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POLYGONAL FUNCTIONAL CALCULUS FOR OPERATORS WITH FINITE PERIPHERAL SPECTRUM

BY

Oualid Bouabdillah and Christian Le Merdy

Laboratoire de Mathématiques de Besançon, UMR 6623, CNRS
Université Franche-Comté, 25030 Besançon Cedex, France
e-mail: oualid.bouabdillah@univ-fcomte.fr, clemerdy@univ-fcomte.fr

ABSTRACT

Let $T\colon X\to X$ be a bounded operator on Banach space, whose spectrum $\sigma(T)$ is included in the closed unit disc $\overline{\mathbb{D}}$. Assume that the peripheral spectrum $\sigma(T)\cap\mathbb{T}$ is finite and that T satisfies a resolvent estimate

$$\|(z-T)^{-1}\|\lesssim \max\{|z-\xi|^{-1} : \xi\in\sigma(T)\cap\mathbb{T}\},\quad z\in\overline{\mathbb{D}}^c.$$

We prove that T admits a bounded polygonal functional calculus, that is, an estimate $\|\phi(T)\|\lesssim \sup\{|\phi(z)| : z\in\Delta\}$ for some polygon $\Delta\subset\mathbb{D}$ and all polynomials ϕ, in each of the following two cases: (i) either $X=L^p$ for some $1<p<\infty$, and $T\colon L^p\to L^p$ is a positive contraction; or (ii) T is polynomially bounded and for all $\xi\in\sigma(T)\cap\mathbb{T}$, there exists a neighborhood $\mathcal{V}$ of ξ such that the set $\{(\xi-z)(z-T)^{-1} : z\in\mathcal{V}\cap\overline{\mathbb{D}}^c\}$ is R-bounded (here X is arbitrary). Each of these two results extends a theorem of de Laubenfels concerning polygonal functional calculus on Hilbert space. Our investigations require the introduction, for any finite set $E\subset\mathbb{T}$, of a notion of Ritt_E operator which generalizes the classical notion of Ritt operator. We study these Ritt_E operators and their natural functional calculus.

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1. Introduction

Let X be a Banach space, let $T\colon X \to X$ be a bounded operator and let $\Omega \subset \mathbb{C}$ be an open set whose closure contains $\sigma(T)$, the spectrum of T. In various situations, an important issue is to determine whether there exists a constant $K \geq 1$ such that:

$$\text{(1)} \qquad \text{For all polynomial } \phi, \quad \|\phi(T)\| \leq K \sup\{|\phi(z)| : z \in \Omega\}.$$

The search for such functional calculus estimates stemmed from the famous von Neumann inequality which asserts that if $X = H$ is a Hilbert space and $\|T\| \leq 1$, then (1) holds true with $\Omega = \mathbb{D}$, the unit disc of $\mathbb{C}$, and $K = 1$. In Hilbertian operator theory, several important topics are related to von Neumann's inequality and to the search for inequalities of the form (1). This includes the study of polynomial boundedness, K-spectral sets and similarity problems, for which we refer to [4, 5, 7, 9, 23, 24] and the references therein.

We recall that $T\colon X \to X$ is called polynomially bounded if there exists a constant $K \geq 1$ such that (1) holds true with $\Omega = \mathbb{D}$. In this paper we are interested in the case when the open set $\Omega \subset \mathbb{C}$ in (1) is a polygon. More explicitly, we say that $T\colon X \to X$ admits a bounded polygonal functional calculus if there exist a (convex, open) polygon $\Delta \subset \mathbb{D}$ such that $\sigma(T) \subset \overline{\Delta}$, and a constant $K \geq 1$ such that (1) holds true with $\Omega = \Delta$.

Let

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

For any T such that $\sigma(T) \subset \overline{\mathbb{D}}$, call $\sigma(T) \cap \mathbb{T}$ the peripheral spectrum of T. It is easy to check (see Remark 3.2) that if $T\colon X \to X$ admits a bounded polygonal functional calculus, then $\sigma(T) \cap \mathbb{T}$ is finite and one has an estimate

$$\text{(2)} \qquad \|(z - T)^{-1}\| \lesssim \max\{|z - \xi|^{-1} : \xi \in \sigma(T) \cap \mathbb{T}\}, \quad z \in \overline{\mathbb{D}}^c.$$

In the Hilbertian case, the following converse was established by de Laubenfels [8, Theorem 4.4, $(a) \Rightarrow (b)$].

Theorem 1.1: *Let H be a Hilbert space and let $T\colon H \to H$ be a polynomially bounded operator. Assume that T has a finite peripheral spectrum and that* (2) *holds true. Then T admits a bounded polygonal functional calculus.*

This remarkable result also follows from [10, Theorem 5.5]. Note that it is already significant when $T\colon H \to H$ is a contraction.

The motivation for this paper is to understand polygonal functional calculus in the Banach space setting.

Our first main result is a Banach space version of Theorem 1.1 relying on the notion of R-boundedness (for which we refer, e.g., to [13, Chapter 8]). We prove (see Corollary 3.10 and Remark 3.8) that if a polynomially bounded operator $T\colon X \to X$ has a finite peripheral spectrum and if for any $\xi \in \sigma(T)\cap\mathbb{T}$, there exists a neighborhood $\mathcal{V}$ of ξ such that the set

$$\{(\xi - z)(z - T)^{-1} : z \in \mathcal{V} \cap \overline{\mathbb{D}}^c\}$$

is R-bounded, then T admits a bounded polygonal functional calculus. We will see that such a bounded functional calculus holds with respect to a polygon $\Delta \subset \mathbb{D}$ such that $\overline{\Delta} \cap \mathbb{T} = \sigma(T) \cap \mathbb{T}$ (see Remark 4.5). In the case when the peripheral spectrum $\sigma(T) \cap \mathbb{T}$ is a singleton, this result reduces to [18, Proposition 7.7], established for Ritt operators.

Our second main result concerns positive contractions on L^p-spaces. Let $1 < p < \infty$, let (S, μ) be a measure space and let $T\colon L^p(S) \to L^p(S)$ be a positive contraction. We show (see Theorem 4.7) that if T has a finite peripheral spectrum and if (2) holds true, then T admits a bounded polygonal functional calculus. Note that in this result, we do not need to assume that T is polynomially bounded. The latter comes as a consequence of the bounded polygonal functional calculus. This result, which holds as well if $T\colon L^p(S) \to L^p(S)$ is a contractively regular operator, should be regarded as an L^p-version of the case when $\|T\| \leq 1$ in Theorem 1.1.

In order to study operators with finite peripheral spectrum and to obtain the aforementioned results, it is relevant to introduce, for a finite set $E = \{\xi_1, \dots, \xi_N\} \subset \mathbb{T}$, the notion of Ritt_E operator; see Definition 2.1 below. This is a natural generalization of Ritt operators, the latter having attracted a lot of attention recently; see, e.g., [2, 3, 6, 11, 14, 18, 19, 20, 21, 22, 27]. Section 2 is devoted to the study of Ritt_E operators. In particular we establish an analogue of the well-known theorem which asserts that $T\colon X \to X$ is a Ritt operator if and only if the two sets

$$\{T^n : n \geq 0\} \quad \text{and} \quad \{n(T^n - T^{n-1}) : n \geq 1\}$$

are bounded. This is achieved in Theorem 2.10. In Sections 3 and 4, we relate the polygonal functional calculus to a natural functional calculus associated with Ritt_E operators, and prove our main results.

2. Ritt$_E$ operators

Let X be a Banach space. We let $B(X)$ denote the Banach algebra of all bounded operators on X, equipped with its usual norm. We denote by I_X the identity operator on X. For any $T \in B(X)$, we let $\sigma(T)$ denote the spectrum of T and we let $R(z,T) = (z - T)^{-1}$ denote the resolvent operator, when z belongs to the resolvent set $\mathbb{C} \setminus \sigma(T)$.

For any $a \in \mathbb{C}$ and any $r > 0$, we let $D(a,r) \subset \mathbb{C}$ denote the open disc of radius r centered at a. We recall that we use the notations $\mathbb{D} = D(0,1)$ and $\mathbb{T}$ for the the open unit disc and for its boundary, respectively.

2.1. Definition and basic facts. We consider distinct complex numbers $\xi_1, \ldots, \xi_N$ of modulus 1, for some $N \geq 1$, and we let

$$E = \{\xi_1, \ldots, \xi_N\} \subset \mathbb{T}. \tag{3}$$

Definition 2.1: Let X be a Banach space and let $T \in B(X)$. We say that T is Ritt$_E$ (or is a Ritt$_E$ operator) if $\sigma(T) \subset \overline{\mathbb{D}}$ and there exists a constant $c > 0$ such that

$$\|R(z,T)\| \leq \frac{c}{\prod_{j=1}^{N} |\xi_j - z|}, \qquad z \in \mathbb{C},\ 1 < |z| < 2. \tag{4}$$

Note that when $N = 1$ and $E = \{1\}$, Ritt$_E$ operators coincide with the usual Ritt operators, for which we refer, e.g., to [2, 3, 6, 11, 14, 18, 19, 20, 21, 22, 27].

Lemma 2.2: *If $T \in B(X)$ is a* Ritt$_E$ *operator, then $\sigma(T) \subset \mathbb{D} \cup E$.*

Proof. Let $z \in \sigma(T) \cap \mathbb{T}$. For any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, we have

$$\|R(\lambda, T)\| \geq \frac{1}{\mathrm{dist}(\lambda, \sigma(T))}.$$

Further, for any integer $t > 0$, $\mathrm{dist}((1+t)z, \sigma(T)) \leq t$. Hence assuming (4), we have

$$\prod_{j=1}^{N} |\xi_j - (1+t)z| \leq ct, \quad t > 0.$$

Letting $t \to 0$, this implies that $z = \xi_j$ for some $j \in \{1, \ldots, N\}$. ∎

Remark 2.3: Let $T \in B(X)$ such that $\sigma(T) \subset \overline{\mathbb{D}}$.

(a) For any $R > 1$, (4) is equivalent to the existence of some constant $c_R > 0$ such that

$$\|R(z,T)\| \leq \frac{c_R}{\prod_{j=1}^{N} |\xi_j - z|}, \quad z \in \mathbb{C},\ 1 < |z| < R.$$

(b) We have $\|R(z,T)\| = O(|z|^{-1})$ when $|z| \to \infty$, hence (4) is equivalent to the existence of some constant $C > 0$ such that

$$\|R(z,T)\| \leq C \max_{j \in \{1,\dots,N\}} \frac{1}{|\xi_j - z|}, \quad z \in \overline{\mathbb{D}}^c.$$

(c) Using Lemma 2.2, we see that the existence of a finite subset $E \subset \mathbb{T}$ such that T is Ritt_E is equivalent to the following:

- The peripheral spectrum $\sigma(T) \cap \mathbb{T}$ is finite and there exists $C > 0$ such that

$$\|R(z,T)\| \leq C \max\left\{\frac{1}{|\xi - z|} : \xi \in \sigma(T) \cap \mathbb{T}\right\}, \quad z \in \overline{\mathbb{D}}^c.$$

2.2. Auxiliary tools. For any $\omega \in (0, \pi)$, we let

$$\Sigma_\omega = \{\lambda \in \mathbb{C}^* : |\mathrm{Arg}(\lambda)| < \omega\}$$

be the open sector of angle 2ω around the positive real axis.

