# SMALL DEVIATION ESTIMATES
# AND SMALL BALL PROBABILITIES
# FOR GEODESICS IN LAST PASSAGE PERCOLATION

BY

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ABSTRACT

For the exactly solvable model of exponential last passage percolation on $\mathbb{Z}^2$, consider the geodesic $\Gamma_n$ joining $(0,0)$ and $(n,n)$ for large $n$. It is well known that the transversal fluctuation of $\Gamma_n$ around the line $x = y$ is $n^{2/3+o(1)}$ with high probability. We obtain the exponent governing the decay of the small ball probability for $\Gamma_n$ and establish that for small $\delta$, the probability that $\Gamma_n$ is contained in a strip of width $\delta n^{2/3}$ around the diagonal is $\exp(-\Theta(\delta^{-3/2}))$ uniformly in high $n$. We also obtain optimal small deviation estimates for the one point distribution of the geodesic showing that for $\frac{t}{2n}$ bounded away from 0 and 1, we have $\mathbb{P}(|x(t) - y(t)| \leq \delta n^{2/3}) = \Theta(\delta)$ uniformly in high $n$, where $(x(t), y(t))$ is the unique point where $\Gamma_n$ intersects the line $x + y = t$. Our methods are expected to go through for other exactly solvable models of planar last passage percolation and also, upon taking the $n \to \infty$ limit, expected to provide analogous estimates for geodesics in the directed landscape.

CONTENTS

## 1. Introduction and statement of main results

Small ball probabilities are fundamental objects associated to stochastic processes where one asks the following question: what is the probability that a stochastic process remains within a ball of small radius (in an appropriate norm) of a fixed function? One extensively studied case in the literature is that of the small ball probabilities of $C[0,1]$ valued Gaussian processes in the sup norm, of which Brownian motion and Brownian bridge are paradigm examples. If $\{B_t\}_{t\in[0,1]}$ denotes a standard Brownian motion or the standard Brownian Bridge, it is well known [16] that

$$\log \mathbb{P}(\sup_{t\in[0,1]} |B(t)| \leq \delta) \sim -\frac{\pi^2}{8}\delta^{-2} \quad \text{as } \delta \to 0.$$

Our objective in this paper is to investigate the corresponding question for the geodesics in planar last passage percolation models in the Kardar–Parisi–Zhang (KPZ) universality class.

We primarily work with the exactly solvable model of exponential last passage percolation on $\mathbb{Z}^2$. Let $\{\omega_v\}_{v\in\mathbb{Z}^2}$ be a configuration of independent and identically distributed rate one exponentials associated with the vertices of $\mathbb{Z}^2$. For any $u, v \in \mathbb{Z}^2$ such that $u$ is co-ordinate wise smaller than $v$, and an up-right path $\gamma$ from $u$ to $v$, we define the passage time of the path $\gamma$, denoted $\ell(\gamma)$, by

$$\ell(\gamma) := \sum_{w\in\gamma\backslash\{u,v\}} \omega_w;$$

i.e., the passage time of a path is the sum of the weights on it excluding the weight of the initial and final vertices.[1] The last passage time between $u$ and $v$, denoted $T_{u,v}$, is the maximum of $\ell(\gamma)$ where the maximum is taken over all up-right paths from $u$ to $v$. The last passage time from $\mathbf{0}$ to $\mathbf{n}$ (we shall denote the vertex $(r,r)$ by $\mathbf{r}$ for $r \in \mathbb{Z}$) will be denoted by $T_n$. Note that by the continuity of the exponential distribution, almost surely, between any two (ordered) vertices $u$ and $v$, there exists a unique path attaining the last passage time $T_{u,v}$; this path, denoted $\Gamma_{u,v}$, will be called the geodesic between $u$ and $v$. The geodesic between $\mathbf{0}$ and $\mathbf{n}$ will be denoted by $\Gamma_n$.

Observe that $\Gamma_n$ defines a stochastic process on $[\![0, 2n]\!]$ ($[\![a, b]\!]$ will denote the discrete interval $[a, b] \cap \mathbb{Z}$) in the following way: For $t \in [\![0, 2n]\!]$, define

$$\Gamma_n(t) := x(t) - y(t),$$

where $(x(t), y(t))$ is the unique point where $\Gamma_n$ intersects the line $\mathbb{L}_t := \{x + y = t\}$; note that the same definition also allows us to define $\Gamma_{u,v}(t)$ for any $u \in \mathbb{L}_0$ and $v \in \mathbb{L}_n$. Clearly, $\Gamma_n(\cdot)$ is an integer valued stochastic process on $[\![0, 2n]\!]$ pinned to 0 at either end and having $\pm 1$ increments, i.e., taking the same realizations as a simple random walk bridge on the same interval. The distribution of $\Gamma_n(\cdot)$, however, is very different. Unlike the $n^{1/2}$ fluctuations in the case of the SRW bridge, $\Gamma_n(\cdot)$ has fluctuations of the order $n^{2/3}$ [32, 14], 2/3 being the characteristic KPZ scaling exponent for correlation length. Hence it is natural to consider the small ball probability that $\sup |\Gamma_n(\cdot)|$, usually referred to as the **transversal fluctuation** of the geodesic, is upper bounded by $\delta n^{2/3}$ for some small positive $\delta$. Our first main theorem identifies the exponent governing the decay of this probability.

THEOREM 1: *There exists $\delta_0 > 0$ and positive constants $C_1, C_2, c_1, c_2$ such that for all $0 < \delta < \delta_0$ and for all $n \geq n_0(\delta)$, we have*

$$C_2 e^{-c_2 \delta^{-3/2}} \leq \mathbb{P}(\sup_{t \in [\![0, 2n]\!]} |\Gamma_n(t)| \leq \delta n^{2/3}) \leq C_1 e^{-c_1 \delta^{-3/2}}.$$

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[1] Note that we are excluding both the initial and final vertices in the computation of $\ell(\gamma)$ contrary to the more standard definition that includes both the endpoints. This is done for certain technical reasons explained later and our main results remain valid under the standard definition. Indeed, one can note that all paths $\gamma \colon u \to v$ share the vertices $u$ and $v$. Hence, the geodesic is not dependent on whether we include the weights at $u$ and $v$ in the definition of $\ell(\gamma)$. In fact, for our purposes, we shall also briefly need to consider a variant of the definition that excludes only the weight of the last vertex.

Our second main result concerns small deviations of the one point distribution $\Gamma_n(\cdot)$. As mentioned before, it is known that if $\frac{t}{2n}$ is bounded away from 0 and 1, then $\Gamma_n(t)$ has fluctuations of the order $n^{2/3}$ (if $t$ is close to 0 or $2n$, the fluctuation is of the order $t^{2/3}$ or $(2n-t)^{2/3}$ respectively, see [13, Theorem 3]) and the following theorem addresses the small deviation question for such values of $t$.

THEOREM 2: *There exists $\delta_0 > 0$ such that for all $\epsilon \in (0,1)$ and $0 < \delta < \delta_0$, there exist positive constants $C_3, c_3$ depending on $\epsilon$ such that for all $n \geq n_0(\delta, \epsilon)$ and $t \in [\![\epsilon n, (2-\epsilon)n]\!]$, we have*

$$c_3\delta \leq \mathbb{P}(|\Gamma_n(t)| \leq \delta n^{2/3}) \leq C_3\delta.$$

Notice that it is not necessary to take $\delta < \delta_0$ in the statements of Theorems 1 and 2; indeed, one can simply adjust the constants to ensure that the estimates hold for all $\delta$. Further, although we have stated the theorems for a fixed $\delta$ and $n \to \infty$, one can also get similar results if $\delta \to 0$ sufficiently slowly with $n$. For Theorem 1, it should suffice to assume $\delta n^{2/3} \to \infty$; see Section 6.2 for a more detailed discussion on this point. It will be clear from our proofs that Theorem 2 holds for any $\delta \geq 2n^{-2/3}$ for all $n$ sufficiently large (depending on $\epsilon$). The factor 2 is needed to handle the case of odd $t$; for even $t$, one gets the same statement for $\delta \geq n^{-2/3}$ (indeed, for $t$ odd, $\mathbb{P}(|\Gamma_n(t)| \leq 1) = 0$). Furthermore, for each $L > 0$ and any interval $I \subseteq [-Ln^{2/3}, Ln^{2/3}]$ with $|I| \geq 2$ (the lower bound of 2 is imposed to make sure that the corresponding probability is not 0 as before), we have that for some positive constants $c_3, C_3$ depending on $\epsilon$ and $L$, $\mathbb{P}(|\Gamma_n(t)| \in I) \in n^{-2/3}|I|[c_3, C_3]$ for $n$ sufficiently large. See Corollary 6.1 for a precise statement.

The $n^{2/3}$ fluctuation suggests the following scaling for $\Gamma_n$ akin to the scaling taking simple random walk to Brownian motion: we define a $C[0,1]$ valued stochastic process $\pi_n$ by setting

$$\pi_n(s) := n^{-2/3}\Gamma_n(2ns)$$

for $s \in [0,1]$ if $2ns \in \mathbb{Z}$ and extending $\pi_n$ by linear interpolation to all of $[0,1]$. One can show using the estimates in [14, 11, 13] that $\pi_n$ is tight in the topology of uniform convergence (see e.g. [30, Theorem 1.1 (a)] for the corresponding result in Poissonian LPP) and one would expect that there exists a $C[0,1]$ valued stochastic process $\pi$ such that $\pi_n \Rightarrow \pi$ weakly in $C[0,1]$, where the

limit $\pi$ corresponds to a geodesic in the universal space-time scaling limit of the last passage percolation landscape.

COROLLARY 3: *Let $\pi$ denote any subsequential weak limit of $\pi_n$ defined above in $C[0,1]$ equipped with the topology of uniform convergence. Then we have the following.*

(i) *There exists $\delta_0 > 0$ and positive constants $C_1, C_2, c_1, c_2$ such that for all $0 < \delta < \delta_0$, we have*

$$C_2 e^{-c_2 \delta^{-3/2}} \leq \mathbb{P}(\sup_{s \in [0,1]} |\pi(s)| \leq \delta) \leq C_1 e^{-c_1 \delta^{-3/2}}.$$

(ii) *There exists $\delta_1 > 0$ such that for all $s \in (0,1)$ and $0 < \delta < \delta_1$ there exists $C_3(s), c_3(s)$ (bounded away from $0$ and $\infty$ as long as $s$ is bounded away from $0$ and $1$) with*

$$c_3 \delta \leq \mathbb{P}(|\pi(s)| \leq \delta) \leq C_3 \delta.$$

The universal space-time scaling limit for the passage time landscape mentioned above has been recently established where the limiting object, a four parameter random field encoding the (scaled) last passage time between the space-time points $(x, s)$ and $(y, t)$ for $s < t$, is called the **directed landscape**. This was initially done in the set-up of the exactly solvable model of Brownian LPP in [17]; we would not define the directed landscape precisely, see [17] for details. Roughly speaking, the directed landscape is obtained from LPP under an affine scaling that scales the time direction by $n$, the space direction is scaled by $n^{2/3}$, the origin is fixed and the point $(n, n)$ is scaled to $(0, 1)$. Under this scaling the geodesic from $(0, 0)$ to $(n, n)$ becomes a geodesic from $(0, 0)$ to $(0, 1)$. The existence of the geodesic in the directed landscape between $(0, 0)$ and $(0, 1)$ in the directed landscape (a random continuous path moving vertically upwards) was shown in [17], and several of its properties (including weak convergence of geodesics in Brownian LPP to the geodesic in the directed landscape) was studied in [17, 18]. The weak convergence of the geodesics in exponential LPP to the geodesic in directed landscape was established in [20, Theorem 1.8]. Using this, one should get the variants of Corollary 3 for geodesics in the directed landscape. However, the space-time scaling considered in [20] is slightly different from the one considered above; we shall therefore refrain from making a precise statement in that set-up.

1.1. BACKGROUND AND RELATED RESULTS. Planar last passage percolation models are believed to belong to the KPZ universality class for a very general class of passage time distributions and are predicted to exhibit the universal scaling exponents 1/3 and 2/3 for the passage time and the transversal fluctuation of the geodesic respectively. Starting with the breakthrough work of Baik–Deift–Johansson [2] which established the exponent 1/3 (and the Tracy–Widom limit) for Poissonian LPP, this area has seen a great flurry of activity which has led to a similar analysis of a number of other exactly solvable models of planar last passage percolation including exponential and geometric LPP [31], and Brownian LPP [7]. Using the connection of exponential LPP with the Totally Asymmetric Simple Exclusion process, the first order behaviour of $T_{m,n}$ was already identified by Rost in [38] who established that for $\gamma$ bounded away from 0 and $\infty$, almost surely,

$$n^{-1}T_{\mathbf{0},(n,\gamma n)} \to (1+\sqrt{\gamma})^2.$$

In [31], it was shown that $T_{m,n}$ has the same law as the largest eigenvalue of a certain random matrix ensemble (Laguerre Unitary Ensemble (LUE)) and that for $\gamma$ bounded away from 0 and $\infty$, one has that

$$\text{(1)} \qquad \gamma^{1/6}(1+\sqrt{\gamma})^{-4/3}n^{-1/3}(T_{\mathbf{0},(n,\gamma n)} - (1+\sqrt{\gamma})^2 n)$$

converges weakly to the GUE Tracy–Widom distribution as $n \to \infty$. For our purposes, we shall need finite $n$ quantitative results, namely, uniform moderate deviation estimates for $n^{-1/3}(T_{\mathbf{0},(n,\gamma n)} - (1+\sqrt{\gamma})^2 n)$. These are provided in [34] using the tridiagonal form of LUE (a non-optimal estimate is also available in [4]).

Using the understanding of the fluctuation of the passage times, one can study the transversal fluctuations of the geodesic. Under some unproven assumptions, an upper bound on the transversal fluctuation exponent was proved for first passage percolation by Newman and co-authors (see e.g. [37]) and a rigorous lower bound was proved in [42, 41] for the related model of Brownian motion in a Poissonian potential; a general argument proving both conditional upper and lower bounds for FPP later appeared in [15]. Using similar arguments together with the moderate deviation estimates from [2], Johansson [32] first proved the 2/3 exponent for geodesics in Poissonian LPP. In particular, he proved that for every $\epsilon > 0$, the probability that either $\sup_t |\Gamma_n(t)| \geq n^{2/3+\epsilon}$ or $\sup_t |\Gamma_n(t)| \leq n^{2/3-\epsilon}$ goes to 0 as $n \to \infty$ (here $|\Gamma_n(\cdot)|$ is defined similarly

as before, but for Poissonian LPP). The same result was proved for Geometric LPP [33] and the same argument would also provide the same result for exponential LPP using the moderate deviation estimates from [34]. The transversal fluctuation exponent of 2/3 for exponential LPP was also established in [5] using a very different approach involving stationary LPP.

One point of note regarding [32] (and other similar results, e.g., [15]) is that the argument for the upper bound for the transversal fluctuation can be made quantitative, while the lower bound cannot. That is, one can tighten the argument in [32] and write down an upper bound for $\mathbb{P}(|\Gamma_n(t)| > xn^{2/3})$ for some large but fixed $x$ (indeed, one would even get the optimal result if one uses the optimal moderate deviation estimates from [32], see [11]). However, the lower bound typically involves a union bound over a discretization that is polynomially large in $n$, and hence the same argument cannot be used to get a bound for $\mathbb{P}(\sup_t |\Gamma_n(t)| < \delta n^{2/3})$ for a small but fixed $\delta$ (a quantitative but non optimal upper bound for $\mathbb{P}(\sup_t |\Gamma_n(t)| \leq n^{2/3-\epsilon})$ appeared in [9]). Theorem 1 therefore requires a somewhat different approach.

This paper falls within the general program of understanding the geodesic geometry in integrable models of last passage percolation using one point moderate deviation estimates together with percolation techniques. This program was initiated in [14] and has been followed up in [13, 12, 30, 43, 10, 24, 22]. Consequences of understanding the geodesic geometry have been further explored in [8, 11]. In [14], among other results, quantitative upper tail bounds for the transversal fluctuation of geodesics in Poissonian LPP were proved using moderate deviation estimates and a chaining argument. Although [14] used sub-optimal moderate deviation estimates from [4], the same argument together with the optimal moderate deviation estimates lead to the optimal upper tail bound for large transversal fluctuations:

$$(2) \qquad\qquad \mathbb{P}(\sup_t |\Gamma_n(t)| > xn^{2/3}) \leq e^{-cx^3}.$$

See [11, Proposition C.9] for the corresponding result (obtained by the same argument) written out in the exponential LPP setting. In the set-up of Poissonian LPP, [30] proved a matching lower bound $\mathbb{P}(\sup_t |\Gamma_n(t)| > xn^{2/3}) \geq e^{-c'x^3}$ establishing that the exponent is indeed optimal. As far as we are aware, this result does not explicitly appear in the literature for exponential LPP, but the arguments are robust and are expected to go through.

There is a separate line of works relevant for the current paper involving the related semi-discrete model of Brownian last passage percolation. Using the one point moderate deviation estimates and a special resampling property (Brownian Gibbs property) exhibited by a line ensemble associated with Brownian LPP, Hammond [29, 28, 27, 26] developed a deep understanding of geodesic geometry in Brownian LPP. Using similar techniques, [17] constructed the scaling limit– the directed landscape, as mentioned before.

1.2. Outline of the proofs and new contributions in this paper. As mentioned before, this paper continues the general program of understanding the geodesic geometry in integrable planar last passage percolation models using the one point moderate deviation estimates. We shall have occasion to use some of the estimates obtained and techniques developed in the prior works in this program, in particular from [14, 13, 12, 11] while also requiring several new ideas and technical ingredients. We provide a sketch of our arguments proving Theorems 1 and 2 in this subsection and point out the primary new contributions of our work. We discuss the upper and lower bounds in each case separately below. In each case, the upper bound turns out to be significantly easier than the lower bound.

1.2.1. *Theorem 1, upper bound.* The basic idea for the upper bound in Theorem 1 is rather simple: we first obtain upper tail estimates for the weight of best path from $\mathbf{0}$ to $\mathbf{n}$ constrained to be in the $\delta n^{2/3}$ strip and show that for small $\delta$, the probability that it is competitive with $T_n$ has the desired upper bound. Namely, if we let $T_n^\delta$ denote the weight of the best path from $\mathbf{0}$ to $\mathbf{n}$ that does not exit the strip $\{|x - y| \leq \delta n^{2/3}\}$, then we have the following proposition:

PROPOSITION 1.1: *There exists $C, c > 0$ such that for all $\delta$ sufficiently small and all $n$ sufficiently large (depending on $\delta$), we have*

$$\mathbb{P}\Big(T_n^\delta \geq 4n - \frac{C}{\delta}n^{1/3}\Big) \leq e^{-c\delta^{-3/2}}.$$

The idea behind Proposition 1.1 is the following: one can approximate $T_n^\delta$ by a sum of $\Theta(\delta^{-3/2})$ many i.i.d. variables, each of which roughly corresponds to the passage time across the two smaller sides of a $\delta^{3/2}n \times \delta n^{2/3}$ rectangle. Owing to the negativity of the mean of the GUE Tracy–Widom distribution, each of these variables have mean $4\delta^{3/2}n - c'\delta^{1/2}n^{1/3}$ and have sub-exponential tails at the scale $\delta^{1/2}n^{1/3}$. Once this is established, the proof of Proposition 1.1

is a simple application of a Bernstein type inequality for i.i.d. sub-exponential variables. With Proposition 1.1 at our disposal, completing the proof of the upper bound in Theorem 1 is easy by using the lower tail estimate for $T_n$. The argument proving Proposition 1.1 has already been used in the literature several times, sometimes with sub-optimal tails (see e.g. [8, 11, 12], and also [10, Proposition 4.2], where the optimal exponent was obtained in a related setting), but its consequence for the upper bound of small ball probability had not been noted before as far as we are aware. In the setting of Poissonian LPP, another relevant work is [21], where the mean, fluctuation and central limit behaviour is studied for an off-scale analogue of $T_n^\delta$, where the strip $\{|x - y| \leq \delta n^{2/3}\}$ in the definition of $T_n^\delta$ is replaced by the off-scale strip $\{|x - y| \leq n^{2/3-\epsilon}\}$.

The proofs of Proposition 1.1 and the upper bound in Theorem 1 are provided in Section 3.

1.2.2. *Theorem 1, lower bound.* This is the most technical part of our arguments and also the heart of major new technical achievements of this paper. Recall that we are trying to show that on an event of probability at least $e^{-c\delta^{-3/2}}$, the geodesic $\Gamma_n$ does not exit the strip $\{|x - y| \leq \delta n^{2/3}\}$. As we only require to prove this for sufficiently small $\delta$, we shall instead consider the following reparametrization for notational convenience. We shall show that there exists an absolute constant $M \geq 1$ such that with probability at least $e^{-c\delta^{-3/2}}$, the geodesic does not exit the strip $\{|x - y| \leq M\delta n^{2/3}\}$.

The main idea is to construct two favourable events. The first one, called **Inside**, shall depend on the inside of the strip $\{|x - y| \leq \delta n^{2/3}\}$ and shall ensure that:

- $T_n^\delta \geq 4n + \frac{C}{\delta}n^{1/3}$.
- The passage time for any points (not necessarily well-separated) inside the strip $\{|x - y| \leq \delta n^{2/3}\}$ is not too much smaller compared to its expectation.

