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HYPERGRAPHS WITH MINIMUM POSITIVE UNIFORM TURÁN DENSITY*

BY

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ABSTRACT

Reiher, Rödl and Schacht showed that the uniform Turán density of every 3-uniform hypergraph is either 0 or at least 1/27, and asked whether there exist 3-uniform hypergraphs with uniform Turán density equal or arbitrarily close to 1/27. We construct 3-uniform hypergraphs with uniform Turán density equal to 1/27.

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1. Introduction

Determining the minimum density of a (large) combinatorial structure required to contain a given (small) substructure is a classical extremal combinatorics problem, which can be traced to the work of Mantel [12] and Turán [24] in the first half of the 20th century. The **Turán density** of a k-uniform hypergraph H, which is denoted by $\pi(H)$, is the infimum over all d such that every sufficiently large host k-uniform hypergraph with edge density at least d contains H as a subhypergraph. It can be shown [10] that the Turán density of H is equal to the limit of the maximum density of a k-uniform n-vertex H-free hypergraph (n tends to infinity); in particular, Katona, Nemetz and Simonovits [10] showed that this sequence of maximum densities is non-increasing and so the limit always exists.

The Turán density of a complete graph K_r of order r is equal to $\frac{r-2}{r-1}$ as determined by Turán [24] himself. Erdős and Stone [6] showed that the Turán density of any r-chromatic graph H is equal to $\frac{r-2}{r-1}$; also see [4]. The situation is more complex already for 3-uniform hypergraphs, which we will call 3-graphs for simplicity, compared to graphs (which are 2-uniform hypergraphs). In particular, determining the Turán density of the complete 4-vertex 3-graph $K_4^{(3)}$ is a major open problem, and likewise determining the Turán density of $K_4^{(3)-}$, defined as $K_4^{(3)}$ with an edge removed, is a challenging open problem [1, 7, 15] despite some recent progress obtained using the flag algebra method of Razborov [14]; also see the survey [11] for further details.

It is well-known that H-free graphs with density close to the Turán density $\pi(H)$ are close to $(r-1)$-partite complete graphs [8, 23], i.e., the edges in such graphs are distributed in a highly non-uniform way. The same applies to conjectured extremal constructions in the setting of 3-graphs [7]. In this paper, we study the notion of uniform Turán density of hypergraphs, which requires the edges in the host hypergraph to be distributed uniformly. This notion was suggested by Erdős and Sós [3, 5] in the 1980s and there is a large amount of recent progress in relation to this notion and to some of its variants [9, 17–21], see also the survey [16]. For example, Glebov, Volec and the second author [9] and Reiher, Rödl and Schacht [20] answered a question raised by Erdős and Sós by showing that the uniform Turán density of $K_4^{(3)-}$ is equal to $1/4$.

The following result of Reiher, Rödl and Schacht [18] is the starting point of our work: the uniform Turán density of every 3-graph is either zero or at least $1/27$. Reiher et al. [18] asked whether there exist 3-graphs with uniform Turán density equal or arbitrarily close to $1/27$. We answer this question in the affirmative by giving a sufficient condition for a 3-graph to have uniform Turán density equal to $1/27$ and finding examples of 3-graphs satisfying this condition.

We next introduce the notation needed to state our results precisely. The **ε-linear** density of an n-vertex hypergraph H is the minimum density of an induced subhypergraph of H with at least εn vertices. The **uniform Turán density** of a hypergraph H_0 is the infimum over all d such that there exists $\varepsilon > 0$ such that every sufficiently large hypergraph H with ε-linear density d contains H_0. We also present an equivalent definition, which is used by Reiher, Rödl and Schacht [17–21]. An n-vertex k-uniform hypergraph H is **(d, ε)-dense** if every subset W of its vertices induces at least $d\binom{|W|}{k} - \varepsilon n^k$ edges. The uniform Turán density of a hypergraph H_0 is the supremum over all d such that for every $\varepsilon > 0$, there exist arbitrarily large H_0-free (d, ε)-dense hypergraphs. It is easy to show that the two definitions are equivalent.

The notion of the uniform Turán density is trivial for graphs as the uniform Turán density of every graph is equal to zero. However, the situation is much more complex already for 3-graphs. As we have already mentioned, the uniform Turán density of $K_4^{(3)-}$ has been determined only recently [9, 20], and the only other 3-graphs with a positive uniform Turán density that has been determined are tight 3-uniform cycles of length not divisible by three [2]—note that for tight cycles divisible by 3 the uniform Turán density is equal to 0. In particular, determining the uniform Turán density of $K_4^{(3)}$ is a challenging open problem though it is believed that the 35-year-old construction of Rödl [22] showing that the uniform Turán density of $K_4^{(3)}$ is at least $1/2$ is optimal [16].

Reiher, Rödl and Schacht [18] gave a characterization of 3-graphs with uniform Turán density equal to zero, which we now present. Let H be a 3-graph with n vertices. We say that an ordering $v_1, \dots, v_n$ of the vertices of H is **vanishing** if the set of pairs (i, j), $1 \le i < j \le n$, can be partitioned to sets L, T and R such that every edge $\{v_i, v_j, v_k\}$ of H, where $i < j < k$, satisfies that $(i, j) \in L$, $(i, k) \in T$ and $(j, k) \in R$. The pairs that belong to L, T and R are referred to as **left**, **top** and **right**, respectively (the reason for this terminology comes from the visualization of the pairs of a triple by arcs over a horizontal line as in Figure 1). We remark that when the vanishing ordering is fixed, the partition to the sets L, T and R is unique up to the pairs (i, j) such that the vertices v_i and v_j are not contained in a common edge, and we can choose all such undetermined pairs to be, say, left. So, we can speak about left, top and right pairs whenever a vanishing ordering is fixed.

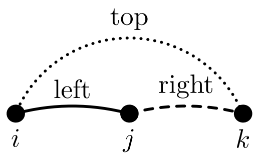

Figure 1. Illustration of left, right and top pairs in an edge of a 3-graph with ordered vertex set $i < j < k$. The left pair is drawn solid, the right pair dashed and the top pair dotted following the convention used later in Figure 3.