Given any $A \in B(X)$, we say that A is sectorial of type ω if $\sigma(A) \subset \overline{\Sigma_\omega}$ and for any $\nu \in (\omega, \pi)$, there exists a constant $K_\nu > 0$ such that

$$\|\lambda R(\lambda, A)\| \leq K_\nu, \quad \lambda \in \overline{\Sigma_\nu}^c.$$

It is well-known that if $A \in B(X)$ is such that $\sigma(A) \subset \overline{\Sigma_{\frac{\pi}{2}}}$ and $\{\lambda R(\lambda, A) : \lambda \in \overline{\Sigma_{\frac{\pi}{2}}}^c\}$ is bounded, then there exists $\omega \in (0, \frac{\pi}{2})$ such that A is sectorial of type ω. We will use this property in the next proof. We refer, e.g., to [13, Section 10] for information on sectorial operators.

Lemma 2.4: *Assume that $T \in B(X)$ is a Ritt_E operator. Then for any $j = 1, \dots, N$, the operator*

$$A_j = I_X - \overline{\xi_j} T \tag{5}$$

is a sectorial operator of type $< \frac{\pi}{2}$.

Proof. Let $\lambda \in \overline{\Sigma_{\frac{\pi}{2}}}^c$. Then $\xi_j(1 - \lambda) \in \overline{\mathbb{D}}^c$ and an elementary computation shows that λ belongs to the resolvent set of A_j, with

$$\lambda R(\lambda, A_j) = -\lambda \xi_j R(\xi_j(1 - \lambda), T). \tag{6}$$

Hence assuming (4), we have

$$\|\lambda R(\lambda, A_j)\| \leq \frac{c|\lambda|}{\prod_{k=1}^{N}|\xi_k - \xi_j(1-\lambda)|} = \frac{c}{\prod_{\substack{1\leq k\leq N\\ k\neq j}}|\xi_k - \xi_j(1-\lambda)|}.$$

Now observe that for any $k \neq j$, the set $\{|\xi_k - \xi_j(1-\lambda)| : \lambda \in \overline{\Sigma_{\frac{\pi}{2}}}^c\}$ is bounded away from 0. We deduce that $\{\lambda R(\lambda, A_j) : \lambda \in \overline{\Sigma_{\frac{\pi}{2}}}^c\}$ is bounded, which implies that A_j is sectorial of type $< \frac{\pi}{2}$. ∎

Remark 2.5: For T as above, (6) is valid for any λ in the resolvent set of A_j. Equivalently, for any z in the resolvent set of T, we have

$$(\xi_j - z)R(z,T) = -\overline{\xi_j}(\xi_j - z)R(\overline{\xi_j}(\xi_j - z), A_j). \tag{7}$$

When one studies Ritt operators and their functional calculus, it is useful to consider, for any angle $\omega \in (0, \frac{\pi}{2})$, the Stolz domain $B_\omega \subset 1 - \Sigma_\omega$ defined as the interior of the convex hull of 1 and the disc $D(0, \sin(\omega))$ (see, e.g., [3, 18]). We will introduce similar domains adapted to the study of Ritt_E operators.

Definition 2.6: Let $E = \{\xi_1, \ldots, \xi_N\}$ as in (3) and let $r \in (0,1)$. We let E_r denote the interior of the convex hull of $D(0,r) \cup E$; see Figure 1.

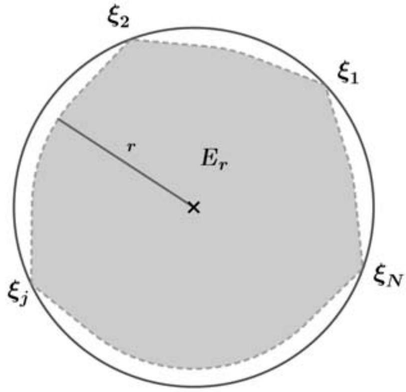

Figure 1. The "generalized" Stolz domain E_r.

Remark 2.7: Assume that $N \geq 2$ and that the sequence $(\xi_1, \xi_2, \ldots, \xi_N)$ is oriented counterclockwise on $\mathbb{T}$. Set $\xi_{N+1} = \xi_1$. We note that if $r \in (0,1)$ is such that $[\xi_j, \xi_{j+1}] \cap \overline{D}(0,r) \neq \emptyset$ for all $j = 1, \ldots N$ (as shown on Figure 1), then for all $r \leq u < s < 1$, we have $\partial E_u \setminus E \subset E_s$. In this case we will say that r is E-large enough.

If $N = 1$, any $r \in (0,1)$ will be called E-large enough.

For any $\xi \in \mathbb{C}^*$ and $\omega \in (0, \frac{\pi}{2})$, we set

$$\Sigma(\xi, \omega) = \xi(1 - \Sigma_\omega). \tag{8}$$

This is the open sector with vertex ξ and angle 2ω around the semi-axis $[\xi, 0)$; see Figure 5.

Lemma 2.8: *An operator $T \in B(X)$ is Ritt_E if and only if there exists $r \in (0,1)$ which is E-large enough (in the sense of Remark 2.7), such that the following two conditions are satisfied:*

(i) $\sigma(T) \subset \overline{E_r}$.

(ii) *For all $s \in (r, 1)$, there exists a constant $c > 0$ such that*

$$\|R(z,T)\| \leq \frac{c}{\prod_{j=1}^N |\xi_j - z|}, \quad z \in D(0,2) \setminus \overline{E_s}.$$

Proof. The 'if' part is obvious. To prove the 'only if' part, let us assume that T is a Ritt_E operator. For any $j = 1, \ldots, N$, consider A_j given by (5). According to Lemma 2.4, we may find $\omega \in (0, \frac{\pi}{2})$ such that A_j is sectorial of type ω for all $j = 1, \ldots, N$. We derive that

$$\sigma(T) \subset \bigcap_{j=1}^N \overline{\Sigma(\xi_j, \omega)}.$$

Choosing ω close enough to $\frac{\pi}{2}$, we may assume that we also have

$$E \subset \overline{\Sigma(\xi_j, \omega)}, \quad j = 1, \ldots N.$$

Choose $\eta > 0$ small enough to ensure that $D(\xi_j, 2\eta) \cap \Sigma(\xi_j, \omega) \subset \mathbb{D}$ for all $j = 1, \ldots, N$. Since $\sigma(T)$ is compact, the set

$$F = \sigma(T) \setminus \left[\bigcup_{j=1}^N D(\xi_j, \eta) \right]$$

is compact as well. Hence there exists $r \in (0,1)$ such that $F \subset \overline{D}(0,r)$. We may and do assume that $r \geq \sin(\omega)$. This ensures that $\sigma(T) \subset \overline{E_r}$ and that r is E-large enough.

We set

$$h(z) = R(z,T) \prod_{k=1}^N (\xi_k - z), \quad z \in \mathbb{C} \setminus \sigma(T).$$

Let $s \in (r, 1)$ and let $\beta = \arcsin(s)$. Then $\beta \in (\omega, \frac{\pi}{2})$. For any $j = 1, \ldots, N$, A_j is sectorial of type ω hence by (7), $(\xi_j - z)R(z,T)$ is bounded on $\overline{\Sigma(\xi_j, \beta)}^c$.

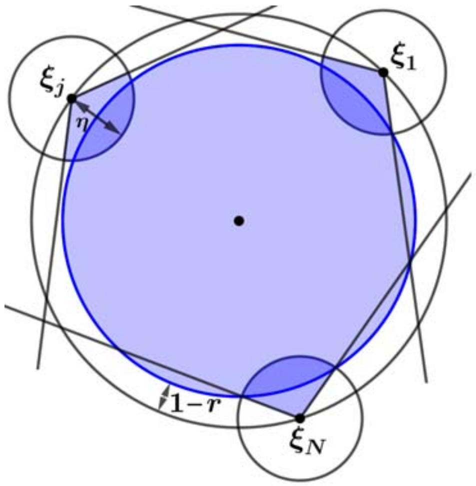

Figure 2. Set containing the spectrum of T.

This implies that h is bounded on $\mathbb{D}\cap\overline{\Sigma(\xi_j,\beta)}^c$. Thus if we let

$$F_0=\bigcup_{j=1}^{N}\overline{\Sigma(\xi_j,\beta)}^c,$$

we obtain that h is bounded on $\mathbb{D}\cap F_0$.

Now observe that by construction,

$$F_1=\Bigl[\Bigl(\bigcap_{j=1}^{N}\overline{\Sigma(\xi_j,\beta)}\Bigr)\bigcap\{s\leq|\lambda|\leq 1\}\Bigr]\setminus\bigcup_{j=1}^{N}D(\xi_j,\cos(\beta)).$$

is a compact subset of $\mathbb{C}\setminus\sigma(T)$. Thus h is bounded on F_1. Finally, we have $\mathbb{D}\setminus\overline{E_s}\subset F_0\cup F_1$, hence h is bounded on $\mathbb{D}\setminus\overline{E_s}$. Since T is Ritt$_E$, this implies (ii). ∎

Remark 2.9: It follows from Lemmas 2.2 and 2.4 and from the proof of Lemma 2.8 that an operator $T\in B(X)$ is Ritt$_E$ if and only if the operators $A_1,\ldots,A_N$ defined by (5) are all sectorial of type $<\frac{\pi}{2}$, and $\sigma(T)\subset\mathbb{D}\cup E$.

2.3. A characterization of Ritt$_E$ operators. It is well-known that an operator $T\in B(X)$ is a Ritt operator if and only if the two sets $\{T^n : n\geq 0\}$ and $\{n(T^n-T^{n-1}) : n\geq 1\}$ are bounded; see [20, 21, 22, 27]. The next theorem is an extension of this result to Ritt$_E$ operators.

Theorem 2.10: *An operator $T \in B(X)$ is* Ritt_E *if and only if the following two conditions hold:*

(i) *T is power bounded, that is, there exists a constant $c_0 \geq 1$ such that*

$$\|T^n\| \leq c_0, \quad n \geq 0.$$

(ii) *There exists a constant $c_1 > 0$ such that*

$$\left\| T^{n-1} \prod_{j=1}^{N} (\xi_j - T) \right\| \leq \frac{c_1}{n}, \quad n \geq 1.$$

In order to prove the 'if' part, we will need the following algebraic factorization property, which is a generalization of a formula used in the proof of [26, Theorem 2.4].