Both these conditions can be shown to hold with probability at least $e^{-c\delta^{-3/2}}$, and as they are both increasing events, the FKG inequality ensures that $\mathbb{P}(\textbf{Inside})$ satisfies a desired probability lower bound.

The second event is a barrier event, called **Bar**, which ensures that there is a barrier straddling both of the longer sides of the rectangle

$$\{0 \leq x + y \leq 2n\} \cap \{|x - y| \leq \delta n^{2/3}\}$$

such that any path that spends a lot of time inside this barrier region incurs a penalty. Though the event **Bar**, as defined, will depend on the entire complement of the above-mentioned rectangle, it essentially puts constraints only in a region of width $O(\delta n^{2/3})$ around the rectangle. One can show that the barrier event **Bar** holds with probability at least $e^{-c\delta^{-3/2}}$, and since, by definition, this is independent of **Inside**, the intersection of the two favourable events have the desired probability lower bound of $e^{-c\delta^{-3/2}}$.

The rest of the argument is to show that on these favourable events, one indeed has that $\Gamma_n$ does not exit the strip $\{|x-y| \leq M\delta n^{2/3}\}$, which is achieved by ruling out both short and long excursions outside this strip. Indeed, if the geodesic has a short excursion (i.e., the starting and ending point of the excursion is separated by $O(\delta^{3/2}n)$ in the time direction) outside the strip $\{|x-y| \leq \delta n^{2/3}\}$ during which it also exits the wider strip $\{|x-y| \leq M\delta n^{2/3}\}$ this segment will then have a very high transversal fluctuation which would make it uncompetitive with the best path between the excursion endpoints restricted to be within the strip $\{|x-y| \leq \delta n^{2/3}\}$. Long excursions are ruled out using the definition of the barrier event and the fact that the best path inside the strip is ensured to be longer than typical.

A superficially similar scheme was adapted in [8, 11] to lower bound correlations between last passage times; however here we are faced with significant new technical challenges. Among other issues, the barrier event has to be suitably defined so that its probability can be appropriately lower bounded as a function of $\delta$, which requires the introduction of a number of new geometric ingredients. This is one of the primary new contributions in this work.

1.2.3. *Theorem 2, upper bound.* This follows quite easily from existing results in the literature, with the idea going back to [12]. In fact, the special case of $t = n$ and $\delta n^{2/3} = 1$ of Theorem 2 was alluded to in [12, Remark 3.13] in connection with the so-called midpoint problem where it was remarked that the probability that $\Gamma_n$ passes through $\frac{\mathbf{n}}{2}$ is $O(n^{-2/3})$. A sketch for the upper bound was provided there, and we adapt the same argument for our purposes.

Let us fix $t \in [\![\epsilon n, (2-\epsilon)n]\!]$ and without loss of generality let us assume $t$ is even. We consider the points $u_i = (i\delta n^{2/3}, -i\delta n^{2/3})$ and $v_i = \mathbf{n} + u_i$ for $i \in [\![-\frac{\delta^{-1}}{2}, \frac{\delta^{-1}}{2}]\!]$. Let $I_0$ denote the line segment on $\mathbb{L}_t$ between the points $(\frac{t}{2} + \frac{\delta}{2}n^{2/3}, \frac{t}{2} - \frac{\delta}{2}n^{2/3})$ and $(\frac{t}{2} - \frac{\delta}{2}n^{2/3}, \frac{t}{2} + \frac{\delta}{2}n^{2/3})$ and let $I_i = I_0 + u_i$. Clearly by translation invariance, for each $i$, $\mathbb{P}(|\Gamma_n(t)| \leq \delta n^{2/3}) = p$ is equal to

the probability $p_i$ that $\Gamma_{u_i,v_i}$ intersects $I_i$. Now clearly, $\sum_i p_i = \delta^{-1}p$ is upper bounded by the expected number of distinct points at which the geodesics $\Gamma_{u_i,v_i}$ can intersect the line $\mathbb{L}_t$. We shall show in Lemma 5.2, following an argument in [12], that the latter number is upper bounded independent of $\delta$, and this would provide the required upper bound for $p$. This proof is completed in Section 5.1.

1.2.4. *Theorem 2, lower bound.* The idea here is similar to the upper bound, but requires several different ingredients. Using the same notations as above, we need to show that $\sum p_i$ is bounded below away from 0 independently of $\delta$. It suffices to show that with probability bounded away from 0 independently of $\delta$, there exists an $i$ such that the geodesic $\Gamma_{u_i,v_i}$ intersects $I_i$. By planarity, and the ordering of geodesics, it is enough to show the following.

PROPOSITION 1.2: *Given $\epsilon \in (0,1)$ and $t \in [\![\epsilon n, (2-\epsilon)n]\!]$, there exists a constant $c > 0$ and a large enough positive choice of $M$ depending on $\epsilon$ such that for all $n > n_0(\epsilon)$, we have*

$$\mathbb{P}(\{\Gamma_{a_1,b_1}(t) = \Gamma_{a_2,b_2}(t)\} \cap \{|\Gamma_{a_1,b_1}(t)| \leq 2Mn^{2/3}\}) \geq c$$

*where $a_1, a_2$ denote the points $(-Mn^{2/3}, Mn^{2/3})$ and $(Mn^{2/3}, -Mn^{2/3})$ respectively and $b_i = a_i + \mathbf{n}$ for $i = 1, 2$.*

The proof of Proposition 1.2 hinges on constructing favourable geometric events which force the geodesics to coalesce. Several new ideas and ingredients are required to ensure that a common point on the geodesic is located in a restricted region in both space and time (on the line segment $\mathbb{L}_t \cap \{|x - y| \leq 2Mn^{2/3}\}$), which requires strong control on the geometry of the geodesics. A related, albeit simpler, result showing coalescence (without control of the coalescence point) was obtained before in [13, Proposition 3.1]. We use Proposition 1.2 to complete the proof of the lower bound of Theorem 2 in Section 5.2 and the proof of Proposition 1.2 is provided in Section 5.3.

ORGANIZATION OF THE PAPER. The rest of this paper is organised as follows. In Section 2, we collect the basic inputs we use in this work including the one point moderate deviation estimates and their consequences that have appeared in the literature. In Sections 3 and 4, we complete the proofs of the upper and lower bounds of Theorem 1 respectively. Section 5 contains the proof of Theorem 2. We finish with a discussion of potential extensions in Section 6.

## 2. Moderate deviation estimates and consequences

In this section, we recall some of the fundamental estimates about exponential LPP and their consequences that have appeared in the literature. These include moderate deviation estimates for the passage times, and estimates on controlling passage times across parallelograms and transversal fluctuations of geodesics. We shall heavily rely on these estimates throughout this paper.

Before starting, we introduce some notation. For a path $\gamma \colon u \to v$, where $u \leq v$ (i.e., $u$ is coordinate-wise smaller than $v$), we use the notation

$$\ell(\gamma) = \sum_{w \in \gamma \setminus \{u,v\}} \omega_w, \tag{3}$$

$$\underline{\ell}(\gamma) = \sum_{w \in \gamma \setminus \{v\}} \omega_w. \tag{4}$$

For a point $(x_1, y_1) \in \mathbb{R}^2$, we will often use the change of co-ordinates

$$\phi((x_1, y_1)) = x_1 + y_1, \tag{5}$$

$$\psi((x_1, y_1)) = x_1 - y_1. \tag{6}$$

Keeping in line with the literature, $\phi(\cdot)$ and $\psi(\cdot)$ will be called the time coordinate and the space coordinate of a point respectively. For points $u, v \in \mathbb{Z}^2$ with $u \leq v$, we use $T_{u,v}$ to denote the last passage time from $u$ to $v$, calculated by using weights given by $\ell$, i.e.,

$$T_{u,v} = \max_{\gamma \colon u \to v} \ell(\gamma).$$

We shall also have brief occasions to use a variant of the above definition of last passage time defined by replacing $\ell$ by $\underline{\ell}$ in the above display: this will be denoted by $\underline{T}_{u,v}$. Clearly, $\underline{T}_{u,v} \geq T_{u,v}$; in fact, we have that $\underline{T}_{u,v} = T_{u,v} + \omega_u$, and this implies that $\mathbb{E}\underline{T}_{u,v} = \mathbb{E}T_{u,v} + 1$.

   We shall use centered passage times; in general, we use a $\widetilde{\cdot}$ symbol over a variable to denote a centered (by its mean) variable, e.g.,

$$(7) \qquad\qquad \widetilde{T}_{u,v} = T_{u,v} - \mathbb{E}T_{u,v}.$$

We will use $\mathbb{L}_t$ to denote the line $\{x + y = t\} \subseteq \mathbb{Z}^2$.

2.1. One point moderate deviation estimates and passage times across parallelograms. As already mentioned, the correspondence between point to point passage times and the largest eigenvalue of LUE was obtained in [31]. The following sharp moderate deviation estimate for the latter has been obtained in [34].[2]

Proposition 2.1 ([34, Theorem 2]): *For each $\eta > 1$, there exist $C, c > 0$ depending on $\eta$ such that for all $m, n$ sufficiently large with $\eta^{-1} < \frac{m}{n} < \eta$ and all $y > 0$, we have the following:*

   (i)  $\mathbb{P}(T_{\mathbf{0},(m,n)} - (\sqrt{m} + \sqrt{n})^2 \geq yn^{1/3}) \leq Ce^{-c\min\{y^{3/2}, yn^{1/3}\}}.$
   (ii)  $\mathbb{P}(T_{\mathbf{0},(m,n)} - (\sqrt{m} + \sqrt{n})^2 \leq -yn^{1/3}) \leq Ce^{-cy^3}.$

   Observe that for $m, n$ as above, Proposition 2.1 implies that

$$(8) \qquad\qquad |\mathbb{E}T_{\mathbf{0},(m,n)} - (\sqrt{m} + \sqrt{n})^2| \leq Cn^{1/3}$$

for some positive constant $C$ (depending only on $\eta$). A similar statement holds for $\mathbb{E}\underline{T}_{\mathbf{0},(m,n)}$ simply because $\mathbb{E}\underline{T}_{\mathbf{0},(m,n)} = \mathbb{E}T_{\mathbf{0},(m,n)} + 1$.

   Proposition 2.1 can be used to control passage times across an on-scale parallelogram (i.e., a parallelogram whose dimensions in the time and space directions are $n$ and $n^{2/3}$ respectively). Such estimates were first obtained in [14] in the context of Poissonian LPP, and the details for the exponential LPP were worked out in [11]; we shall quote the latter source. We need to set up some further notation before stating the results.

---

[2] The correspondence to LUE holds when the last passage time includes the weights of the endpoints. However, for $m, n$ large the contribution of the endpoints is negligible and Proposition 2.1 holds for our definition of $T_{\mathbf{0},(m,n)}$ (and also $\underline{T}_{\mathbf{0},(m,n)}$).

We use $U_\delta^n$ for the rectangle which is defined by

$$(9) \qquad U_\delta^n = \{-\delta n^{2/3} \le \psi(u) \le \delta n^{2/3}\} \cap \{0 \le \phi(u) \le 2n\}.$$

In general, we suppress the dependence on $n$ and simply write $U_\delta$ for $U_\delta^n$. We will still use the latter notation in case we need to use the notation with some parameter other than $n$. We shall use notations $U_\delta^{n,\mathrm{L}}$ and $U_\delta^{n,\mathrm{R}}$ to denote the left and right line segments of $U_\delta^n$ respectively.[3] That is, we define

$$(10) \qquad \begin{aligned} U_\delta^{n,\mathrm{L}} &= U_\delta^n \cap \{\psi(u) = -\delta n^{2/3}\}, \\ U_\delta^{n,\mathrm{R}} &= U_\delta^n \cap \{\psi(u) = \delta n^{2/3}\}. \end{aligned}$$

To reduce clutter, we usually abbreviate these to just $U_\delta^{\mathrm{L}}$ and $U_\delta^{\mathrm{R}}$. Similarly, the two short sides of the parallelogram $U_\delta^n$ are denoted by $\underline{U}_\delta^n$ and $\overline{U}_\delta^n$ respectively. That is, we define

$$(11) \qquad \begin{aligned} \underline{U}_\delta^n &= U_\delta^n \cap \mathbb{L}_0, \\ \overline{U}_\delta^n &= U_\delta^n \cap \mathbb{L}_{2n}. \end{aligned}$$

These are similarly abbreviated to $\underline{U}_\delta$ and $\overline{U}_\delta$ respectively.

We will in general be quoting results from [11] for passage times across the parallelograms $U_\Delta$ for any fixed $\Delta > 0$. These results were originally written for $\Delta = 1$ but all the proofs straightforwardly generalize for any $\Delta$ (see [11, Lemma C.3, Lemma C.15]). Thus, we will directly quote such results for general $\Delta$ and not comment further. The following result controlling the tails of the maximum and minimum passage time from $\underline{U}_\Delta$ to $\overline{U}_\Delta$ for any fixed $\Delta > 0$ will be crucial for us.

PROPOSITION 2.2 ([11, Theorem 4.2]): *For any $\Delta > 0$, there exist constants $C_1, C_2, c_1, c_2$ depending on $\Delta$ such that for all $r, n$ large enough, we have:*
  (1) $\mathbb{P}(\sup_{u \in \underline{U}_\Delta, v \in \overline{U}_\Delta} \widetilde{T}_{u,v} \ge rn^{1/3}) \le C_1 e^{-c_1 \min\{r^{3/2}, rn^{1/3}\}}$.
  (2) $\mathbb{P}(\inf_{u \in \underline{U}_\Delta, v \in \overline{U}_\Delta} \widetilde{T}_{u,v} \le -rn^{1/3}) \le C_2 e^{-c_2 r^3}$.

We would like to point out that there is another slight discrepancy between Proposition 2.2 stated as above and the corresponding statement in [11], and the same is true for the other results below quoted from the same source. Indeed, [11] proves Proposition 2.2 with $T$ above replaced by $\underline{T}$. As we have

---

[3] The standard convention of rotating the picture by 45 degrees counter-clockwise so that time direction moves vertically upwards will often guide our choice of defining "left" and "right".

pointed out above, the exclusion of one of the endpoints does not change the estimates. For the sake of completeness, we shall explain, just this once, how to get Proposition 2.2 from the corresponding result in [11]. We shall ignore this issue for the subsequent results quoted in this section with the understanding that similar minor adaptations can be made to work in each of the cases.

As mentioned earlier, observe first that for any $u, v \in \mathbb{Z}^2, \underline{T}_{u,v} = T_{u,v} + \omega_u$ and hence $\mathbb{E}\underline{T}_{u,v} = \mathbb{E}T_{u,v} + 1$. It therefore follows that $\sup_{u,v} \widetilde{T}_{u,v} \leq \sup_{u,v} \widetilde{\underline{T}}_{u,v} + 1$ and item (1) of Proposition 2.2 is immediate from the corresponding result for $\underline{T}$. For item (2), let $U_*$ denote the line segment

$$\{u : \phi(u) = 1, |\psi(u)| \leq \Delta n^{2/3} + 1\}.$$

By using (8), one has that $\mathbb{E}\underline{T}_{u,v}, \mathbb{E}T_{u,v} \in (4n - Cn^{1/3}, 4n + Cn^{1/3})$ for large enough $n$ and all $u \in U_* \cup \underline{U}_\Delta, v \in \overline{U}_\Delta$. Clearly, this implies the crude bound

$$\inf_{u \in \underline{U}_\Delta, v \in \overline{U}_\Delta} \widetilde{T}_{u,v} \geq \inf_{u \in U_*, v \in \overline{U}_\Delta} \widetilde{\underline{T}}_{u,v} - 2Cn^{1/3}$$

and applying [11, Theorem 4.2] to the RHS above immediately gives item (2).

We will also require a version of Proposition 2.2 for passage times of highest weight paths restricted to be in some parallelogram. For any $u \leq v$ and a region $G \subseteq \mathbb{Z}^2$ satisfying $u, v \in G \cup \partial G$, we define the constrained passage time

$$(12) \qquad\qquad T_{u,v}^G = \sup_{\gamma: u \to v, \gamma \setminus \{u,v\} \subseteq G} \ell(\gamma).$$

For constrained last passage times, we define the centered version

$$\widetilde{T}_{u,v}^G = T_{u,v}^G - \mathbb{E}T_{u,v}.$$

Note that this notation is a slight deviation from (7): the centering here is done with the mean of the unrestricted passage time $T_{u,v}$ instead of $T_{u,v}^G$. We similarly define $\underline{T}_{u,v}^G$ and $\widetilde{\underline{T}}_{u,v}^G = \underline{T}_{u,v}^G - \mathbb{E}\underline{T}_{u,v}$. Since we will be using the terms $T_{u,v}^{U_\delta^n}$ and $\widetilde{T}_{u,v}^{U_\delta^n}$ very often, to reduce notational clutter, we introduce the shorthand notations

$$T_{u,v}^\delta = T_{u,v}^{U_\delta^n},$$
$$\widetilde{T}_{u,v}^\delta = \widetilde{T}_{u,v}^{U_\delta^n}.$$

We need the following tail estimates for constrained passage times between well separated points in an $n \times n^{2/3}$ rectangle.

PROPOSITION 2.3 ([11, Theorem 4.2]): *For any $\Delta > 0$, there exist constants* $C_1, C_2, c_1, c_2$ *depending on $\Delta$ such that for any $L > 0$ and all $r, n$ large enough depending on $L$, we have:*

(1) $\mathbb{P}(\sup_{u,v \in U_\Delta, \phi(v)-\phi(u) \geq \frac{n}{L}} \widetilde{T}_{u,v}^\Delta \geq rn^{1/3}) \leq C_1 e^{-c_1 \min\{r^{3/2}, rn^{1/3}\}}$.

(2) $\mathbb{P}(\inf_{u,v \in U_\Delta, \phi(v)-\phi(u) \geq \frac{n}{L}} \widetilde{T}_{u,v}^\Delta \leq -rn^{1/3}) \leq C_2 e^{-c_2 r}$.

In Proposition 2.2, we allowed the two points $u, v$ to vary on the shorter sides of the parallelogram $U_\Delta$. We now state an analogous result from [11] where the points $u, v$ vary on the long sides of $U_\Delta$ (i.e., $U_\Delta^{\text{L}}$ or $U_\Delta^{\text{R}}$), and are thus allowed to be arbitrarily close to each other in the time direction.

PROPOSITION 2.4 ([11, Lemma C.16]): *For any $\Delta > 0$, there exist constants* $C, c$ *depending on $\Delta$ such that for all $r, n$ large enough, we have*

$$\mathbb{P}(\inf_{u,v \in U_\Delta^{\text{L}}, \phi(u) \leq \phi(v)} \widetilde{T}_{u,v}^\Delta \leq -rn^{1/3}) \leq Ce^{-cr}.$$

*The same holds if $U_\Delta^{\text{L}}$ is replaced by $U_\Delta^{\text{R}}$.*

2.2. TRANSVERSAL FLUCTUATION ESTIMATES. For both the small ball and the one point estimates, we will require strong estimates on the upper tail of the transversal fluctuation of the point-to-point geodesic. The following result from [11] states that paths from $\mathbf{0}$ to $\mathbf{n}$ with a transversal fluctuation larger than $Mn^{2/3}$ incur a loss of order $M^2 n^{1/3}$ in weight with large probability.

PROPOSITION 2.5 ([11, Proposition 4.7]): *There exist constants $\xi, c_1 > 0$ such that for all $M$ sufficiently large, and all $n$ sufficiently large, the event (denoted by $\mathcal{G}$) that there exists a path $\gamma$ from $\mathbf{0}$ to $\mathbf{n}$ satisfying $\gamma \nsubseteq U_M$ and $\ell(\gamma) \geq 4n - \xi M^2 n^{1/3}$ satisfies*

$$\mathbb{P}(\mathcal{G}) \leq e^{-c_1 M^3}.$$

2.3. LOWER BOUNDS. To obtain the lower bounds in Theorem 1 and Theorem 2, one needs to show that the probability of certain unlikely events are nonetheless uniformly bounded away from 0. We need two results: one for the upper tail and one for the lower tail. The first result, which is a strengthening of [11, Lemma 4.9], shows that with probability bounded away from 0, last passage times (and constrained last passage times) can be arbitrarily larger than typical at the fluctuation scale. Although it is a rather straightforward consequence

of [11, Lemma 4.9], we believe that it can be potentially useful in other settings and hence state the following lemma separately.

