ISRAEL JOURNAL OF MATHEMATICS **261** (2024), 791–807
DOI: 10.1007/s11856-023-2596-3

A REMARK ON DISCRETE BRUNN–MINKOWSKI TYPE INEQUALITIES VIA TRANSPORTATION OF MEASURE

BY

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ABSTRACT

We give an alternative proof for discrete Brunn–Minkowski type inequalities, recently obtained by Halikias, Klartag and the author. This proof also implies somewhat stronger weighted versions of these inequalities. Our approach generalizes ideas of Gozlan, Roberto, Samson and Tetali from the theory of measure transportation and provides new displacement convexity of entropy type inequalities on the n-dimensional integer lattice.

1. Introduction

1.1. Brunn–Minkowski type inequalities. Denote the n-dimensional Lebesgue volume in $\mathbb{R}^n$ by $\mathrm{vol}(\cdot)$, and the Minkowski sum of two sets $A, B \subseteq \mathbb{R}^n$ by $A + B = \{a + b :, a \in A, b \in B\}$. The classical Brunn–Minkowski inequality states that for any non-empty Borel measurable sets $A, B \subseteq \mathbb{R}^n$,

$$\mathrm{vol}(A+B)^{\frac{1}{n}} \geq \mathrm{vol}(A)^{\frac{1}{n}} + \mathrm{vol}(B)^{\frac{1}{n}}. \tag{1}$$

By homogeneity, its equivalent dimension-free form tells us that for all $\lambda \in [0,1]$,

$$\mathrm{vol}(\lambda A+(1-\lambda)B) \geq (\lambda \mathrm{vol}(A)^{1/n}+(1-\lambda)\mathrm{vol}(B)^{1/n})^n \geq \mathrm{vol}(A)^{\lambda}\mathrm{vol}(B)^{1-\lambda}. \tag{2}$$

Received July 4, 2021 and in revised form July 3, 2022

A functional form of this inequality is known as the Prékopa–Leindler inequality which states that for any Borel functions $f, g, h : \mathbb{R}^n \to [0, \infty)$ and $\lambda \in [0, 1]$ such that $h(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}$ for all $x, y \in \mathbb{R}^n$, one has

$$\int_{\mathbb{R}^n} h(x)dx \geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^\lambda \left(\int_{\mathbb{R}^n} g(x)dx\right)^{1-\lambda}. \tag{3}$$

For proofs and historical comments, we refer the reader to [3] and references therein.

In the discrete setting, one natural problem, which has been frequently studied, is finding bounds, in the spirit of (1), for the cardinality $|A + B|$ of the sumset of finite subsets A, B of the integer lattice $\mathbb{Z}^n$ (or other discrete groups). The elementary bound $|A + B| \geq |A| + |B| - 1$ is a simple form of the Cauchy–Davenport inequality; see, e.g., [22, Ch. 5]. More results in this direction are included in the works of Ruzsa [20, 19], Gardner and Gronchi [6] and Hernández Cifre, Iglesias and Yepes Nicolás [11].

The problem of bounding the size of averages of two discrete point sets, as in (2), has also been considered. Making this problem meaningful for integer point sets, naturally requires some modifications to the notions of "size" or "average" as, for one, averages of the form $\lambda A + (1 - \lambda)B$ are usually not integer point sets. One possibility is to consider different average operators. One result in this direction was established for the discrete cube $\{0, 1\}^n$ by Ollivier and Villani [18]. They showed that for $A, B \subseteq \{0, 1\}^n$, one has

$$|M(A, B)| \geq \sqrt{|A| \cdot |B|}, \tag{4}$$

where $M(A, B)$ is the set of midpoints of all pairs of points $(a, b) \in A \times B$, with respect to the hamming distance. More precisely, $x = (x_1, \dots, x_n) \in \{0, 1\}^n$ is a midpoint of $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ in $\{0, 1\}^n$ if $x_i = a_i$ whenever $a_i = b_i$ and, for some $\varepsilon \in \{0, 1\}$, $\#\{i : x_i \neq a_i\} = \#\{i : x_i \neq b_i\} + \varepsilon$. We remark that they also proved a stronger inequality which includes a positive curvature term in the right hand side.

Another possibility to obtain discrete analogues of (2) was explored by Iglesias, Yepes Nicolás and Zvavitch [13]. They considered the lattice point enumerator G_n, defined for a compact set $K \subseteq \mathbb{R}^n$ by $G_n(K) = |K \cap \mathbb{Z}^n|$, and proved the following sharp inequality, for all $\lambda \in [0, 1]$ and any compact $K, L \subseteq \mathbb{R}^n$:

$$G_n(\lambda K + (1 - \lambda)L + [-1, 1]^n) \geq \lambda G_n(K)^{1/n} + (1 - \lambda)G_n(L)^{1/n}. \tag{5}$$

In fact, they also established a discrete form of the Prékopa–Leindler inequality (and more generally, discrete Borell–Brascamp–Lieb inequalities).

Other discrete forms of the Prékopa–Leindler inequality can be found in the works of Gozlan, Roberto, Samson and Tetali [7], Green, Matolcsi, Ruzsa, Shakan and Zhelezov [9] and Marsiglietti and Melbourne [15]. For additional related works concerning discrete analogues for various results in convexity theory, see [2, 5, 12, 16, 17, 21].