Lemma 2.11: *There exists a two-variable complex polynomial Q such that for all $T \in B(X)$, all $\lambda \in \mathbb{C}^*$ and all $n \geq 1$,*

$$\begin{aligned} &\lambda^n \prod_{j=1}^{N} (\xi_j - \lambda) - T^n \prod_{j=1}^{N} (\xi_j - T) \\ &\qquad = (\lambda - T) \prod_{j=1}^{N} (\xi_j - \lambda) \lambda^{n-1} \sum_{k=0}^{n-1} \lambda^{-k} T^k + T^n (\lambda - T) Q(\lambda, T). \end{aligned}$$

Proof. We have

$$\begin{aligned} &\lambda^n \prod_{j=1}^{N} (\xi_j - \lambda) - T^n \prod_{j=1}^{N} (\xi_j - T) \\ &\quad = \lambda^n \prod_{j=1}^{N} (\xi_j - \lambda) - T^n \prod_{j=1}^{N} (\xi_j - \lambda) + T^n \prod_{j=1}^{N} (\xi_j - \lambda) - T^n \prod_{j=1}^{N} (\xi_j - T) \\ &\quad = (\lambda^n - T^n) \prod_{j=1}^{N} (\xi_j - \lambda) + T^n \Bigg(\prod_{j=1}^{N} (\xi_j - \lambda) - \prod_{j=1}^{N} (\xi_j - T) \Bigg). \end{aligned}$$

Set

$$P(\lambda, z) = \prod_{j=1}^{N} (\xi_j - \lambda) - \prod_{j=1}^{N} (\xi_j - z), \quad \lambda, z \in \mathbb{C}.$$

Then we have

$$P(\lambda, z) = (\lambda - z) Q(\lambda, z)$$

for some polynomial Q. Then

$$\prod_{j=1}^{N}(\xi_j-\lambda)-\prod_{j=1}^{N}(\xi_j-T)=(\lambda-T)Q(\lambda,T).$$

We deduce that

$$\begin{aligned}\lambda^n\prod_{j=1}^{N}(\xi_j-\lambda)-T^n\prod_{j=1}^{N}(\xi_j-T)&\\ &=(\lambda^n-T^n)\prod_{j=1}^{N}(\xi_j-\lambda)+T^n(\lambda-T)Q(\lambda,T)\\ &=(\lambda-T)\prod_{j=1}^{N}(\xi_j-\lambda)\sum_{k=0}^{n-1}\lambda^{n-1-k}T^k+T^n(\lambda-T)Q(\lambda,T),\end{aligned}$$

which yields the desired identity. ∎

Proof of Theorem 2.10. $\Rightarrow$: We assume that the operator T is Ritt_E and we will show (i) and (ii). The argument is an adaptation of the one used for Ritt operators in [27].

Let $r\in(0,1)$ such that T satisfies Lemma 2.8. Let $s\in(r,1)$, let $\alpha=\arcsin(s)$ and note that

$$|1-\cos(\alpha)e^{\pm i\alpha}|=\sin(\alpha)=s.$$

For convenience we assume that the sequence $(\xi_1,\xi_2,\dots,\xi_N)$ is oriented counterclockwise on $\mathbb{T}$ and we set $\xi_{N+1}=\xi_1$. Let $n\geq 1$. For any $j=1,\dots,N$, we define the following four paths (whose definitions depend on n, although this is not reflected by the notation), see Figure 3:

- γ_{j+} is the oriented segment $[\xi_j(1-\frac{\cos(\alpha)}{n}e^{-i\alpha}),\xi_j(1-\cos(\alpha)e^{-i\alpha})]$;
- γ_{j-} is the oriented segment $[\xi_j(1-\cos(\alpha)e^{i\alpha}),\xi_j(1-\frac{\cos(\alpha)}{n}e^{i\alpha})]$;
- $\gamma_{j,j+1}$ is the simple path going from $\xi_j(1-\cos(\alpha)e^{-i\alpha})$ to $\xi_{j+1}(1-\cos(\alpha)e^{i\alpha})$ counterclockwise along the circle of center 0 and radius s;
- γ_j is the simple path going from $\xi_j(1-\frac{\cos(\alpha)}{n}e^{i\alpha})$ to $\xi_j(1-\frac{\cos(\alpha)}{n}e^{-i\alpha})$ counterclockwise along the circle of center ξ_j and radius $\frac{\cos(\alpha)}{n}$.

For s close enough to 1 and n large enough, the concatenation of these $4N$ paths, that we denote by Γ_n, is a Jordan contour enveloping $\sigma(T)$, and the support of Γ_n is included in E_s^c.

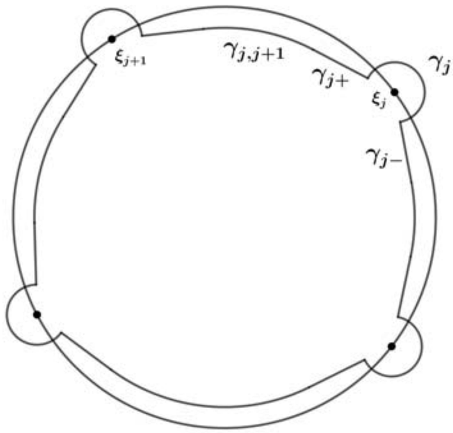

Figure 3. Integration contour.

We fix $s < 1$ and $n_0 \geq 1$ such that we are in the above case for all $n \geq n_0$. Then applying the Dunford–Riesz functional calculus, we have

$$T^n = \frac{1}{2\pi i} \int_{\Gamma_n} \lambda^n R(\lambda, T) d\lambda,$$

for all $n \geq n_0$. The upper bound in Lemma 2.8(ii), provides an estimate

$$\|T^n\| \leq \frac{1}{2\pi} \int_{\Gamma_n} |\lambda^n| \|R(\lambda, T)\| |d\lambda| \leq \frac{c}{2\pi} \int_{\Gamma_n} \frac{|\lambda|^n}{\prod_{j=1}^N |\lambda - \xi_j|} |d\lambda|.$$

We will show that the integral in the right-hand side is uniformly bounded for $n \geq n_0$.

Fix $j_0 \in \{1, \ldots, N\}$. For any $\lambda \in \gamma_{j_0, j_0+1}$, we have $|\lambda| = s$ and hence

$$|\lambda - \xi_j| \geq 1 - s$$

for all $j = 1, \ldots, N$. Consequently,

$$\int_{\gamma_{j_0, j_0+1}} \frac{|\lambda|^n}{\prod_{j=1}^N |\lambda - \xi_j|} |d\lambda| \leq \int_{\gamma_{j_0, j_0+1}} \frac{s^n}{(1-s)^N} |d\lambda| \longrightarrow 0,$$

when $n \to \infty$.

Next observe that

$$K = \sup\left\{\frac{1}{\prod_{\substack{1\leq j\leq N\\ j\neq j_0}}|\lambda-\xi_j|} : |\lambda-\xi_{j_0}|\leq \frac{\cos(\alpha)}{n}\right\} < \infty.$$

For all $\lambda\in\gamma_{j_0}$, we have $|\lambda-\xi_{j_0}| = \frac{\cos(\alpha)}{n}$ and hence $|\lambda|\leq 1+\frac{1}{n}$. Consequently,

$$\begin{aligned}\int_{\gamma_{j_0}}\frac{|\lambda|^n}{\prod_{j=1}^N|\lambda-\xi_j|}|d\lambda| &= \frac{1}{\cos(\alpha)}\int_{\gamma_{j_0}}\frac{n|\lambda|^n}{\prod_{\substack{1\leq j\leq N\\ j\neq j_0}}|\lambda-\xi_j|}|d\lambda|\\ &\leq \frac{K}{\cos(\alpha)}\int_{\gamma_{j_0}} n\Big(1+\frac{1}{n}\Big)^n|d\lambda|\\ &\leq 2\pi e K.\end{aligned}$$

Let us now focus on the integral over $\gamma_{j_0,+}$. We may assume that $\xi_{j_0}=1$ (otherwise, change T into $\overline{\xi_{j_0}}T$.) Observe that

$$C = \sup\left\{\frac{1}{\prod_{\substack{1\leq j\leq N\\ j\neq j_0}}|\lambda-\xi_j|} : \lambda\in[1,1-\cos(\alpha)e^{-i\alpha}]\right\} < \infty.$$

For all $t\in[0,\cos(\alpha)]$, we have $t^2\leq t\cos(\alpha)$, hence

$$|1-te^{-i\alpha}|^2 = 1+t^2-2t\cos(\alpha)\leq 1-t\cos(\alpha).$$

We derive the estimate

$$\int_{\gamma_{j_0,+}}\frac{|\lambda|^n}{\prod_{j=1}^N|\lambda-\xi_j|}|d\lambda|\leq C\int_{\frac{\cos(\alpha)}{n}}^{\cos(\alpha)}\frac{(1-t\cos(\alpha))^{\frac{n}{2}}}{t}dt.$$

Using the change of variable $t\to t\cos(\alpha)$ and the inequality $1-t\leq e^{-t}$, we have

$$\int_{\frac{\cos(\alpha)}{n}}^{\cos(\alpha)}\frac{(1-t\cos(\alpha))^{\frac{n}{2}}}{t}dt\leq\int_{\frac{\cos^2(\alpha)}{n}}^{+\infty}\frac{e^{-\frac{nt}{2}}}{t}dt = \int_{\frac{\cos^2(\alpha)}{2}}^{+\infty}\frac{e^{-t}}{t}dt.$$

This proves that the integrals $\int_{\gamma_{j_0,+}}\frac{|\lambda|^n}{\prod_{j=1}^N|\lambda-\xi_j|}|d\lambda|$ are uniformly bounded for $n\geq n_0$. The same holds true if we replace $\gamma_{j_0,+}$ by $\gamma_{j_0,-}$. Thus we have proved that

$$\sup_{n\geq n_0}\int_{\Gamma_n}\frac{|\lambda|^n}{\prod_{j=1}^N|\lambda-\xi_j|}|d\lambda| < \infty.$$

This implies that $\{T^n : n\geq 0\}$ is bounded, hence (i) is proved.

Let us now prove (ii), using Γ_n as above. Applying the Dunford–Riesz functional calculus again, we have

$$nT^{n-1}\prod_{j=1}^{N}(\xi_j - T) = \frac{n}{2\pi i}\int_{\Gamma_n} \lambda^{n-1}\prod_{j=1}^{N}(\xi_j-\lambda)R(\lambda,T)d\lambda,$$

for all $n \geq n_0$. This implies

$$\left\|nT^{n-1}\prod_{j=1}^{N}(\xi_j - T)\right\| \leq \frac{cn}{2\pi}\int_{\Gamma_n}|\lambda|^{n-1}|d\lambda|.$$

Thus it suffices to show that the integrals in the right-hand side are uniformly bounded for $n \geq n_0$.

As before we fix $j_0 \in \{1,\ldots,N\}$. For all $\lambda \in \gamma_{j_0,j_0+1}$, we have $|\lambda| = s$ hence

$$n\int_{\gamma_{j_0,j_0+1}}|\lambda|^{n-1}|d\lambda| \leq 2\pi n s^{n-1} \longrightarrow 0,$$

when $n \to \infty$.

Next for all $\lambda \in \gamma_{j_0}$, we have $|\lambda| \leq 1+\frac{1}{n}$ hence

$$n\int_{\gamma_{j_0}}|\lambda|^{n-1}|d\lambda| \leq n\Big(1+\frac{1}{n}\Big)^{n-1}|\gamma_{j_0}| \leq 2\pi e.$$

Finally assume as before that $\xi_{j_0} = 1$. Then arguing as above we have

$$\begin{aligned} n\int_{\gamma_{j_0,+}}|\lambda|^{n-1}|d\lambda| &\leq n\int_{\frac{\cos(\alpha)}{n}}^{\cos(\alpha)}(1-t\cos(\alpha))^{\frac{n-1}{2}}dt \\ &\leq \frac{n}{\cos(\alpha)}\int_{\frac{\cos^2(\alpha)}{n}}^{\infty}(1-t)^{\frac{n-1}{2}}dt \\ &\leq \frac{n}{\cos(\alpha)}\int_{\frac{\cos^2(\alpha)}{n}}^{\infty}e^{\frac{-t(n-1)}{2}}dt \\ &\leq \frac{4}{\cos(\alpha)}\int_0^{\infty}e^{-t}dt = \frac{4}{\cos(\alpha)}. \end{aligned}$$

We obtain that $n\int_{\gamma_{j_0,+}}|\lambda|^{n-1}|d\lambda|$ has a uniform bound. The same holds true with $\gamma_{j_0,-}$ in place of $\gamma_{j_0,+}$. These estimates imply (ii).