LEMMA 2.6: *For any $\Delta > 0$, there exist constants $C, c > 0$ depending on $\Delta$ such that for every $x > 0$, we have for all sufficiently large $n$ (depending on $x$)*

$$\mathbb{P}\big(\inf_{u \in \underline{U}_\Delta, v \in \overline{U}_\Delta} T_{u,v}^\Delta \geq 4n + xn^{1/3}\big) \geq Ce^{-cx^{3/2}}.$$

*Proof.* We will define three independent events $A_1, A_2, A_3$ such that the event in question is a sub-event of $A_1 \cap A_2 \cap A_3$ and $\mathbb{P}(A_1 \cap A_2 \cap A_3) \geq Ce^{-cx^{3/2}}$. Define

$$A_1 = \Big\{ \inf_{u \in \underline{U}_\Delta} T_{u,\mathbf{n}/4}^\Delta \geq n - \frac{x}{2}n^{1/3} \Big\},$$

$$A_2 = \Big\{ T_{\mathbf{n}/4, 3\mathbf{n}/4}^\Delta \geq 2n + 2xn^{1/3} \Big\},$$

$$A_3 = \Big\{ \inf_{v \in \overline{U}_\Delta} T_{3\mathbf{n}/4, v}^\Delta \geq n - \frac{x}{2}n^{1/3} \Big\}.$$

The independence of $A_1, A_2, A_3$ is clear by definition. Also, by [11, Lemma 4.9] we have $\mathbb{P}(A_2) \geq C_1 e^{-c_1 x^{3/2}}$ for $n$ large enough depending on $x$. Observe now that by (8), we have that there exists a large enough constant $C_2$ such that

$$A_1 \supseteq \Big\{ \inf_{u \in \underline{U}_\Delta} \widetilde{T}_{u,\mathbf{n}/4}^\Delta \geq -\big(\frac{x}{2} - C_2\big)n^{1/3} \Big\}$$

and analogously

$$A_3 \supseteq \Big\{ \inf_{v \in \overline{U}_\Delta} \widetilde{T}_{3\mathbf{n}/4, v}^\Delta \geq -\big(\frac{x}{2} - C_2\big)n^{1/3} \Big\}.$$

Now on using Proposition 2.3, we get high probability lower bounds for $\mathbb{P}(A_1)$ and $\mathbb{P}(A_3)$. Combining this with the independence of $A_1, A_2, A_3$ along with $\mathbb{P}(A_2) \geq Ce^{-cx^{3/2}}$, we get the required lower bound for the probability of $\mathbb{P}(A_1 \cap A_2 \cap A_3)$. The fact that $A_1 \cap A_2 \cap A_3$ is a sub-event of the event in question is straightforward. Indeed, we have that

$$T_{u,v}^\Delta \geq T_{u,\mathbf{n}/4}^\Delta + T_{\mathbf{n}/4, 3\mathbf{n}/4}^\Delta + T_{3\mathbf{n}/4, v}^\Delta$$

deterministically for any $u, v$ in the respective line segments due to the exclusion of the endpoints in the definition of $\ell$. ∎

The next result, quoted from [11], provides a lower bound for unlikely events in the lower tail.

LEMMA 2.7 ([11, Lemma 4.10]): *For any $\Delta, M > 0$, we have that there exists a constant $c > 0$ (depending on $\Delta, M$) such that for all $n$ sufficiently large depending on $\Delta, M$, we have*

$$\mathbb{P}(\sup_{u \in \underline{U}_\Delta, v \in \overline{U}_\Delta} T_{u,v} \le 4n - Mn^{1/3}) \ge c.$$

## 3. Upper bound for the small ball probability

In this section, we provide the proof of Proposition 1.1 and use it to establish the upper bound in Theorem 1. The first step of the proof of Proposition 1.1 is to divide the rectangle $U_\delta$ into sub-rectangles of size $O(\delta^{3/2}n) \times \delta n^{2/3}$.[4]

Define for any $A > 0$ and for all $i \in \mathbb{Z}$,

$$(13) \quad U_{\delta,A,i}^n = \{-\delta n^{2/3} \le \psi(u) \le \delta n^{2/3}\} \cap \{2iA\delta^{3/2}n \le \phi(u) \le 2(i+1)A\delta^{3/2}n\}.$$

Note that $U_\delta = \bigcup_{i=0}^{\delta^{-3/2}/A-1} U_{\delta,A,i}^n$. The abbreviations $U_{\delta,A,i}, \underline{U}_{\delta,A,i}, \overline{U}_{\delta,A,i}$ are defined analogously to the abbreviations for the corresponding quantities of the parallelogram $U_\delta^n$.

The basic idea for the proof of Proposition 1.1 is that for small $\delta$, $T_n^\delta$ can be approximated by sums of passage times across $U_{\delta,A,i}$. Indeed, for $A$ sufficiently large, to be chosen appropriately later, we define

$$(14) \qquad\qquad Y_i = \sup_{u \in \underline{U}_{\delta,A,i}, v \in \overline{U}_{\delta,A,i}} \underline{T}_{u,v}^{U_{\delta,A,i}}.$$

Clearly, the $Y_i$ are i.i.d. across $i$. Let us define i.i.d. variables

$$Z_i = \frac{Y_i - 4A\delta^{3/2}n}{A^{1/3}\delta^{1/2}n^{1/3}}.$$

The next result gives information about the mean and upper tail of $Z_i$.

LEMMA 3.1: *For $A$ sufficiently large, there exist positive constants $c_2, c_3, C_3$ (independent of $\delta$) such that for all $n$ sufficiently large depending on $\delta$, we have:*

  (i) $\mathbb{E}Z_i \le -c_2$.
  (ii) $\mathbb{P}(Z_i \ge r) \le C_3 e^{-c_3 r}$ *for each $r > 0$.*

---

[4] Throughout the paper, we shall assume without loss of generality that $\delta n^{2/3}$ and $\delta^{3/2}n$ are even integers and ignore rounding issues arising from this, and also several other divisibility issues. This is done merely to reduce notational overhead and the reader can verify that the same arguments go through in the general case, with appropriate additions of floor and ceiling signs.

*Proof.* Observe first that in the expression (14), each of the terms satisfy $\underline{T}_{u,v}^{U_{\delta,A,i}} \leq \underline{T}_{u,v}$. In view of this, (ii) is an immediate consequence of Proposition 2.2, (i) and (8). Note that Proposition 2.2, as stated, is for the variables $T$ and not the variables $\underline{T}$. However, as remarked in a footnote earlier, the effect of the endpoints is negligible and Proposition 2.2 (and also (8)) also hold for the variables $\underline{T}$.

To prove (i), we show that $(A\delta^{3/2}n)^{-1/3}(\mathbb{E}Y_i - \mathbb{E}T_{A\delta^{3/2}n})$ can be made arbitrarily small by taking $A$ sufficiently large. Since

$$\mathbb{E}T_{A\delta^{3/2}n} \leq 4A\delta^{3/2}n - cA^{1/3}\delta^{1/2}n^{1/3}$$

(this is a consequence of the distributional convergence as in (1) and the fact that GUE Tracy–Widom distribution has negative mean, see [10, Lemma A.4]) for some $c > 0$ and $n$ sufficiently large, (i) follows by choosing $A$ appropriately large. See [10, Lemma 4.1] or [12, Lemma 2.4] for a complete argument. ∎

We can now give the proof of Proposition 1.1.

*Proof of Proposition 1.1.* Notice first that

$$(15) \qquad T_n^\delta \leq \sum_{i=0}^{\delta^{-3/2}/A-1} Y_i.$$

Indeed, this is the reason we used $\underline{T}$ instead of $T$ in the definition of $Y_i$. Thus, it suffices to show that for some constant $c_1$, we have

$$(16) \qquad \mathbb{P}\left( \sum_{i=0}^{\delta^{-3/2}/A-1} Y_i \geq 4n - \frac{c_1}{\delta}n^{1/3} \right) \leq Ce^{-c\delta^{-3/2}}.$$

Recalling the definition of $Z_i$, (16) is equivalent to showing

$$(17) \qquad \mathbb{P}\left( \frac{1}{\delta^{-3/2}/A} \sum Z_i \geq -c_1 A^{2/3} \right) \leq Ce^{-c\delta^{-3/2}}$$

for some choice of the constants $A, c_1$. We now use Lemma 3.1, by first fixing $A$ to be large enough and then choosing $c_1$ small enough to ensure that $c_1 A^{2/3} = c_2/2$ where $c_2$ is as in Lemma 3.1. This yields that the variables $Z_i + c_1 A^{2/3}$ are subexponential with negative mean. Now, (17) is an easy consequence of a Bernstein type concentration inequality for sums of i.i.d. variables with sub-exponential tails; see e.g. [40, Corollary 2.8.3]. This completes the proof of the proposition. ∎

In view of Proposition 1.1, the proof of the upper bound in Theorem 1 is almost immediate.

*Proof of Theorem 1, upper bound.* With $c_1$ as in the statement of Proposition 1.1, we know that for some constants $C, c, C', c'$ (independent of $\delta$), we have

$$
(18) \quad
\begin{aligned}
\mathbb{P}(\Gamma_n \subseteq U_\delta) &\leq \mathbb{P}\left(T_n^\delta \geq 4n - \frac{c_1}{\delta}n^{1/3}\right) + \mathbb{P}\left(T_n \leq 4n - \frac{c_1}{\delta}n^{1/3}\right) \\
&\leq Ce^{-c\delta^{-3/2}} + C'e^{-c'\delta^{-3}} \leq 2Ce^{-c\delta^{-3/2}}
\end{aligned}
$$

where in the second inequality above, we have used Proposition 1.1 along with the second part of Proposition 2.1. This completes the proof of the upper bound in Theorem 1. ∎

Before completing this section, let us give a sketch of an alternative proof of the upper bound in Theorem 1. This argument hinges on having a lower bound of the probability that geodesics have large transversal fluctuation (at the scale $n^{2/3}$). Such a result is known in Poissonian LPP; see [30, Proposition 1.4]. Even though the same argument should work for exponential LPP with minor modifications, we did not find the result in the literature for exponential LPP and hence will not attempt to write down a complete proof.

For $i \in \{0, 1, \ldots, \delta^{-3/2}\}$, define the points

$$
a_i = i\delta^{3/2}\mathbf{n} - \delta n^{2/3}(1, -1).
$$

Let $\Gamma_{a_i, a_{i+1}}$ denote the geodesic joining $a_i$ and $a_{i+1}$. On the event $\{\Gamma_n \subseteq U_\delta\}$, by the planar ordering of the geodesics, we know that each $\Gamma_{a_i, a_{i+1}}$ lies to the left of the geodesic $\Gamma_n$, and hence we have

$$
(19) \quad
\begin{aligned}
\{\Gamma_n \subseteq U_\delta\} &\subseteq \bigcap_{i=0}^{\delta^{-3/2}-1} \{\sup_{u \in \Gamma_{a_i, a_{i+1}}} \psi(u) \leq \delta n^{2/3}\} \\
&= \bigcap_{i=0}^{\delta^{-3/2}-1} \{\sup_{u \in \Gamma_{a_i, a_{i+1}}} \psi(u) - (-2\delta n^{2/3}) \leq 3\delta n^{2/3}\}.
\end{aligned}
$$

Note that the events on the right hand side are independent across $i$ and each event has probability bounded away from 1 by [30, Proposition 1.4] adapted to the exponential case. The upper bound in Theorem 1 follows.

## 4. Lower bound for the small ball probability

In this section, we will obtain the lower bound in Theorem 1. As discussed in the introduction, the strategy is to construct a favourable event with the requisite lower bound on its probability, on which the small ball event holds. We first define the favourable events, and state the probability lower bounds for them. Then we complete the proof of the lower bound in Theorem 1 assuming these. The proofs of the probability bounds are provided at the end of the section.

4.1. CONSTRUCTION OF FAVOURABLE EVENTS. Before coming to the construction of the needed events, we introduce the following notation:

$$
\begin{aligned}
\texttt{Left}_\delta^n &= \{u \in \mathbb{Z}^2 : \psi(u) < -\delta n^{2/3}\}, \\
\texttt{Right}_\delta^n &= \{u \in \mathbb{Z}^2 : \psi(u) > \delta n^{2/3}\}.
\end{aligned}
\tag{20}
$$

We shall use $\texttt{Left}_\delta$ and $\texttt{Right}_\delta$ as shorthands for the above, but in case we need to make use of the notations for something other than $n$, we shall use the more general notation.

We define three independent events $\mathbf{Bar}^\mathtt{L}$, $\mathbf{Inside}$ and $\mathbf{Bar}^\mathtt{R}$ (we use $\mathbf{Bar}$ to denote the event $\mathbf{Bar}^\mathtt{R} \cap \mathbf{Bar}^\mathtt{L}$) measurable with respect to the vertex weights in the regions $\texttt{Left}_\delta$, $U_\delta$ and $\texttt{Right}_\delta$ respectively. As already alluded to in the introduction, for some fixed large constant $M$, we shall show that on the event $\mathcal{E} = \mathbf{Bar}^\mathtt{L} \cap \mathbf{Inside} \cap \mathbf{Bar}^\mathtt{R}$, we have that $\Gamma_n \subseteq U_{M\delta}$, and obtain $\mathbb{P}(\mathcal{E}) \geq Ce^{-c\delta^{-3/2}}$ by lower bounding the probabilities of each constituent event separately; see Figure 1 for an illustration.

4.1.1. *Choice of parameters.* We shall fix $\delta$ to be sufficiently small throughout this section. Note that $K, k_0, k_1, \beta, M > 0$ will be constants which will appear in the definitions and will be fixed later in this section; all of them will be independent of $\delta$.

We now explicitly point out how the constants are fixed to prevent confusion later.

- $K$, a large positive constant is obtained by invoking Lemma 4.1.
- $k_0$, a large positive constant is fixed by invoking Lemma 4.8.
- $k_1$ is now defined to be $\max\{K, k_0\} + 2$.
- $\beta$ is an absolute constant not depending on any of the other parameters and is obtained from the statement of Lemma 4.9. In fact, we have $\beta = 2^{-2/3}\xi$, where $\xi$ appears in the statement of Proposition 2.5.
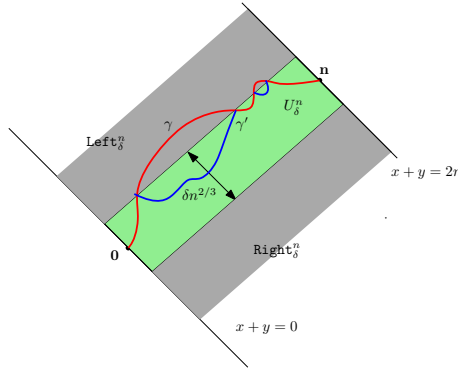
Figure 1. The basic set-up for the proof of the lower bound in
Theorem 1: The region $U_\delta$ is shaded in green. The event **Inside**
is a function of the vertex weights in $U_\delta$ and ensures that any
two points $u, v$ in $U_\delta$ that differ in the time co-ordinate by a
large constant times $\delta^{3/2}n$ have large $\widetilde{T}_{u,v}^\delta$, and even for points
$u, v$ that are not very well separated in the time coordinate, $\widetilde{T}_{u,v}^\delta$
is not too small. The events $\mathbf{Bar^L}$ and $\mathbf{Bar^R}$ are functions of the
vertex weights on the strips to the left and right of $U_\delta$ (denoted
$\mathtt{Left}_\delta$ and $\mathtt{Right}_\delta$) respectively and ensure that for any path $\gamma$
(marked in red) that has either a short or a long excursion
outside $U_\delta$ that exits $U_{M\delta}$ also, the excursion can be replaced
by a path in the interior of $U_\delta$ (marked in blue) that has higher
weight. Thus we ensure that on $\mathbf{Bar^L} \cap \mathbf{Inside} \cap \mathbf{Bar^R}$, one has
$\Gamma_n \subseteq U_{M\delta}$.

- The constant $M$ is fixed to be large enough compared to all the other
  parameters fixed so far so that the conclusion of Lemma 4.9 holds and
  (53) in the proof of Lemma 4.2 holds.

Note that all the above constants will be independent of $\delta$, and $n$ will be taken
to be sufficiently large depending on $\delta$ (and all the other parameters). Actually,
it will be clear from the proofs that it suffices to take $n \gg \delta^{-3/2}$; we will
comment more about this in Section 6. For the rest of this section, we shall
work with a fixed choice of parameters as described above, $\delta$ sufficiently small
and $n$ sufficiently large.

4.1.2. *The event* **Inside**. The event **Inside** is composed of two parts: (i) **Far**, which asks that for any two points $u, v \in U_\delta^{\mathtt{L}}$ (resp. $U_\delta^{\mathtt{R}}$) which are well separated in the time direction, $T_{u,v}^\delta$ is sufficiently larger than typical; and (ii) **Close**, which asks that for any two points $u, v$ in $U_\delta^{\mathtt{L}}$ (resp. $U_\delta^{\mathtt{R}}$) (not necessarily well separated) the constrained centered passage time $\widetilde{T}_{u,v}^\delta$ is not much smaller than typical. Let us now move towards defining the events formally.

*Definition of* **Far***:* For any positive integer $K$, let **Far** denote the event that for any two points $u, v \in U_\delta^{\mathtt{L}}$ (or both in $U_\delta^{\mathtt{R}}$) satisfying $\phi(v) - \phi(u) \geq K\delta^{3/2}n$, we have

$$(21) \qquad T_{u,v}^\delta \geq 2(\phi(v) - \phi(u)) + \frac{\phi(v) - \phi(u)}{\delta^{3/2}n}(\sqrt{\delta}n^{1/3}).$$

The following lemma provides a lower bound for $\mathbb{P}(\textbf{Far})$.

LEMMA 4.1: *For all $\delta$ sufficiently small, there exists a positive integer $K$ (independent of $\delta$) and positive constants $C, c$ (independent of $\delta$) such that*

$$\mathbb{P}(\textbf{Far}) \geq Ce^{-c\delta^{-3/2}}$$

*for all $n$ large enough depending on $\delta$.*

The proof of Lemma 4.1 has been postponed to Section 4.3.

*Definition of* **Close***:* The event **Close**, as described above, will control the $T_{u,v}^\delta$ where both $u$ and $v$ are close by points either on $U_\delta^{\mathtt{L}}$ or on $U_\delta^{\mathtt{R}}$. This will be defined as the intersection of several events indexed by $\mathtt{L}$ or $\mathtt{R}$ depending on which of the boundaries are being considered and also by $i$, which controls the location of the points in the time direction. For $k_1$ sufficiently large (compared to $K$ obtained from Lemma 4.1), a positive absolute constant $\beta$ and $i \in \{0, 1, \ldots, \frac{1}{k_1\delta^{3/2}} - 2\}$, we define the events $\textbf{Close}_i^{\mathtt{L}}$ and $\textbf{Close}_i^{\mathtt{R}}$ by setting

$$(\textbf{Close}_i^{\mathtt{L}})^c = \{ \inf_{\substack{u,v\in U_\delta^{\mathtt{L}}, \\ 2ik_1\delta^{3/2}n \leq \phi(u) \leq \phi(v) \leq 2(i+2)k_1\delta^{3/2}n}} \{T_{u,v}^\delta - 2(\phi(v) - \phi(u))\}$$

$$(22) \qquad\qquad\qquad\qquad\qquad\qquad \leq -\beta M^2(2k_1\delta^{3/2}n)^{1/3}\},$$

$$(\textbf{Close}_i^{\mathtt{R}})^c = \{ \inf_{\substack{u,v\in U_\delta^{\mathtt{R}}, \\ 2ik_1\delta^{3/2}n \leq \phi(u) \leq \phi(v) \leq 2(i+2)k_1\delta^{3/2}n}} \{T_{u,v}^\delta - 2(\phi(v) - \phi(u))\}$$

$$(23) \qquad\qquad\qquad\qquad\qquad\qquad \leq -\beta M^2(2k_1\delta^{3/2}n)^{1/3}\}.$$

We set $\mathbf{Close}_i = \mathbf{Close}_i^{\mathrm{L}} \cap \mathbf{Close}_i^{\mathrm{R}}$ and define

$$(24) \qquad \mathbf{Close} = \bigcap_{i=0}^{\frac{1}{k_1 \delta^{3/2}} - 2} \mathbf{Close}_i.$$

We have the following lower bound on $\mathbb{P}(\mathbf{Close})$.

LEMMA 4.2: *There exist positive constants $C, c$ (independent of $\delta$) such that for all $\delta$ sufficiently small, and for the parameters being chosen as in Section 4.1.1, we have*

$$\mathbb{P}(\mathbf{Close}) \geq Ce^{-c\delta^{-3/2}}.$$

The proof of Lemma 4.2 has been postponed to the end of Section 4.3.

Finally, we define the event $\mathbf{Inside}$ by

$$(25) \qquad\qquad \mathbf{Inside} = \mathbf{Far} \cap \mathbf{Close}.$$

The following lower bound on $\mathbb{P}(\mathbf{Inside})$ easily follows from Lemma 4.1 and Lemma 4.2.

LEMMA 4.3: *For all $\delta$ sufficiently small, and for the parameters chosen as in Section 4.1.1, we have that there exist positive constants $C, c$ (independent of $\delta$) such that for all $n$ large enough (depending on $\delta$),*

$$\mathbb{P}(\mathbf{Inside}) \geq Ce^{-c\delta^{-3/2}}.$$

*Proof.* Observe that both $\mathbf{Far}$ and $\mathbf{Close}$ are increasing events (i.e., for two weight configurations that are point-wise ordered, the event being satisfied for the smaller weight configuration implies that it is also satisfied for the larger one), and hence by the FKG inequality,

$$\mathbb{P}(\mathbf{Inside}) \geq \mathbb{P}(\mathbf{Far})\mathbb{P}(\mathbf{Close}).$$

The lemma immediately follows from Lemma 4.1 and Lemma 4.2.  ∎

4.1.3. *The event* $\mathbf{Bar}$. Before proceeding, we first remark that the events $\mathbf{Bar}^{\mathrm{L}}$ and $\mathbf{Bar}^{\mathrm{R}}$ will be defined symmetrically about the line $\{\psi(u) = 0\}$. Hence, it suffices to give the details of the construction of $\mathbf{Bar}^{\mathrm{L}}$. Our motivation while defining $\mathbf{Bar}^{\mathrm{L}}$ is to obtain an environment where paths from $\mathbf{0}$ to $\mathbf{n}$ which enter the region $\mathtt{Left}_\delta$ incur a loss in weight. To achieve this, $\mathbf{Bar}^{\mathrm{L}}$ will consist of two events $\mathbf{Short}^{\mathrm{L}}$ and $\mathbf{Long}^{\mathrm{L}}$ which will give the necessary weight deficits for short and long excursions into the region $\mathtt{Left}_\delta$ respectively.

