_k\psi_k(\bar{x}^k)) = 0$, and hence

$$\nabla(w^{\tau_0}(\bar{x}^k)) \to 0, \quad \text{as } k \to \infty. \tag{4.30}$$

Since

$$0 > w^{\tau_0}(\bar{x}^k) \geq w^{\tau_0}(x^k) + \varepsilon_k\psi(x^k) - \varepsilon_k\psi(\bar{x}^k) \geq w^{\tau_0}(x^k) \to 0, \quad \text{as } k \to \infty,$$

hence $w^{\tau_0}(\bar{x}^k) \to 0$, as $k \to \infty$. By the continuity of f, we have

$$\begin{aligned} &f(\bar{x}^k, u(\bar{x}^k), \nabla u(\bar{x}^k)) - f((\bar{x}^k)', \bar{x}^k_n + \tau_0 e_n, u^{\tau_0}(x^k), \nabla u^{\tau_0}(x^k)) \\ &\leq f(\bar{x}^k, u(\bar{x}^k), \nabla u(\bar{x}^k)) - f(\bar{x}^k, u^{\tau_0}(x^k), \nabla u^{\tau_0}(x^k)) \to 0, \quad \text{as } k \to \infty. \end{aligned} \tag{4.31}$$

On the one hand, in view of (4.30) and (4.31), we have

$$
\begin{aligned}
&-(\Delta+\lambda)^{\frac{\alpha}{2}}(w^{\tau_0}+\varepsilon_k\psi)(\bar{x}^k)\\
&\quad=-(\Delta+\lambda)^{\frac{\alpha}{2}}w^{\tau_0}(\bar{x}^k)+(\Delta+\lambda)^{\frac{\alpha}{2}}(\varepsilon_k\psi)(\bar{x}^k)\\
&\quad=f(\bar{x}^k,u(\bar{x}^k),\nabla u(\bar{x}^k))-f((\bar{x}^k)',\bar{x}^k_n+\tau_0e_n,u^{\tau_0}(x^k),\nabla u^{\tau_0}(x^k))\\
&\qquad+\varepsilon_k(\Delta+\lambda)^{\frac{\alpha}{2}}\psi(\bar{x}^k)\\
&\quad\le f(\bar{x}^k,u(\bar{x}^k),\nabla u(\bar{x}^k))-f(\bar{x}^k,u^{\tau_0}(x^k),\nabla u^{\tau_0}(x^k))+C\varepsilon_k\\
&\quad=-c(\bar{x}^k)w^{\tau_0}(\bar{x}^k)-\sum_{i=1}^{n}c_i(\bar{x}^k)w_i^{\tau_0}(\bar{x}^k)+C\varepsilon_k\to 0,\quad \text{as } k\to\infty.
\end{aligned}
\tag{4.32}
$$

Moreover, it follows from (4.29) that

$$
\begin{aligned}
&-(\Delta+\lambda)^{\frac{\alpha}{2}}(w^{\tau_0}+\varepsilon_k\psi)(\bar{x}^k)\\
&\quad=c_{n,\alpha}P.V.\int_{\mathbb{R}^n}\frac{w^{\tau_0}(\bar{x}^k)+\varepsilon_k\psi(\bar{x}^k)-w^{\tau_0}(y)-\varepsilon_k\psi(y)}{e^{\lambda|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}dy\\
&\quad\ge C\int_{B^c_\delta(\bar{x}^k)}\frac{w^{\tau_0}(\bar{x}^k)+\varepsilon_k\psi(\bar{x}^k)-w^{\tau_0}(y)}{e^{\lambda|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}dy\\
&\quad\ge C\int_{B^c_\delta(\bar{x}^k)}\frac{-w^{\tau_0}(y)}{e^{\lambda|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}dy\\
&\quad=C\int_{B^c_\delta(\bar{x}^k)}\frac{|w^{\tau_0}(y)|}{e^{\lambda|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}dy\\
&\quad\to C\int_{B^c_\delta(0)}\frac{|w^{\tau_0}(z+\bar{x}^k)|}{e^{\lambda|z|}|z|^{n+\alpha}}dz,\quad \text{as } k\to\infty.
\end{aligned}
\tag{4.33}
$$