This paper directly pertains to several other Brunn–Minkowski type inequalities, which we describe next. To that end, and for the sake of brevity, it will be convenient to use the following definition.

Definition: We say that an operation $T : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ admits a Brunn–Minkowski inequality if for all functions $f, g, h, k : \mathbb{Z}^n \to [0, \infty)$ such that

$$f(x)g(y) \le h(T(x,y))k(x+y-T(x,y)) \quad \forall x, y \in \mathbb{Z}^n,$$

it follows that

$$\Big(\sum_{x\in\mathbb{Z}^n} f(x)\Big)\Big(\sum_{x\in\mathbb{Z}^n} g(x)\Big) \le \Big(\sum_{x\in\mathbb{Z}^n} h(x)\Big)\Big(\sum_{x\in\mathbb{Z}^n} k(x)\Big).$$

Denote the lower and upper integer parts of $r \in \mathbb{R}$ by

$$\lfloor r \rfloor = \max\{m \in \mathbb{Z}\,;\, m \le r\} \quad \text{and} \quad \lceil r \rceil = \min\{m \in \mathbb{Z}\,;\, r \le m\}.$$

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, denote

$$\lfloor x \rfloor = (\lfloor x_1 \rfloor, \ldots \lfloor x_n \rfloor) \quad \text{and} \quad \lceil x \rceil = (\lceil x_1 \rceil, \ldots, \lceil x_n \rceil).$$

Also denote the characteristic function of $A \subseteq \mathbb{R}^n$ by $\mathbb{1}_A$.

In [14, Theorem 1.4], Klartag and Lehec proved that the operation

$$T(x,y) = \lfloor (x+y)/2 \rfloor$$

admits a Brunn-Minkowsi inequalty. Note that here $x+y-T(x,y) = \lceil (x+y)/2 \rceil$. To see its direct relation to Brunn–Minkowski inequality, let $K, L \subseteq \mathbb{R}^n$ be non-empty compact sets, and consider the sets

$$\lfloor (K+L)/2 \rfloor = \{\lfloor (x+y)/2 \rfloor : x \in K, y \in L\}$$

and, similarly, $\lceil (K+L)/2 \rceil$. Applying their result to the functions $f = \mathbb{1}_K$, $g = \mathbb{1}_L$, $h = \mathbb{1}_{\lfloor \frac{K+L}{2} \rfloor}$ and $k = \mathbb{1}_{\lceil \frac{K+L}{2} \rceil}$, we obtain the inequality

$$G_n\Big(\Big\lfloor \frac{K+L}{2} \Big\rfloor\Big) G_n\Big(\Big\lceil \frac{K+L}{2} \Big\rceil\Big) \ge G_n(K)G_n(L).$$

Using a standard limiting argument, this inequality yields the multiplicative form of the classical Brunn–Minkowski inequality (2) with $\lambda = 1/2$ (see [8, Section 2.3]). One can also apply the same result with $f = \mathbb{1}_k, g = \mathbb{1}_L$ and $h = k = \mathbb{1}_{\frac{K+L}{2}+[-1,1]^n}$ to obtain the inequality

$$G_n((K+L)/2 + [-1,1]^n) \geq \sqrt{G_n(K)G_n(L)},$$

which is a particular weaker instance of (5) (which also implies (2), as shown in [13]).

Another example for an operation that admits a discrete Brunn–Minkowski inequality is $T(x,y) = x \wedge y = (\min(x_1,y_1),\dots,\min(x_n,y_n))$, for which

$$x + y - T(x,y) = x \vee y = (\max(x_1,y_1),\dots,\max(x_n,y_n)).$$

This fact is known as the four functions theorem, due to Ahlswede and Daykin [1].

The first to link between the four functions theorem of Ahlswede and Daykin and the discrete Brunn-Minkowksi inequality of Klartag and Lehec were Gozlan, Roberto, Samson and Tetali [8]. They provided alternative proofs for both of these results, which are based on ideas from the theory of measure transportation.

Recently, a unified elementary proof for both of the above results was given in [10]. This proof applies to all operations $T : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ sharing two common properties:

(P1) **Translation equivariance**: $T(x+z, y+z) = T(x,y) + z$ for all $z \in \mathbb{Z}^n$.

(P2) **Monotonicity in the sense of Knothe**: there exists a decomposition of $\mathbb{Z}^n$ into a direct sum of groups $\mathbb{Z}^n = G_1 \times \cdots \times G_k$ such that for each $i \in \{1,\dots,k\}$:

- (i) $T_i : (G_1 \times \cdots \times G_i) \times (G_1 \times \cdots \times G_i) \to G_i$ where $T = (T_1,\dots,T_k)$. In other words, $T_i(x,y)$ depends only on the first i coordinates of its arguments $x, y \in G_1 \times \cdots \times G_k$, so that T is triangular.
- (ii) There exists a total additive ordering $\preceq_i$ on G_i such that $T_i^{(a,b)} : G_i \times G_i \to G_i$ defined by

$$T_i^{(a,b)}(x,y) = T_i((a,x),(b,y))$$

for $a, b \in G_1 \times \cdots \times G_{i-1}$ satisfies

$$x_1 \preceq_i x_2,\ y_1 \preceq_i y_2 \implies T_i^{(a,b)}(x_1,y_1) \preceq_i T_i^{(a,b)}(x_2,y_2)$$

for all $a, b \in G_1 \times \cdots \times G_{i-1}$ and $x_1, x_2, y_1, y_2 \in G_i$.