*Definition of* **Long**$^{\text{L}}$*:* For $k_1$ as before, we define the event **Long**$^{\text{L}}$ as

(26)  $$\mathbf{Long}^{\text{L}} = \{ \sup_{\substack{u,v \in U_\delta^{\text{L}}: \\ \phi(v) - \phi(u) \geq k_1 \delta^{3/2} n}} \{ T_{u,v}^{\mathtt{Left}_\delta} - 2(\phi(v) - \phi(u)) \} < \frac{(\phi(v) - \phi(u))}{\delta^{3/2} n} (\sqrt{\delta} n^{1/3}) \}.$$

We have the following lower bound of the probability of the above event.

LEMMA 4.4: *For all $\delta$ sufficiently small, and for the parameters chosen as in Section 4.1.1, we have that there exist constants $C, c$ (independent of $\delta$) such that for all large enough $n$ (depending on $\delta$),*

$$\mathbb{P}(\mathbf{Long}^{\text{L}}) \geq Ce^{-c\delta^{-3/2}}.$$

The proof of Lemma 4.4 is postponed to Section 4.4.1.

*Definition of* **Short**$^{\text{L}}$*:* Similar to the definition of **Close**$^{\text{L}}$, **Short**$^{\text{L}}$ will also be defined as the intersection of **Short**$_i^{\text{L}}$, where $i$ shall index the location of the short excursion.

For $i \in \{0, 1, \ldots, \frac{1}{k_1 \delta^{3/2}} - 2\}$, we define **Short**$_i^{\text{L}}$ by setting $(\mathbf{Short}_i^{\text{L}})^c$ to be the event that for some $u, v \in U_\delta^{\text{L}}$ with

$$2ik_1 \delta^{3/2} n \leq \phi(u) \leq \phi(v) \leq 2(i+2)k_1 \delta^{3/2} n,$$

there exists $\gamma : u \to v$ satisfying $\gamma \setminus \{u, v\} \subseteq \mathtt{Left}_\delta$ and $\gamma \not\subseteq (\mathtt{Left}_{M\delta})^c$ such that

(27)  $$\ell(\gamma) > 2(\phi(v) - \phi(u)) - \beta M^2 (2k_1 \delta^{3/2} n)^{1/3}.$$

Having defined the events **Short**$_i^{\text{L}}$, we simply define

(28)  $$\mathbf{Short}^{\text{L}} = \bigcap_{i=0}^{\frac{1}{k_1 \delta^{3/2}} - 2} \mathbf{Short}_i^{\text{L}}.$$

We have the following lower bound for $\mathbb{P}(\mathbf{Short}^{\text{L}})$.

LEMMA 4.5: *For all $\delta$ sufficiently small, and for the parameters chosen as in Section 4.1.1, we have that there exist positive constants $C, c$ (independent of $\delta$) such that for $n$ large enough depending on $\delta$,*

$$\mathbb{P}(\mathbf{Short}^{\text{L}}) \geq Ce^{-c\delta^{-3/2}}.$$

The proof of Lemma 4.5 is postponed to Section 4.4.2.

Having completed the construction of $\mathbf{Bar}^{\mathrm{L}}$ by setting

$$\mathbf{Bar}^{\mathrm{L}} := \mathbf{Long}^{\mathrm{L}} \cap \mathbf{Short}^{\mathrm{L}}$$

we define $\mathbf{Bar}^{\mathrm{R}}$ by symmetry about the line $\{\psi(u) = 0\}$. We will not repeat the details for $\mathbf{Bar}^{\mathrm{R}}$, but would like to record that just in the same way as $\mathbf{Bar}^{\mathrm{L}}$, we also have

(29) $$\mathbf{Bar}^{\mathrm{R}} = \mathbf{Long}^{\mathrm{R}} \cap \mathbf{Short}^{\mathrm{R}}$$

for analogously defined events $\mathbf{Long}^{\mathrm{R}}$ and $\mathbf{Short}^{\mathrm{R}}$.

We have the following lower bound for the probability of $\mathbf{Bar} := \mathbf{Bar}^{\mathrm{L}} \cap \mathbf{Bar}^{\mathrm{R}}$.

LEMMA 4.6: *For all $\delta$ small enough, and for the parameters chosen as in Section 4.1.1, we have that there exist positive constants $C, c$ (independent of $\delta$) such that for all $n$ large enough (depending on $\delta$),*

$$\mathbb{P}(\mathbf{Bar}) \geq Ce^{-c\delta^{-3/2}}.$$

*Proof.* Since $\mathbf{Bar}^{\mathrm{L}} = \mathbf{Short}^{\mathrm{L}} \cap \mathbf{Long}^{\mathrm{L}}$, where both the events are decreasing, by Lemma 4.4, Lemma 4.5 and the FKG inequality, we have that for $n$ large enough depending on $\delta$,

$$\mathbb{P}(\mathbf{Bar}^{\mathrm{L}}) \geq C_1 e^{-c_1 \delta^{-3/2}}.$$

By the symmetry about the line $\{\psi(u) = 0\}$, we also obtain that

$$\mathbb{P}(\mathbf{Bar}^{\mathrm{R}}) \geq C_1 e^{-c_1 \delta^{-3/2}}.$$

The lemma now follows immediately by using the independence of $\mathbf{Bar}^{\mathrm{L}}$ and $\mathbf{Bar}^{\mathrm{R}}$.  ∎

4.2. PROOFS OF THEOREM 1, LOWER BOUND AND COROLLARY 3(i). As mentioned before, our main interest is in the event $\mathcal{E}$ defined by

(30) $$\mathcal{E} = \mathbf{Inside} \cap \mathbf{Bar}.$$

We first show that a small ball event is indeed satisfied on the event $\mathcal{E}$.

LEMMA 4.7: *We have the deterministic inclusion $\mathcal{E} \subseteq \{\Gamma_n \subseteq U_{M\delta}\}$, where the parameters are chosen as in Section 4.1.1.*

*Proof.* We prove by contradiction. Clearly if $\Gamma_n \subseteq U_\delta$, we are done, so let us suppose that for some weight configuration in the event $\mathcal{E}$, we have that $\Gamma_n \not\subseteq U_\delta$.

First consider the case that $\Gamma_n \cap \text{Left}_\delta \neq \emptyset$. In this case, there must exist $t_1, t_2 \in \mathbb{N}$ satisfying $0 \leq t_1 < t_2 \leq 2n$ such that

$$\Gamma_n|_{[t_1+1, t_2-1]} \subseteq \text{Left}_\delta \quad \text{and} \quad \Gamma_n(t_1) = \Gamma_n(t_2) = -\delta n^{2/3}.$$

Let us define the points $u, v$ by

$$(\phi(u), \psi(u)) = (t_1, \Gamma_n(t_1)) \quad \text{and} (\phi(v), \psi(v)) = (t_2, \Gamma_n(t_2)),$$

that is, $u$ and $v$ are the locations of the geodesics at times $t_1$ and $t_2$ respectively. We need to consider two separate cases (refer to Figure 1).

CASE 1: $(\phi(v) - \phi(u)) \geq k_1 \delta^{3/2} n$.

In this case we reach a contradiction due to the definitions of the events $\mathbf{Long}^{\text{L}}$ and $\mathbf{Far}$. Indeed, $\mathbf{Long}^{\text{L}}$ implies that

$$(31) \qquad \ell(\Gamma_n|_{[t_1, t_2]}) < 2(\phi(v) - \phi(u)) + \frac{\phi(v) - \phi(u)}{\delta^{3/2} n}(\sqrt{\delta} n^{1/3})$$

while $\mathbf{Far}$ implies (note that $k_1 > K$ due to our choice of the parameters) that there exists a path $\gamma \colon u \to v$ such that $\gamma \subseteq U_\delta$ satisfying

$$(32) \qquad \ell(\gamma) \geq 2(\phi(v) - \phi(u)) + \frac{\phi(v) - \phi(u)}{\delta^{3/2} n}(\sqrt{\delta} n^{1/3}).$$

It is clear that (31) and (32) contradict the fact that $\Gamma_n|_{[t_1, t_2]}$ is a geodesic from $u$ to $v$.

CASE 2: $(\phi(v) - \phi(u)) < k_1 \delta^{3/2} n$.

In this scenario, there exists an $i_0 \in \{0, 1, \ldots, \frac{1}{k_1 \delta^{3/2}} - 2\}$ satisfying

$$2 i_0 k_1 \delta^{3/2} n \leq \phi(u) \leq \phi(v) \leq 2(i_0 + 2) k_1 \delta^{3/2} n.$$

Now, the definition of the event $\mathbf{Short}_{i_0}^{\text{L}}$ and $\mathbf{Close}$ forces $\Gamma_n|_{[t_1, t_2]} \subseteq U_{M\delta}$. Indeed, if we had $\Gamma_n|_{[t_1, t_2]} \cap \text{Left}_{M\delta} \neq \emptyset$, the event $\mathbf{Short}_{i_0}^{\text{L}}$ would imply that

$$(33) \qquad \ell(\Gamma_n|_{[t_1, t_2]}) \leq 2(\phi(v) - \phi(u)) - \beta M^2 (2 k_1 \delta^{3/2} n)^{1/3}.$$

On the other hand, the event $\mathbf{Close}_{i_0}^{\text{L}}$ would imply that there exists $\gamma \colon u \to v$ satisfying $\gamma \subseteq U_\delta$ along with

$$(34) \qquad \ell(\gamma) > 2(\phi(v) - \phi(u)) - \beta M^2 (2 k_1 \delta^{3/2} n)^{1/3},$$

thereby contradicting that $\Gamma_n|_{[t_1, t_2]}$ is a geodesic between $u$ and $v$. In effect, we have shown that $\Gamma_n|_{[t_1, t_2]} \subseteq U_{M\delta}$.

By an identical reasoning and the symmetric definition of the event $\mathbf{Bar}^{\text{R}}$, we can handle the case $\Gamma_n \cap \text{Right}_\delta \neq \emptyset$, and this completes the proof. ∎

We are now ready to complete the proof of the lower bound in Theorem 1.

*Proof of Theorem 1, lower bound.* In view of Lemma 4.7 (we are using that $M$ is a fixed constant), we need only show that for all $\delta$ small enough, we have that there exist positive constants $C, c$ (independent of $\delta$) such that for all $n$ large enough depending on $\delta$,

$$
(35) \qquad \mathbb{P}(\mathcal{E}) \geq C e^{-c\delta^{-3/2}},
$$

where all the parameters are obtained as described in Section 4.1.1. By the definition of the event $\mathcal{E}$ and the independence of **Bar** and **Inside**, we have that

$$
(36) \qquad \mathbb{P}(\mathcal{E}) = \mathbb{P}(\textbf{Bar})\mathbb{P}(\textbf{Inside}).
$$

The lower bound for $\mathbb{P}(\textbf{Inside})$ follows from Lemma 4.3 and the lower bound for $\mathbb{P}(\textbf{Bar})$ follows from Lemma 4.6. Piecing these ingredients together, we obtain that for $n$ large enough depending on $\delta$, (35) holds. ∎

Before completing the proofs postponed earlier in this section, we quickly complete the straightforward proof of Corollary 3(i) using Theorem 1.

*Proof of Corollary 3(i).* By using the definition $\pi_n(s) := n^{-2/3}\Gamma_n(2ns)$ for $2ns \in \mathbb{Z}$ along with Theorem 1, we have

$$
C_2 e^{-c_2\delta^{-3/2}} \leq \mathbb{P}(\sup_{s\in[0,1], 2ns\in\mathbb{Z}} |\pi_n(s)| \leq \delta) \leq C_1 e^{-c_1\delta^{-3/2}}.
$$

Since $\pi_n(s)$ is linearly interpolated to all values $s \in [0,1]$ by using the values of $\pi_n(s)$ for $s \in [0,1] \cap \frac{1}{2n}\mathbb{Z}$, the above immediately implies

$$
C_2 e^{-c_2\delta^{-3/2}} \leq \mathbb{P}(\sup_{s\in[0,1]} |\pi_n(s)| \leq \delta) \leq C_1 e^{-c_1\delta^{-3/2}}.
$$

Now, we just need to pass to the limit. To do this, first note that the mapping $f \mapsto \sup_{s\in[0,1]} f(s)$ is a continuous map from $C[0,1]$ to $\mathbb{R}^+$, where the former is equipped with the topology of uniform convergence and the latter is equipped with the Euclidean topology. Consider $\pi$, a subsequential weak limit $\pi$ of $\pi_n$, that is, $\pi_{n_i} \Rightarrow \pi$ as $i \to \infty$ ($\Rightarrow$ denotes weak convergence) for some subsequence $\{n_i\}$. Then by the continuous mapping theorem, we have that

$$
\sup_{s\in[0,1]} \pi_{n_i}(s) \Rightarrow \sup_{s\in[0,1]} \pi(s)
$$

as $i \to \infty$, and we get the result by applying the Portmanteau theorem. ∎

The rest of this section is devoted to the proofs of Lemma 4.1, Lemma 4.2, Lemma 4.4 and Lemma 4.5. Before proceeding with the proofs, we introduce the notation $U_{\delta,i}^n$ for the $i$th rectangle when the strip $U_\delta^n$ divides into $\delta^{-3/2}$ many smaller rectangles. That is, let

$$(37) \quad U_{\delta,i}^n = \{-\delta n^{2/3} \leq \psi(u) \leq \delta n^{2/3}\} \cap \{2i\delta^{3/2}n \leq \phi(u) \leq 2(i+1)\delta^{3/2}n\}.$$

For convenience, we make the above definition for all $i \in \mathbb{Z}$ instead of just $i \in \{0, \ldots, \delta^{-3/2} - 1\}$, though only the values in $\{0, \ldots, \delta^{-3/2} - 1\}$ correspond to subrectangles in $U_\delta^n$. Again, we will simply write $U_{\delta,i}^n$ as $U_{\delta,i}$. Similar to the definitions (10) for the left and right sides of $U_\delta^n$, we denote the left and right sides of $U_{\delta,i}^n$ by $U_{\delta,i}^{n,\mathrm{L}}$ and $U_{\delta,i}^{n,\mathrm{R}}$ respectively. These will be abbreviated to $U_{\delta,i}^{\mathrm{L}}$ and $U_{\delta,i}^{\mathrm{R}}$.

4.3. LOWER BOUNDS FOR EVENTS INSIDE $U_\delta$. This subsection is devoted to the proofs of Lemma 4.1 and Lemma 4.2, i.e., we prove the lower bounds for the probabilities of the events **Far** and **Close**. The first one is more involved and will take up most of this subsection.

*Proof of Lemma 4.1.* We locally define $\mathbf{Far}_\alpha$ to be the event that for any two points $u, v \in U_\delta^{\mathrm{L}}$ (or both in $U_\delta^{\mathrm{R}}$) satisfying $\phi(v) - \phi(u) \geq K\delta^{3/2}n$, we have

$$T_{u,v}^\delta \geq 2(\phi(v) - \phi(u)) + \frac{\phi(v) - \phi(u)}{\delta^{3/2}n}(\alpha\sqrt{\delta}n^{1/3}).$$

It suffices to prove that there exists a positive integer $K$, positive constants $C, c$ and some $\alpha > 1$ (all independent of $\delta$) such that

$$\mathbb{P}(\mathbf{Far}_\alpha) \geq Ce^{-c\delta^{-3/2}}.$$

Indeed, this is because $\mathbb{P}(\mathbf{Far}_\alpha)$ is decreasing in $\alpha$, and $\mathbb{P}(\mathbf{Far})$ corresponds to $\alpha = 1$. We shall construct an event $\mathcal{C}$ with the requisite probability lower bound such that $\mathcal{C}$ implies $\mathbf{Far}_\alpha$ for some large $\alpha$ whose value will be chosen later.

To illustrate our strategy, let us consider two points $u, v$ on $U_\delta^{\mathrm{L}}$ which satisfy $\phi(v) - \phi(u) \geq K\delta^{3/2}n$, where $K$ will be chosen large later (the case $u, v \in U_\delta^{\mathrm{R}}$ can be handled by an identical argument). It is immediate that we have

$$(38) \qquad \left\lfloor \frac{\phi(v) - \phi(u)}{2\delta^{3/2}n} \right\rfloor \geq K/2 - 1,$$

Thus, there exists a positive integer $K_{u,v}$ such that

$$(39) \qquad\qquad K_{u,v} \geq K/2 - 3$$

and a positive integer $a$ depending on $u, v$ satisfying

$$2(a-1) \leq \frac{\phi(u)}{\delta^{3/2}n} \leq 2a$$
$$\leq 2(a+1)$$

$$(40) \qquad\qquad \vdots$$

$$\leq 2(a + K_{u,v}) \leq \frac{\phi(v)}{\delta^{3/2}n}$$
$$\leq 2(a + K_{u,v} + 1).$$

Let $w_i$ denote the point $i\delta^{3/2}\mathbf{n}$. This gives that

$$(41) \qquad T^\delta_{u,v} \geq T^\delta_{u,w_{a+1}} + \sum_{j=2}^{K_{u,v}-1} T^\delta_{w_{a+j-1},w_{a+j}} + T^\delta_{w_{a+K_{u,v}-1},v}.$$

Our aim now is to construct an event having probability at least $Ce^{-c\delta^{-3/2}}$, on which we have that the right-hand side in (41) is larger than typical for each pair $u, v$ satisfying the conditions in the statement of the lemma. We will do this by ensuring that the terms $T^\delta_{w_{a+j-1},w_{a+j}}$ in (41) are larger than typical for $j = 2$ to $K_{u,v} - 3$ while the terms $T^\delta_{u,w_{a+1}}$ and $T^\delta_{w_{a+K_{u,v}-3},v}$ are not too small compared to their typical value; see Figure 2.

For each $i \in \{0, \ldots, \delta^{-3/2} - 2\}$, define the events

$$(42) \qquad \mathcal{A}_i = \{T^\delta_{w_i,w_{i+1}} \geq 2(2\delta^{3/2}n) + 4(\alpha\sqrt{\delta}n^{1/3})\}.$$

Note that the $\mathcal{A}_i$ satisfy

$$(43) \qquad\qquad \mathbb{P}(\mathcal{A}_i) \geq Ce^{-c\alpha^{3/2}} > c_3$$

for some $C, c$ (independent of $\delta$) coming from Lemma 2.6.

We now define the events $\mathcal{B}_i$ as follows:

$$(44) \qquad \mathcal{B}_i = \{ \inf_{u,v \in U_{\delta,i} \cup U_{\delta,i+1}, \phi(v)-\phi(u) \geq 2\delta^{3/2}n} \widetilde{T}^\delta_{u,v} \geq -\alpha\sqrt{\delta}n^{1/3}\}.$$

Note that by an application of Proposition 2.3, we have that for some constants $C_2, c_2$,

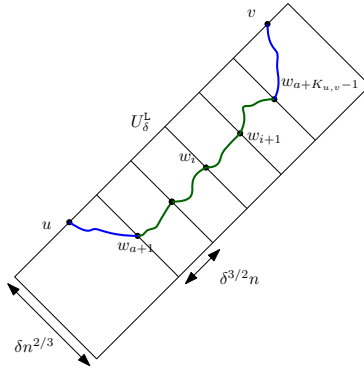$$(45) \qquad\qquad \mathbb{P}(\mathcal{B}_i) \geq 1 - C_2 e^{-c_2\alpha} > \frac{1}{2}$$

Figure 2. Proof of Lemma 4.1: for $u, v \in U_\delta^{\mathsf{L}}$ with $\phi(v) - \phi(u) \geq K\delta^{3/2}n$, we lower bound $T_{u,v}^\delta$ by the weight of the concatenated path showed in the figure. The initial and final segments (marked in blue) denote the paths attaining weights $T_{u,w_{a+1}}^\delta$ and $T_{w_{a+K_{u,v}-1},v}^\delta$ respectively; both these paths are ensured to be not too small compared to typical. The intermediate green segments denote the paths attaining $T_{w_i,w_{i+1}}^\delta$ and these are ensured to be larger than typical. For $K_{u,v}$ large, these two conditions ensure that $T_{u,v}^\delta$ is larger than typical, as required.

for all $\alpha$ sufficiently large. Finally, we define the event $\mathcal{C}$ by

$$(46) \qquad \mathcal{C} = \bigcap_{i=0}^{\delta^{-3/2}-2} (\mathcal{A}_i \cap \mathcal{B}_i).$$

Notice now that the events $\mathcal{A}_i$ and $\mathcal{B}_i$ are all increasing events measurable with respect to the vertex weights in $U_\delta$. By using the FKG inequality along with (43) and (45), we have

$$(47) \qquad \begin{aligned} \mathbb{P}(\mathcal{C}) &\geq \left( \prod_i \mathbb{P}(\mathcal{A}_i) \prod_i \mathbb{P}(\mathcal{B}_i) \right) \\ &\geq (c_3/2)^{\delta^{-3/2}-2} \\ &\geq e^{-c_5 \delta^{-3/2}} \end{aligned}$$

for some $c_5 > 0$.