Denote

$$
u_k(x)=u(x+\bar{x}^k)\quad\text{and}\quad w_k^{\tau_0}(x)=w^{\tau_0}(x+\bar{x}^k).
$$

Since f is bounded, one can derive that $u(x)$ is at least uniformly Hölder continuous, therefore $u(x)$ is uniformly continuous, by the Arzelà–Ascoli theorem, up to extraction of a subsequence, so one has

$$
u_k(x)\to u_\infty(x),\quad x\in\mathbb{R}^n_+,\quad \text{as } k\to\infty.
$$

Putting together (4.32) and (4.33), we can deduce that

$$
w_k^{\tau_0}(x)\to 0,\quad x\in B^c_\delta(0),\ \text{as } k\to\infty.
$$

Therefore, it follows that

$$
u_\infty(x)=u_\infty^{\tau_0}(x),\quad x\in B^c_\delta(0).
$$

Recalling that $u>0$ in $\mathbb{R}^n_+$ and $u(x)\equiv 0$, $x\in\mathbb{R}^n\backslash\mathbb{R}^n_+$. Since $x^k\in\mathbb{R}^{n-1}\times(0,M_0+1]$, we choose $x^0\in\mathbb{R}^n\backslash\mathbb{R}^n_+$ (where x^0_n is sufficiently negative) such that $u_\infty(x^0)=0$. Then by (1.11),

$$\begin{aligned}0&=u_\infty(x^0)=u_\infty^{\tau_0}(x^0)=u_\infty((x^0)',x_n^0+\tau_0)\\&=u^0_\infty((x^0)',x^0_n+2\tau_0)\\&\ \ \vdots\\&=\lim_{k\to\infty}u_\infty((x^0)',x^0_n+k\tau_0)=A.\end{aligned}$$

which is absurd. Hence (4.27) must hold.

(II) Secondly, we claim that

$$\sup_{\mathbb{R}^n} w^\tau(x)\le 0,\quad \forall\tau\in(\tau_0-\epsilon,\tau_0]. \tag{4.34}$$

Equation (4.27) implies that there exists an ϵ small enough such that

$$\sup_{\mathbb{R}^{n-1}\times(0,M_1+1)} w^\tau(x)< 0,\quad \forall\tau\in(\tau_0-\epsilon,\tau_0]. \tag{4.35}$$

Consequently, we only need to prove that

$$\sup_{\mathbb{R}^{n-1}\times(M_1+1,+\infty)} w^\tau(x)\le 0,\quad \forall\tau\in(\tau_0-\epsilon,\tau_0]. \tag{4.36}$$

For convenience, we denote $D_M=:\mathbb{R}^{n-1}\times(M_1+1,+\infty)$. Note that D_M is an unbounded domain. Suppose that equation (4.36) is not valid, and then, through a direct calculation, at points $x\in D_M$ where $w^\tau(x)>0$, we have

$$\begin{aligned}-(\Delta+\lambda)^{\frac{\alpha}{2}}w^\tau(x)&=f(x,u(x),(\nabla_x u)(x))-f(x',x_n+\tau e_n,u^\tau(x),(\nabla_x u^\tau)(x))\\&\le f(x,u(x),(\nabla_x u)(x))-f(x,u^\tau(x),(\nabla_x u^\tau)(x))\\&=:-c(x)w^\tau(x)-\sum_{i=1}^n c_i(x)w_i^\tau(x),\end{aligned} \tag{4.37}$$

provided that

$$c(x)=-\frac{f(x,u,\nabla_x u)-f(x,u^\tau,\nabla_x u)}{u-u^\tau}\ge 0,$$

since $u(x)>u^\tau(x)\ge A-\delta$, for any $x\in D_M$.

Using (4.35) and (4.37), we can deduce by using Theorem 4.2 that for any $\tau\in(\tau_0-\epsilon,\tau_0]$, $\forall x\in D$, we obtain that $w^\tau(x)\le 0$ Thus we justify claim (4.34), which again is a contradiction with the definition of τ_0, and hence $\tau_0=0$.