Recall that a total ordering $\preceq$ on an abelian group $G \approx \mathbb{Z}^l$ is a binary relation which is reflexive, anti-symmetric and transitive, such that for any distinct x, y, either $x \preceq y$ or else $y \preceq x$. An ordering $\preceq$ is additive if for all x, y, z, we have

$$x \preceq y \implies x + z \preceq y + z.$$

Examples for additive, total orderings on $\mathbb{Z}^n$ (or on $\mathbb{R}^n$) are the standard lexicographic order relation and invertible linear images thereof.

Theorem 1.1 ([10, Theorem 1.3]): *Every translation equivariant operation $T : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ which is monotone in the sense of Knothe admits a Brunn–Minkowski inequality.*

In addition to the four functions theorem and the Brunn–Minkowski inequality of Klartag and Lehec, Theorem 1.1 implies various other inequalities, including an improvement of (4), a result Cordero-Erausquin and Maurey [4] and additional new inequalities. For a more detailed account of these implications, see [10].

Our first main result is the following extension of Theorem 1.1:

Theorem 1.2: *Let $\alpha, \beta, \gamma, \delta > 0$ such that $\max\{\alpha, \beta\} \le \min\{\gamma, \delta\}$. Let $T : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ satisfy properties* (P1) *and (P2) and suppose that $f, g, h, k : \mathbb{Z}^n \to [0, \infty)$ satisfy*

$$f^\alpha(x) g^\beta(y) \le h^\gamma(T(x,y)) k^\delta(x + y - T(x,y)) \quad \forall x, y \in \mathbb{Z}^n.$$

Then

$$\Big(\sum_{x\in\mathbb{Z}^n} f(x)\Big)^\alpha \Big(\sum_{x\in\mathbb{Z}^n} g(x)\Big)^\beta \le \Big(\sum_{x\in\mathbb{Z}^n} h(x)\Big)^\gamma \Big(\sum_{x\in\mathbb{Z}^n} k(x)\Big)^\delta.$$

Note that if an operation $T : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ satisfies properties (P1) and (P2), then so does the operation $x + y - T(x, y)$. In the sequel, we shall say that $T_\pm : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ are **complementing operations** if they satisfy the above relation, i.e., $T_-(x, y) + T_+(x, y) = x + y$.

1.2. An entropic version. Our approach is based on the work of Gozlan, Roberto, Samson and Tetali [8] who proved the next displacement convexity of entropy result for the counting measure on $\mathbb{Z}$ and the operations

$$T_-(x,y) = \lfloor (x+y)/2 \rfloor \quad \text{and} \quad T_+(x,y) = \lceil (x+y)/2 \rceil.$$

Denote the counting measure on $\mathbb{Z}^n$ by m_n. Given a probability measure μ on $\mathbb{Z}^n$, the relative entropy of μ with respect to m_n is given by

$$H(\mu|m_n) = \sum_{x\in\mathbb{Z}^n} \mu(x)\log(\mu(x)).$$

Recall that a coupling between two probability measures μ and ν on $\mathbb{Z}^n$ is a probability measure π on $\mathbb{Z}^n \times \mathbb{Z}^n$ whose coordinate marginals are μ and ν, i.e., for all $A, B \subseteq \mathbb{Z}^n$, we have $\pi(A\times\mathbb{Z}^n) = \mu(A)$ and $\pi(\mathbb{Z}^n \times B) = \nu(B)$. The push forward of π by a map $T : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ is the probability measure $\pi \circ T^{-1}$ on $\mathbb{Z}^n$, defined by

$$(\pi \circ T^{-1})(A) = \pi(T^{-1}(A))$$

for all $A \subseteq \mathbb{Z}^n$.

Theorem 1.3 ([8, Theorem 8]): *Let $T_\pm : \mathbb{Z} \to \mathbb{Z}$ be the maps defined by $T_-(x, y) = \lfloor (x + y)/2\rfloor$ and $T_+(x, y) = \lceil (x + y)/2\rceil$ for all $x, y \in \mathbb{Z}$. Suppose that μ and ν are finitely supported probability measures on $\mathbb{Z}$. Then there exists a coupling π between μ and ν such that*

$$H(\mu|m_1) + H(\nu|m_1) \geq H(\pi \circ T_-^{-1} \,|\, m_1) + H(\pi \circ T_+^{-1} \,|\, m_1). \tag{6}$$

We remark that the coupling for which (6) holds is the monotone coupling (see Section 2.1 for the definition).

As shown in [8], using a duality argument, Theorem 1.3 implies the discrete Brunn–Minkowski inequality of Klartag and Lehec [14, Theorem 1.4].

Using the same argument, we shall deduce Theorem 1.2 from the next generalization of Theorem 1.3, which is our second main result:

Theorem 1.4: *Let $T_\pm : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ be complementing operations satisfying properties* (P1) *and (P2). Suppose that μ, ν are finitely supported probability measures on $\mathbb{Z}^n$. Then there exists a coupling π between μ and ν such that*

$$H(\mu|m_n) + H(\nu|m_n) \geq H(\pi \circ T_-^{-1}|m_n) + H(\pi \circ T_+^{-1}|m_n). \tag{7}$$

The coupling π for which (7) holds is the Knothe–Rosenblatt coupling which disintegrates into a product of monotone couplings with respect to the decomposition $\mathbb{Z}^n = G_1 \times \cdots \times G_k$ given in property (P2). The construction of this coupling is described in Section 2.2.