It remains to prove that $\mathcal{C} \subseteq \mathbf{Far}_\alpha$ for some $\alpha$ sufficiently large. We shall only consider the case of $u, v \in U_\delta^{\mathsf{L}}$, the case of $u, v \in U_\delta^{\mathsf{R}}$ can be handled by an identical argument. On the event $\mathcal{C}$, we have that in (41), for any $u, v$ satisfying the conditions in the statement of the lemma,

$$(48) \qquad T_{w_{a+j-1}, w_{a+j}}^\delta \geq 2(2\delta^{3/2}n) + 4(\alpha\sqrt{\delta}n^{1/3})$$

for all $j \in \{2, \ldots, K_{u,v} - 1\}$. This is a straightforward consequence of the definition of the events $\mathcal{A}_i$. By using the definition of $\mathcal{B}_i$, we have that on $\mathcal{C}$,

$$(49) \qquad T_{u, w_{a+1}}^\delta \geq \mathbb{E}T_{u, w_{a+1}} - \alpha\sqrt{\delta}n^{1/3}.$$

By using (8), we have that for all $\alpha$ large enough, on $\mathcal{C}$,

$$(50) \qquad T_{u, w_{a+1}}^\delta \geq 2(\phi(w_{a+1}) - \phi(u)) - \frac{3\alpha}{2}\sqrt{\delta}n^{1/3}.$$

By an analogous argument, we obtain that on $\mathcal{C}$, we have for all $\alpha$ large enough,

$$(51) \qquad T_{w_{a+K_{u,v}-1}, v}^\delta \geq 2(\phi(v) - \phi(w_{a+K_{u,v}-1})) - \frac{3\alpha}{2}\sqrt{\delta}n^{1/3}.$$

On combining (48), (50) and (51) with (41), we deduce that on $\mathcal{C}$, for all $\alpha$ large enough

$$(52) \qquad \begin{aligned} T_{u,v}^\delta - 2(\phi(v) - \phi(u)) &\geq (4K_{u,v} - 11)\alpha\sqrt{\delta}n^{1/3} \\ &\geq 3K_{u,v}\alpha\sqrt{\delta}n^{1/3} \\ &\geq \frac{\phi(v) - \phi(u)}{\delta^{3/2}n}(\alpha\sqrt{\delta}n^{1/3}), \end{aligned}$$

and we now fix $\alpha$ to be one such value which in addition satisfies $\alpha > 1$. Note that we have used (39) along with the fact that $K$ can be fixed to be large to obtain the last two inequalities. Thus, we have established that $\mathcal{C} \subseteq \mathbf{Far}_\alpha$, which together with (47) completes the proof. ∎

We shall now prove Lemma 4.2.

*Proof of Lemma 4.2.* Recall the parameter $M$ which we have been using; it appears in the definitions of $\mathbf{Close}_i^{\mathsf{L}}$ and $\mathbf{Close}_i^{\mathsf{R}}$ in (22), (23) and in Lemma 4.7. Using Proposition 2.4 along with the fact that $M$ is fixed to be much larger than all the other parameters in Section 4.1.1, it is clear that for $n$ large enough depending on $\delta$, we have

$$(53) \qquad \begin{aligned} \mathbb{P}(\mathbf{Close}_i^{\mathsf{L}}) &\geq 1 - C_1 e^{-c_1 M^2} \geq 1/2, \\ \mathbb{P}(\mathbf{Close}_i^{\mathsf{R}}) &\geq 1 - C_1 e^{-c_1 M^2} \geq 1/2. \end{aligned}$$

Since $\mathbf{Close}_i^{\mathtt{L}}$ and $\mathbf{Close}_i^{\mathtt{R}}$ are increasing events, we have by the FKG inequality

$$(54) \qquad\qquad \mathbb{P}(\mathbf{Close}_i) \geq 1/4.$$

Again, the $\mathbf{Close}_i$ are all increasing events, and by the FKG inequality, we have

$$(55) \qquad\qquad \mathbb{P}(\mathbf{Close}) \geq (1/4)^{\frac{\delta^{-3/2}}{k_1}-1} \geq C_2 e^{-c_2\delta^{-3/2}},$$

completing the proof of the lemma. ∎

4.4. LOWER BOUNDS FOR THE BARRIER EVENTS. This subsection is devoted to the proofs of Lemma 4.4 and Lemma 4.5, i.e., we prove the lower bounds for the probabilities of the events $\mathbf{Long}^{\mathtt{L}}$ and $\mathbf{Short}^{\mathtt{L}}$.

4.4.1. *Lower bound for the event* $\mathbf{Long}^{\mathtt{L}}$. We need the following result to prove Lemma 4.4. Recall the notation $U_{\delta,i}$ and $U_{\delta,i}^{\mathtt{L}}$ for all $i \in \mathbb{Z}$ from (37).

LEMMA 4.8: *For all $\delta$ sufficiently small, there exist constants $C, c$ (independent of $\delta$) and a positive integer $k_0$ (independent of $\delta$) such that for any integers $i, j$ with $j - i = k \geq k_0$, for all $n$ large enough depending on $\delta$, we have that*

$$\mathbb{P}\Big(\sup_{u\in U_{\delta,i}^{\mathtt{L}},v\in U_{\delta,j}^{\mathtt{L}}} \{T_{u,v}^{\mathtt{Left}_\delta}-2(\phi(v)-\phi(u))\} \geq \frac{(\phi(v)-\phi(u))}{\delta^{3/2}n}(\sqrt{\delta}n^{1/3})\Big) \leq Ce^{-ck} < 1.$$

*Proof.* Observe that the event in question is measurable with respect to the vertex weights in the region $\mathtt{Left}_\delta$ (indeed, this is one of the reasons we chose the definition of $T$ to exclude the weights of the endpoints). Condition on the occurrence of the event in question and refer to it locally in this proof as $\mathcal{A}$. Thus on $\mathcal{A}$, there exist $u \in U_{\delta,i}^{\mathtt{L}}, v \in U_{\delta,j}^{\mathtt{L}}$ such that

$$T_{u,v}^{\mathtt{Left}_\delta} - 2(\phi(v)-\phi(u)) \geq \frac{(\phi(v)-\phi(u))}{\delta^{3/2}n}(\sqrt{\delta}n^{1/3}).$$

Define the points

$$w_1 = ((i-1)\delta^{3/2}n, (i-1)\delta^{3/2}n) \quad \text{and} \quad w_2 = ((j+2)\delta^{3/2}n, (j+2)\delta^{3/2}n).$$

Let $\mathcal{A}_1$ be defined by

$$\mathcal{A}_1 = \Big\{T_{w_1,u}^{(\mathtt{Left}_\delta)^c} - 2(\phi(u)-\phi(w_1)) \geq -\frac{1}{4}\frac{(\phi(v)-\phi(u))}{\delta^{3/2}n}(\sqrt{\delta}n^{1/3})\Big\}$$

and define $\mathcal{A}_2$ by

$$\mathcal{A}_2 = \Big\{T_{v,w_2}^{(\mathtt{Left}_\delta)^c} - 2(\phi(w_2)-\phi(v)) \geq -\frac{1}{4}\frac{(\phi(v)-\phi(u))}{\delta^{3/2}n}(\sqrt{\delta}n^{1/3})\Big\}.$$

Note that $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent of each other and the vertex weights in the region $\texttt{Left}_\delta$. Also, $\mathcal{A}_1$ and $\mathcal{A}_2$ together with the conditioning imply the event $\mathcal{A}_3$ which is defined by

$$(56) \qquad \mathcal{A}_3 = \left\{ T_{w_1,w_2} - 2(\phi(w_2) - \phi(w_1)) \geq \frac{1}{2} \frac{(\phi(v) - \phi(u))}{\delta^{3/2}n} (\sqrt{\delta}n^{1/3}) \right\}.$$

Indeed, this is simply a consequence of $T_{w_1,w_2} \geq T_{w_1,u}^{(\texttt{Left}_\delta)^c} + T_{u,v}^{\texttt{Left}_\delta} + T_{v,w_2}^{(\texttt{Left}_\delta)^c}$, which holds because both the endpoints are excluded in the definition of $\ell$. Note that $\phi(w_2) - \phi(w_1) = 2(k+3)\delta^{3/2}n$. From this discussion, it is clear that

$$(57) \qquad \min_{u \in U_{\delta,i}^{\texttt{L}}} \mathbb{P}(\mathcal{A}_1) \min_{v \in U_{\delta,j}^{\texttt{L}}} \mathbb{P}(\mathcal{A}_2)\mathbb{P}(\mathcal{A}) \leq \mathbb{P}(\mathcal{A}_3) \leq Ce^{-c\left(\frac{1}{(k+3)^{1/3}} \frac{\phi(v)-\phi(u)}{\delta^{3/2}n}\right)^{3/2}}$$
$$\leq C_1 e^{-c_1 k}.$$

The second inequality follows by using Proposition 2.1 and the third inequality follows by taking $k_0$ large, observing that $\phi(v) - \phi(u) \geq 2(k-2)\delta^{3/2}n$.

It remains to establish good lower bounds for $\min_u \mathbb{P}(\mathcal{A}_1)$ and $\min_v \mathbb{P}(\mathcal{A}_2)$; by symmetry, we only deal with the former. Note that by translation invariance and Proposition 2.3, we have that for $n$ large enough depending on $\delta$ and for some constants $C_2, c_2$ (independent of $u$ and $\delta$),

$$(58) \qquad \mathbb{P}(\mathcal{A}_1^c) \leq C_2 e^{-c_2 \frac{\phi(v)-\phi(u)}{\delta^{3/2}n} \frac{\sqrt{\delta}n^{1/3}}{(\phi(u)-\phi(w_1))^{1/3}}} \leq C_3 e^{-c_3 k} \leq \frac{1}{2}$$

by taking $k_0$ to be large enough. Indeed, (58) is obtained by using Proposition 2.3 on the parallelogram $U_{\delta,i-1} \cup U_{\delta,i}$, using (8) to change the centering from $\mathbb{E}T_{w_1,u}$ to $2(\phi(u) - \phi(w_1))$ and noting that

$$(59) \qquad T_{w_1,u}^{(\texttt{Left}_\delta)^c} \geq T_{w_1,u}^{U_{\delta,i-1} \cup U_{\delta,i}}.$$

The second inequality in (58) follows by using that $(\phi(v)-\phi(u)) \geq 2(k-2)\delta^{3/2}n$ along with $2\delta^{3/2}n \leq \phi(u) - \phi(w_1) \leq 4\delta^{3/2}n$. Thus, (57), (58) and an analogous lower bound on $\min_v \mathbb{P}(\mathcal{A}_2)$ immediately yields

$$(60) \qquad \mathbb{P}(\mathcal{A}) \leq 4C_1 e^{-c_1 k}$$

if we take $k_0$ to be large enough.  ∎

We are now ready to prove Lemma 4.4.

*Proof of Lemma 4.4.* Denote by $\mathcal{A}_{i,j}$ the event whose probability is considered in Lemma 4.8. By the choice of $k_1$ and $k_0$ in Section 4.1.1, we have that for $u, v$ as in the definition of $\mathbf{Long}^{\mathrm{L}}$ we have $\phi(v) - \phi(u) \geq k_1 \delta^{3/2} n \geq (k_0 + 2) \delta^{3/2} n$, and this implies that $u \in U_{\delta,i}^{\mathrm{L}}, v \in U_{\delta,j}^{\mathrm{L}}$ for some $i, j$ satisfying $0 \leq i \leq j \leq \delta^{-3/2} - 1$ and $j - i \geq k_0$. Thus, we have that

$$
(61) \qquad \mathbb{P}\Big( \sup_{\substack{u,v \in U_\delta^{\mathrm{L}}: \\ \phi(v) - \phi(u) \geq k_1 \delta^{3/2} n}} \{T_{u,v}^{\mathtt{Left}_\delta} - 2(\phi(v) - \phi(u))\} < \frac{(\phi(v) - \phi(u))}{\delta^{3/2} n}(\sqrt{\delta} n^{1/3}) \Big)
$$

$$
\geq \mathbb{P}\Big( \bigcap_{i,j : j - i \geq k_0} \mathcal{A}_{i,j}^c \Big).
$$

It thus suffices to show that the latter expression is at least $C' e^{-c' \delta^{-3/2}}$ for some constants $C', c'$. If we let $k$ be the variable for $j - i$, observe that for a fixed value of $k$, there are $\delta^{-3/2} - k$ pairs of $(i,j)$ satisfying $j - i = k$ and $0 \leq i \leq j \leq \delta^{-3/2} - 1$. Also, by translation invariance, we have that $\mathbb{P}(\mathcal{A}_{i,j})$ depends only on $k = j - i$. By the FKG inequality for the decreasing events $\mathcal{A}_{i,j}^c$, we have

$$
(62) \qquad
\begin{aligned}
\mathbb{P}\Big( \bigcap_{i,j : j - i \geq k_0} \mathcal{A}_{i,j}^c \Big) &\geq \prod_{i,j : j - i \geq k_0} \mathbb{P}(\mathcal{A}_{i,j}^c) \\
&\geq \prod_{k_0 \leq k \leq \delta^{-3/2} - 1} (1 - Ce^{-ck})^{\delta^{-3/2} - k} \\
&\geq \prod_{k \geq k_0} (1 - Ce^{-ck})^{\delta^{-3/2}}.
\end{aligned}
$$

Noting that $\prod_{k \geq k_0}(1 - Ce^{-ck}) > 0$ because $\sum_{k \geq 1} e^{-ck} < \infty$, we have the needed result on using (61). ∎

### 4.4.2. *Lower bound for the event* $\mathbf{Short}^{\mathrm{L}}$.

In this subsection, we will obtain the required lower bound for $\mathbb{P}(\mathbf{Short}^{\mathrm{L}})$ in Lemma 4.5. We first need the following result.

LEMMA 4.9:    *For any* $k_1 > 0$ *fixed, all* $\delta$ *sufficiently small, and all* $i \in \{0, 1, \ldots, \frac{1}{k_1 \delta^{3/2}} - 2\}$, *we have that there exists an absolute constant* $\beta$ *and positive constants* $C, c$ *(independent of* $\delta$*) such that for* $M$ *large enough (independent of* $\delta$*) and for* $n$ *large enough depending on* $\delta$,

$$
\mathbb{P}((\mathbf{Short}_i^{\mathrm{L}})^c) \leq Ce^{-cM^3} < 1/2.
$$

*Proof.* By translation invariance, we can restrict to $i = 0$. Define

$$w_1 = \left( -\frac{1}{2}\delta n^{2/3}, \frac{1}{2}\delta n^{2/3} \right) - k_1 \delta^{3/2}\mathbf{n}$$

and

$$w_2 = 2k_1\delta^{3/2}\mathbf{n} + \left( -\frac{1}{2}\delta n^{2/3}, \frac{1}{2}\delta n^{2/3} \right) + k_1\delta^{3/2}\mathbf{n};$$

see Figure 3. Now, condition on the occurrence of the event $(\mathbf{Short}_0^{\mathsf{L}})^c$. Thus, there exist $u, v \in U_\delta^{\mathsf{L}}$ along with a path $\gamma\colon u \to v$ satisfying (27) along with $0 \leq \phi(u) \leq \phi(v) \leq 4k_1\delta^{3/2}n$. Analogous to $\mathcal{A}_1$ and $\mathcal{A}_2$ in the proof of Lemma 4.8, consider the events

$$\mathcal{B}_1 = \{T_{w_1,u}^{(\mathtt{Left}_\delta)^c} - 2(\phi(u) - \phi(w_1)) \geq -\frac{M^2}{2}\beta(2k_1\delta^{3/2}n)^{1/3}\}$$

and

$$\mathcal{B}_2 = \{T_{v,w_2}^{(\mathtt{Left}_\delta)^c} - 2(\phi(w_2) - \phi(v)) \geq -\frac{M^2}{2}\beta(2k_1\delta^{3/2}n)^{1/3}\}.$$

Note that $\mathcal{B}_1$ and $\mathcal{B}_2$ are independent of each other and the vertex weights in $\mathtt{Left}_\delta$. Also, $\mathcal{B}_1$ and $\mathcal{B}_2$ together with the conditioning imply the event $\mathcal{B}_3$ which is defined by

(63)
$$\mathcal{B}_3 = \{\exists\gamma_1\colon w_1 \to w_2 \text{ with } \gamma_1 \cap \mathtt{Left}_{M\delta} \neq \emptyset$$
$$\text{and } \ell(\gamma_1) - 2(\phi(w_2) - \phi(w_1)) \geq -2\beta M^2(2k_1\delta^{3/2}n)^{1/3}\}.$$

Indeed, if $\chi_1\colon w_1 \to u$ is the path attaining $T_{w_1,u}^{(\mathtt{Left}_\delta)^c}$ and $\chi_2\colon v \to w_2$ is the path attaining $T_{v,w_2}^{(\mathtt{Left}_\delta)^c}$, then we can define $\gamma_1\colon w_1 \to w_2$ as the concatenation of $\chi_1, \gamma, \chi_2$ and use $\ell(\gamma_1) \geq \ell(\chi_1) + \ell(\gamma) + \ell(\chi_2)$ which holds because both the endpoints were excluded in the definition of $\ell$.

Since $k_1$ is some fixed constant (independent of $\delta$), we can invoke Proposition 2.5 to say that there exists an absolute constant $\beta$ such that we have

(64)
$$\mathbb{P}(\mathcal{B}_3) \leq Ce^{-cM^3}$$

for large enough $n$. Indeed, we can just define $\beta = 2^{-2/3}\xi$, where $\xi$ is in the statement of Lemma 2.5. Using this, we have that

(65)
$$\min_u \mathbb{P}(\mathcal{B}_1) \min_v \mathbb{P}(\mathcal{B}_2)\mathbb{P}((\mathbf{Short}_0^{\mathsf{L}})^c) \leq \mathbb{P}(\mathcal{B}_3) \leq Ce^{-cM^3},$$

where both the minimum's in the above equation are taken over the set

$$\{z \in U_\delta^{\mathsf{L}} : \phi(z) \in [0, 4k_1\delta^{3/2}n]\}.$$
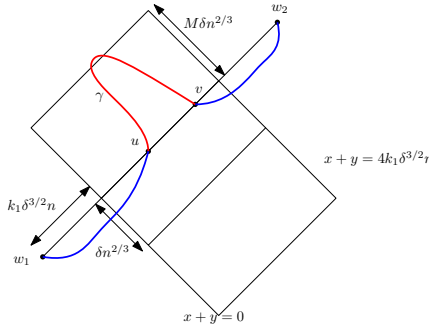
Figure 3. Proof of Lemma 4.9: we show that it is unlikely for there to be a short excursion $\gamma$ (marked in red) between $u$ and $v$ outside $U_\delta$ that exits $U_{M\delta}$ and has weight not too small. To show this, on the event of existence of such a $\gamma$, we construct a path $\gamma_1$ (the concatenation of the paths marked in blue with $\gamma$) such that $\gamma_1$ is a path between two fixed points, well separated in the time direction, such that $\ell(\gamma_1)$ is not too small. Notice that $\gamma_1$ has large transversal fluctuation and hence the latter event is shown to be unlikely using Proposition 2.5.

By the symmetry of the setting, we only prove a lower bound for $\min_u \mathbb{P}(\mathcal{B}_1)$ and omit the proof of the corresponding lower bound for $\min_v \mathbb{P}(\mathcal{B}_2)$. Note that for all possible points $u$, we have that $2k_1\delta^{3/2}n \leq \phi(u) - \phi(w_1) \leq 6k_1\delta^{3/2}n$, and thus $\frac{(\phi(u)-\phi(w_1))^{1/3}}{2\beta(2k_1\delta^{3/2}n)^{1/3}}$ is bounded away from 0 and $\infty$. Hence, on noting that $T_{w_1,u}^{(\mathbf{Left}_\delta)^c} \geq T_{w_1,u}^V$ for the rectangle $V$ defined by

$$V = \{-\delta n^{2/3} \leq \psi(u) \leq \delta n^{2/3}\} \cap \{-2k_1\delta^{3/2}n \leq \phi(u) \leq 6k_1\delta^{3/2}n\}$$

and by using Proposition 2.3 for $V$, along with (8) to change the centering from $\mathbb{E}T_{w_1,u}$ to $2(\phi(u) - \phi(w_1))$, we get that for $M$ large enough (independent of $\delta$),

$$(66) \qquad \min_u \mathbb{P}(\mathcal{B}_1) \geq 1 - C_2 e^{-c_2 M^2} \geq \frac{1}{2}.$$

On combining this and the analogous lower bound of $\min_v \mathbb{P}(\mathcal{B}_2)$ with (65), we get that for large enough $M$ (independent of $\delta$), we have

$$(67) \qquad \mathbb{P}((\mathbf{Short}_0^\mathrm{L})^c) \leq 4Ce^{-cM^3}$$

provided that $n$ is large enough depending on $\delta$.  ∎

Instead of the conditioning argument presented above, another slightly different way to prove Lemma 4.9 would be to define $\mathcal{B}$ by

$$\mathcal{B} = \Big\{ \inf_{\substack{u',v'\in V^{\mathrm{L}}, \\ \phi(v')-\phi(u')\geq 2k_1\delta^{3/2}n}} \{T_{u',v'}^{V} - 2(\phi(v')-\phi(u'))\} \geq -\frac{M^2}{2}\beta(2k_1\delta^{3/2}n)^{1/3}\Big\},$$

where $V^{\mathrm{L}}$ refers to the left long side of the rectangle $V$. One can now observe that $(\mathbf{Short}_0^{\mathrm{L}})^c\cap\mathcal{B}\subseteq\mathcal{B}_3$, where the events on the left hand side are independent. $\mathbb{P}(\mathcal{B})$ can be lower bounded by Proposition 2.3 and (8), and the rest of the proof follows similarly.