The characterization of 3-graphs with uniform Turán density equal to zero reads as follows.

Theorem 1 (Reiher, Rödl and Schacht [18]): *Let H be a 3-graph. The uniform Turán density of H is zero if and only if H has a vanishing ordering of its vertices.*

If a 3-graph H has no vanishing ordering, then the uniform Turán density of H is at least $1/27$ because of the following construction from [18]. Indeed, fix a 3-graph H with no vanishing ordering and construct a random n-vertex 3-graph H_n as follows: let $v_1, \dots, v_n$ be the vertices of H_n, randomly partition all pairs of those vertices to sets L, T and R, and include $\{v_i, v_j, v_k\}$, $1 \le i < j < k \le n$, as an edge of H_n if $(i,j) \in L$, $(i,k) \in T$ and $(j,k) \in R$. Observe that H cannot be a subhypergraph of H_n (as H has no vanishing ordering). On the other hand, for every $\varepsilon > 0$ and $\delta > 0$, there exists n_0 such that the density of every subset of at least εn vertices of H_n for $n \ge n_0$ is at least $1/27 - \delta$ with positive probability. It follows that the uniform Turán density of H is at least $1/27$ as claimed. Hence, Theorem 1 implies the following.

Corollary 2: *The uniform Turán density of every 3-graph is either zero or at least $1/27$.*

Reiher, Rödl and Schacht [18] asked whether there exist 3-graphs with uniform Turán density equal or arbitrarily close to $1/27$. In this paper we answer this question in the affirmative.

THEOREM 3: *There exists an infinite family of* 3*-graphs with uniform Turán density equal to* 1/27*.*

Theorem 3 is implied by the following. In Theorem 15, we give a sufficient condition for a 3-graph to have uniform Turán density equal to 1/27, we then present a 7-vertex 9-edge 3-graph (Theorem 17) and an infinite family of 3-graphs (Theorem 18), whose smallest element has 8 vertices and 9 edges, that satisfy the condition given in Theorem 15. We remark that we have verified by a computer that there is no such 3-graph with six or fewer vertices; in fact, we have been able to show that every 3-graph with six or fewer vertices has Turán density either equal to zero or at least 1/8.

2. Notation

In this section, we introduce the notation used throughout the paper. We write $[n]$ for the set of the first n positive integers, i.e., $[n] = \{1, \dots, n\}$. An **n-partitioned hypergraph** H is a 3-graph such that its vertex set is partitioned to sets V_{ij}, $1 \le i < j \le n$, and every edge e of H satisfies that there exist indices $1 \le i < j < k \le n$ such that one vertex of e belongs to V_{ij}, one to V_{ik} and one to V_{jk}. The set of all edges of H with a vertex from V_{ij}, one from V_{ik} and one from V_{jk} is called an **(i, j, k)-triad**, and the **density** of an (i, j, k)-triad is the number of edges forming the triad divided by $|V_{ij}| \cdot |V_{ik}| \cdot |V_{jk}|$. Finally, the **density** of an n-partitioned hypergraph H is the minimum density of a triad of H.

We will use the following convention to simplify our notation used throughout the paper. If H is an n-partitioned hypergraph, we write V_{ij}, $1 \le i < j \le n$, for its vertex parts, and if H' is an n'-partitioned hypergraph, we write V'_{ij}, $1 \le i < j \le n'$, for its vertex parts, i.e., we use the same mathematical accents when denoting a hypergraph as we do for its vertex parts without specifying the relation explicitly. The **reverse** of an n-partitioned hypergraph H is an n-partitioned hypergraph H' with the same vertex set and the same edge set as H but with the partition of vertices given by $V'_{ij} = V_{n-j+1,n-i+1}$ for $1 \le i < j \le n$.

Let H be an n-partitioned hypergraph. We say that H' is an **induced** subhypergraph of H if there exists $I \subseteq [n]$ such that H' is an $|I|$-partitioned hypergraph, its vertex parts are the parts V_{ij} of H such that $i, j \in I$ and H'

contains all edges of H with vertices in the vertex parts forming H'. We will refer to H' as the subhypergraph of H **induced** by the index set I.

We next define several notions of a normalized degree of a vertex of an n-partitioned hypergraph H. Fix $1 \le i < j < k \le n$ and define

- $d_{ij\to k}(v)$ for $v \in V_{ij}$ to be the number of edges of the (i,j,k)-triad containing v divided by $|V_{ik}| \cdot |V_{jk}|$,
- $d_{ik\to j}(v)$ for $v \in V_{ik}$ to be the number of edges of the (i,j,k)-triad containing v divided by $|V_{ij}| \cdot |V_{jk}|$, and
- $d_{jk\to i}(v)$ for $v \in V_{jk}$ to be the number of edges of the (i,j,k)-triad containing v divided by $|V_{ij}| \cdot |V_{ik}|$.

Note that the arrow in the notation indicates to which part of the triad v belongs. Further, $d_{ij,ik}(v,v')$ for $v \in V_{ij}$ and $v' \in V_{ik}$ is the number of edges of the (i,j,k)-triad containing v and v' divided by $|V_{jk}|$; we analogously use $d_{ij,jk}(v,v')$ for $v \in V_{ij}$ and $v' \in V_{jk}$ and $d_{ik,jk}(v,v')$ for $v \in V_{ik}$ and $v' \in V_{jk}$. The considered hypergraph H when using the just introduced notation will always be clear from the context, so we decided not to include it as a part of the notation to keep the notation simpler.

An N-partitioned hypergraph H **embeds** an n-vertex hypergraph H_0 if it is possible to choose distinct $1 \le a_1, \ldots, a_n \le N$ corresponding to the vertices of H_0 and vertices $v_{ij} \in V_{a_i a_j}$ for $1 \le i < j \le n$ such that if the i-th, j-th and k-th vertex of H_0 form an edge, then $\{v_{ij}, v_{ik}, v_{jk}\}$ is an edge of H.