$\Leftarrow$: Assume (i) and (ii). The fact that T is power bounded implies that $\sigma(T) \subset \overline{\mathbb{D}}$. For any $\lambda \in \mathbb{C}$, with $1 < |\lambda| < 2$, we multiply the identity of

Lemma 2.11 by $R(\lambda,T)$ to obtain

$$\lambda^n \prod_{j=1}^{N}(\xi_j-\lambda)R(\lambda,T)$$
$$= T^n\prod_{j=1}^{N}(\xi_j-T)R(\lambda,T)+\prod_{j=1}^{N}(\xi_j-\lambda)\lambda^{n-1}\sum_{k=0}^{n-1}\lambda^{-k}T^k+T^nQ(\lambda,T).$$

This implies

$$|\lambda|^n\prod_{j=1}^{N}|\xi_j-\lambda|\|R(\lambda,T)\|\leq\left\|T^n\prod_{j=1}^{N}(\xi_j-T)\right\|\|R(\lambda,T)\|$$
$$+\prod_{j=1}^{N}|\xi_j-\lambda||\lambda|^{n-1}\sum_{k=0}^{n-1}|\lambda|^{-k}\|T^k\|+\|Q(\lambda,T)\|\|T^n\|.$$

By continuity of polynomials, there exists a constant $c>0$ such that

$$\|Q(\lambda,T)\|\leq c,\quad \lambda\in D(0,2).$$

We derive the following estimate:

$$|\lambda|^n\prod_{j=1}^{N}|\xi_j-\lambda|\|R(\lambda,T)\|\leq\frac{c_1}{n}\|R(\lambda,T)\|+c_0\prod_{j=1}^{N}|\xi_j-\lambda||\lambda|^{n-1}\sum_{k=0}^{n-1}|\lambda|^{-k}+cc_0.$$

Dividing both sides by $|\lambda|^n$, and using the fact that $|\lambda|>1$, we derive

$$\prod_{j=1}^{N}|\xi_j-\lambda|\|R(\lambda,T)\|\leq\frac{c_1}{n}\|R(\lambda,T)\|+c_0\left(n\prod_{j=1}^{N}|\xi_j-\lambda|+c\right). \tag{9}$$

Now for a given $\lambda\in D(0,2)\setminus\overline{\mathbb{D}}$, we apply this estimate with $n-1$ equal to the integer part of $\frac{2c_1}{\prod_{j=1}^{N}|\xi_j-\lambda|}$. Thus

$$\frac{c_1}{n}\leq\frac{1}{2}\prod_{j=1}^{N}|\xi_j-\lambda|\quad\text{and}\quad n\prod_{j=1}^{N}|\xi_j-\lambda|\leq 2c_1+\prod_{j=1}^{N}|\xi_j-\lambda|\leq 2c_1+3^N.$$

Then (9) implies that

$$\prod_{j=1}^{N}|\xi_j-\lambda|\|R(\lambda,T)\|\leq\frac{1}{2}\prod_{j=1}^{N}|\xi_j-\lambda|\|R(\lambda,T)\|\leq+c_0\left(n\prod_{j=1}^{N}|\xi_j-\lambda|+c\right),$$

and hence

$$\prod_{j=1}^{N}|\xi_j-\lambda|\|R(\lambda,T)\|\leq 2c_0\left(n\prod_{j=1}^{N}|\xi_j-\lambda|+c\right)\leq 2c_0(2c_1+3^N+c).$$

This proves that T is Ritt_E. ∎

3. Polygonal functional calculus and R-Ritt$_E$ operators

In this section we define polygonal functional calculus and $H^\infty(E_s)$ functional calculus, as well as R-Ritt$_E$ operators. Our main results, Theorem 3.9 and Corollary 3.10, assert that for any R-Ritt$_E$ operator, polynomial boundedness implies (and hence is equivalent to) either bounded polygonal functional calculus or a bounded $H^\infty(E_s)$ functional calculus. The proofs rely on a decomposition principle that we establish in Subsection 3.2.

3.1. Functional calculi. For any non-empty open set $\Omega \subset \mathbb{C}$, we let $H^\infty(\Omega)$ denote the Banach algebra of all bounded holomorphic functions $\phi\colon \Omega \to \mathbb{C}$, equipped with

$$\|\phi\|_{\infty,\Omega} = \sup\{|\phi(\lambda)| : \lambda \in \Omega\}.$$

Let $\mathcal{P}$ denote the algebra of all one-variable complex polynomials.

Let X be a Banach space and let $T \in B(X)$ such that $\sigma(T) \subset \overline{\mathbb{D}}$. Following classical terminology, we say that T is polynomially bounded if there exists a constant $K \geq 1$ such that

$$\|\phi(T)\| \leq K\|\phi\|_{\infty,\mathbb{D}}, \quad \phi \in \mathcal{P}. \tag{10}$$

We recall the obvious fact that any polynomially bounded operator is power bounded.

In the sequel, the name "polygon" is reserved for open convex polygons, that is, bounded finite intersections of open half-planes.

Definition 3.1: We say that T admits a bounded polygonal functional calculus if there exist a polygon $\Delta \subset \mathbb{D}$ such that $\sigma(T) \subset \overline{\Delta}$, and a constant $K \geq 1$ such that

$$\|\phi(T)\| \leq K\|\phi\|_{\infty,\Delta}, \quad \phi \in \mathcal{P}. \tag{11}$$

It is plain that T is polynomially bounded if it admits a bounded polygonal functional calculus. It essentially follows from [16] that the converse is wrong (see Remark 4.6).

Remark 3.2: Assume that T satisfies Definition 3.1 for some polygon $\Delta \subset \mathbb{D}$. Let $E = \overline{\Delta} \cap \mathbb{T}$. This is a finite set, containing the peripheral spectrum $\sigma(T) \cap \mathbb{T}$.

Assume that $\sigma(T) \cap \mathbb{T} \neq \emptyset$. Let

$$\mathcal{D}_{1,2} = \{z \in \mathbb{C} : 1 < |z| < 2\}$$

and define

$$h(z,\lambda) = (z-\lambda)^{-1}\prod_{\xi\in E}(\xi - z), \quad z\in \mathcal{D}_{1,2},\ \lambda\in\mathbb{D}.$$

For any $\xi\in E$, the quotient $\frac{\xi-z}{z-\lambda}$ is bounded when z,λ are close to ξ, $z\in\mathcal{D}_{1,2}$ and $\lambda\in\Delta$.

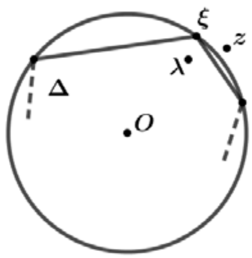

Figure 4. Positions of ξ, z, λ.

This implies that $C = \sup\{\|h(z,\cdot)\|_{\infty,\Delta} : z\in\mathcal{D}_{1,2}\} < \infty$. Then we derive from (11) that $h(z,T) = R(z,T)\prod_{\xi\in E}(\xi - z)$ satisfies $\|h(z,T)\|\leq KC$ for all $z\in\mathcal{D}_{1,2}$. This implies that T is Ritt_E. Equivalently (see Remark 2.3(b)) T satisfies a resolvent estimate

$$\|R(z,T)\| \lesssim \max\{|z-\xi|^{-1} : \xi\in E\}, \quad z\in\overline{\mathbb{D}}^c. \tag{12}$$

Since $R(z,T)$ is bounded in the neighborhood of any $\xi\in E$ not belonging to $\sigma(T)$, this implies an estimate (2) as well.

Of course if $\sigma(T)\cap\mathbb{T}=\emptyset$, then $R(z,T)$ is bounded on $\overline{\mathbb{D}}^c$.

Let $E=\{\xi_1,\ldots,\xi_N\}\subset\mathbb{T}$ as in Section 2. Assume that T is a Ritt_E operator. In the sequel we will say that T is a Ritt_E operator of type $r\in(0,1)$ provided that it satisfies the conclusion of Lemma 2.8.

For any $s\in(0,1)$, we let $H^\infty_0(E_s)$ be the subspace of all $\phi\in H^\infty(E_s)$ for which there exist positive real numbers $c, s_1,\ldots,s_n>0$ such that

$$|\phi(\lambda)| \leq c\prod_{j=1}^N |\xi_j-\lambda|^{s_i}, \tag{13}$$

for all $\lambda\in E_s$.

Assume that T is a Ritt_E operator of type $r \in (0,1)$ and let $s \in (r,1)$. For any $\phi \in H_0^\infty(E_s)$, we set

$$\phi(T) = \frac{1}{2\pi i} \int_{\partial E_u} \phi(\lambda) R(\lambda, T) d\lambda,$$

where $u \in (r,s)$ and the boundary ∂E_u of E_u is oriented counterclockwise. According to Remark 2.7, we have $\partial E_u \setminus E \subset E_s$ hence integration of $\phi(\lambda)R(\lambda,T)$ along ∂E_u makes sense. This integral is therefore absolutely convergent, hence well-defined. Indeed this follows from (13) and the estimate (ii) in Lemma 2.8. Furthermore by Cauchy's theorem, the value of this integral does not depend on the choice of u. It is easy to check that $H_0^\infty(E_s)$ is a subalgebra of $H^\infty(E_s)$ and that the mapping

$$H_0^\infty(E_s) \longrightarrow B(X), \quad \phi \mapsto \phi(T),$$

is a homomorphism.

Definition 3.3: Let T be a Ritt_E operator of type $r \in (0,1)$ and let $s \in (r,1)$ We say that T admits a bounded $H^\infty(E_s)$ functional calculus if there exists a constant $K \geq 1$ such that

$$\|\phi(T)\| \leq K \|\phi\|_{\infty, E_s}, \quad \phi \in H_0^\infty(E_s). \tag{14}$$

The above definitions are natural extensions of the ones considered in [18] for Ritt operators. In this spirit, the following is an analogue of [18, Proposition 2.5].

PROPOSITION 3.4: *Let T be a Ritt_E operator of type $r \in (0,1)$ and let $s \in (r,1)$. Then T has a bounded $H^\infty(E_s)$ functional calculus if and only if there exists a constant $K \geq 1$ such that*

$$\|\phi(T)\| \leq K \|\phi\|_{\infty, E_s}, \quad \phi \in \mathcal{P}.$$

Proof. The proof of the 'if' part is identical to that of [18, Proposition 2.5] so we skip it.