We end this section by proving Lemma 4.5.

*Proof of Lemma 4.5.* Note that the $\mathbf{Short}_i^{\mathrm{L}}$ are all decreasing events. Using the FKG inequality along with Lemma 4.9 immediately gives that for $n$ large enough depending on $\delta$,

$$(68) \qquad \mathbb{P}(\mathbf{Short}^{\mathrm{L}}) \geq (1 - C_1 e^{-c_1 M^3})^{\frac{\delta^{-3/2}}{k_1}-1} \geq (1/2)^{c_2\delta^{-3/2}} = Ce^{-c\delta^{-3/2}}.$$

Note that as mentioned in Section 4.1.1, $M$ is fixed to be large enough compared to the other parameters so as to satisfy the conclusion of Lemma 4.9. ∎

## 5. One point small deviation estimates for the geodesic

In this section, we will provide the proof of Theorem 2 and the proof of Corollary 3(ii). We start with the upper bound in Theorem 2, which is easier.

5.1. PROOF OF THEOREM 2, UPPER BOUND. As already mentioned in the introduction, we shall complete the proof of the upper bound using the idea outlined in [12, Remark 3.13]. The main ingredient that we require, an estimate of the number of disjoint geodesics between two intervals of size $n^{2/3}$ on $\mathbb{L}_0$ and $\mathbb{L}_{2n}$, is quoted from [12]. Recall that we use $\mathbb{L}_t$ to denote the line $\{x+y=t\}\subseteq\mathbb{Z}^2$. Also, recall from (9) that for any $\Delta > 0$, we use the notation $U_\Delta$ for the rectangle

$$\{-\Delta n^{2/3}\leq\psi(u)\leq\Delta n^{2/3}\}\cap\{0\leq\phi(u)\leq 2n\}.$$

We also recall the notation $\underline{U}_\Delta$ and $\overline{U}_\Delta$ from (11) for the lower and upper short sides of $U_\Delta$.

PROPOSITION 5.1 ([12, Proposition 3.1]): *For any $\Delta > 0$, let $\mathcal{M}_l$ denote the event that there exist points $\{f_1, f_2, \ldots, f_l\}$ on $\underline{U}_\Delta$ and points $\{g_1, g_2, \ldots, g_l\}$ on $\overline{U}_\Delta$ satisfying $\psi(f_1) > \psi(f_2) > \cdots > \psi(f_l)$ and $\psi(g_1) > \psi(g_2) > \cdots > \psi(g_l)$ such that the geodesics $\Gamma_{f_i, g_i}$ are all pairwise disjoint. Then there exist positive constants $n_0, l_0$ such that for all $n > n_0$ and all $l_0 \le l \le n^{0.01}$, we have*

$$\mathbb{P}(\mathcal{M}_l) \le e^{-c_1 l^{1/4}}$$

*for some positive constant $c_1$ depending on $\Delta$.*

Now fix an $\epsilon > 0$ and let $t \in [\![\epsilon n, (2-\epsilon)n]\!]$ as in the setting of Theorem 2. Let $\mathcal{L}_t$ denote the maximum number of points $\{h_1, h_2, \ldots, h_l\}$ on $\mathbb{L}_t \cap U_1$ strictly decreasing in $\psi(\cdot)$ such that there exist points $\{f_1, f_2, \ldots, f_l\}$ on $\underline{U}_1$ and points $\{g_1, g_2, \ldots, g_l\}$ on $\overline{U}_1$ strictly decreasing in $\psi(\cdot)$ such that $h_i \in \Gamma_{f_i, g_i}$. We now use Proposition 5.1 to show that $\mathbb{E}\mathcal{L}_t$ is upper bounded uniformly in $n$.

LEMMA 5.2: *Fix an $\epsilon \in (0, 1)$. There exists a constant $C > 0$ such that for all $n$ large and all $t \in [\![\epsilon n, (2-\epsilon)n]\!]$, we have*

$$\mathbb{E}\mathcal{L}_t \le C.$$

*Proof.* For a collection $\{f_1, f_2, \ldots, f_l\}$, $\{g_1, g_2, \ldots, g_l\}$ and $\{h_1, h_2, \ldots, h_l\}$ satisfying the conditions in the definition of $\mathcal{L}_t$, we have that any two geodesics $\Gamma_{f_i, g_i}$ and $\Gamma_{f_j, g_j}$ must be disjoint when restricted to at least one of the time intervals $[0, \epsilon n]$ and $[(2-\epsilon)n, 2n]$. Indeed, this follows by the ordering (and uniqueness) of geodesics along with the fact that $h_i \ne h_j$. Let $F$ be the size of the largest subset of $\{1, \ldots, l\}$ such that all the geodesics $\{\Gamma_{f_i, g_i}\}_{i \in I_1}$ are pairwise disjoint when restricted to the time interval $[0, \epsilon n]$. Let $G$ be defined analogously for the time interval $[(2-\epsilon)n, 2n]$. It is easy to see that $\max\{F, G\} \ge l/2$. Hence, by a union bound along with Proposition 5.1, we have

$$(69) \qquad\qquad \mathbb{P}(\mathcal{L}_t \ge l) \le e^{-c_2 l^{1/4}}$$

for all $l \le n^{0.01}$ and some constant $c_2$ depending on $\epsilon$. The far end of the tail of $\mathcal{L}_t$ can be bounded by using that

$$(70) \qquad\qquad \mathcal{L}_t \le |\mathbb{L}_t \cap U_1^n| = n^{2/3} + 1$$

deterministically, and on using (69) with $l = n^{0.01}$, this yields

$$(71) \qquad\qquad \mathbb{E}[\mathbb{1}_{\mathcal{L}_t \ge n^{0.01}} \mathcal{L}_t] \le e^{-c_2 n^{0.01/4}}(n^{2/3} + 1) < C_2$$

for some positive constant $C_2$ for all $n$. The proof is completed using (69) and (71). ∎

We now complete the proof of the upper bound in Theorem 2 by using Lemma 5.2.

*Proof of Theorem 2, upper bound.* We consider the points $u_i = (i\delta n^{2/3}, -i\delta n^{2/3})$ and $v_i = \mathbf{n} + u_i$ for $i \in [\![-\frac{\delta^{-1}}{2}, \frac{\delta^{-1}}{2}]\!]$. By translation invariance, we have that the probability

$$(72) \qquad \mathbb{P}(|\Gamma_{u_i,v_i}(t) - \psi(u_i)| < \delta n^{2/3})$$

is independent of $i$. Also, by looking at the definition of $\mathcal{L}_t$ in Lemma 5.2, we can deduce that

$$(73) \qquad \begin{aligned} \sum_{i=-\delta^{-1}/4}^{\delta^{-1}/4} & \mathbb{P}(|\Gamma_{u_i,v_i}(t) - \psi(u_i)| < \delta n^{2/3}) \\ &= \mathbb{E}\left[\sum_{i=-\delta^{-1}/4}^{\delta^{-1}/4} \mathbb{1}(|\Gamma_{u_i,v_i}(t) - \psi(u_i)| < \delta n^{2/3})\right] \leq \mathbb{E}[\mathcal{L}_t] \leq C, \end{aligned}$$

where $C_2$ is obtained from Lemma 5.2. (72) and (73) together imply

$$(74) \qquad \mathbb{P}(|\Gamma_{u_0,v_0}(t) - \psi(u_0)| < \delta n^{2/3}) = \mathbb{P}(|\Gamma_n(t)| < \delta n^{2/3}) \leq 2C\delta,$$

and the result follows immediately. ∎

5.2. PROOF OF THE LOWER BOUND IN THEOREM 2. We shall prove the lower bound in Theorem 2 in this subsection. We first complete the proof of the lower bound using Proposition 1.2 and prove the latter result, which is the heart of the technical content in this section, in the next subsection.

We recall the notation from the statement of Proposition 1.2: for some $M > 0$ which will be taken to be large later, we define the points $a_1, a_2$ by

$$(75) \qquad \begin{aligned} a_1 &= (-Mn^{2/3}, Mn^{2/3}), \\ a_2 &= (Mn^{2/3}, -Mn^{2/3}), \end{aligned}$$

and let $b_1 = a_1 + \mathbf{n}$ and $b_2 = a_2 + \mathbf{n}$. Note that the parameter $M$ used in this section is completely independent of the parameter $M$ used in Section 4. Proposition 1.2 roughly states that the geodesics $\Gamma_{a_1,b_1}$ and $\Gamma_{a_2,b_2}$ coalesce with positive probability, while not straying more than $Mn^{2/3}$ distance away in the transversal direction.
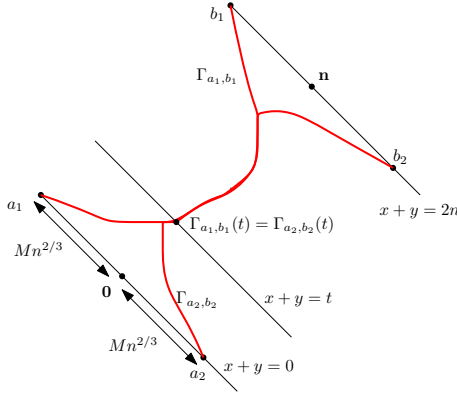
Figure 4. Proof of the lower bound in Theorem 2: Proposition 1.2 implies that with probability bounded away from 0, the geodesics $\Gamma_{a_1,b_1}$ and $\Gamma_{a_2,b_2}$ intersect the line $x + y = t$ at the same point with the point's $\psi$-coordinate bounded by $M\delta n^{2/3}$ in the absolute value. By the planar ordering of geodesics, this implies that for every point $u$ between $a_1$ and $a_2$ and for every point $v$ between $b_1$ and $b_2$, $\Gamma_{u,v}$ intersects the line $x + y = t$ at the same point. In particular, this implies that there exists $u_i = (i\delta n^{2/3}, -i\delta n^{2/3})$ and $v_i = u_i + \mathbf{n}$ such that $|\Gamma_{u_i,v_i} - \psi(u_i)| \le \delta n^{2/3}$, and hence the expected number of such $i$ is bounded away from 0. The proof of the lower bound in Theorem 2 follows by a translation invariance argument.

*Proof of the lower bound in Theorem 2.* The basic strategy is similar to the proof of the upper bound. We consider points $u_i = (i\delta n^{2/3}, -i\delta n^{2/3})$ and $v_i = \mathbf{n} + u_i$ for $i \in [\![-M\delta^{-1}, M\delta^{-1}]\!]$. By the same reasoning as in (72), we have that

(76) $$\mathbb{P}(|\Gamma_{u_i,v_i}(t) - \psi(u_i)| \le \delta n^{2/3})$$

is independent of $i$. Similarly, we also have that

(77) $$\sum_{i=-M/\delta}^{M/\delta} \mathbb{P}(|\Gamma_{u_i,v_i}(t) - \psi(u_i)| \le \delta n^{2/3}) = \mathbb{E}\left[\sum_{i=-M/\delta}^{M/\delta} \mathbb{1}(|\Gamma_{u_i,v_i}(t) - \psi(u_i)| \le \delta n^{2/3})\right].$$

We locally refer to the event defined in the statement of Proposition 1.2 as $\mathcal{A}$. Note that by the ordering of geodesics, we have that the value $\Gamma_{u_i,v_i}(t)$ is equal for all $i$ on the event $\mathcal{A}$. This in turn implies that on the event $\mathcal{A}$, the expression

$$\mathbb{1}(|\Gamma_{u_i,v_i}(t) - \psi(u_i)| \leq \delta n^{2/3})$$

must be 1 for at least one $i$. Thus, we have

(78)
$$\mathbb{E}\Bigg[ \sum_{i=-M/\delta}^{M/\delta} \mathbb{1}(|\Gamma_{u_i,v_i}(t) - \psi(u_i)| \leq \delta n^{2/3}) \Bigg]$$

$$\geq \mathbb{E}\Bigg[ \sum_{i=-M/\delta}^{M/\delta} \mathbb{1}(|\Gamma_{u_i,v_i}(t) - \psi(u_i)| \leq \delta n^{2/3}); \mathcal{A} \Bigg]$$

$$\geq \mathbb{P}(\mathcal{A}) \geq c,$$

where $c$ comes from Proposition 1.2. Finally, by combining (76), (77) and (78), we obtain

(79)     $$\mathbb{P}(|\Gamma_n| \leq \delta n^{2/3}) = \mathbb{P}(|\Gamma_{u_0,v_0}(t) - \psi(u_0)| \leq \delta n^{2/3}) \geq \frac{c}{2M}\delta,$$

which finishes the proof since $\frac{c}{2M}$ is a constant.   ∎

Before moving onto the proof of Proposition 1.2, we quickly finish the proof of Corollary 3 (ii) by using Theorem 2.

*Proof of Corollary 3(ii).*     We use Theorem 2 along with the definition $\pi_n(s) = n^{-2/3}\Gamma_n(2ns)$ for $2ns \in \mathbb{Z}$. Indeed, for $n$ large enough, we have that for any fixed $s \in (0,1)$, $\frac{\lceil 2ns \rceil}{2n}$ is bounded away from 0 and 1 and now Theorem 2 implies that for some constants $c, C$, we have

$$c\delta \leq \mathbb{P}(|\Gamma_n(\lceil 2ns \rceil)| \leq \delta n^{2/3}) \leq C\delta.$$

Using that $\Gamma_n(\cdot)$ has $\pm 1$ increments, and that $\pi_n(s)$ is defined for all $s \in [0,1]$ by interpolating the values on $[0,1] \cap \frac{1}{2n}\mathbb{Z}$, we obtain that

$$\frac{c}{2}\delta \leq \mathbb{P}(|\pi_n(s)| \leq \delta) \leq 2C\delta$$

for all $n$ large enough. The mapping $f \mapsto |f(s)|$ is continuous as a map from $C[0,1]$ to $\mathbb{R}^+$ with the uniform convergence topology and Euclidean topology respectively. Thus by the continuous mapping theorem, if $\pi_{n_i} \Rightarrow \pi$ as $i \to \infty$ for a subsequence $\{n_i\}$, we have that $|\pi_{n_i}(s)| \Rightarrow |\pi(s)|$. We can now use the

Portmanteau theorem in the same way as in the Proof of Corollary 3 (i) to obtain

$$\frac{c}{4}\delta \leq \mathbb{P}(|\pi(s)| \leq \delta) \leq 4C\delta$$

for any fixed $s \in (0, 1)$. The uniformity of $c, C$ as long as $s$ is bounded away from 0 and 1 is evident from the proof. ∎

5.3. PROOF OF PROPOSITION 1.2. To prove Proposition 1.2, we shall again construct a host of favourable events whose intersection holds with positive probability and on which the geodesics are forced to coalesce at the required location. These events will be defined with a parameter $M$ (note that this $M$ has nothing to do with the parameter used in the proof of the lower bound in Theorem 1 from the previous sections). We start by outlining how the choice of $M$ is fixed, and then define the relevant events.

5.3.1. *Choice of the parameter $M$.* All the constructions will be made with a parameter $M$ which will be fixed to be a large constant at the end, and this fixed constant is the one appearing in the statement of Proposition 1.2. The specific choice of $M$ is fixed large enough to satisfy the conclusions of Lemma 5.3, Lemma 5.4 and Proposition 5.9. Using this value of $M$ in Lemma 5.7 provides us with a corresponding value of $\beta$ and the final probability lower bound in (105) is in terms of $\beta$.

Observe that if we fix $\Delta$ and take $n$ large depending on $\Delta$, then the slope of all lines joining points in $\underline{U}_\Delta$ to those in $\overline{U}_\Delta$ can be made arbitrarily close to 1. Thus as a consequence of (8) along with $\mathbb{E}\underline{T}_{u,v} = \mathbb{E}T_{u,v} + 1$, we have that for all $\Delta$ large, for all $n$ large depending on $\Delta$, and for all $u \in \underline{U}_\Delta$ and $v \in \overline{U}_\Delta$,

(80)
$$-C\Delta^2 n^{1/3} - C'n^{1/3} \leq \mathbb{E}T_{u,v} - 4n \leq C'n^{1/3},$$
$$-C\Delta^2 n^{1/3} - C'n^{1/3} \leq \mathbb{E}\underline{T}_{u,v} - 4n \leq C'n^{1/3}$$

for some constants $C, C'$ not depending on $\Delta$. We will repeatedly use (80) with $\Delta$ being equal to $M^{3/4}, M$ and $M^2$ to change the centering in the estimates.

5.3.2. *Events favourable for coalescence.* Recall from (9) that for any $\Delta > 0$, we use the notation $U_\Delta$ for the rectangle

$$\{-\Delta n^{2/3} \leq \psi(u) \leq \Delta n^{2/3}\} \cap \{0 \leq \phi(u) \leq 2n\}.$$

Recall the fixed parameter $\epsilon \in (0,1)$ from the statement of Proposition 1.2. Let the rectangles $R_1$ and $R_2$ be defined by

$$(81) \qquad R_1 = U_{M^2} \cap \left\{ \frac{\epsilon}{4} n \leq \phi(u) \leq \frac{3\epsilon}{4} n \right\},$$

$$(82) \qquad R_2 = U_{M^2} \cap \left\{ 2n - \frac{3\epsilon}{4} n \leq \phi(u) \leq 2n - \frac{\epsilon}{4} n \right\}.$$

Let $\underline{R}_1$ and $\overline{R}_1$ denote the short sides of $R_1$ lying on the lines $\mathbb{L}_{\frac{\epsilon n}{4}}$ and $\mathbb{L}_{\frac{3\epsilon n}{4}}$ respectively. Similarly, let $\underline{R}_2$ and $\overline{R}_2$ denote the short sides of $R_2$ lying on the lines $\mathbb{L}_{2n-\frac{3\epsilon n}{4}}$ and $\mathbb{L}_{2n-\frac{\epsilon n}{4}}$ respectively; see Figure 5. We now define some high probability events which will be used later. For a point $u \in \mathbb{L}_0$, and any $x > 0$, define the event $\mathbf{Restr}_{u,x}$ by

$$(83) \qquad \begin{aligned} \mathbf{Restr}_{u,x} = \Big\{ &\text{All } \gamma \colon u \to u + \mathbf{n} \text{ satisfying } \gamma \cap (U_x)^c \neq \emptyset \\ &\text{have } \ell(\gamma) \leq 4n - \frac{\xi}{2} x^2 n^{1/3} \Big\}, \end{aligned}$$

where in this section, $\xi$ always denotes the constant from Proposition 2.5.

We define the event $\mathbf{Restr}$ by

$$(84) \qquad \mathbf{Restr} = \mathbf{Restr}_{0,2M} \cap \mathbf{Restr}_{a_1,M^2} \cap \mathbf{Restr}_{a_2,M^2},$$

where $a_1 = (-Mn^{2/3}, Mn^{2/3})$ and $a_2 = (Mn^{2/3}, -Mn^{2/3})$ as defined in (75). Also, recall that we have defined $b_1 = a_1 + \mathbf{n}$ and $b_2 = a_2 + \mathbf{n}$.

We now define some more events which will be used in our construction.

- $\mathcal{W}_1$: the event that for any $u \in \mathbb{L}_0 \cap U_{M^2}$ and $v \in \underline{R}_1$, we have $|\widetilde{T}_{u,v}| \leq Mn^{1/3}$.
- $\mathcal{W}_2$: the event that for any $u \in \overline{R}_1$ and $v \in \underline{R}_2$, we have $|\widetilde{\underline{T}}_{u,v}| \leq Mn^{1/3}$.
- $\mathcal{W}_3$: the event that for any $u \in \overline{R}_2$ and $v \in \mathbb{L}_{2n} \cap U_{M^2}$, we have $|\widetilde{\underline{T}}_{u,v}| \leq Mn^{1/3}$.
- $\mathcal{S}_1$: the event that for any $u \in \underline{R}_1$ and $v \in \overline{R}_1$, we have $|\widetilde{\underline{T}}_{u,v}^{R_1}| \leq Mn^{1/3}$.
- $\mathcal{S}_2$: the event that for any $u \in \underline{R}_2$ and $v \in \overline{R}_2$, we have $|\widetilde{\underline{T}}_{u,v}^{R_2}| \leq Mn^{1/3}$.
- $\mathcal{S}_1'$: the event that for any $u \in \underline{R}_1$ and $v \in \overline{R}_1$, we have $\widetilde{\underline{T}}_{u,v}^{R_1} \leq Mn^{1/3}$.
- $\mathcal{S}_2'$: the event that for any $u \in \underline{R}_2$ and $v \in \overline{R}_2$, we have $\widetilde{\underline{T}}_{u,v}^{R_2} \leq Mn^{1/3}$.