In [16], Reiher gave a general theorem that relates computing the uniform Turán density of 3-graphs to embeddings in partitioned hypergraphs. In our notation, the theorem reads as follows.

Proposition 4 (Reiher [16, Theorem 3.3]): *Let H be a 3-graph and $d \in [0,1]$. Suppose that for every $\delta > 0$ there exists N such that every N-partitioned hypergraph with density at least $d+\delta$ embeds H. Then, the uniform Turán density of H is at most d.*

Some of our arguments use the classical Ramsey theorem for multicolored hypergraphs, which we state below for reference.

Theorem 5 (Ramsey [13]): *For all $k, r, n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that every k-edge-coloring of a complete r-uniform hypergraph with N vertices contains a monochromatic complete r-uniform hypergraph with n vertices.*

3. Preprocessing steps

In this section, we present two lemmas that we use to tame a given partitioned hypergraph before we can apply our main arguments. The first lemma says that we can find a subhypergraph of a partitioned hypergraph such that the proportions of left, top and right vertices with non-negligible degrees in all triads are approximately the same.

Lemma 6: *For every $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that the following holds. For every N-partitioned hypergraph H, there exist reals ℓ, t and r, and an n-partitioned induced subhypergraph H' such that for all $1 \le i < j < k \le n$*

$$\begin{aligned} \ell|V'_{ij}| &\le |\{v \in V'_{ij},\, d_{ij\to k}(v) \ge \varepsilon\}| \le (\ell+\varepsilon)|V'_{ij}|, \\ t|V'_{ik}| &\le |\{v \in V'_{ik},\, d_{ik\to j}(v) \ge \varepsilon\}| \le (t+\varepsilon)|V'_{ik}|, \\ r|V'_{jk}| &\le |\{v \in V'_{jk},\, d_{jk\to i}(v) \ge \varepsilon\}| \le (r+\varepsilon)|V'_{jk}|. \end{aligned}$$

Proof. Apply Theorem 5 for 3-graphs with $k_R = (\lfloor \varepsilon^{-1} \rfloor + 1)^3$ and $n_R = n$ to get N (the variables on the left of the equalities are named as in the statement of Theorem 5 but with the subscript R added). Let H be an N-partitioned hypergraph. Consider the following k_R-edge-coloring of the complete 3-graph with vertex set $[N]$: for $1 \le i < j < k \le N$, let L be the set of vertices $v \in V_{ij}$ such that $d_{ij\to k}(v) \ge \varepsilon$, T the set of $v \in V_{ik}$ such that $d_{ik\to j}(v) \ge \varepsilon$, and R the set of $v \in V_{jk}$ such that $d_{jk\to i}(v) \ge \varepsilon$, and color the edge $\{i, j, k\}$ with the triple

$$\Big(\Big\lfloor \frac{|L|}{\varepsilon|V_{ij}|} \Big\rfloor, \Big\lfloor \frac{|T|}{\varepsilon|V_{ik}|} \Big\rfloor, \Big\lfloor \frac{|R|}{\varepsilon|V_{jk}|} \Big\rfloor\Big).$$

Theorem 5 implies that there exists a subset $I \subseteq [N]$ such that all edges with vertices in I have the same color, say (ℓ', t', r'). The n-partitioned subhypergraph H' of H induced by the index set I satisfies the statement of the lemma with $\ell = \varepsilon\ell'$, $t = \varepsilon t'$ and $r = \varepsilon r'$. ∎

The next lemma concerns partitioned hypergraphs with density larger than $1/27$, and yields that every such hypergraph contains an induced subhypergraph with one of the three properties described in the lemma. We will refer to the first of these properties as the case of **horizontal intersection** and the other as the case of **vertical intersection** (the second and third cases are symmetric by reversing the order of the parts of the partitioned hypergraph). The case of horizontal intersection corresponds to the existence of an edge in a (k', i, j)-triad and an edge in an (i, j, k)-triad, $k' < i < j < k$, that share a common vertex of V_{ij} (the adjective horizontal comes from the fact that the (k', i, j)-triad and the (i, j, k)-triad can be visualized by being drawn as overlapping edges following each other on a horizontal line). The case of vertical intersection corresponds to the existence of an edge in an (i, k', j)-triad and an edge in an (i, j, k), $i < k' < j < k$, that share a common vertex of V_{ij} (the adjective vertical comes from the fact that the two triads cannot be visualized as in the previous case as the edges are nested) or the existence of an edge in a (j, k', k)-triad and an edge in an (i, j, k), $i < j < k' < k$. Using this terminology, the next lemma asserts that every partitioned hypergraph with density larger than $1/27$ has a subhypergraph such that all triads have non-trivial horizontal intersection or all triads have non-trivial vertical intersection. We will show that hypergraphs that we construct later can be embedded in every partitioned hypergraph where all triads have non-trivial horizontal intersection and in every partitioned hypergraph where all triads have non-trivial vertical intersection, which are the two cases corresponding to the two possible outcomes of Lemma 14.

Lemma 7: *For every $\delta > 0$, there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that the following holds. For every N-partitioned hypergraph H with density at least $1/27 + \delta$, there exists an n-partitioned induced subhypergraph H' of H such that at least one of the following holds:*

- *For all $1 \leq k' < i < j < k \leq n$, the set V'_{ij} contains at least $\varepsilon|V'_{ij}|$ vertices v such that $d_{ij\to k}(v) \geq \varepsilon$ and $d_{ij\to k'}(v) \geq \varepsilon$.*
- *For all $1 \leq i < k' < j < k \leq n$, the set V'_{ij} contains at least $\varepsilon|V'_{ij}|$ vertices v such that $d_{ij\to k}(v) \geq \varepsilon$ and $d_{ij\to k'}(v) \geq \varepsilon$.*
- *For all $1 \leq i < j < k' < k \leq n$, the set V'_{jk} contains at least $\varepsilon|V'_{jk}|$ vertices v such that $d_{jk\to i}(v) \geq \varepsilon$ and $d_{jk\to k'}(v) \geq \varepsilon$.*

Proof. We can assume that $\delta \leq 1/2$ without loss of generality. Set $\varepsilon = \delta/9$ and $\varepsilon_0 = \delta/3$ and suppose that n is given. Apply Theorem 5 with $k_R = 3$, $r_R = 5$ and $n_R = 2n + 1$ to get N_0 and apply Lemma 6 with ε_0 and N_0 to get N. Let H be an N-partitioned hypergraph with density at least $1/27 + \delta$, and let H_0 be the induced N_0-partitioned subhypergraph provided by Lemma 6 along with the reals ℓ, t and r with the properties given in the statement of Lemma 6.