For the 'only if' part, assume (14). Consider (Lagrange) polynomials $L_1, \ldots, L_N \in \mathcal{P}$ satisfying $L_i(\xi_j) = \delta_{i,j}$, for all $1 \leq i,j \leq N$. Let $\psi \in \mathcal{P}$ and write $\psi = \psi_0 + \psi_1$, with

$$\psi_0 = \sum_{j=1}^N \psi(\xi_j) L_j \quad \text{and} \quad \psi_1 = \psi - \sum_{j=1}^N \psi(\xi_j) L_j.$$

Then

$$\psi_1(\xi_j) = 0 \quad \text{for all } j = 1, \ldots, N,$$

hence $\psi_1 \in H_0^\infty(E_s)$. Writing $\psi(T) = \psi_0(T) + \psi_1(T)$, and using

$$\psi_0(T) = \sum_j \psi(\xi_j) L_j(T),$$

we infer

$$\|\psi(T)\| \leq \|\psi_0(T)\| + \|\psi_1(T)\| \leq \sum_{j=1}^{N} |\psi(\xi_j)| \|L_j(T)\| + K \|\psi_1\|_{\infty, E_s}.$$

Further, $\|\psi_1\|_{\infty,E_s} \leq \|\psi\|_{\infty,E_s} + \|\psi_0\|_{\infty,E_s}$ and

$$\|\psi_0\|_{\infty,E_s} \leq \sum_{j=1}^{N} |\psi(\xi_j)| \|L_j\|_{\infty,E_s} \leq \Big(\sum_{j=1}^{N} \|L_j\|_{\infty,E_s}\Big) \|\psi\|_{\infty,E_s}.$$

We derive that

$$\|\psi(T)\| \leq \bigg(K\Big(1 + \sum_{j=1}^{N} \|L_j\|_{\infty,E_s}\Big) + \sum_{j=1}^{N} \|L_j(T)\|\bigg) \|\psi\|_{\infty,E_s},$$

which proves the result. ∎

Definitions 3.1 and 3.3 are connected by the following property. See also Remark 4.5 for more on this.

Proposition 3.5: *For any $T \in B(X)$, the following assertions are equivalent:*

(i) *The operator T admits a bounded polygonal functional calculus.*

(ii) *There exist a finite subset $E \subset \mathbb{T}$ and $s \in (0,1)$ such that T is* Ritt_E *and T admits a bounded $H^\infty(E_s)$ functional calculus.*

Proof. Assume (i), that is, T satisfies Definition 3.1 for some polygon $\Delta \subset \mathbb{D}$. We may assume that $E = \overline{\Delta} \cap \mathbb{T}$ is non-empty. Then T is Ritt_E by Remark 3.2. Furthermore for $s \in (0,1)$ large enough, we have $\Delta \subset E_s$. Hence by Proposition 3.4, the estimate (11) implies that T admits a bounded $H^\infty(E_s)$ functional calculus.

Assume (ii). It is plain that there exists a polygon $\Delta \subset \mathbb{D}$ such that

$$E_s \subset \Delta \subset \mathbb{D}.$$

Hence applying Proposition 3.4, we obtain that T satisfies an estimate (11). ∎

3.2. Decomposition of unity. We fix $E = \{\xi_1, \dots, \xi_N\} \subset \mathbb{T}$ as in Section 2. We let $H^\infty_{0,E}(\mathbb{D})$ be the space of all $\phi \in H^\infty(\mathbb{D})$ for which there exist positive real numbers $c, s_1, \dots, s_n > 0$ such that (13) holds true for all $\lambda \in \mathbb{D}$.

Proposition 3.6: *There exist three sequences $(\theta_i)_{i\geq 1}$, $(\phi_i)_{i\geq 1}$ and $(\psi_i)_{i\geq 1}$ of $H^\infty_{0,E}(\mathbb{D})$ such that:*

(i) $$\sup_{z\in\mathbb{D}} \sum_{i=1}^{\infty} |\phi_i(z)| < \infty \quad \textit{and} \quad \sup_{z\in\mathbb{D}} \sum_{i=1}^{\infty} |\psi_i(z)| < \infty;$$

(ii) $$\sup_{i\geq 1} \sup_{z\in\mathbb{D}} |\theta_i(z)| < \infty;$$

(iii) $$\forall r \in (0,1), \quad \sup_{i\geq 1} \int_{\partial E_r} \frac{|\theta_i(z)|}{\prod_{k=1}^N |\xi_k - z|} |dz| < \infty;$$

(iv) $$\forall z \in \mathbb{D}, \quad 1 = \sum_{i=1}^{\infty} \theta_i(z)\phi_i(z)\psi_i(z).$$

Proof. Let $H^\infty_{00}(\Sigma_{\frac{\pi}{2}})$ be the space of all $\Phi \in H^\infty(\Sigma_{\frac{\pi}{2}})$ for which there exist $c, s > 0$ such that

$$|\Phi(\lambda)| \leq c|\lambda|^s, \quad \lambda \in \Sigma_{\frac{\pi}{2}}. \tag{15}$$

Then according to [1, Proposition 6.3], there exist three sequences $(\Theta_i)_{i\geq 1}$, $(\Phi_i)_{i\geq 1}$ and $(\Psi_i)_{i\geq 1}$ of $H^\infty_{00}(\Sigma_{\frac{\pi}{2}})$ such that

$$\sup_{\lambda\in\Sigma_{\frac{\pi}{2}}} \sum_{i=1}^{\infty} |\Phi_i(\lambda)| < \infty \quad \text{and} \quad \sup_{\lambda\in\Sigma_{\frac{\pi}{2}}} \sum_{i=1}^{\infty} |\Psi_i(\lambda)| < \infty, \tag{16}$$

$$\sup_{i\geq 1} \sup_{\lambda\in\Sigma_{\frac{\pi}{2}}} |\Theta_i(\lambda)| < \infty, \tag{17}$$

$$\forall \nu \in (0, \tfrac{\pi}{2}), \quad \sup_{i\geq 1} \int_{\partial\Sigma_\nu} |\Theta_i(\lambda)| \frac{|d\lambda|}{|\lambda|} < \infty, \tag{18}$$

and

$$\forall \lambda \in \Sigma_{\frac{\pi}{2}}, \quad 1 = \sum_{i=1}^{\infty} \Theta_i(\lambda)\Phi_i(\lambda)\Psi_i(\lambda). \tag{19}$$

For any multi-index $\iota = (i_1, i_2, \dots, i_N) \in \mathbb{N}^N$, define $\theta_\iota, \phi_\iota, \psi_\iota \colon \mathbb{D} \to \mathbb{C}$ by

$$\theta_\iota(z) = \Theta_{i_1}(1 - \overline{\xi_1}z)\Theta_{i_2}(1 - \overline{\xi_2}z) \cdots \Theta_{i_N}(1 - \overline{\xi_N}z),$$

and similarly

$$\phi_\iota(z) = \prod_{j=1}^N \Phi_{i_j}(1-\overline{\xi_j}z) \quad \text{and} \quad \psi_\iota(z) = \prod_{j=1}^N \Psi_{i_j}(1-\overline{\xi_j}z).$$

This is well-defined since for any $z \in \mathbb{D}$, we have $\overline{\xi_j}z \in \mathbb{D}$ hence $1-\overline{\xi_j}z \in \Sigma_{\frac{\pi}{2}}$. Moreover

$$|\theta_\iota(z)| \leq c^N \prod_{j=1}^N |1-\overline{\xi_j}z|^s = c^N \prod_{j=1}^N |\xi_j - z|^s$$

for any $z \in \mathbb{D}$, by (15). Hence all the functions θ_ι belong to $H^\infty_{0,E}(\mathbb{D})$. Likewise, all the functions ϕ_ι and ψ_ι belong to $H^\infty_{0,E}(\mathbb{D})$. Since we can re-index the families $(\theta_\iota)_{\iota\in\mathbb{N}^N}$, $(\phi_\iota)_{\iota\in\mathbb{N}^N}$ and $(\psi_\iota)_{\iota\in\mathbb{N}^N}$ as sequences, it suffices to show that they satisfy the properties (i)–(iv) of the statement.

For any $z \in \mathbb{D}$,

$$\sum_{\iota=(i_1,\ldots,i_N)\in\mathbb{N}^N} |\phi_\iota(z)| = \prod_{j=1}^N \Biggl(\sum_{i_j=1}^\infty |\Phi_{i_j}(1-\overline{\xi_j}z)|\Biggr) \leq \Biggl(\sup_{\lambda\in\Sigma_{\frac{\pi}{2}}} \sum_{i=1}^\infty |\Phi_i(\lambda)|\Biggr)^N.$$

Hence by (16), the family $(\phi_\iota)_{\iota\in\mathbb{N}^N}$ satisfies (i). Likewise, the family $(\psi_\iota)_{\iota\in\mathbb{N}^N}$ satisfies (i), and $(\theta_\iota)_{\iota\in\mathbb{N}^N}$ satisfies (ii), by (17). Further (iv) holds true by (19), since we have

$$\begin{aligned} 1 &= \prod_{j=1}^N \Biggl(\sum_{i_j=1}^\infty \Theta_{i_j}(1-\overline{\xi_j}z)\Phi_{i_j}(1-\overline{\xi_j}z)\Psi_{i_j}(1-\overline{\xi_j}z)\Biggr) \\ &= \sum_{\iota=(i_1,\ldots,i_N)\in\mathbb{N}^N} \theta_\iota(z)\phi_\iota(z)\psi_\iota(z), \end{aligned}$$

for all $z \in \mathbb{D}$.

It therefore remains to check (iii). We fix $r \in (0,1)$. It follows from Definition 2.6 that there exist $C, \delta > 0$, $\rho \in (0,1)$, as well as $2N$ angles $\nu_1, \nu_1', \ldots, \nu_N, \nu_N'$ in $(0, \frac{\pi}{2})$ so that the following holds true:

(a) For all $j = 1, \ldots, N$, $\partial E_r \cap D(\xi_j, \delta)$ is the concatenation of the oriented segments $[\xi_j(1-\delta e^{i\nu_j}), \xi_j]$ and $[\xi_j, \xi_j(1-\delta e^{-i\nu_j'})]$.

(b) For all $z \in \partial E_r \setminus \cup_{j=1}^N D(\xi_j, \delta)$, we have $|z| \leq \rho$.

(c) For all $1 \leq j \neq k \leq N$, for all $z \in D(\xi_j, \delta)$, we have $|z - \xi_k| \geq C$.