Theorem 1.4 is an immediate consequence of the following extension of [8, Theorem 9], which was used to obtain Theorem 1.3 in the same manner:

THEOREM 1.5: *Let $T_{\pm} : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ be complementing operations satisfying properties* (P1) *and (P2). Suppose that μ, ν are finitely supported probability measures on $\mathbb{Z}^n$. Then there exists a coupling π between μ and ν such that, denoting by $\kappa_{\pm}$ the push forward of π by $T_{\pm}$, we have*

$$\sum_{(x,y)\in\mathbb{Z}^n\times\mathbb{Z}^n} \frac{\kappa_-(T_-(x,y))\kappa_+(T_+(x,y))}{\mu(x)\nu(y)}\,\pi(x,y) \leq 1.$$

The remainder of the paper is organized as follows. In Section 2 we recall the monotone and Knothe–Rosenblatt couplings, which are needed for the proofs of Theorems 1.4 and 1.5. Section 3 is devoted for the proof of Theorem 1.5 in the case where $T_{\pm}$ are monotone with respect to some total ordering on $\mathbb{Z}^n$. In Section 4 we complete the proof of Theorem 1.5 and derive Theorems 1.4 and 1.2.

ACKNOWLEDGEMENTS. I am grateful to Bo'az Klartag for fruitful conversions and for his advice and comments. I would like to thank Shiri Artstein and the anonymous referee for their useful remarks and suggestions. I would also like to thank the anonymous referee of the paper [10] for suggesting to pursue this direction of research. Supported by ISF grant 784/20.

2. Two useful couplings

In this section we give the construction of two well-known couplings which shall be needed in the sequel; the monotone coupling and the Knothe–Rosenblatt coupling.

We refer the reader to [23] for further reading on these couplings and their uses. Also see [7, 8] for applications which are similar to that presented here.

2.1. THE MONOTONE COUPLING. Let $G \approx \mathbb{Z}^n$ be a finitely generated group, endowed with a totally additive ordering $\preceq$. Given a probability measure μ on G, the cumulative distribution of μ with respect to $\preceq$ is defined by

$$F_\mu(x) = \mu((-\infty, x]) = \mu\{g \in G\,;\, g \preceq x\} \quad \forall x \in G.$$

Similarly, the generalized inverse of F_μ at a point $t \in (0,1)$ is given by

$$F_\mu^{-1}(t) = \inf\{x \in G\,;\, F_\mu(x) \geq t\}.$$

Given two finitely supported probability measures μ, ν on G and a random variable U, uniformly distributed on $(0,1)$, the monotone coupling between μ and ν with respect to the ordering $\preceq$ is defined as the probability distribution of the random vector $(F_\mu^{-1}(U), F_\nu^{-1}(U))$.

One can check that the support of π is monotone with respect to $\preceq \times \preceq$. That is, if $(a,b),(c,d) \in \mathrm{supp}(\pi)$ then either $a \preceq c$ and $b \preceq d$ or vice versa, $c \preceq a$ and $d \preceq b$.

2.2. The Knothe–Rosenblatt coupling. Suppose that $\mathbb{Z}^n = G_1 \times \cdots \times G_k$ is given as a direct sum of groups $G_1, \ldots, G_k$, each of which equipped with a total additive ordering $\preceq_i$. Denote this decomposition by $\mathbb{Z}^n = (G_i, \preceq_i)_{1:k}$.

For each $(x_1, \ldots, x_k) \in (G_1, \ldots, G_k)$ and $i \in \{1, \ldots, k\}$, denote by $x_{1:i}$ the sub-vector $(x_1, \ldots, x_i) \in (G_1, \ldots, G_i)$. The disintegration formula for a measure κ on $\mathbb{Z}^n$ with respect to the decomposition $\mathbb{Z}^n = G_1 \times \cdots \times G_k$ is given by

$$\kappa(x_1, \ldots, x_k) = \kappa^1(x_1)\kappa^2(x_2|x_1)\cdots\kappa^k(x_k|x_{1:k-1}), \quad \forall x, y \in \mathbb{Z}^n$$

where κ^1 is the marginal of κ onto G_1, $\kappa^2(\cdot\,|\,x_1)$ is the marginal of $\kappa(\cdot\,|\,x_1)$ onto G_2, etc.

Let μ and ν be probability measures on $\mathbb{Z}^n$. The Knothe–Rosenblatt coupling between μ and ν with respect to the decomposition $\mathbb{Z}^n = (G_i, \preceq_i)_{1:k}$ is defined by

$$\pi(x,y) = \pi^1(x_1,y_1)\pi^2(x_2,y_2\,|\,x_1,y_1)\cdots\pi^k(x_k,y_k\,|\,x_{1:k-1},y_{1:k-1}), \quad \forall x,y \in \mathbb{Z}^n$$

where $\pi^i(\cdot,\cdot\,|\,x_{1:i-1},y_{1:i-1})$ is the monotone coupling between $\mu^i(\cdot\,|\,x_{1:i-1})$ and $\nu^i(\cdot\,|\,y_{1:i-1})$.