We first show the following

Claim: $\ell + t + r \geq 1 + \varepsilon_0$.

Proof of Claim. Suppose that $\ell + t + r < 1 + \varepsilon_0$ and choose arbitrary i, j and k such that $1 \leq i < j < k \leq N_0$. Let L be the set of vertices $v \in V_{ij}$ such that $d_{ij\to k}(v) \geq \varepsilon$, T the set of vertices $v \in V_{ik}$ such that $d_{ik\to j}(v) \geq \varepsilon$, and R the set of vertices $v \in V_{jk}$ such that $d_{jk\to i}(v) \geq \varepsilon$. Observe that the number of edges of the (i,j,k)-triad that contain a particular vertex $v \in V_{ij} \setminus L$ is at most $\varepsilon \cdot |V_{ik}| \cdot |V_{jk}|$, the number of edges that contain a particular vertex $v \in V_{ik} \setminus T$ is at most $\varepsilon \cdot |V_{ij}| \cdot |V_{jk}|$, and the number of edges that contain a particular vertex $v \in V_{jk} \setminus R$ is at most $\varepsilon \cdot |V_{ij}| \cdot |V_{ik}|$. Hence, the (i,j,k)-triad has at most $3\varepsilon \cdot |V_{ij}| \cdot |V_{jk}| \cdot |V_{ik}|$ edges in addition to the edges with a vertex from L, a vertex from T and a vertex from R. The number of the edges of the latter type is at most $|L| \cdot |T| \cdot |R|$. We derive using $\ell + t + r < 1 + \varepsilon_0$ that the density of the (i,j,k)-triad is at most

$$(\ell + \varepsilon_0)(t + \varepsilon_0)(r + \varepsilon_0) + 3\varepsilon < \left(\frac{1 + 4\varepsilon_0}{3}\right)^3 + 3\varepsilon < \frac{1}{27} + 3\varepsilon_0,$$

which contradicts that the density of the (i,j,k)-triad is at least $\frac{1}{27} + \delta$. ∎

We next construct an auxiliary 3-edge-coloring of the complete 5-uniform hypergraph with vertex set $[N_0]$. Consider $1 \leq k < i < k' < j < k'' \leq N_0$ and let R be the set of vertices v of V_{ij} such that $d_{ij\to k}(v) \geq \varepsilon$, T the set of vertices v of V_{ij} such that $d_{ij\to k'}(v) \geq \varepsilon$, and L the set of vertices v of V_{ij} such that $d_{ij\to k''}(v) \geq \varepsilon$. If $|R \cap L| \geq \frac{\varepsilon_0}{3}|V_{ij}|$, we color the edge $\{k, i, k', j, k''\}$ with the color red; otherwise, if $|L \cap T| \geq \frac{\varepsilon_0}{3}|V_{ij}|$, we color the edge $\{k, i, k', j, k''\}$ with the color green; otherwise, if $|R \cap T| \geq \frac{\varepsilon_0}{3}|V_{ij}|$, we color the edge $\{k, i, k', j, k''\}$ with the color blue. If neither of the three cases applied, it would hold that each of the sets $R \cap T$, $R \cap L$ and $L \cap T$ has fewer than $\frac{\varepsilon_0}{3}|V_{ij}|$ vertices; this would imply that

$$|L \cup T \cup R| \geq |R| + |L| + |T| - |R \cap T| - |R \cap L| - |L \cap T| > (\ell + t + r - \varepsilon_0)|V_{ij}| \geq |V_{ij}|,$$

which is impossible since $L \cup T \cup R$ is a subset of V_{ij}. Hence, one of the three cases always applies and so each edge gets a color. Theorem 5 yields that there exists a subset $I_0 \subseteq [N_0]$ of size $2n + 1$ such that all edges with vertices from I_0 have the same color.

Let $I_0 = \{a_1, a_2, a_3, \ldots, a_{2n+1}\}$ and let $I = \{b_1, \ldots, b_n\}$ where $b_i = a_{2i}$ for $i = 1, \ldots, n$. We define the n-partitioned hypergraph H' as the subhypergraph of H_0 induced by I, where the vertex set V_{ij} of H' is identified with the vertex set $V_{b_ib_j}$ of H_0. We claim that the n-partitioned hypergraph H' has one of the three properties described in the statement of the lemma. We distinguish three cases based on the common color of the edges of the complete 5-uniform hypergraph induced by I_0. If the common color is red, we will show that the first property holds, i.e., we obtain the case of the horizontal intersection. Indeed, for any integers $1 \le k' < i < j < k \le n$, consider $\{a_{2k'}, a_{2i}, a_{2i+1}, a_{2j}, a_{2k}\}$ and the sets L and R from the definition of the color of this edge. Observe that the set $L \cap R$ contains vertices v such that $d_{ij\to k'}(v) \ge \varepsilon$ and $d_{ij\to k}(v) \ge \varepsilon$ in the n-partitioned hypergraph H', which are vertices v such that $d_{a_{2i}a_{2j}\to a_{2k'}}(v) \ge \varepsilon$ and $d_{a_{2i}a_{2j}\to a_{2k}}(v) \ge \varepsilon$ in H.