Let $\iota = (i_1, \ldots, i_N) \in \mathbb{N}^N$. Using (b), we have

$$\int_{\partial E_r} \frac{|\theta_\iota(z)|}{\prod_{k=1}^N |\xi_k - z|} |dz| \\ \leq \frac{1}{(1-\rho)^N} \int_{\partial E_r} |\theta_\iota(z)||dz| + \sum_{j=1}^N \int_{\partial E_r \cap D(\xi_j,\delta)} \frac{|\theta_\iota(z)|}{\prod_{k=1}^N |\xi_k - z|} |dz|.$$

By (ii), the first term in the right-hand side is uniformly bounded. According to (a), it therefore suffices to show that for any $j = 1, \ldots, N$, the integrals

$$\int_{[\xi_j(1-\delta e^{i\nu_j}),\xi_j]} \frac{|\theta_\iota(z)|}{\prod_{k=1}^N |\xi_k - z|} |dz| \quad \text{and} \quad \int_{[\xi_j,\xi_j(1-\delta e^{-i\nu'_j})]} \frac{|\theta_\iota(z)|}{\prod_{k=1}^N |\xi_k - z|} |dz| \tag{20}$$

are uniformly bounded. Let $J_{\iota,j}$ be the first of these two integrals. Let

$$K = \sup_i \|\Theta_i\|_{\infty,\frac{\pi}{2}},$$

given by (17). Then by (c), we have

$$J_{\iota,j} \leq \Big(\frac{K}{C}\Big)^{N-1} \int_{[\xi_j(1-\delta e^{i\nu_j}),\xi_j]} \frac{|\Theta_{i_j}(1-\overline{\xi_j}z)|}{|\xi_j - z|} |dz|.$$

Moreover the change of variable $\lambda = 1 - \overline{\xi_j}z$ leads to

$$\int_{[\xi_j(1-\delta e^{i\nu_j}),\xi_j]} \frac{|\Theta_{i_j}(1-\overline{\xi_j}z)|}{|\xi_j - z|} |dz| \leq \int_{\partial\Sigma_{\nu_j}} |\Theta_{i_j}(\lambda)| \frac{|d\lambda|}{|\lambda|}.$$

Property (18) ensures that these integrals are uniformly bounded, hence the $J_{\iota,j}$ are uniformly bounded. Likewise, the second integrals in (20) are uniformly bounded, which concludes the proof. ∎

3.3. From polynomial boundedness to bounded polygonal functional calculus. In the sequel we will use R-boundedness. We refer to [13, Chapter 8] for general information on this notion. We only recall basic notations and the main definition. We let $(\epsilon_i)_{i\geq 1}$ be a family of independent Rademacher variables of some probability space $(\mathcal{M}, \mathbb{P})$. For any finite family $x_1, \ldots, x_n$ in X, we set

$$\Big\| \sum_{i=1}^n \epsilon_i \otimes x_i \Big\|_{\mathrm{Rad}(X)} = \Big(\int_{\mathcal{M}} \Big\| \sum_{i=1}^n \epsilon_i(t) x_i \Big\|_X^2 d\mathbb{P}(t) \Big)^{\frac{1}{2}}.$$

Then we say that a subset $F \subset B(X)$ is R-bounded if there exists a constant $K \geq 0$ such that, for any $n \geq 1$, any $T_1, \ldots, T_n$ in F and any $x_1, \ldots, x_n$ in X,

$$\Big\| \sum_{i=1}^{n} \epsilon_i \otimes T_i(x_i) \Big\|_{\mathrm{Rad}(X)} \leq K \Big\| \sum_{i=1}^{n} \epsilon_i \otimes x_i \Big\|_{\mathrm{Rad}(X)}.$$

In this case, we let $\mathcal{R}(F)$ denote the smallest possible $K \geq 0$ verifying this property.

In the sequel we fix again $E = \{\xi_1, \ldots, \xi_N\} \subset \mathbb{T}$ as in Section 2.

Definition 3.7: We say that an operator $T \in B(X)$ is R-Ritt$_E$ if $\sigma(T) \subset \overline{\mathbb{D}}$ and the set

$$\Big\{ R(z,T) \prod_{j=1}^{N} (\xi_j - z) : z \in \mathbb{C},\ 1 < |z| < 2 \Big\}$$

is R-bounded.

Remark 3.8: Let $T \in B(X)$ such that $\sigma(T) \subset \overline{\mathbb{D}}$.

(a) For each $j = 1, \ldots, N$, let $\mathcal{V}_j$ be an open neighborhood of ξ_j such that $\overline{\mathcal{V}_j} \cap \overline{\mathcal{V}_k} = \emptyset$ if $j \neq k$. If $\sigma(T) \cap \mathbb{T} \subset E$, then the set

$$W = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\} \setminus \bigcup_{j=1}^{N} \mathcal{V}_j$$

is compact and $W \cap \sigma(T) = \emptyset$. Hence

$$\Big\{ R(z,T) \prod_{j=1}^{N} (\xi_j - z) : z \in W \Big\}$$

is R-bounded, by [28, Proposition 2.6]. Thus T is R-Ritt$_E$ if and only if for all $j = 1, \ldots, N$, the set

$$\{(\xi_j - z) R(z,T) : 1 < |z| < 2,\ z \in \mathcal{V}_j\}$$

is R-bounded. Furthermore by [28, Proposition 2.8], the sets

$$\{(\xi_j - z) R(z,T) : |z| \geq 2\}$$

are R-bounded. (Here we use the disjointness of the $\overline{\mathcal{V}_j}$.) Hence T is R-Ritt$_E$ if and only if the sets

$$\{(\xi_j - z) R(z,T) : z \in \mathcal{V}_j \cap \overline{\mathbb{D}}^c\}, \quad j = 1, \ldots, N,$$

are R-bounded.

(b) Arguing as in Lemma 2.8, one also obtains that T is R-Ritt$_E$ if and only if there exists $r \in (0,1)$ such that

$$\sigma(T) \subset \overline{E_r}$$

and for all $s \in (r,1)$, the set

$$\left\{ R(z,T) \prod_{j=1}^{N} (\xi_j - z) : z \in D(0,2) \setminus \overline{E_s} \right\}$$

is R-bounded.

Theorem 3.9: *Let $T \in B(X)$ be an R-Ritt$_E$ operator. If T is polynomially bounded, then T admits a bounded $H^\infty(E_s)$ functional calculus for some $s \in (0,1)$.*

Combining this theorem with Proposition 3.5, we immediately deduce the following result.

Corollary 3.10: *Let $T \in B(X)$ be an R-Ritt$_E$ operator. If T is polynomially bounded, then T admits a bounded polygonal functional calculus.*

Proof of Theorem 3.9. Let $r \in (0,1)$ such that T satisfies Remark 3.8(b), and fix some $s \in (r,1)$. Consider the three sequences $(\theta_i)_{i \geq 1}$, $(\phi_i)_{i \geq 1}$ and $(\psi_i)_{i \geq 1}$ provided by Proposition 3.6. Let $h \in H_0^\infty(E_s)$. Applying part (iv) of Proposition 3.6, we have

$$h(z) = \sum_{i=1}^{\infty} h(z)\theta_i(z)\phi_i(z)\psi_i(z), \quad z \in \mathbb{D}.$$

Fix some $u \in (r,s)$. According to parts (i) and (ii) of Proposition 3.6, and the fact that $h \in H_0^\infty(E_s)$, we have

$$\sum_{i=1}^{\infty} \int_{\partial E_u} |h(z)\theta_i(z)\phi_i(z)\psi_i(z)| \|R(z,T)\| |dz| < \infty.$$

Hence

$$h(T) = \sum_{i=1}^{\infty} (h\theta_i\phi_i\psi_i)(T) = \sum_{i=1}^{\infty} h(T)\theta_i(T)\phi_i(T)\psi_i(T).$$

For any $i \geq 1$, we may write

$$h(T)\theta_i(T) = \frac{1}{2\pi i}\int_{\partial E_u} h(z)\theta_i(z)R(z,T)dz$$
$$= \frac{1}{2\pi i}\int_{\partial E_u}\left(\frac{h(z)\theta_i(z)}{\prod_{j=1}^N(\xi_j - z)}\right)\prod_{j=1}^N(\xi_j - z)R(z,T)dz.$$

Further we have an estimate

$$\int_{\partial E_u}\left|\frac{h(z)\theta_i(z)}{\prod_{j=1}^N(\xi_j - z)}\right||dz| \lesssim \|h\|_{\infty,E_s},$$

by part (iii) of Proposition 3.6. Applying [13, Theorem 8.5.2], we deduce that the set of all $h(T)\theta_i(T)$ is R-bounded, with

$$\mathcal{R}(\{h(T)\theta_i(T) : i \geq 1\}) \lesssim \|h\|_{\infty,E_s}. \tag{21}$$

Fix $x \in X$ and $y \in X^*$. Set $h_n = \sum_{i=1}^n h\theta_i\phi_i\psi_i$ for any $n \geq 1$. Then we have

$$\langle h_n(T)x, y\rangle = \sum_{i=1}^n \langle h(T)\theta_i(T)\phi_i(T)x, \psi_i(T)^*y\rangle$$
$$\times \int_{\mathcal{M}}\left\langle \sum_{i=1}^n \epsilon_i(t)h(T)\theta_i(T)\phi_i(T)x, \sum_{i=1}^n \epsilon_i(t)\psi_i(T)^*y\right\rangle d\mathbb{P}(t).$$

Applying the Cauchy–Schwarz inequality, we deduce the inequality

$$|\langle h_n(T)x,y\rangle| \leq \left\|\sum_{i=1}^n \epsilon_i\otimes h(T)\theta_i(T)\phi_i(T)x\right\|_{\mathrm{Rad}(X)}\left\|\sum_{i=1}^n \epsilon_i\otimes\psi_i(T)^*y\right\|_{\mathrm{Rad}(X^*)}.$$

Now applying (21), this implies

$$|\langle h_n(T)x,y\rangle| \lesssim \|h\|_{\infty,E_s}\left\|\sum_{i=1}^n \epsilon_i\otimes \phi_i(T)x\right\|_{\mathrm{Rad}(X)}\left\|\sum_{i=1}^n \epsilon_i\otimes\psi_i(T)^*y\right\|_{\mathrm{Rad}(X^*)}.$$

By assumption, T satisfies (10) for some $K \geq 1$. Arguing as in [18, Proposition 2.5], this implies that

$$\|\phi(T)\| \leq K\|\phi\|_{\infty,\mathbb{D}}, \quad \phi \in H_0^\infty(E_s).$$

Hence applying part (i) of Proposition 3.6, we have an estimate

$$\left\|\sum_{i=1}^n \epsilon_i(t)\phi_i(T)\right\| \lesssim 1, \quad t \in \mathcal{M}, n \geq 1.$$

This readily implies

$$\Big\|\sum_{i=1}^{n}\epsilon_i\otimes\phi_i(T)x\Big\|_{\mathrm{Rad}(X)}\lesssim\|x\|.$$

Similarly we have

$$\Big\|\sum_{i=1}^{n}\epsilon_i\otimes\psi_i(T)^*y\Big\|_{\mathrm{Rad}(X^*)}\lesssim\|y\|.$$

Thus we have an estimate

$$|\langle h_n(T)x,y\rangle|\lesssim\|h\|_{\infty,E_s}\|x\|\|y\|.$$

Since $h_n(T)\to h(T)$ when $n\to\infty$, we deduce the estimate

$$\|h(T)\|\lesssim\|h\|_{\infty,E_s},$$

which proves the bounded $H^\infty(E_s)$ functional calculus. ∎

Theorem 3.9 above was stated and proved for Ritt operators in [18, Proposition 7.7]. The multi-point version considered here required a different proof.

Remark 3.11: Recall that if $X=H$ is a Hilbert space, any bounded subset of $B(H)$ is automatically R-bounded. Hence any Ritt_E operator $T\in B(H)$ is automatically R-Ritt_E. It therefore follows from Corollary 3.10 that if $T\in B(H)$ is a polynomially bounded Ritt_E operator, then it admits a bounded polygonal functional calculus. This Hilbertian case, which was the motivation to undertake this work, is due to de Laubenfels [8, Theorem 4.4, (a)⇒(b)]. It is also implicit in the Franks–McIntosh result [10, Theorem 5.5]. Note that Proposition 3.6 relies on [1, Proposition 6.3], which is itself a consequence of a construction devised in [10]. Thus in spirit, the proof of Corollary 3.10 is closer to [10] than to [8].