Note that one may replace the monotone couplings in this construction with any given couplings, to produce different Knothe–Rosenblatt type couplings. However, in this paper, we shall strictly refer to the Knothe–Rosenblatt coupling as the one defined above.

3. The monotone case

The purpose of this section is to prove Theorem 1.5 in the case where T_- (equivalently T_+) itself is monotone in each of its two entries with respect to some total additive ordering $\preceq$ on $\mathbb{Z}^n$. That is,

$$x_1 \preceq x_2, y_1 \preceq y_2 \implies T_\pm(x_1,y_1) \preceq T_\pm(x_2,y_2). \tag{8}$$

Let us state this result precisely as the following proposition:

Proposition 3.1: *Let G be a finitely generated group equipped with a total additive ordering $\preceq$, let $T_\pm : G \times G \to G$ be complementing operations satisfying* (P1) *and* (8), *and let μ, ν be finitely supported probability measures on G. Suppose that π is the monotone coupling between μ, ν and $\kappa_\pm = \pi \circ {T_\pm}^{-1}$ is the push forward of π by $T_\pm$. Then*

$$\sum_{(x,y)\in G\times G} \frac{\kappa_-(T_-(x,y))\kappa_+(T_+(x,y))}{\mu(x)\nu(y)}\,\pi(x,y) \leq 1.$$

The core ideas of the proof are drawn from the proof of [8, Theorem 1.3], which is also a particular case of the proposition. However, we manage to simplify some of its key steps.

In preparation for the proof of Proposition 3.1, we need the following lemmata about the structure of the support of a monotone coupling under monotone complementing operations.

For the remainder of this section, let $T_\pm$ and π be the complementing monotone operations and the monotone coupling given in Proposition 3.1. Denote the support of π by $\mathrm{supp}(\pi)$.

Lemma 3.2: *Suppose $(x_1, y_1) \neq (x_2, y_2)$, $x_1 \preceq x_2, y_1 \preceq y_2$ and $T_-(x_1, y_1) = T_-(x_2, y_2)$. Then, either $x_1 = x_2$ or $y_1 = y_2$. Moreover, we have*

$$T_+(x_1, y_1) \prec T_+(x_2, y_2) = T_+(x_1, y_1) + [(x_2 - x_1) + (y_2 - y_1)].$$

Proof. Assume that $y_1 \prec y_2$ and $x_1 \prec x_2$. Then, denoting $a = \min\{x_2 - x_1, y_2 - y_1\}$, we have

$$T_-(x_1, y_1) \prec a + T(x_1, y_1) = T_-(x_1 + a, y_1 + a) \preceq T_-(x_2, y_2),$$

a contradiction. Thus, either $x_1 = x_2$ or $y_1 = y_2$. The relation

$$T_-(x, y) + T_+(x, y) = x + y$$

implies that

$$T_+(x_2, y_2) = x_2 + y_2 - T_-(x_2, y_2) = x_2 + y_2 - (x_1 + y_1 - T_+(x_1, y_1)),$$

as claimed. ∎

Note that Lemma 3.2 and its proof both hold if one interchanges T_- with T_+, due to the symmetry between them.

For $a \in G$, define $S_\pm(a) = \{(x,y) \in \mathrm{supp}(\pi)\,;\, T_\pm(x,y) = a\}$.

Lemma 3.3: *For every $a \in G$ we have $S_-(a) = \{(x_0,y_0),(x_1,y_1),\ldots,(x_k,y_k)\}$ where either $x_0 = x_1 = \cdots = x_k$ and $y_0 \prec y_1 \prec \cdots \prec y_k$ or $x_0 \prec x_1 \prec \cdots \prec x_k$ and $y_0 = y_1 = \cdots = y_k$.*

Similarly, we have $S_+(a) = \{(x'_0,y'_0),(x'_1,y'_1),\ldots,(x'_m,y'_m)\}$ where either $x'_0 = x'_1 = \cdots = x'_m$ and $y'_0 \prec y'_1 \prec \cdots \prec y'_m$ or $x'_0 \prec x'_1 \prec \cdots \prec x'_m$ and $y'_0 = y'_1 = \cdots = y'_m$.

Proof. By the monotonicity of $\mathrm{supp}(\pi)$, we have

$$S_-(a) = \{(x_0,y_0),(x_1,y_1),\ldots,(x_k,y_k)\}$$

where $x_0 \preceq \cdots \preceq x_k$ and $y_0 \preceq \cdots \preceq y_k$. Moreover, by Lemma 3.2, for every $i,j \in \{0,\ldots,k\}$ such that $i \neq j$, we have either $x_i = x_j$ or $y_i = y_j$, from which it follows that either $x_0 = x_1 = \cdots = x_k$ or $y_0 = y_1 = \cdots = y_k$.