If the common color is green, we will show that the second property holds. Indeed, for any integers $1 \le i < k' < j < k \le n$, consider the edge $\{a_{2i-1}, a_{2i}, a_{2k'}, a_{2j}, a_{2k}\}$. The sets T and L from the definition of the color of the edge have the property that the set $L \cap T$ contains vertices v such that $d_{ij\to k'}(v) \ge \varepsilon$ and $d_{ij\to k}(v) \ge \varepsilon$ in H'. Finally, if the common color is blue, we conclude using an argument analogous to the just analyzed case that H' has the third property given in the statement of the lemma. ∎

4. Embedding lemma

In this section, we prove Lemma 14 which asserts that every partitioned hypergraph with density larger than 1/27 contains one of two specific general substructures that can be used to embed our considered hypergraphs. We remark that Lemmas 8–10 are implicitly contained in [18] where they were proven using an iterative approach; we prove them using Ramsey type arguments and extend them to a more general setting (Lemmas 11 and 12) which is needed to deal with two possible outcomes of Lemma 7.

We start with stating and proving Lemma 8.

Lemma 8: *For every n and $\varepsilon > 0$, there exists N such that the following holds. If H is an N-partitioned hypergraph and for each $1 \le i < j < k \le N$ a subset S_{ijk} of V_{ij} with at least $\varepsilon|V_{ij}|$ vertices is given, then there exist a subset $I \subseteq [N]$ of size n and vertices s_{ij}, $i, j \in I$, $i < j$, such that $s_{ij} \in S_{ijk}$ for all $i, j, k \in I$, $i < j < k$.*

Proof. Apply Theorem 5 with $k_R = 2$, $r_R = n$ and $n_R = \max\{2n, 2+\lceil n/\varepsilon\rceil\}$ to get N (the variables on the left of the equalities are named as in the statement of Theorem 5 with the subscript R added to distinguish them from the variables in the statement of the lemma). Let H be an N-partitioned hypergraph and sets $S_{ijk} \subseteq V_{ij}$ be as described in the statement of the lemma. We construct an auxiliary 2-edge-coloring of the complete n-uniform hypergraph on the vertex set $[N]$ as follows: an n-tuple $a_1 < a_2 < \cdots < a_n$ is colored blue if the $n-2$ sets $S_{a_1a_2a_3}, S_{a_1a_2a_4}, \ldots, S_{a_1a_2a_n}$ have a common vertex, and it is colored red otherwise. By Theorem 5, there exist $a_1, \ldots, a_{n_R} \in [N]$, $a_1 < a_2 < \cdots < a_{n_R}$, such that all n-tuples of $a_1, \ldots, a_{n_R}$ have the same color. We next distinguish two cases depending on the common color of those n-tuples.

If the common color of the n-tuples is blue, then we set $I = \{a_1, \ldots, a_n\}$ and let $s_{a_ia_j}$ for $1 \le i < j \le n$ be any element contained in the intersection of the sets $S_{a_ia_ja_{j+1}}, S_{a_ia_ja_{j+2}}, \ldots, S_{a_ia_ja_n}$.

Suppose that the common color for the n-tuples is red. Since each of the sets $S_{a_1a_2a_\ell}$ for $\ell = 3, \ldots, 2+\lceil n/\varepsilon\rceil$ contains at least $\varepsilon|V_{a_1a_2}|$ elements of $V_{a_1a_2}$ and the number of choices for ℓ is $\lceil n/\varepsilon\rceil$, there exist an element $s \in V_{a_1a_2}$ and $J \subseteq \{a_3, \ldots, a_{2+\lceil n/\varepsilon\rceil}\}$, $|J| \ge n$, such that $s \in S_{a_1a_2a}$ for every $a \in J$. This implies that the n-tuple formed by a_1, a_2 and any $n-2$ elements of J should be blue, which contradicts that the common color for the n-tuples formed by $a_1, \ldots, a_{n_R}$ is red. ∎

The first of the next two lemmas follows from Lemma 8 by applying it to the reverse of H, however, for later use it is beneficial to state it explicitly; the proof of the second lemma follows along the lines of Lemma 8 and we only include its sketch for completeness.

Lemma 9: *For every n and $\varepsilon > 0$, there exists N such that the following holds. If H is an N-partitioned hypergraph and for each $1 \le i < j < k \le N$ a subset S_{ijk} of V_{jk} with at least $\varepsilon|V_{jk}|$ vertices is given, then there exist a subset $I \subseteq [N]$ of size n and vertices s_{jk}, $j,k \in I$, $j < k$, such that $s_{jk} \in S_{ijk}$ for all $i,j,k \in I$, $i < j < k$.*

Lemma 10: *For every n and $\varepsilon > 0$, there exists N such that the following holds. If H is an N-partitioned hypergraph and for each $1 \le i < j < k \le N$ a subset S_{ijk} of V_{ik} with at least $\varepsilon|V_{ik}|$ vertices is given, then there exist a subset $I \subseteq [N]$ of size n and vertices s_{ik}, $i,k \in I$, $i < k$, such that $s_{ik} \in S_{ijk}$ for all $i,j,k \in I$, $i < j < k$.*

Proof. As we have mentioned, we only sketch the proof as it follows the lines of the proof of Lemma 8. We first apply Theorem 5 with $k_R = 2$, $r_R = n$ and $n_R = \max\{n^2, 2 + \lceil n/\varepsilon \rceil\}$ to get N. Suppose that an N-partitioned hypergraph H and sets $S_{ijk} \subseteq V_{ik}$ are given. We construct an auxiliary 2-edge-coloring of the complete n-uniform hypergraph on the vertex set $[N]$ as follows: an n-tuple $a_1 < a_2 < \cdots < a_n$ is colored blue if the $n-2$ sets $S_{a_1 a_2 a_n}, S_{a_1 a_3 a_n}, \ldots, S_{a_1 a_{n-1} a_n}$ have a common vertex, and it is colored red otherwise. By Theorem 5, there exist $a_1, \ldots, a_{n_R} \in [N]$, $a_1 < a_2 < \cdots < a_{n_R}$, such that all n-tuples of $a_1, \ldots, a_{n_R}$ have the same color. If the common color is blue, we set $I = \{a_1, a_{n+1}, \ldots, a_{n^2-n+1}\}$. If the common color is red, we argue as in the proof of Lemma 8 that there is an element $s \in V_{a_1 a_{\lceil n/\varepsilon \rceil+2}}$ contained in at least n sets $S_{a_1 a_\ell a_{\lceil n/\varepsilon \rceil+2}}$ where ℓ ranges between 2 and $\lceil n/\varepsilon \rceil + 1$. Hence, the n-tuple formed by a_1, $a_{\lceil n/\varepsilon \rceil+2}$ and $n-2$ choices of ℓ with this property should be blue, which contradicts that the common color of the n-tuples formed by $a_1, \ldots, a_{n_R}$ is red. ∎