Remark 3.12: Let T be a Ritt_E operator and recall the operators A_j defined by (5). For $s\in(0,1)$ close enough to 1, we have an inclusion

$$E_s\subset\Sigma(\xi_j,\arcsin(s))$$

for all $j=1,\dots,N$. It easily follows that if T admits a bounded $H^\infty(E_s)$ functional calculus, then each A_j admits a bounded $H^\infty(\Sigma_{\arcsin(s)})$ functional calculus.

Assume that X has the triangular contraction property, in the sense of [13, Section 7.5.b]. If T admits a bounded $H^\infty(E_s)$ functional calculus for some $s \in (0,1)$, it follows from above and from [15, Theorem 5.3] that the A_j are R-sectorial of R-type $< \frac{\pi}{2}$. Using (7) and applying Remark 3.8, we deduce that T is R-Ritt$_E$.

Combining the above paragraph and Theorem 3.9, we deduce that if X has the triangular contraction property, then T admits a bounded $H^\infty(E_s)$ functional calculus for some $s \in (0,1)$ if and only if T is both polynomially bounded and R-Ritt$_E$.

4. Ritt$_E$ contractively regular operators on L^p-spaces

Our aim is to show that for any $1 < p < \infty$, contractively regular operators on L^p admit a bounded polygonal functional calculus if they are Ritt$_E$ for some E. This will be achieved in Subsection 4.2. In the previous Subsection 4.1, we establish a result of independent interest (valid on all Banach spaces) connecting $H^\infty(E_s)$ functional calculus to the classical H^∞ functional calculus of sectorial operators.

4.1. Intersection of sectorial functional calculi. In Subsection 2.2 we recalled the definition of a sectorial operator A of type $\omega \in (0,\pi)$ in the case when A is bounded (we will not need unbounded sectorial operators in this paper). We will now use the notion of bounded $H^\infty(\Sigma_\theta)$ functional calculus for any A as above and $\theta \in (\omega,\pi)$. We refer to [29, Section 2], [12, Chapter 5] or [13, Section 10.2.b] for the relevant definitions. These three references provide comprehensive information on the H^∞ functional calculus of sectorial operators.

The following simple fact is given by [13, Proposition 10.2.21].

Lemma 4.1: *Let $A \in B(X)$ and assume that $\sigma(A) \subset \Sigma_\omega$ for some $\omega \in (0,\pi)$. Then A is sectorial of type ω and for any $\theta \in (\omega,\pi)$, A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

For the next statement, we note that if $A \in B(X)$ is sectorial of type $\omega \in (0,\pi)$, then for all $\rho \in (0,1)$, we have

$$\sigma((1-\rho)I_X + \rho A) \subset \Sigma_\omega.$$

Hence for all $g \in H^\infty(\Sigma_\omega)$, the operator $g((1-\rho)I_X + \rho A)$ is well-defined by the Dunford–Riesz functional calculus.

LEMMA 4.2: *Let $A \in B(X)$ be a sectorial operator and assume that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \pi)$. Then there exists a constant $K > 0$ such that for all $g \in H^\infty(\Sigma_\theta)$ and for all $\rho \in (0, 1)$,*

$$\|g((1-\rho)I_X + \rho A)\| \leq K\|g\|_{\infty,\Sigma_\theta}. \tag{22}$$

Proof. Let $\mathrm{Rat}_\theta \subset H^\infty(\Sigma_\theta)$ be the algebra of all rational functions with non-positive degree and poles off $\overline{\Sigma_\theta}$. Since A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus, there exists a constant $K > 0$ such that

$$\|\phi(A)\| \leq K\|\phi\|_{\infty,\Sigma_\theta}, \quad \phi \in \mathrm{Rat}_\theta.$$

Let $\rho \in (0, 1)$. Set

$$\varphi_\rho(z) = (1-\rho) + \rho z$$

and observe that $\varphi_\rho(\Sigma_\theta) \subset \Sigma_\theta$. We set $A_\rho = \varphi_\rho(A)$. For any $\psi \in \mathrm{Rat}_\theta$, we have $\psi(A_\rho) = (\psi \circ \varphi_\rho)(A)$, hence

$$\|\psi(A_\rho)\| \leq K\|\psi \circ \varphi_\rho\|_{\infty,\Sigma_\theta} \leq K\|\psi\|_{\infty,\Sigma_\theta}.$$

Thus (22) is satisfied by any element of Rat_θ. By an entirely classical argument (see [17] or [12, Section 5.3.4]) , we deduce (22) for all $g \in H^\infty(\Sigma_\theta)$. ∎

In the sequel, we fix a finite set $E = \{\xi_1, \ldots, \xi_N\} \subset \mathbb{T}$ as in Section 2.

THEOREM 4.3: *Let $T \in B(X)$ be a Ritt_E operator. For any $j = 1, \ldots, N$, let $A_j = I_X - \overline{\xi_j}T$ and assume that there exists $\theta_j \in (0, \frac{\pi}{2})$ such that A_j admits a bounded $H^\infty(\Sigma_{\theta_j})$ functional calculus. Then,*

- (i) *There exists $s \in (0, 1)$ such that T admits a bounded $H^\infty(E_s)$ functional calculus.*
- (ii) *T admits a bounded polygonal functional calculus.*

Proof. We are going to build a (convex, open) polygon $\Delta \subset \mathbb{D}$ with the following three properties:

- (•) There exists a finite set $E' \subset \mathbb{D}$ such that the set of vertices of Δ is equal to $E \cup E'$.
- (••) $\sigma(T) \subset \overline{\Delta}$.
- (•••) There exists a constant $K \geq 1$ such that $\|\phi(T)\| \leq K\|\phi\|_{\infty,\Delta}$ for all $\phi \in \mathcal{P}$.

This will obviously prove part (ii) of the statement. This will also prove part (i), since for any polygon Δ satisfying (•), there exists $s \in (0, 1)$ such that $\Delta \subset E_s$.

Recall $\Sigma(\xi, \omega)$ defined by (8) for any $\xi \in \mathbb{C}^*$ and any $\omega \in (0, \frac{\pi}{2})$. We introduce

$$\partial\Sigma(\xi, \omega)_+ = \{\xi(1 - te^{-i\omega}) : t > 0\}$$

and

$$\partial\Sigma(\xi, \omega)_- = \{\xi(1 - te^{i\omega}) : t > 0\};$$

see Figure 5.

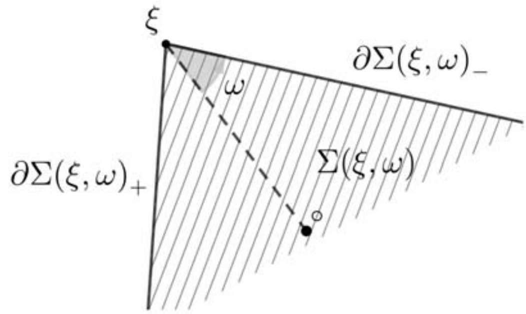

Figure 5. The set $\Sigma(\xi, \omega)$ and its boundary.

Assume that $N \geq 2$ (it is easy to adapt the proof to the case $N = 1$). For convenience we assume that the sequence $(\xi_1, \xi_2, \ldots, \xi_N)$ is oriented counterclockwise on $\mathbb{T}$ and we set $\xi_{N+1} = \xi_1$. We may choose $\theta \in (0, \frac{\pi}{2})$ close enough to $\frac{\pi}{2}$ so that: for all $j = 1, \ldots, N$, the operators A_j have a bounded $H^\infty(\Sigma_\theta)$ functional calculus; for all $1 \leq j \neq j' \leq N$, $\xi_{j'} \in \Sigma(\xi_j, \theta)$; for all $j = 1, \ldots, N$, the half-lines $\partial\Sigma(\xi_j, \theta)_+$ and $\partial\Sigma(\xi_{j+1}, \theta)_-$ do not meet in $\overline{\mathbb{D}}$.

Let us momentarily focus on the couple (ξ_1, ξ_2); see Figure 6. Let $\Gamma_{1,2}$ be the closed arc of $\mathbb{T}$ joining the points where $\Sigma(\xi_1, \theta)_+$ and $\Sigma(\xi_2, \theta)_-$ meet $\mathbb{T}$. Then $\text{dist}(\Gamma_{1,2}, \sigma(T)) > 0$ hence by compactness, we can find $r \in (0, 1)$, $\theta' \in (0, \frac{\pi}{2})$ and points $z_1, \ldots, z_p$ on $r\Gamma_{1,2}$, ordered counterclockwise, such that:

- For all $i = 1, \ldots, p$, $\sigma(T) \subset \Sigma(z_i, \theta')$.
- The half-lines $\partial\Sigma(\xi_1, \theta)_+$ and $\partial\Sigma(z_1, \theta')_-$ meet in $\mathbb{D} \setminus \{0\}$.
- The half-lines $\partial\Sigma(z_p, \theta')_+$ and $\partial\Sigma(\xi_2, \theta)_-$ meet in $\mathbb{D} \setminus \{0\}$.
- For all $i = 1, \ldots, p-1$, $\partial\Sigma(z_i, \theta')_+$ and $\partial\Sigma(z_{i+1}, \theta')_-$ meet in $\mathbb{D} \setminus \{0\}$.

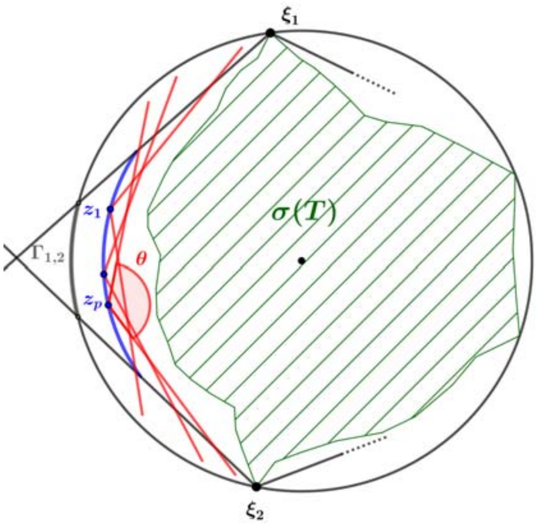

Figure 6. Construction of the polygon.

Then we apply the same process to the couples $(\xi_2, \xi_3), \ldots, (\xi_N, \xi_1)$. Putting together the points ξ_j and the intermediate points z_i, we therefore obtain a finite sequence $(\zeta_1, \zeta_2, \ldots, \zeta_m)$ of distinct elements of $\overline{\mathbb{D}} \setminus \{0\}$, ordered counterclockwise, as well as angles $\mu_1, \ldots, \mu_m$ in $(0, \frac{\pi}{2})$, verifying the following properties:

(a) We have

$$\{\zeta_1, \zeta_2, \ldots, \zeta_m\} \cap \mathbb{T} = E.$$

(b) For any $i = 1, \ldots, m$:
  (b1) If there exists $j \in \{1, \ldots, N\}$ such that $\zeta_i = \xi_j$, then $\mu_i = \theta$.
  (b2) If $\zeta_i \notin E$, then $\sigma(T) \subset \Sigma(\zeta_i, \mu_i)$.