Note that both $T$ and $\underline{T}$ are used to define the above events, and the minor technical reason for doing this will become clear in the proof of Proposition 5.9.
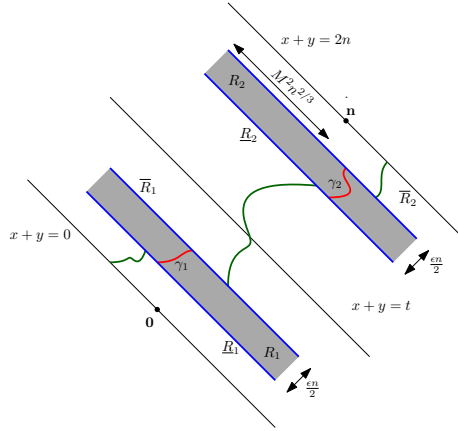
Figure 5. The basic elements for constructing favourable events to force coalescence of geodesics in the proof of Proposition 1.2: The rectangles $R_1$ and $R_2$, marked in grey, act as barriers, on either side of the lines $x+y = t$. All paths across $R_1$ and $R_2$ (except for those that intersect two specially designated paths $\gamma_1$ and $\gamma_2$, marked in red) are forced to be much shorter than typical. This is the content of the event $\mathcal{P}^{\gamma_1,\gamma_2}$. The other events $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{W}$ are typical; $\mathcal{W}$ ensures that the paths through the regions below $R_1$, between $R_1$ and $R_2$, and above $R_2$ (some instances of such paths are marked in green) have typical weights, whereas $\mathcal{S}'$ ensures that any path through $R_1$ and $R_2$ does not have atypically large weight. The event **Restr**, not shown in the figure, ensures that paths have typical transversal fluctuation.

We will be using the events

$$\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2 \cap \mathcal{W}_3, \tag{85}$$

$$\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2, \tag{86}$$

$$\mathcal{S}' = \mathcal{S}'_1 \cap \mathcal{S}'_2. \tag{87}$$

Note that the reason for defining $\mathcal{S}'$ and $\mathcal{S}$ separately is that $\mathcal{S}'$ is a decreasing event, and this will allow us to apply the FKG inequality in Proposition 5.8. We now show that all the above events occur with high probability.

LEMMA 5.3: *For all $M$ large, and all $n$ large depending on $M$, we have*

$$\mathbb{P}(\mathcal{W}) \geq 0.99, \quad \mathbb{P}(\mathcal{S}) \geq 0.99, \quad \mathbb{P}(\mathcal{S}') \geq 0.99 \quad and \quad \mathbb{P}(\mathbf{Restr}) \geq 0.99.$$

*Thus, we have*

$$\mathbb{P}(\mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W}) \geq 0.97.$$

*Proof.* The lower bounds for $\mathbb{P}(\mathcal{W}), \mathbb{P}(\mathcal{S})$ and $\mathbb{P}(\mathcal{S}')$ are very simple consequences of a union bound, Proposition 2.2 and Proposition 2.3. We just illustrate the proof by proving the bound for $\mathbb{P}(\mathcal{S})$. By symmetry and a union bound, we only need to lower bound $\mathbb{P}(\mathcal{S}_1)$. Now, divide $\underline{R}_1$ into $M^2$ disjoint segments of length $n^{2/3}$, and call these segments $(\underline{R}_1)_i$ where $i \in \{1, \ldots, M^2\}$; do the same for $\overline{R}_1$ and call the segments $(\overline{R}_1)_j$. Let $P_{i,j}$ denote the parallelogram with $(\underline{R}_1)_i$ and $(\overline{R}_1)_j$ as the short sides. By using a slightly generalized version of Proposition 2.3 for parallelograms whose long sides have slopes other than 1 (see [11, Theorem 4.2(ii)]), we have

$$(88) \quad \mathbb{P}(\inf_{u \in (\underline{R}_1)_i, v \in (\overline{R}_1)_j} \widetilde{T}_{u,v}^{R_1} \geq -Mn^{1/3}) \geq \mathbb{P}(\inf_{u \in (\underline{R}_1)_i, v \in (\overline{R}_1)_j} \widetilde{T}_{u,v}^{P_{i,j}} \geq -Mn^{1/3})$$
$$\geq 1 - Ce^{-cM}.$$

By Proposition 2.2, we have

$$(89) \quad \mathbb{P}(\sup_{u \in (\underline{R}_1)_i, v \in (\overline{R}_1)_j} \widetilde{T}_{u,v}^{R_1} \leq Mn^{1/3}) \geq \mathbb{P}(\sup_{u \in (\underline{R}_1)_i, v \in (\overline{R}_1)_j} \widetilde{T}_{u,v} \leq Mn^{1/3})$$
$$\geq 1 - C_1 e^{-c_1 M^{3/2}}.$$

To complete the proof, one just needs to take a union bound over the polynomially many (in $M$) choices of $i, j$, and we are done by choosing $M$ sufficiently large because of the stretched exponential decay in the probability.

The lower bound for $\mathbb{P}(\mathbf{Restr})$ is an immediate application of Proposition 2.5 and a union bound. Finally, we obtain

$$(90) \qquad\qquad \mathbb{P}(\mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W}) \geq 0.97$$

by another union bound. ∎

We now use the events that we have defined to show that with high probability, the geodesic $\Gamma_n$ is constrained inside $U_{M^{3/4}}$ while ensuring that its different segments have weights close to typical.

LEMMA 5.4: *On the event* $\mathbf{Restr}_{\mathbf{0},M^{3/4}} \cap \mathcal{W} \cap \mathcal{S}$, *we have that the geodesic* $\Gamma_n \subseteq U_{M^{3/4}}$ *for all* $M$ *large, and* $n$ *large depending on* $M$. *We also have*

$$\mathbb{P}(\mathbf{Restr}_{\mathbf{0},M^{3/4}} \cap \mathcal{W} \cap \mathcal{S}) \geq 0.97.$$

*Proof.* On $\mathcal{W} \cap \mathcal{S}$, we have that for some constant $C$,

(91)
$$\begin{aligned} T_{\mathbf{0},\mathbf{n}} &\geq T_{\mathbf{0},\frac{\epsilon}{8}\mathbf{n}} + T^{R_1}_{\frac{\epsilon}{8}\mathbf{n},\frac{3\epsilon}{8}\mathbf{n}} + T_{\frac{3\epsilon}{8}\mathbf{n},(1-\frac{3\epsilon}{8})\mathbf{n}} + T^{R_2}_{(1-\frac{3\epsilon}{8})\mathbf{n},(1-\frac{\epsilon}{8})\mathbf{n}} + T_{(1-\frac{\epsilon}{8})\mathbf{n},\mathbf{n}} \\ &\geq 4n - (5M + 5C)n^{1/3}, \end{aligned}$$

where the constant $C$ comes from switching the centering by using (8).

Using the definition of $\mathbf{Restr}_{\mathbf{0},M^{3/4}}$ and the fact that $\xi M^{3/2}/2 > M + C$ for $M$ large, it is clear from (91) that $\Gamma_n \subseteq U_{M^{3/4}}$.

The probability lower bound is an easy consequence of Proposition 2.5 and Lemma 5.3. Indeed, we just need to individually bound fro below $\mathbb{P}(\mathbf{Rest}_{\mathbf{0},M^{3/4}})$, $\mathbb{P}(\mathcal{W})$ and $\mathbb{P}(\mathcal{S})$ by 0.99 and use a union bound. ∎

At this point, we introduce some notation. For a given path $\gamma \colon u \to v$, we define

(92)
$$\begin{aligned} \widetilde{\ell}(\gamma) &= \ell(\gamma) - \mathbb{E}T_{u,v}, \\ \widetilde{\underline{\ell}}(\gamma) &= \underline{\ell}(\gamma) - \mathbb{E}\underline{T}_{u,v}. \end{aligned}$$

From now on, for a fixed $M$, we say that a path $\gamma$ is $\theta$-typical if it satisfies $\gamma \subseteq U_\theta$ and $|\widetilde{\underline{\ell}}(\gamma)| \leq Mn^{1/3}$.

LEMMA 5.5: *Let* $\mathcal{T}$ *be the event that* $\Gamma_n|_{R_1}$ *and* $\Gamma_n|_{R_2}$ *are* $M^{3/4}$-*typical. Then*

$$\mathbb{P}(\mathcal{T}) \geq 0.97$$

*for all* $M$ *large, and* $n$ *large depending on* $M$ *as in the statement of Lemma 5.4.*

*Proof.* By Lemma 5.4, on the event $\mathbf{Restr}_{\mathbf{0},M^{3/4}} \cap \mathcal{W} \cap \mathcal{S}$, we have that the geodesic $\Gamma_n \subseteq U_{M^{3/4}}$, which in particular implies that $\Gamma_n|_{R_1}, \Gamma_n|_{R_2} \subseteq U_{M^{3/4}}$. Also, the definition of the event $\mathcal{S}$ ensures that on $\mathbf{Restr}_{\mathbf{0},M^{3/4}} \cap \mathcal{W} \cap \mathcal{S}$, we have $|\widetilde{\underline{\ell}}(\Gamma_n|_{R_1})| \leq Mn^{1/3}$ and $|\widetilde{\underline{\ell}}(\Gamma_n|_{R_2})| \leq Mn^{1/3}$. Thus, we have that

$$\mathbb{P}(\mathcal{T}) \geq \mathbb{P}(\mathbf{Restr}_{\mathbf{0},M^{3/4}} \cap \mathcal{W} \cap \mathcal{S}) \geq 0.97$$

by Lemma 5.4. ∎

Let $\mathscr{I}_1$ and $\mathscr{I}_2$ denote the set of all $M^{3/4}$-typical paths from $\underline{R}_1$ to $\overline{R}_1$ and $\underline{R}_2$ to $\overline{R}_2$ respectively. By Lemma 5.5, it is clear that

$$(93) \qquad\qquad \mathbb{P}(\Gamma_n|_{R_1} \in \mathscr{I}_1, \Gamma_n|_{R_2} \in \mathscr{I}_2) \geq 0.97.$$

Let $\mathscr{I}$ denote the set of pairs of paths $(\gamma_1, \gamma_2)$ such that $\gamma_1 \in \mathscr{I}_1$, $\gamma_2 \in \mathscr{I}_2$ and

$$(94) \qquad \mathbb{P}(\mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W} | \Gamma_n|_{R_1} = \gamma_1, \Gamma_n|_{R_2} = \gamma_2) \geq \frac{1}{2}.$$

Notice that $\mathscr{I}_1, \mathscr{I}_2$, $\mathscr{I}$ depend both on the path and the weights on the path; for notational convenience, in the following, whenever we fix $\gamma_1, \gamma_2$, it will be assumed that both the path and its weight has been fixed, similar convention will also be adapted whenever conditioning on events such as $\{\Gamma_n|_{R_1} = \gamma_1, \Gamma_n|_{R_2} = \gamma_2\}$ as above. In view of Lemma 5.3, we have the following lower bound.

LEMMA 5.6: $\mathbb{P}((\Gamma_n|_{R_1}, \Gamma_n|_{R_2}) \in \mathscr{I}) \geq 0.9$ for all $M$ large and $n$ large depending on $M$ as in the statement of Lemma 5.4.

Proof. Let $\mathscr{P}$ be the set of pairs of paths $(\gamma_1, \gamma_2)$ such that $\gamma_1 \in \mathscr{I}_1$ and $\gamma_2 \in \mathscr{I}_2$. Note that we have

$$\mathbb{P}(\mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W})$$
$$= 1 - \mathbb{P}((\mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W})^c)$$
$$\leq 1 - \mathbb{E}_{\gamma_1, \gamma_2}[\mathbb{P}((\mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W})^c | \Gamma_n|_{R_1} = \gamma_1, \Gamma_n|_{R_2} = \gamma_2) \mathbb{1}_{(\gamma_1, \gamma_2) \in \mathscr{P} \setminus \mathscr{I}}]$$
$$\leq 1 - \frac{1}{2} \mathbb{P}((\Gamma_n|_{R_1}, \Gamma_n|_{R_2}) \in \mathscr{P} \setminus \mathscr{I})$$

by using the definition of $\mathscr{I}$ as in (94). Using (90) along with the above, we obtain

$$(95) \qquad\qquad \mathbb{P}((\Gamma_n|_{R_1}, \Gamma_n|_{R_2}) \in \mathscr{P} \setminus \mathscr{I}) \leq 0.06.$$

Finally, we have that

$$\mathbb{P}((\Gamma_n|_{R_1}, \Gamma_n|_{R_2}) \in \mathscr{I}) = \mathbb{P}((\Gamma_n|_{R_1}, \Gamma_n|_{R_2}) \in \mathscr{P}) - \mathbb{P}((\Gamma_n|_{R_1}, \Gamma_n|_{R_2}) \in \mathscr{P} \setminus \mathscr{I})$$
$$\geq 0.97 - 0.06 = 0.91$$

by using (93) and (95) and this completes the proof. ∎

We will finally want to condition on the event $\{(\Gamma_n|_{R_1}, \Gamma_n|_{R_2}) = (\gamma_1, \gamma_2)\}$ for some $(\gamma_1, \gamma_2) \in \mathscr{I}$, and then show that in this conditional environment, we have that both the geodesics $\Gamma_{a_1, b_1}$ and $\Gamma_{a_2, b_2}$ meet $\gamma_1$ and $\gamma_2$ with positive probability. We will ensure this by "decreasing" the background vertex weights in the region $R_1 \setminus \gamma_1$ and $R_2 \setminus \gamma_2$, and we define some more events which will help us in achieving this.

For any path $\gamma_1 \subseteq R_1$ from $\underline{R}_1$ to $\overline{R}_1$, and any path $\gamma_2 \subseteq R_2$ from $\underline{R}_2$ to $\overline{R}_2$, we will consider the following events:

- $\mathcal{P}^{\gamma_1}$: the event that any path $\gamma \subseteq R_1$ from $\underline{R}_1$ to $\overline{R}_1$ disjoint from $\gamma_1$ satisfies $\widetilde{\underline{\ell}}(\gamma) \leq -M^4 n^{1/3}$.
- $\mathcal{P}^{\gamma_2}$: the event that any path $\gamma \subseteq R_2$ from $\underline{R}_2$ to $\overline{R}_2$ disjoint from $\gamma_2$ satisfies $\widetilde{\underline{\ell}}(\gamma) \leq -M^4 n^{1/3}$.

In the above setting, we define the event $\mathcal{P}^{\gamma_1,\gamma_2}$ by

$$\mathcal{P}^{\gamma_1,\gamma_2} = \mathcal{P}^{\gamma_1} \cap \mathcal{P}^{\gamma_2}.$$

Note that the event $\mathcal{P}^{\gamma_1,\gamma_2}$ depends only on the vertex weights inside

$$(R_1 \setminus (\gamma_1 \cup \overline{R}_1)) \cup (R_2 \setminus (\gamma_2 \cup \overline{R}_2)).$$

Note that by using (80) along with Lemma 2.7, we have that once we fix some large constant $M$, then for some constant $c$ not depending on $\gamma_1$ or $\gamma_2$, we have that the probabilities of each of the above events are lower bounded by some constant $c$ for all $n$ large enough. Using the independence between the vertex weights in $R_1$ and $R_2$, we have that $\mathcal{P}^{\gamma_1}$ and $\mathcal{P}^{\gamma_2}$ are independent and this immediately implies the following lemma:

LEMMA 5.7: *For any path $\gamma_1 \subseteq R_1$ from $\underline{R}_1$ to $\overline{R}_1$ and any path $\gamma_2 \subseteq R_2$ from $\underline{R}_2$ to $\overline{R}_2$, we have that for any $M > 0$, there exists a positive constant $\beta$ (depending on the choice of $M$, independent of the choice of $\gamma_1, \gamma_2$) such that for all $n$ large enough,*

$$(96) \qquad\qquad \mathbb{P}(\mathcal{P}^{\gamma_1,\gamma_2}) \geq \beta.$$

*Proof.* The lemma follows by using (80), Lemma 2.7, and the independence of $\mathcal{P}^{\gamma_1}$ and $\mathcal{P}^{\gamma_2}$ as explained above. Indeed, by (80), we have that for any path $\gamma \subset R_1$ from $\underline{R}_1$ to $\overline{R}_1$, and for some positive constants $C_1, C_2$ not depending on $M$,

$$\{\widetilde{\underline{\ell}}(\gamma) \leq -M^4 n^{1/3}\} \supseteq \{\underline{\ell}(\gamma) - \epsilon n \leq (-C_1 - 1)M^4 n^{1/3} - C_2 n^{1/3}\},$$

and the $M$-dependent lower bound for $\mathbb{P}(\mathcal{P}^{\gamma_1})$ now follows by using Lemma 2.7 for the rectangle $R_1$. Similar considerations for $R_2$ yield the lower bound for $\mathbb{P}(\mathcal{P}^{\gamma_2})$.    ■

Again, note that the $\beta$ and $M$ in this section have no relation to the ones in Section 4.1.1.

PROPOSITION 5.8: *For any $M > 0$ fixed, let $\beta = \beta(M)$ be the constant obtained from Lemma 5.7. For any path $\gamma_1 \subseteq R_1$ from $\underline{R}_1$ to $\overline{R}_1$ and any path $\gamma_2 \subseteq R_2$ from $\underline{R}_2$ to $\overline{R}_2$, we have*

$$\mathbb{P}(\mathcal{P}^{\gamma_1,\gamma_2}|\mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W} \cap \{\Gamma_n|_{R_1} = \gamma_1\} \cap \{\Gamma_n|_{R_2} = \gamma_2\}) \geq \beta$$

*for all $n$ large enough.*

*Proof.* Note that after conditioning on the configuration in

$$((R_1 \setminus (\gamma_1 \cup \overline{R}_1)) \cup (R_2 \setminus (\gamma_2 \cup \overline{R}_2)))^c,$$

we have that $\mathcal{S}', \mathbf{Restr}, \mathcal{P}^{\gamma_1,\gamma_2}$ and $\{\Gamma_n|_{R_1} = \gamma_1\} \cap \{\Gamma_n|_{R_2} = \gamma_2\}$ are decreasing events on $(R_1 \setminus (\gamma_1 \cup \overline{R}_1)) \cup (R_2 \setminus (\gamma_2 \cup \overline{R}_2))$. Let $\mathscr{F}_{\gamma_1,\gamma_2}$ denote the $\sigma$-algebra generated by the vertex weights in $((R_1 \setminus (\gamma_1 \cup \overline{R}_1)) \cup (R_2 \setminus (\gamma_2 \cup \overline{R}_2)))^c$. Using the FKG inequality and the dependence of $\mathcal{P}^{\gamma_1,\gamma_2}$ only on the vertex weights in $(R_1 \setminus (\gamma_1 \cup \overline{R}_1)) \cup (R_2 \setminus (\gamma_2 \cup \overline{R}_2))$ (thus, $\mathcal{P}^{\gamma_1,\gamma_2}$ is independent of $\mathscr{F}_{\gamma_1,\gamma_2}$), we have that

$$(97) \quad \begin{aligned} \mathbb{P}(\mathcal{P}^{\gamma_1,\gamma_2} &\cap \mathcal{S}' \cap \mathbf{Restr} \cap \{\Gamma_n|_{R_1} = \gamma_1\} \cap \{\Gamma_n|_{R_2} = \gamma_2\}|\mathscr{F}_{\gamma_1,\gamma_2}) \\ &\geq \mathbb{P}(\mathcal{P}^{\gamma_1,\gamma_2})\mathbb{P}(\mathcal{S}' \cap \mathbf{Restr} \cap \{\Gamma_n|_{R_1} = \gamma_1\} \cap \{\Gamma_n|_{R_2} = \gamma_2\}|\mathscr{F}_{\gamma_1,\gamma_2}). \end{aligned}$$

Noting that the event $\mathcal{W}$ is measurable with respect to $\mathscr{F}_{\gamma_1,\gamma_2}$, we have

$$(98) \quad \begin{aligned} \mathbb{P}(\mathcal{P}^{\gamma_1,\gamma_2} &\cap \mathcal{S}' \cap \mathbf{Restr} \cap \{\Gamma_n|_{R_1} = \gamma_1\} \cap \{\Gamma_n|_{R_2} = \gamma_2\}|\mathcal{W}) \\ &\geq \mathbb{P}(\mathcal{P}^{\gamma_1,\gamma_2})\mathbb{P}(\mathcal{S}' \cap \mathbf{Restr} \cap \{\Gamma_n|_{R_1} = \gamma_1\} \cap \{\Gamma_n|_{R_2} = \gamma_2\}|\mathcal{W}) \end{aligned}$$

and this in turn yields

$$(99) \quad \mathbb{P}(\mathcal{P}^{\gamma_1,\gamma_2}|\mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W} \cap \{\Gamma_n|_{R_1} = \gamma_1\} \cap \{\Gamma_n|_{R_2} = \gamma_2\}) \geq \mathbb{P}(\mathcal{P}^{\gamma_1,\gamma_2}) \geq \beta,$$

thereby completing the proof. ∎

5.3.3. *Forcing the geodesics to coalesce on the favourable events.* We shall now show that the positive probability event constructed above implies that $\Gamma_{a_1,b_1}$ and $\Gamma_{a_2,b_2}$ coincide between $\overline{R}_1$ and $\underline{R}_2$.