We now extend Lemmas 8–10 to the setting needed to prove Lemma 14.

LEMMA 11: *For every n and $\varepsilon > 0$, there exists N such that the following holds. If H is an N-partitioned hypergraph and for each $1 \le i < j < k \le N$ subsets S_{ijk} of V_{ij} and S'_{ijk} of V_{jk} are given such that the intersection $S'_{k'ij} \cap S_{ijk}$ has at least $\varepsilon|V_{ij}|$ elements for all $1 \le k' < i < j < k \le N$, then there exists a subset $I \subseteq [N]$ of size n and vertices s_{ij}, $i, j \in I$, $i < j$, such that $s_{ij} \in S'_{k'ij}$ and $s_{ij} \in S_{ijk}$ for all $k' < i < j < k$ with $i, j, k, k' \in I$.*

Proof. Apply Theorem 5 with $k_R = 2$, $r_R = 2n-2$ and $n_R = \max\{3n, 2 + 2\lceil n/\varepsilon \rceil\}$ to get N. Let H be an N-partitioned hypergraph and sets $S_{ijk}, S'_{ijk} \subseteq V_{ij}$ as described in the statement of the lemma. We construct an auxiliary 2-edge-coloring of the complete $(2n-2)$-uniform hypergraph on the vertex set $[N]$ as follows: an $(2n-2)$-tuple $a_1 < a_2 < \cdots < a_{2n-2}$ is colored blue if the $n-2$ sets $S'_{a_1 a_{n-1} a_n}, S'_{a_2 a_{n-1} a_n}, \ldots, S'_{a_{n-2} a_{n-1} a_n}$ and the $n-2$ sets $S_{a_{n-1} a_n a_{n+1}}, S_{a_{n-1} a_n a_{n+2}}, \ldots, S_{a_{n-1} a_n a_{2n-2}}$ have a common vertex, and it is colored red otherwise. By Theorem 5, there exist $a_1, \ldots, a_{n_R} \in [N]$, $a_1 < a_2 < \cdots < a_{n_R}$, such that all $(2n-2)$-tuples of $a_1, \ldots, a_{n_R}$ have the same color. We next distinguish two cases depending on the common color of these $(2n-2)$-tuples.

If the common color of the $(2n-2)$-tuples is blue, we set $I = \{a_{n-1}, \ldots, a_{2n-2}\}$ and let $s_{a_i a_j}$ for $n-1 \le i < j \le 2n-2$ be any element contained in the intersection of the sets $S'_{a_{i-(n-2)} a_i a_j}, \ldots, S'_{a_{i-1} a_i a_j}$ and $S_{a_i a_j a_{j+1}}, \ldots, S_{a_i a_j a_{j+n-2}}$.

Suppose that the common color for the $(2n-2)$-tuples is red. Since each of the $m := \lceil n/\varepsilon \rceil$ many sets $S'_{a_\ell a_{m+1} a_{m+2}} \cap S_{a_{m+1} a_{m+2} a_{m+2+\ell}}$ for $\ell = 1, \ldots, m$ contains at least $\varepsilon |V_{a_{m+1} a_{m+2}}|$ elements of $V_{a_{m+1} a_{m+2}}$, there exist an element $s \in V_{a_{m+1} a_{m+2}}$ and $J \subseteq \{1, \ldots, m\}$, $|J| = n-2$, such that

$$s \in S'_{a_\ell a_{m+1} a_{m+2}} \cap S_{a_{m+1} a_{m+2} a_{m+2+\ell}}$$

for every $\ell \in J$, i.e., $s \in S'_{a_\ell a_{m+1} a_{m+2}}$ and $s \in S_{a_{m+1} a_{m+2} a_{m+2+\ell}}$ for every $\ell \in J$. It follows that the $(2n-2)$-tuple formed by the indices a_{m+1}, a_{m+2}, a_ℓ and $a_{m+2+\ell}$, $\ell \in J$, should be colored blue. This contradicts that the common color for the $(2n-2)$-tuples formed by elements of I is red. ∎

The proof of the next lemma follows along the lines of the proof of Lemma 11 but since it is not completely analogous, we decided to include its sketch for completeness.

Lemma 12: *For every n and $\varepsilon > 0$, there exists N such that the following holds. If H is an N-partitioned hypergraph and for each $1 \le i < j < k \le N$ subsets S_{ijk} of V_{ij} and S'_{ijk} of V_{ik} are given such that the intersection $S'_{ik'j} \cap S_{ijk}$ has at least $\varepsilon |V_{ij}|$ elements for all $1 \le i < k' < j < k \le N$, then there exists a subset $I \subseteq [N]$ of size n and vertices s_{ij}, $i, j \in I$, $i < j$, such that $s_{ij} \in S'_{ik'j}$ and $s_{ij} \in S_{ijk}$ for all $i < k' < j < k$ with $i, j, k, k' \in I$.*