(c) Setting $\zeta_{m+1} = \zeta_1$, the half-lines $\partial\Sigma(\zeta_i, \mu_i)_+$ and $\partial\Sigma(\zeta_{i+1}, \mu_{i+1})_-$ meet exactly at one point $c_i \in \mathbb{D} \setminus \{0\}$, for all $i = 1, \ldots, m$.

Finally we set

$$d_i = \frac{1}{2}(c_i + c_i|c_i|^{-1}), \quad i = 1, \ldots, m.$$

We let Δ_0 be the open polygon with vertices $\{\zeta_1, c_1, \zeta_2, c_2, \ldots, \zeta_m, c_m\}$. We may assume that it is convex. Likewise, we let Δ be the open polygon with vertices $\{\zeta_1, d_1, \zeta_2, d_2, \dots, \zeta_m, d_m\}$. By construction we have

$$\Delta_0 = \bigcap_{i=1}^{m} \Sigma(\zeta_i, \mu_i)$$

and

$$\Delta_0 \subset \Delta.$$

According to (c), all the d_i belong to $\mathbb{D}$. Hence it follows from (a) that the polygon Δ satisfies (•). It further satisfies (••), by (b).

Let us now show that the polygon Δ satisfies (• • •). We set $d_0 = d_m$ for convenience. For all $i = 1, \dots, m$, we let γ_i be the path obtained as the concatenation of the oriented segments $[d_{i-1}, \zeta_i]$ and $[\zeta_i, d_i]$. Next we fix $\epsilon > 0$ small enough so that the sets $V_i = \Delta_0 \cap D(\zeta_i, \epsilon)$ are pairwise disjoint; see Figure 7.

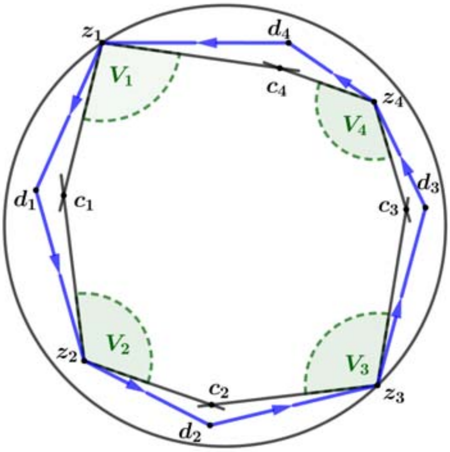

Figure 7. Polygons Δ_0 and Δ with $E = \{z_1, z_3\}$ and additional points z_2, z_4.

Let ϕ be a polynomial. For all $i = 1, \dots, m$, we define $\phi_i : \mathbb{C} \setminus \gamma_i \to \mathbb{C}$ by setting

$$\phi_i(z) = \frac{1}{2\pi i} \int_{\gamma_i} \frac{\phi(\lambda)}{\lambda - z} d\lambda.$$

These functions are holomorphic. Moreover by Cauchy's theorem, we have

$$\forall z \in \Delta, \quad \phi(z) = \sum_{i=1}^{m} \phi_i(z). \tag{23}$$

Since the distance from γ_i to $\Sigma(\zeta_i, \mu_i) \setminus V_i$ is positive, we have an estimate

$$|\phi_i(z)| \lesssim \|\phi\|_{\infty,\Delta}, \quad z \in \Sigma(\zeta_i, \mu_i) \setminus V_i, \tag{24}$$

for all $i = 1, \ldots, m$. Next, observe that since the set V_1 is disjoint from each V_i, with $i \geq 2$, we also have estimates

$$|\phi_i(z)| \lesssim \|\phi\|_{\infty,\Delta}, \quad z \in V_1, i \geq 2.$$

We have $V_1 \subset \Delta$ hence by (23),

$$\phi_1(z) = \phi(z) - \sum_{i=2}^{m} \phi_i(z) \quad \text{for all } z \in V_1.$$

We deduce an estimate

$$|\phi_1(z)| \lesssim \|\phi\|_{\infty,\Delta}, \quad z \in V_1.$$

Combining with (24) for $i = 1$, we obtain that ϕ_1 belongs to $H^\infty(\Sigma(\zeta_1, \mu_1))$ and satisfies an estimate

$$\|\phi_1\|_{\infty,\Sigma(\zeta_1,\mu_1)} \lesssim \|\phi\|_{\infty,\Delta}.$$

A similar argument shows that for all $i = 1, \ldots, m$,

$$\phi_i \in H^\infty(\Sigma(\zeta_i, \mu_i)) \quad \text{and} \quad \|\phi_i\|_{\infty,\Sigma(\zeta_i,\mu_i)} \lesssim \|\phi\|_{\infty,\Delta}. \tag{25}$$

Now let $\rho \in (0, 1)$. Since (••) holds true, we have $\sigma(\rho T) \subset \Delta$. We can therefore define operators $\phi_i(\rho T)$ by the Dunford–Riesz functional calculus and we have

$$\phi(\rho T) = \sum_{i=1}^{m} \phi_i(\rho T), \tag{26}$$

by (23). (Here we use ρT instead of T because the $\phi_i(T)$ are a priori not defined.)

We shall now apply (b). Let $i \in \{1, \ldots, m\}$ and assume first that there exists $j \in \{1, \ldots, N\}$ such that $\zeta_i = \xi_j$. Let $g \colon \Sigma_\theta \to \mathbb{C}$ be defined by

$$g(z) = \phi_i(\xi_j(1 - z)).$$

This function is well defined, holomorphic and bounded, by (25). We further have

$$\|g\|_{\infty,\Sigma_\theta} = \|\phi_i\|_{\infty,\Sigma(\xi_j,\theta)} = \|\phi_i\|_{\infty,\Sigma(\zeta_i,\mu_i)}.$$

Since $A_j = I_X - \overline{\xi_j}T$, we have

$$g((1-\rho)I_X + \rho A_j) = \phi_i(\rho T).$$

Applying Lemma 4.2, we obtain an estimate

$$\|\phi_i(\rho T)\| \lesssim \|\phi_i\|_{\infty,\Sigma(z_i,\mu_i)}.$$

Appealing to (25), we deduce an estimate

$$\|\phi_i(\rho T)\| \lesssim \|\phi\|_{\infty,\Delta}. \tag{27}$$

Otherwise, $\zeta_i \notin E$ hence $\sigma(T) \subset \Sigma(z_i,\mu_i)$. Arguing as above and using Lemma 4.1 instead of Lemma 4.2, we obtain an estimate (27) as well.

We have $\|\phi(\rho T)\| \leq \sum_{i=1}^m \|\phi_i(\rho T)\|$, by (26), hence the estimates (27), for $i = 1,\ldots,m$, yield

$$\|\phi(\rho T)\| \lesssim \|\phi\|_{\infty,\Delta}.$$

Since this estimate does not depend on ρ and $\phi(\rho T) \to \phi(T)$ when $\rho \to 1$, we obtain property $(\bullet\bullet\bullet)$. ∎

Remark 4.4: It follows from the first paragraph of Remark 3.12 and Theorem 4.3 that given a Ritt_E operator $T \in B(X)$, there exists $s \in (0,1)$ such that T admits a bounded $H^\infty(E_s)$ functional calculus if and only if for each $j = 1,\ldots,N$, there exists $\theta_j \in (0,\frac{\pi}{2})$ such that A_j admits a bounded $H^\infty(\Sigma_{\theta_j})$ functional calculus.

Remark 4.5: We give here a complement to Proposition 3.5. Let $T \in B(X)$ be a Ritt_E operator. It follows from the previous remark and the proof of Theorem 4.3 that if T admits a bounded $H^\infty(E_s)$ functional calculus for some $s \in (0,1)$, then there exists a polygon $\Delta \subset \mathbb{D}$ such that T admits a bounded functional calculus with respect to Δ and the set of vertices of Δ belonging to $\mathbb{T}$ coincides with E. This new condition on vertices is sharp.

Remark 4.6: As a complement to Corollary 3.10 and Remark 3.11, we mention that there exist a Banach space X and a Ritt_E operator $T \in B(X)$ such that T is polynomially bounded but T does not admit any bounded polygonal functional calculus. To check this, assume (as we may do) that $1 \in E$. Recall the Stolz domains

$$B_\omega = \overset{\circ}{\text{Conv}}(1, D(0,\sin(\omega))),$$

for $\omega \in (0, \frac{\pi}{2})$. According to [16, Theorem 3.2], there exists a Ritt operator T on some X which is polynomially bounded although it does not admit any bounded $H^\infty(B_\omega)$ functional calculus. The operator T is Ritt_E. Assume that T admits a bounded functional calculus with respect to some polygon $\Delta \subset \mathbb{D}$. Then $1 \in \Delta$ and the argument in Remark 4.4 implies that $I_X - T$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$. This implies, by [18, Proposition 4.1], that T admits a bounded $H^\infty(B_\omega)$ functional calculus for some $\omega \in (0, \frac{\pi}{2})$, whence a contradiction.

4.2. Ritt_E CONTRACTIVELY REGULAR OPERATORS ON L^p SPACES. We consider a measure space (S, μ), we let $1 < p < \infty$ and we consider operators acting on $L^p(S)$.

We recall that a bounded operator $T\colon L^p(S) \to L^p(S)$ is called regular if there exists a constant $C \geq 0$ such that for all $n \geq 1$ and all $x_1, \ldots, x_n$ in $L^p(S)$, we have

$$\| \sup_{1\leq i\leq n} |T(x_i)| \|_p \leq C \| \sup_{1\leq i\leq n} |x_i| \|_p.$$

In this case, the smallest $C \geq 0$ verifying this property is called the regular norm of T and is denoted by $\|T\|_r$. We say that T is contractively regular if

$$\|T\|_r \leq 1.$$

If $T\colon L^p(S) \to L^p(S)$ is positive, then T is regular and $\|T\|_r = \|T\|$. Thus positive contractions are contractively regular.

It is plain that the set $B_{reg}(L^p(S))$ of all regular operators equipped with $\|\cdot\|_r$ is a Banach algebra.

We refer, e.g., to [25, Section 1] for information on regular operators.

THEOREM 4.7: *Let $T\colon L^p(S) \to L^p(S)$ be a* Ritt_E *contractively regular operator, with $1 < p < \infty$. Then:*

(i) *There exists $s \in (0, 1)$ such that T admits a bounded $H^\infty(E_s)$ functional calculus;*
(ii) *T admits a bounded polygonal functional calculus.*

Proof. Consider $A_j = I_X - \overline{\xi_j} T$ for all $j = 1, \ldots, N$ and define $T_{j,t} := e^{-tA_j}$ for all $t \geq 0$. These operators are all contractively regular. Indeed,

$$\|T_{j,t}\|_r = e^{-t} \|e^{t\overline{\xi_j}T}\|_r \leq e^{-t} \sum_{k=0}^{\infty} \frac{t^k \|\overline{\xi_j}T\|_r^k}{k!} \leq e^{-t} e^{t\|\overline{\xi_j}T\|_r} \leq 1.$$

Then, according to [19, proposition 2.2], there exists, for all $j = 1, \dots, N$, some $\theta_j \in (0, \frac{\pi}{2})$ such that A_j admits a $H^\infty(\Sigma_{\theta_j})$ bounded functional calculus. We conclude by applying Theorem 4.3. ∎

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