The second part of the lemma regarding $S_+(a)$ is proven in exactly the same way. ∎

Lemma 3.4: *Let $a \in G$ and $k \geq 2$. Suppose $S_-(a) = \{(x_0,y_0),\ldots,(x_k,y_k)\}$ with $x_0 \preceq \cdots \preceq x_k$ and $y_0 \preceq \cdots \preceq y_k$. Then, $S_+(T_+(x_i,y_i)) = \{(x_i,y_i)\}$ for all $0 < i < k$.*

Proof. By Lemma 3.2, $T_+(x_0,y_0) \prec \cdots \prec T_+(x_k,y_k)$. Since T_+ is monotone, it follows that

$$T_+(x_0,y_0) \prec \cdots \prec T_+(x_k,y_k) \preceq T_+(x,y)$$

whenever $x_k \preceq x$ and $y_k \preceq y$. In particular, we have

$$(x,y) \notin S_+(T_+(x_i,y_i))$$

for all $x_k \preceq x$, $y_k \preceq y$ and $0 < i < k$. Similarly, one shows that

$$(x,y) \notin S_+(T_+(x_i,y_i))$$

for all $x \preceq x_k$, $y \preceq y_k$ and $0 < i < k$. Since T_- is monotone, if $x_0 \preceq x \preceq x_k$ and $y_0 \preceq y \preceq y_k$ then $T_-(x,y) = a$, and so either $(x,y) = (x_i,y_i)$ for some $0 \leq i \leq k$ or $(x,y) \notin \mathrm{supp}(\pi)$. Therefore, by the monotonicity of $\mathrm{supp}(\pi)$, we have $S_+(T + (x_i,y_i)) = \{(x_i,y_i)\}$ for all $0 < i < k$. ∎

Proof of Proposition 3.1. Fix $a \in G$ for which $S_-(a) \neq \emptyset$. It is sufficient to show that

$$\sum_{(x,y)\in S_-(a)} \frac{\kappa_-(a)\kappa_+(T_+(x,y))}{\mu(x)\nu(y)}\pi(x,y) \leq \sum_{(x,y)\in S_-(a)} \pi(x,y) = \kappa_-(a). \tag{9}$$

By Lemma 3.4, we have

$$S_-(a) = \{(x_0,y_0),\dots,(x_k,y_k)\}$$

where either $x_0 = x_1 = \cdots = x_k$ and $y_0 \prec y_1 \prec \cdots \prec y_k$ or $x_0 \prec x_1 \prec \cdots \prec x_k$ and $y_0 = y_1 = \cdots = y_k$. By interchanging the first and second coordinates, we may assume without loss of generality that the first possibility holds. Therefore, (9) is reduced to

$$\sum_{j=0}^{k} \frac{\kappa_+(T_+(x_0,y_j))}{\nu(y_j)}\pi(x_0,y_j) \leq \mu(x_0). \tag{10}$$

By Lemma 3.3, we have

$$S_+(T_+(x_0,y_0)) = \{(x_0,y_0),(x'_0,y'_0),\dots,(x'_l,y'_l)\}$$

such that either $x'_0 = \cdots = x'_l = x_0$ or $y'_0 = \cdots = y'_l = y_0$. In particular, it follows that

$$\kappa_+(T_+(x_0,y_0)) \leq \max\{\mu(x_0),\nu(y_0)\}. \tag{11}$$

Similarly, Lemma 3.3 tells us that

$$S_+(T_+(x_0,y_k)) = \{(x_0,y_k),(x''_0,y''_0),\dots,(x''_m,y''_m)\}$$

such that either $x''_0 = \cdots = x''_m = x_0$ or $y''_0 = \cdots = y''_m = y_k$. Note that, by Lemma 3.2, the points

$$(x_0,y_0),\dots,(x_0,y_k),(x'_0,y'_0),\dots,(x'_l,y'_l),(x''_0,y''_0),\dots,(x''_m,y''_m)$$

are all distinct.

Case 1: $k = 0$. Using (11) and the fact that

$$\pi(x_0,y_0) \leq \min\{\mu(x_0),\nu(y_0)\}$$

directly yields (10).

Case 2: $k \geq 1$. By Lemma 3.4, we have $S_+(T_+(x_0, y_i))) = \{(x_0, y_i)\}$ for all $0 < i < k$, and hence $\kappa_+(T_+(x_0, y_i)) = \pi(x_0, y_i) \leq \nu(y_i)$. We can thus rewrite (10) as follows:

$$\begin{aligned} \frac{\kappa_+(T_+(x_0, y_0))}{\nu(y_0)}\pi(x_0, y_0) + \sum_{i=1}^{k-1} \frac{\pi(x_0, y_i)}{\nu(y_i)}\pi(x_0, y_i) \\ + \frac{\kappa_+(T_+(x_0, y_k))}{\nu(y_k)}\pi(x_0, y_k) \leq \mu(x_0). \end{aligned} \tag{12}$$

We split the proof of (12) into four simple subcases, as follows.

Case 2.1: $x'_0 = \cdots = x'_l = x_0$ and $x''_0 = \cdots = x''_m = x_0$. Since $\pi(x, y) \leq \nu(y)$ for all $x, y \in G$, it is enough to show that

$$\kappa_+(T_+(x_0, y_0)) + \sum_{i=1}^{k-1} \pi(x_0, y_i) + \kappa_+(T_+(x_0, y_k)) \leq \mu(x_0)$$

which clearly holds as

$$\kappa_+(T_+(x_0, y_0)) = \sum_{j=0}^{l} \pi(x_0, y'_j) \quad \text{and} \quad \kappa_+(T_+(x_0, y_k)) = \sum_{j=0}^{m} \pi(x_0, y''_j).$$

Case 2.2: $x'_0 = \cdots = x'_l = x_0$ and $y''_0 = \cdots = y''_m = y_k$. Since $\pi(x_0, y_0) \leq \nu(y_0)$, $\pi(x_0, y_i) \preceq \nu(y_i)$ for all $0 < i < k$ and

$$\kappa_+(T_+(x_0, y_k)) = \sum_{j=0}^{m} \pi(x''_j, y_k) \leq \nu(y_k),$$

it is enough to show that

$$\kappa_+(T_+(x_0, y_0)) + \sum_{i=1}^{k-1} \pi(x_0, y_i) + \pi(x_0, y_k) \leq \mu(x_0)$$

which clearly holds as $\kappa_+(T_+(x_0, y_0)) = \sum_{j=0}^{l} \pi(x_0, y'_j)$.