PROPOSITION 5.9: *Take any $(\gamma_1, \gamma_2) \in \mathscr{I}$. On the event*

$$\mathcal{P}^{\gamma_1,\gamma_2} \cap \mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W},$$

*for $M$ large enough, and $n$ large depending on $M$ we have that $\Gamma_{a_1,b_1}, \Gamma_{a_2,b_2} \subseteq U_{M^2}$, $\Gamma_n \subseteq U_{2M}$ and both $\Gamma_{a_1,b_1}$ and $\Gamma_{a_2,b_2}$ intersect both $\gamma_1$ and $\gamma_2$.*

*Proof.* Let $u_1, v_1$ (resp. $u_2, v_2$) be the starting and ending points of $\gamma_1$ (resp. $\gamma_2$). Let $\chi_1$ be the concatenation of the paths $\Gamma_{a_1,u_1}, \gamma_1, \Gamma_{v_1,u_2}, \gamma_2$ and $\Gamma_{v_2,b_1}$. By using that the paths $\gamma_1$ and $\gamma_2$ are $M^{3/4}$-typical along with the definition of the events **Restr** and $\mathcal{W}$, we have that for some absolute constant $c$,

$$(100) \qquad\qquad \ell(\chi_1) - 4n \geq -cM^2 n^{1/3}.$$

Here, we have used that $\ell(\chi_1) = \ell(\Gamma_{a_1,u_1}) + \underline{\ell}(\gamma_1) + \underline{\ell}(\Gamma_{v_1,u_2}) + \underline{\ell}(\gamma_2) + \underline{\ell}(\Gamma_{v_2,b_1})$. The recentering in (100) with respect to $4n$ is done by using (80) for the end-points of each of the five segments making up $\chi_1$, and using that all the end-points lie inside $U_{2M}$. Indeed, this recentering leads to the $M^2$ term in (100). Now, using that $\xi M^4/2 > cM^2$ for large $M$, we have that $\chi_1 \subseteq U_{M^2}$ and consequently $\Gamma_{a_1,b_1} \subseteq U_{M^2}$ since $\textbf{Restr}_{a_1,M^2} \supseteq \textbf{Restr}$. By a symmetric argument, one also obtains that $\Gamma_{a_2,b_2} \subseteq U_{M^2}$.

Now, let $\chi_2$ be the concatenation of the paths $\Gamma_{\mathbf{0},u_1}, \gamma_1, \Gamma_{v_1,u_2}, \gamma_2$ and $\Gamma_{v_2,\mathbf{n}}$. By using that $\gamma_1$ and $\gamma_2$ are $M^{3/4}$-typical along with the definition of the events **Restr** and $\mathcal{W}$, we have that for some absolute constant $c_1$,

$$(101) \qquad\qquad \ell(\chi_2) - 4n \geq -c_1 M^{3/2} n^{1/3}.$$

Again, we have used (80) for the endpoints of the five segments of $\chi_2$ along with $\ell(\chi_2) = \ell(\Gamma_{\mathbf{0},u_1}) + \underline{\ell}(\gamma_1) + \underline{\ell}(\Gamma_{v_1,u_2}) + \underline{\ell}(\gamma_2) + \underline{\ell}(\Gamma_{v_2,\mathbf{n}})$. Here, we obtained $M^{3/2}$ since $\mathbf{0}, u_1, v_1, u_2, v_2, \mathbf{n} \subseteq U_{M^{3/4}}$ and this is because $\gamma_1$ and $\gamma_2$ are $M^{3/4}$-typical as in the definition of $\mathscr{I}$. Using that $\xi(2M)^2/2 > c_1 M^{3/2}$ for large $M$, and the definition of the event $\textbf{Restr}_{\mathbf{0},2M} \supseteq \textbf{Restr}$, we obtain that $\Gamma_n \subseteq U_{2M}$.

Now, consider any path $\gamma \subseteq U_{M^2}$ from $a_1$ to $b_1$ which does not intersect at least one of $\gamma_1$ and $\gamma_2$. By the definition of the event $\mathcal{P}^{\gamma_1,\gamma_2}$, $\mathcal{W}$ and $\mathcal{S}'$, we have that

$$(102) \qquad\qquad \ell(\gamma) - 4n \leq 4Mn^{1/3} - M^4 n^{1/3} + c_2 n^{1/3}.$$

Here, one divides $\gamma$ into five segments which we call $\sigma_1, \sigma_2, \ldots, \sigma_5$ and uses that $\ell(\gamma) = \ell(\sigma_1) + \underline{\ell}(\sigma_2) + \underline{\ell}(\sigma_3) + \underline{\ell}(\sigma_4) + \underline{\ell}(\sigma_5)$. The recentering in (102) is done by using the right hand side part of (80) and this leads to $c_2$, a positive constant. Indeed, one uses (80) once for each segment of $\gamma$, and $c_2$ is the cumulative effect of these five applications. By noting that $M^4 - 4M - c_2 > cM^2$ for large $M$ and by using (100) along with (102), we have that $\Gamma_{a_1,b_1}$ must intersect both $\gamma_1$ and $\gamma_2$. By entirely analogous arguments, one observes that $\Gamma_{a_2,b_2}$ must intersect both $\gamma_1$ and $\gamma_2$; see Figure 6 for an illustration of this argument. ∎
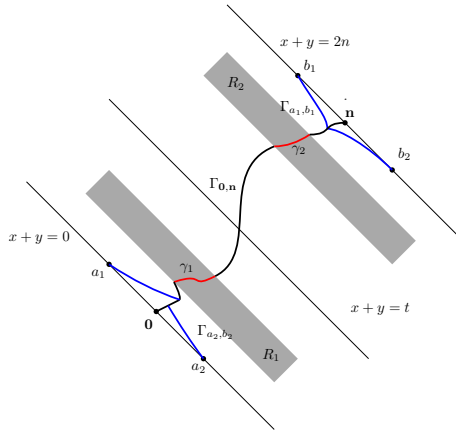
Figure 6. Coalescence of geodesics $\Gamma_{a_1,b_1}, \Gamma_{a_2,b_2}$ on the favourable events: we show that for a pair of paths $\gamma_1$ and $\gamma_2$ across $R_1$ and $R_2$ respectively satisfying the required regularity conditions, on the intersection of event $\mathcal{P}^{\gamma_1,\gamma_2}$ and the typical event $\mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W}$, $\Gamma_{a_1,b_1}$ and $\Gamma_{a_2,b_2}$ each intersect both $\gamma_1$ and $\gamma_2$. This is done by using the definition of $\mathcal{P}^{\gamma_1,\gamma_2}$ which ensures that any path crossing $R_1$ or $R_2$ and disjoint with $\gamma_1$ or $\gamma_2$ respectively incurs a heavy penalty in weight. Finally, the event $\{\Gamma_n|_{R_1} = \gamma_1\} \cap \{\Gamma_n|_{R_2} = \gamma_2\}$ ensures the required coalescence.

We are finally ready to prove Proposition 1.2.

*Proof of Proposition 1.2.* We first show that we can fix $M$ large enough such that if we take any $(\gamma_1, \gamma_2) \in \mathscr{I}$, we have

$$
\begin{aligned}
(103) \quad & \mathcal{P}^{\gamma_1,\gamma_2} \cap \mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W} \cap \{\Gamma_n|_{R_1} = \gamma_1\} \cap \{\Gamma_n|_{R_2} = \gamma_2\} \\
& \subseteq \{\Gamma_{a_1,b_1}(t) = \Gamma_{a_2,b_2}(t)\} \cap \{|\Gamma_{a_1,b_1}(t)| \leq 2Mn^{2/3}\}
\end{aligned}
$$

for all $n$ large enough. As mentioned in Section 5.3.1, fix $M$ large so as to satisfy the conclusions of Lemma 5.3, Lemma 5.4 and Proposition 5.9 for all large $n$. By using Proposition 5.9, we have that $\Gamma_{a_1,b_1}$ intersects $\gamma_1$ and $\gamma_2$. Since $\Gamma_n$ is a geodesic, this implies that $\Gamma_{a_1,b_1} = \Gamma_n$ in the region between $\overline{R}_1$ and $\underline{R}_2$. Using an analogous argument, we have that $\Gamma_{a_1,b_1} = \Gamma_{a_2,b_2} = \Gamma_n$ in the region between $\overline{R}_1$ and $\underline{R}_2$, and note that the line $\mathbb{L}_t$ lies in this region.

Since $\Gamma_n \subseteq U_{2M}$ by Proposition 5.9, it is clear that $|\Gamma_n(t)| \leq 2Mn^{2/3}$. This proves (103).

We now use Lemma 5.7 to obtain $\beta = \beta(M)$ for the $M$ which was just fixed. Now, by using (103), we have that for any $(\gamma_1, \gamma_2) \in \mathscr{I}$,

$$\mathbb{P}(\{\Gamma_{a_1,b_1}(t){=}\Gamma_{a_2,b_2}(t)\}\cap\{|\Gamma_{a_1,b_1}(t)|{\leq}2Mn^{2/3}\}|\{\Gamma_n|_{R_1}{=}\gamma_1\}\cap\{\Gamma_n|_{R_2}{=}\gamma_2\})$$
$$\geq \mathbb{P}(\mathcal{P}^{\gamma_1,\gamma_2} \cap \mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W}|\{\Gamma_n|_{R_1} = \gamma_1\} \cap \{\Gamma_n|_{R_2} = \gamma_2\})$$
$$(104) \quad \geq \mathbb{P}(\mathcal{P}^{\gamma_1,\gamma_2}|\mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W} \cap \{\Gamma_n|_{R_1} = \gamma_1\} \cap \{\Gamma_n|_{R_2} = \gamma_2\})$$
$$\times \mathbb{P}(\mathcal{S}' \cap \mathbf{Restr} \cap \mathcal{W}|\Gamma_n|_{R_1} = \gamma_1, \Gamma_n|_{R_2} = \gamma_2)$$
$$\geq \beta/2.$$

Note that we used Proposition 5.8 and the definition of $\mathscr{I}$ (as in (94)) to obtain the last inequality. Now, observe that

$$\mathbb{P}(\{\Gamma_{a_1,b_1}(t) = \Gamma_{a_2,b_2}(t)\} \cap \{|\Gamma_{a_1,b_1}(t)| \leq 2Mn^{2/3}\})$$
$$\geq \mathbb{P}(\{\Gamma_{a_1,b_1}(t){=}\Gamma_{a_2,b_2}(t)\}\cap\{|\Gamma_{a_1,b_1}(t)|{\leq}2Mn^{2/3}\}\cap\{(\Gamma_n|_{R_1},\Gamma_n|_{R_2}){\in}\mathscr{I}\})$$
$$= \mathbb{E}_{\gamma_1,\gamma_2}[\mathbb{P}(\{\Gamma_{a_1,b_1}(t) = \Gamma_{a_2,b_2}(t)\} \cap \{|\Gamma_{a_1,b_1}(t)| \leq 2Mn^{2/3}\}$$
$$(105) \qquad \qquad |\{\Gamma_n|_{R_1} = \gamma_1\} \cap \{\Gamma_n|_{R_2} = \gamma_2\})\mathbb{1}_{(\gamma_1,\gamma_2)\in\mathscr{I}}]$$
$$\geq \frac{\beta}{2}\mathbb{P}((\Gamma_n|_{R_1}, \Gamma_n|_{R_2}) \in \mathscr{I})$$
$$\geq \frac{0.9\beta}{2}.$$

Note that we have used (104) and Lemma 5.6 to obtain the last two inequalities. This completes the proof of the proposition. ∎

## 6. Concluding remarks and possible extensions

As we have remarked throughout, our objective in this paper was to focus on one of the simplest settings to maintain maximum clarity of exposition. However, we expect that the methods illustrated here have broader applicability and we conclude with a discussion of possible extensions of the results presented in this paper. We shall not attempt to make this discussion precise; working out these details will be taken up elsewhere.

There are primarily two directions of possible generalizations we will discuss: the first one will focus on applicability of our results beyond the exponential LPP model. The other will focus on the set up of exponential LPP itself but will look at the geometry of parts of the geodesics at macroscopic or mesoscopic scales.

6.1. BEYOND EXPONENTIAL LPP. As alluded to before, we recall the reader's attention to the fact that even though we worked with the specific model of exponential LPP, our arguments depended only on the one-point estimates (Proposition 2.1) and its consequences about passage times across the parallelogram (primarily Proposition 2.2, Proposition 2.3) together with some basic tools of percolation like the FKG inequality. An attempt to formalize this axiomatic study of last passage percolation on $\mathbb{Z}^2$ was made in [10], and one expects that the results in this paper will continue to hold under the same set of assumptions (see [10, Section 1.1, page 1245 and Appendix A]). In particular, the analogues of Proposition 2.2 and Proposition 2.3 for Geometric LPP were verified in [10] based on the one point convergence and moderate deviation estimates from [31, 3] (see [10, Sections B.1 and B.3]). Hence, one expects straightforward modifications of Theorem 1 to hold for Geometric LPP, after one modifies the statement to deal with the non-uniqueness of the geodesic (by, e.g., fixing the left most geodesic). Note that our proof of Theorem 2 used the uniqueness of the geodesic in a slightly more crucial way, and hence a direct adaptation of the same to the geometric LPP setting would not work. Even though we believe that an appropriate variant of Theorem 2 would hold for geometric LPP, we shall not comment on this here.

One point convergence and moderate deviation estimates are known for two non-lattice exactly solvable models of planar last passage percolation as well. The first of these is Poissonian LPP on $\mathbb{R}^2$, where the underlying randomness is a homogeneous rate one Poisson point process on $\mathbb{R}^2$ and the last passage time between two ordered points is the maximum number of Poisson points that can be collected in an up/right journey from the "smaller" to the "larger" point. The analogue of Proposition 2.1 for Poissonian LPP was established in [35, 36] and using these, the parallelogram estimates (analogues of Propositions 2.2 and 2.3) were established in [14]. Using these, we expect that our arguments could be used to extend Theorem 1 to Poissonian LPP as well (dealing with the non-uniqueness issue as before).

The second model, the semi-discrete Brownian LPP is defined as follows: let $\{B_i(\cdot)\}_{i\in\mathbb{Z}}$ denote a sequence of two sided standard Brownian motions on $\mathbb{R}$. For a non-decreasing function $\phi : [0,n] \to \{0,1,2,\ldots,n\}$ with $\phi(0)=0$ and $\phi(n)=n$, let us define

$$E(\phi) := \sum_{i=0}^{n} B_i(\phi_{i+1}) - B_i(\phi_i)$$

where $[\phi_i, \phi_{i+1}]$ is the smallest closed interval containing the set $\{x : \phi(x) = i\}$. The last passage time from $(0,0)$ to $(n,n)$ is defined to be the maximum of $E(\phi)$ over all $\phi$. Using the correspondence between passage times in Brownian LPP and the largest eigenvalue of Gaussian Unitary Ensemble (GUE) [7], the one point estimates for passage times in Brownian LPP can be obtained from [34] (see also [1]), and the convergence to the GUE Tracy–Widom distribution is proved in [25]. An inspection of the arguments in [14, 11] shows that the arguments are sufficiently robust and use only the curvature of limit shape and one point moderate deviation estimates, and hence one expects to establish analogues of Proposition 2.2 and Proposition 2.3 for Brownian last passage percolation as well. Although the parallelogram estimates for Brownian LPP has not been worked out anywhere in the literature as far as we are aware, some similar estimates have appeared in the works [29, 28, 27, 17, 19, 18, 39, 23] using the Brownian Gibbs property of a line ensemble associated to Brownian LPP. Recall that we also needed the FKG inequality. An appropriate variant of the FKG inequality on Wiener space that is applicable in the Brownian LPP set-up can be found in [6, Theorem 4].

Again, we shall not work out the details, but for the above explained reasons, we believe that the analogue of Theorem 1 can be proved for Brownian LPP as well.

6.2. GEOMETRY OF GEODESICS AT A FINER SCALE. Here we shall bring our focus back to the exponential LPP model and discuss the applicability of our results to (i) parts of the geodesic $\Gamma_n$ (both macroscopic and mesoscopic), (ii) the scenario when $\delta$ is allowed to go to 0 with $n$.

6.2.1. *Macroscopic segments of the geodesic.* Notice that in Theorem 1, we only considered the small ball probability for the whole geodesic $\Gamma_n$. However, by following our arguments, one can easily also derive the same result for any macroscopic segment of the geodesic. More precisely, for $\epsilon > 0$ fixed, let $[\![t_1, t_2]\!]$

denote a sub-interval of $[\![0, 2n]\!]$ such that $t_2 - t_1 \geq \epsilon n$. We have the following analogue of Theorem 1:

$$(106) \qquad C_2 e^{-c_2 \delta^{-3/2}} \leq \mathbb{P}(\sup_{t \in [\![t_1, t_2]\!]} |\Gamma_n(t)| \leq \delta n^{2/3}) \leq C_1 e^{-c_1 \delta^{-3/2}}.$$

Notice that the lower bound here is immediate from Theorem 1 whereas for the upper bound, one needs to redo the argument restricted to the interval $[\![t_1, t_2]\!]$.

6.2.2. *Mesoscopic segments of the geodesic at either end.* A more interesting question is to consider the segment of geodesic $\Gamma_n$ restricted to the interval $[\![0, r]\!]$ or $[\![2n - r, 2n]\!]$ for some $1 \ll r \ll n$. One of the advantages of working with a pre-limiting model, rather than a limiting model such as the directed land-scape, is that these mesoscopic statistics cannot be read off from the limiting model. It is known that the transversal fluctuation of $\Gamma_n$ at scale $r$ is $O(r^{2/3})$ (see [13, Theorem 3]), hence the natural question is to ask for the probability that $\sup_{t \in [0, r]} |\Gamma_n(t)| \leq \delta r^{2/3}$. We believe that the argument in this paper together with [13, Theorem 3] and [8, Theorem 3] can be used to show that for $r$ sufficiently large and $\delta$ small, we have

$$(107) \qquad C_2 e^{-c_2 \delta^{-3/2}} \leq \mathbb{P}(\sup_{t \in [0, r]} |\Gamma_n(t)| \leq \delta r^{2/3}) \leq C_1 e^{-c_1 \delta^{-3/2}}.$$

One also expects a similar estimate to hold for the semi-infinite geodesic from $\mathbf{0}$ in the direction $(1, 1)$.

The analogue of Theorem 2 is expected to hold at the scale $r \ll n$ as well. In particular, one expects that appropriate modifications of our estimates will yield that

$$(108) \qquad\qquad c\delta \leq \mathbb{P}(|\Gamma_n(r)| \leq \delta r^{2/3}) \leq C\delta.$$

The translation invariance in the proof of the upper bound in Theorem 2 will directly give the upper bound in (108), but the lower bound requires significant modifications in the argument for it to work and would be taken up elsewhere.

6.2.3. *The case of vanishing $\delta$ and small deviations away from the diagonal.* Finally, we want to point out that even though for the sake of notational convenience we have stated our results for a fixed but small $\delta$ while letting $n$ become arbitrarily large, our arguments are sufficiently robust to handle the case when $\delta$ is allowed to go to 0 with $n$ sufficiently slowly. This is rather transparent for Theorem 2, where the key estimates Proposition 5.1 and Proposition 1.2 did not depend on $\delta$ at all and the role of $\delta$ was merely in setting up the appropriate

translations. A moment's thought should convince the reader that Theorem 2 holds for all $\delta$ such that $\delta n^{2/3} \geq 1$ which ensures that the translations can be made sense of in the lattice.

As already mentioned in the introduction, Theorem 2 can also be strengthened by considering $\mathbb{P}(\Gamma_n(\cdot) \in I)$ for any compact interval $I$ of length $\delta n^{2/3} \geq 2$ (the lower bound is imposed to make sure that $\{\Gamma_n(\cdot) \in I\}$ is not vacuously empty).

COROLLARY 6.1: *For each $L \geq 0$ and for all $\epsilon \in (0,1)$, there exist positive constants $C_3, c_3$ depending on $\epsilon$ and $L$ such that for all $n \geq n_0(\epsilon, L)$ and $t \in [\![\epsilon n, (2-\epsilon)n]\!]$, and for all intervals $I \subseteq [-Ln^{2/3}, Ln^{2/3}]$ with $|I| = \delta n^{2/3} \geq 2$, we have*

$$c_3 \delta \leq \mathbb{P}(\Gamma_n(t) \in I) \leq C_3 \delta.$$

*Proof.* We shall discuss only the case of even $t$, an identical argument works for $t$ odd. For $f \in [\![-Ln^{2/3}, Ln^{2/3}]\!]$, it is easy to check that for $t$ even, $i \in \mathbb{Z}$ and $u_i := (i, -i)$ we have $\mathbb{P}(\Gamma_n(t) = 2f) = \mathbb{P}(\Gamma_{u_i, u_i + \mathbf{n}}(t) - \psi(u_i) = 2f)$. Clearly it suffices to show that for some $L'$ (which will be chosen to be a large but fixed number depending on $L'$) we have

(109)
$$\sum_{|i| \leq L'n^{2/3}} \mathbb{P}(\Gamma_{u_i, u_i + \mathbf{n}}(t) - \psi(u_i) = 2f)$$

is bounded away from 0 and $\infty$ uniformly in $n$. Let $\mathcal{L}(t)$ denote the number of points in $\mathbb{L}_t \cap U_{L'+2L}$ that lie on some geodesic $\Gamma_{u_i, u_i + \mathbf{n}}$ for some $|i| < L'n^{2/3}$. The same argument as in Lemma 5.2 shows that $\mathbb{E}\mathcal{L}_t$ is bounded above. The same argument as in the proof of the upper bound of Theorem 2 now shows a uniform (in $n$) upper bound of (109). The lower bound for (109) follows by observing that Proposition 1.2 is true for arbitrarily large choices of $M$ and repeating the proof of the lower bound in Theorem 2.   ∎

Working through the steps of the proof of Theorem 1 in the case of $\delta \to 0$ requires a little more work, but observe that whenever we have applied estimates like Proposition 2.2 or Proposition 2.3 to a rectangle or parallelogram whose dimensions involved $\delta$, it was applied to a parallelogram of size $O(\delta^{3/2}n) \times \delta n^{2/3}$. Application of these parallelogram estimates only requires that the dimensions of the parameters be sufficiently large and hence it is expected that the proofs will all go through as long as $\delta n^{2/3} \to \infty$.

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