STEP 3. We claim that u is strictly increasing with respect to x_n, $u(x)$ depends only on x_n and uniqueness.

In Step 2, we have already proved that $w^\tau(x) \leq 0$ in $\mathbb{R}^n_+, \forall \tau > 0$. Next, we shall prove that

$$w^\tau(x) < 0, \quad \text{in } \mathbb{R}^n_+, \quad \forall \tau > 0. \tag{4.38}$$

If not, then there exists $\tau_2 > 0$ and $\bar{x} \in \mathbb{R}^n_+$ such that $w^{\tau_2}(\bar{x}) = 0$, which implies $\bar{x}$ is the maximum point of $w(x)$, that is $\nabla w^{\tau_2}(\bar{x}) = 0$. In view of (1.10), we know that

$$\begin{aligned}-(\Delta+\lambda)^{\frac{\alpha}{2}} w^{\tau_2}(\bar{x}) &= f(\bar{x}, u(\bar{x}), (\nabla_x u)(\bar{x})) - f(\bar{x}', \bar{x}_n + \tau_2 e_n, u^{\tau_2}(\bar{x}), (\nabla_x u^{\tau_2})(\bar{x})) \\ &\leq f(\bar{x}, u(\bar{x}), (\nabla_x u)(\bar{x})) - f(\bar{x}, u^{\tau_1}(\bar{x}), (\nabla_x u^{\tau_2})(\bar{x})) = 0.\end{aligned}$$

On the other hand, due to $w^{\tau_2} \not\equiv 0$, we have

$$\begin{aligned}-(\Delta+\lambda)^{\frac{\alpha}{2}} w^{\tau_2}(\bar{x}) &= c_{n.\alpha} P.V. \int_{\mathbb{R}^n} \frac{w(\bar{x}) - w(y)}{e^{\lambda|\bar{x}-y|}|\bar{x}-y|^{n+\alpha}} dy \\ &= -c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{w(y)}{e^{\lambda|\bar{x}-y|}|\bar{x}-y|^{n+\alpha}} dy > 0,\end{aligned}$$

which is clearly a contradiction. Hence u is strictly increasing with respect to x_n.

Next, we claim that $u(x)$ depends only on x_n.

In fact, it can be seen from the above process that the argument still holds if we replace $u^\tau(x) := u(x + \tau\nu)$, where $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ with $\nu_n > 0$ being an arbitrary vector pointing upward. Applying similar sliding methods as above, we can get, for each such vector ν,

$$u(x + \tau\nu) > u(x), \quad \text{in } \mathbb{R}^n_+, \ \forall \tau > 0.$$

Accordingly, taking the limit $\nu_n \to 0$, again with the continuity of u, we deduce that

$$u(x + \tau\nu) \geq u(x),$$

for arbitrary vector ν with $\nu_n = 0$. If we replace ν by $-\nu$, we can also obtain $u(x + \tau\nu) = u(x)$, for arbitrary vector ν with $\nu_n = 0$.

Finally, we claim uniqueness for $u(x)$.

If not, we assume that u and v are two bounded solutions of system (1.10); we denote

$$\tilde{w}^\tau(x) = v(x) - u^\tau(x).$$

After a process similar to Step 1 and Step 2, we can get $\tilde{w}_k^{\tau}(x) \to 0$ (as $k \to \infty$) in $B_\delta^c(0)$, that is,

$$v_\infty(x) - u_\infty^{\tau_0}(x) \equiv 0, \quad x \in B_\delta^c(0).$$

Now we can choose a point $\hat{x}$ such that $\hat{x} + x^k \in B_\delta^c(0) \cap \partial\mathbb{R}_+^n$ (where x^k is the same as in Step 2), thus for any $\tau_0 > 0$, we have $\hat{x} + \tau_0 e_n + x^k \in \mathbb{R}_+^n$. Therefore, $v_\infty(\hat{x}) = 0$, $u_\infty^{\tau_0}(x) > 0$, and this gives a contradiction.

This concludes our proof of Theorem 1.8. ∎

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