Case 2.3: $y'_0 = \cdots = y'_l = y_0$ and $x''_0 = \cdots = x''_m = x_0$. Since

$$\kappa_+(T_+(x_0, y_0)) = \sum_{j=0}^{l} \pi(x'_j, y_0) \leq \nu(y_0),$$

$\pi(x_0, y_i) \preceq \nu(y_i)$ for all $0 < i < k$ and $\pi(x_0, y_k) \leq \nu(y_k)$, it is enough to show that

$$\pi(x_0, y_0) + \sum_{i=1}^{k-1} \pi(x_0, y_i) + \kappa_+(T_+(x_0, y_k)) \leq \mu(x_0)$$

which clearly holds as $\kappa_+(T_+(x_0, y_k)) = \sum_{j=0}^{m} \pi(x_0, y''_j)$.

Case 2.4: $y'_0 = \cdots = y'_l = y_0$ and $y''_0 = \cdots = y''_m = y_k$. Since

$$\kappa_+(T_+(x_0, y_0)) = \sum_{j=0}^{l} \pi(x'_j, y_0) \leq \nu(y_0),$$

$\pi(x_0, y_i) \preceq \nu(y_i)$ for all $0 < i < k$ and $\kappa_+(T_+(x_0, y_k)) = \sum_{j=0}^{m} \pi(x''_j, y_k) \leq \nu(y_k)$, it is enough to show that

$$\pi(x_0, y_0) + \sum_{i=1}^{k-1} \pi(x_0, y_i) + \pi(x_0, y_k) \leq \mu(x_0)$$

which clearly holds. This establishes (12) and completes the proof. ∎

4. Proofs of the main results

Proof of Theorem 1.5. Recall that since $T_\pm : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ are monotone in the sense of Knothe with respect to the decomposition $\mathbb{Z}^n = (G_i, \preceq_i)_{1:k}$, we have

$$T_\pm(x, y) = (T^1_\pm(x_1, y_1), T^2_\pm(x_1, x_2, y_1, y_2) \ldots, T^k_\pm(x_{1:k-1}, y_{1:k-1}))$$

for all $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in (G_1, \ldots, G_k)$. Moreover, for each $i \in \{1, \ldots, k\}$, the operations $(T^i_\pm)^{(x_{1:i-1}, y_{1:i-1})} : G_i \times G_i \to G_i$, defined by

$$(T^i_\pm)^{(x_{1:i-1}, y_{1:i-1})}(x_i, y_i) = T^i_\pm(x_{1:i}, y_{1:i})$$

for all $x_{1:i-1}, y_{1:i-1} \in G_1 \times \cdots \times G_{i-1}$, are increasing in each of their two entries.

Let π be the Knothe coupling between μ and ν with respect to the same decomposition $\mathbb{Z}^n = (G_i, \preceq_i)_{1:k}$, and recall that $\kappa_\pm = \pi \circ T_\pm{}^{-1}$. We have

$$\begin{aligned} P &:= \sum_{(x,y) \in \mathbb{Z}^n \times \mathbb{Z}^n} \frac{\kappa_-(T_-(x, y))\kappa_+(T_+(x, y))}{\mu(x)\nu(y)} \pi(x, y) \\ &= \sum_{(x_1,y_1) \in G_1 \times G_1} \sum_{(x_2,y_2) \in G_2 \times G_2} \cdots \sum_{(x_k,y_k) \in G_k \times G_k} \frac{\kappa_-(T_-(x, y))\kappa_+(T_+(x, y))}{\mu(x)\nu(y)} \pi(x, y). \end{aligned}$$

Using the disintegration of $\kappa_\pm$ and the fact that $T_\pm$ are triangular with respect to the given decomposition of $\mathbb{Z}^n$, we have

$$\begin{aligned} &\kappa_\pm(T_\pm(x, y)) \\ &\quad = \kappa^1_\pm(T^1_\pm(x_1, y_1))\kappa^2_\pm(T^2_\pm(x_2, y_2)|x_1, y_1) \cdots \kappa^k_\pm(T^k_\pm(x_k, y_k)|x_{1:k-1}, y_{1:k-1}) \end{aligned}$$

where, for brevity, the expression $T^i_\pm(x_i,y_i)$ within $\kappa^i_\pm(T^i_\pm(x_i,y_i)|x_{1:i-1},y_{1:i-1})$ is understood as $(T^i_\pm)^{(x_{1:i-1},y_{1:i-1})}(x_i,y_i)$. Combined with the disintegration of μ, ν, and π with respect to the given decomposition of $\mathbb{Z}^n$, we obtain that

$$\begin{aligned}P = &\sum_{(x_1,y_1)\in G_1\times G_1} A_1(x_1,y_1)\\ &\times \sum_{(x_2,y_2)\in G_2\times G_2} A_2^{(x_1,y_1)}(x_2,y_2) \ \cdots \sum_{(x_k,y_k)\in G_k\times G_k} A_k^{(x_{1:k-1},y_{1:k-1})}(x_k,y_k),\end{aligned}$$

where $A_i^{(x_{1:i-1},y_{1:i-1})}(x_i,y_i)$ is given by

$$\begin{aligned}&\frac{(\kappa^i_-(T^i_-(x_i,y_i)|x_{1:i-1},y_{1:i-1}))(\kappa^i_+(T^i_+(x_i,y_i)|x_{1:i-1},y_{1:i-1}))}{(\mu^i(x_i\,|\,x_{1:i-1}))(\nu^i(y_i|y_{1:i-1}))}\\ &\qquad\qquad\qquad\qquad\times \pi^i(x_i,y_i\,|\,x_{1:i-1},y_{1:i-1}).\end{aligned}$$

Finally, we apply Proposition 3.1 iteratively to each sum separately to obtain that

$$\sum_{(x_i,y_i)\in G_i} A_i^{(x_{1:i-1},y_{1:i-1})}(x_i,y_i) \le 1$$

for all $i \in \{1,\ldots,k\}$ and $x_{1:i-1},y_{1:i-1}\in G_1\times\cdots\times G_{i-1}$. This completes the proof. ∎

Proof of Theorem 1.4. Let π be the coupling between μ and ν, given in Theorem 1.5, and $\kappa_\pm$ denote the push forward of π by $T_\pm$. By Jensen's inequality, applied to the logarithm function, Theorem 1.5 implies that

$$H := \sum_{(x,y)\in\mathbb{Z}^n\times\mathbb{Z}^n} \log\Big(\frac{\kappa_-(T_-(x,y))\kappa_+(T_+(x,y))}{\mu(x)\nu(y)}\Big)\pi(x,y) \le 0.$$

By the definition of π, κ_- and κ_+, it follows that

$$\begin{aligned}&\sum_{z\in\mathbb{Z}^n}\log(\kappa_-(z))\kappa_-(z) + \sum_{z\in\mathbb{Z}^n}\log(\kappa_+(z))\kappa_+(z)\\ &\qquad\qquad - \sum_{z\in\mathbb{Z}^n}\log(\mu(z))\mu(z) - \sum_{z\in\mathbb{Z}^n}\log(\nu(z))\nu(z) \le 0,\end{aligned}$$

or, equivalently, that

$$H(\kappa_-|m_n) + H(\kappa_+|m_n) - H(\mu|m_n) - H(\nu|m_n) \le 0. \qquad ∎$$

Proof of Theorem 1.2. As in [8], we use the log-Laplace transform of any bounded function φ:

$$\log \int e^{\varphi}\, dm_n = \sup_{\nu} \Big\{ \int \varphi\, d\nu - H(\nu|m_n) \Big\}. \tag{13}$$

Let f, g, h, k satisfy

$$f^{\alpha}(x) g^{\beta}(y) \le h^{\gamma}(T_-(x,y)) k^{\delta}(T_+(x,y)) \quad \forall x, y \in \mathbb{Z}^n. \tag{14}$$

If either h or k is not bounded from above then the statement holds trivially. Otherwise, it follows from (14) that f, g, h, k are all bounded from above. Given $\varepsilon > 0$ and setting $f_\varepsilon = \max(\varepsilon, f(x))$, one may check that the above inequality is equivalent to

$$\alpha \log f_\varepsilon(x) + \beta \log g_\varepsilon(y) \le \gamma \log h_\varepsilon(T_-(x,y)) + \delta \log k_\varepsilon(T_+(x,y)).$$

Integrating this inequality with respect to the Knothe coupling π between finitely supported probability measures μ, ν on $\mathbb{Z}^n$, as given in Theorem 1.4, we have

$$\begin{aligned} \alpha \int \log f_\varepsilon d\mu + \beta \int \log g_\varepsilon d\nu &\le \gamma \int \log(h_\varepsilon \circ T_-) d\pi + \delta \int \log(k_\varepsilon \circ T_+) d\pi \\ &= \gamma \int \log h_\varepsilon d\kappa_- + \delta \int \log k_\varepsilon d\kappa_+, \end{aligned}$$

where $\kappa_\pm = \pi \circ {T_\pm}^{-1}$. Applying Theorem 1.4 and (13) we thus get

$$\begin{aligned} \alpha\Big(\int \log f_\varepsilon d\mu - H(\mu|m_n)\Big) &+ \beta\Big(\int \log g_\varepsilon d\nu - H(\nu|m_n)\Big) \\ &\le \gamma\Big(\int \log h_\varepsilon d\kappa_- - H(\kappa_-|m_n)\Big) + \delta\Big(\int \log k_\varepsilon d\kappa_+ - H(\kappa_+|m_n)\Big) \\ &\le \gamma \log \int h_\varepsilon dm_n + \delta \log \int k_\varepsilon dm_n. \end{aligned}$$

Optimizing over all μ and ν, we get

$$\alpha \log \int f_\varepsilon dm_n + \beta \log \int g_\varepsilon dm_n \le \gamma \log \int h_\varepsilon dm_n + \delta \log \int k_\varepsilon dm_n.$$

We conclude the proof by taking $\varepsilon \to 0$ and applying the monotone convergence theorem. ∎

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