Proof. First apply Theorem 5 with $k_R = 2$, $r_R = 2n-2$ and $n_R = \max\{n^2, 2 + 2\lceil n/\varepsilon \rceil\}$ to get N. Consider an N-partitioned hypergraph H and sets S_{ijk} and S'_{ijk} as given in the statement. We construct an auxiliary 2-edge-coloring of the complete $(2n-2)$-uniform hypergraph on the vertex set $[N]$ as follows: an $(2n-2)$-tuple $a_1 < a_2 < \cdots < a_{2n-2}$ is colored blue if the $n-2$ sets $S'_{a_1 a_2 a_n}, S'_{a_1 a_3 a_n}, \ldots, S'_{a_1 a_{n-1} a_n}$ and the $n-2$ sets $S_{a_{n-1} a_n a_{n+1}}, S_{a_{n-1} a_n a_{n+2}}, \ldots, S_{a_{n-1} a_n a_{2n-2}}$ have a common vertex, and it is colored red otherwise. By Theorem 5, we get $a_1, \ldots, a_{n_R} \in [N]$, $a_1 < a_2 < \cdots < a_{n_R}$, such that all $(2n-2)$-tuples of $a_1, \ldots, a_{n_R}$ have the same color. If the common color of the $(2n-2)$-tuples is blue, we set

$$I = \{a_1, a_{n+1}, \ldots, a_{n^2-n+1}\};$$

the existence of s_{ij} follows as all $(2n-2)$-tuples are blue. Suppose that the common color for the $(2n-2)$-tuples is red. Similarly to the proof of Lemma 11, we consider intersections $S'_{a_1 a_{1+\ell} a_{\lceil n/\varepsilon\rceil+2}} \cap S_{a_1 a_{\lceil n/\varepsilon\rceil+2} a_{\lceil n/\varepsilon\rceil+\ell+2}}$ where ℓ ranges between 1 and $\lceil n/\varepsilon\rceil$ and argue that there exist n of these intersections that have a vertex in common; this implies that one of $(2n-2)$-tuples should be blue. ∎

To prove Lemma 14, we need an additional auxiliary lemma.

Lemma 13: *The following holds for every tripartite hypergraph G with parts A, B and C and every $\varepsilon > 0$: If a vertex a of A is contained in at least $\varepsilon|B|\cdot|C|$ edges of G, then there exist at least $\varepsilon|B|/2$ vertices b of B such that a and b are contained together in at least $\varepsilon|C|/2$ edges of G.*

Proof. Let $B' \subseteq B$ be the subset of vertices b of B which are contained together with a in at least $\varepsilon|C|/2$ edges of G. If $|B'| < \varepsilon|B|/2$, then there are less than $\varepsilon|B|\cdot|C|/2$ edges containing the vertex a and a vertex $b \in B'$. Since any vertex $b \in B\setminus B'$ is contained together with a in less than $\varepsilon|C|/2$ edges of G, the number of edges containing the vertex a is less than $\varepsilon|B|\cdot|C|$, which contradicts the assumption of the lemma. ∎

We are now ready to prove the embedding lemma, which is the main result of this section. The lemma will be used to upper bound the uniform Turán density of hypergraphs constructed in the next section.

Lemma 14: *For every $\delta > 0$ and $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that the following holds. For every N-partitioned hypergraph H with density at least $1/27+\delta$, there exists an n-partitioned induced subhypergraph H' of H and vertices $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \beta'_{ij}, \gamma'_{ij} \in V_{ij}$ for all $1 \le i < j \le n$ such that $\{\alpha_{ij}, \beta_{jk}, \gamma_{ik}\}$ is an edge of H' for all $1 \le i < j < k \le n$ and at least one of the following holds:*

- *For all $1 \le i < j < k \le n$, $\{\beta_{ij}, \beta'_{jk}, \gamma'_{ik}\}$ is an edge of H'.*
- *For all $1 \le i < j < k \le n$, $\{\gamma_{ij}, \beta'_{jk}, \gamma'_{ik}\}$ is an edge of H'.*
- *For all $1 \le i < j < k \le n$, $\{\beta'_{ij}, \gamma_{jk}, \gamma'_{ik}\}$ is an edge of H'.*

Proof. Fix $\delta > 0$ and $n \in \mathbb{N}$. Apply Lemma 7 with δ to get $\varepsilon > 0$. Apply Lemma 8 with n and $\varepsilon/2$ to get n_1, then Lemma 10 with n_1 and $\varepsilon/2$ to get n_2, then Lemma 9 with n_2 and $\varepsilon/2$ to get n_3, then Lemma 10 with n_3 and $\varepsilon/2$ to get n_4, and finally Lemma 11 with n_4+2 and ε to get N_h. Next apply Lemma 9 with n_1 and $\varepsilon/2$ to get n_2', then Lemma 9 again with n_2' and $\varepsilon/2$ to get n_3', then Lemma 10 with n_3' and $\varepsilon/2$ to get n_4', and finally Lemma 12 with $n_4'+2$ and ε to get N_v. We obtain N by applying Lemma 7 with $\max\{N_h, N_v\}$ (with δ and ε as fixed earlier).

Let H be an N-partitioned hypergraph with density at least $1/27+\delta$. By Lemma 7, there exists a $\max\{N_h, N_v\}$-partitioned induced subhypergraph H^5 of H that satisfies one of the three properties given in the statement of Lemma 7. We start with analyzing the case that the first property holds, i.e., the case of horizontal intersection; this case results in the first case described in the statement of the lemma. The sought hypergraph H' and the vertices $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \beta'_{ij}, \gamma'_{ij}$ are obtained as follows:

- For $1 \le k' < i < j < k \le N_h$, let $S'_{k'ij}$ be the set of the vertices $v \in V_{ij}$ such that $d_{ij\to k'}(v) \ge \varepsilon$ and let S_{ijk} be the set of the vertices $v \in V_{ij}$ such that $d_{ij\to k}(v) \ge \varepsilon$. By assumption we are in the horizontal intersection case and the first outcome of Lemma 7 applies and therefore

$$|S'_{k'ij} \cap S_{ijk}| \ge \varepsilon |V_{ij}|$$

for all $1 \le k' < i < j < k \le N_h$. Hence, Lemma 11 yields that there exists an (n_4+2)-partitioned induced subhypergraph of H^5 and vertices β_{ij} such that $d_{ij\to k'}(\beta_{ij}) \ge \varepsilon$ and $d_{ij\to k}(\beta_{ij}) \ge \varepsilon$ for all $1 \le k' < i < j < k \le n_4+2$; removing the first and the last part yields an n_4-partitioned induced subhypergraph H^4 of H^5 such that

$$d_{jk\to i}(\beta_{jk}) \ge \varepsilon \quad \text{and} \quad d_{ij\to k}(\beta_{ij}) \ge \varepsilon$$

for all $1 \le i < j < k \le n_4$.
- For $1 \le i < j < k \le n_4$, let S_{ijk} be the set of vertices $v \in V_{ik}$ such that $d_{ij,ik}(\beta_{ij}, v) \ge \varepsilon/2$; observe that each of the sets S_{ijk} contains at least $\varepsilon|V_{ik}|/2$ elements by Lemma 13. Lemma 10 yields that there exists an n_3-partitioned induced subhypergraph H^3 of H^4 and vertices γ'_{ik} such that $d_{ij,ik}(\beta_{ij}, \gamma'_{ik}) \ge \varepsilon/2$ for all $1 \le i < j < k \le n_3$.
- For $1 \le i < j < k \le n_3$, let S_{ijk} be the set of vertices $v \in V_{jk}$ that form an edge together with β_{ij} and γ'_{ik} in the (i,j,k)-triad of H^3, and apply Lemma 9 to get an n_2-partitioned induced subhypergraph H^2 of H^3 and vertices β'_{jk} such that $\{\beta_{ij}, \beta'_{jk}, \gamma'_{ik}\}$ is an edge of H^2 for all $1 \le i < j < k \le n_2$.

- For $1 \le i < j < k \le n_2$, let S_{ijk} be the set of vertices $v \in V_{ik}$ such that $d_{jk,ik}(\beta_{jk}, v) \ge \varepsilon/2$; observe that each of the sets S_{ijk} contains at least $\varepsilon|V_{ik}|/2$ elements by Lemma 13. So, Lemma 10 yields that there exists an n_1-partitioned induced subhypergraph H^1 of H^2 and vertices γ_{ik} such that $d_{jk,ik}(\beta_{jk}, \gamma_{ik}) \ge \varepsilon/2$ for all $1 \le i < j < k \le n_1$.
- For $1 \le i < j < k \le n_1$, let S_{ijk} be the set of vertices $v \in V_{ij}$ that form an edge with β_{jk} and γ_{ik} in H^1. Lemma 8 yields that there exists an n-partitioned induced subhypergraph H' of H^1 and vertices α_{ij} such that $\{\alpha_{ij}, \beta_{jk}, \gamma_{ik}\}$ is an edge of H' for $1 \le i < j < k \le n$.

The hypergraph H' together with the vertices $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \beta'_{ij}, \gamma'_{ij}$ satisfies the first case of the lemma.

The case of vertical intersection from Lemma 7 is analyzed in an analogous way. We next sketch the steps resulting in the sought hypergraph H' and the vertices $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \beta'_{ij}, \gamma'_{ij}$ if the second property in the statement of Lemma 7 applies.

- Lemma 12 is used to obtain an $(n'_4 + 2)$-partitioned induced subhypergraph of H^5 and vertices γ_{ij} such that $d_{ij\to k}(\gamma_{ij}) \ge \varepsilon$ and $d_{ij\to k'}(\gamma_{ij}) \ge \varepsilon$ for all $1 \le i < k' < j < k \le n_4 + 2$; removing the first and the last part yields an n'_4-partitioned induced subhypergraph H^4 of H^5.
- Lemma 10 is used to obtain an n'_3-partitioned induced subhypergraph of H^3 of H^4 and vertices γ'_{ik} such that $d_{ij,ik}(\gamma_{ij}, \gamma'_{ik}) \ge \varepsilon/2$ for all $1 \le i < j < k \le n_3$.
- Lemma 9 is used to obtain an n'_2-partitioned induced subhypergraph of H^2 of H^3 and vertices β'_{jk} such that $\{\gamma_{ij}, \beta'_{jk}, \gamma'_{ik}\}$ is an edge of H^2.
- Lemma 9 is used to obtain an n'_1-partitioned induced subhypergraph of H^1 of H^2 and vertices β_{jk} such that $d_{jk,ik}(\beta_{jk}, \gamma_{ik}) \ge \varepsilon/2$ for all $1 \le i < j < k \le n_1$.
- Finally, Lemma 8 is used to obtain an n-partitioned induced subhypergraph of H' of H^1 and vertices α_{ij} such that $\{\alpha_{ij}, \beta_{jk}, \gamma_{ik}\}$ is an edge of H' for $1 \le i < j < k \le n$.

The obtained hypergraph H' together with the vertices $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \beta'_{ij}, \gamma'_{ij}$ satisfies the second case of the lemma. The case of the third property in the statement of Lemma 7 applies is completely symmetric and yields the third case of the lemma. ∎

5. Main theorem and examples

We are now ready to prove our main theorem. This is done by transferring our result about n-partitioned hypergraphs contained in Lemma 14 back to the original setting of uniformly dense hypergraphs by using Proposition 4. The second and third properties in the statement of the theorem correspond to the cases of horizontal and vertical intersection, respectively, as described in Lemmas 7 and 14; note that the second and third cases in the two lemmas are symmetric (by reversing the order of the parts) and so are associated with the case of vertical intersection. The definition of a vanishing ordering can be found in Section 1.

Theorem 15: *Let H_0 be an n-vertex 3-graph that*

- *has no vanishing ordering of its vertices,*
- *can be partitioned into two spanning subhypergraphs H_1 and H_2 such that there exists an ordering of the vertices that is vanishing for both H_1 and H_2 and if e_1 is an edge of H_1 and e_2 is an edge of H_2 such that $|e_1 \cap e_2| = 2$, then the pair $e_1 \cap e_2$ is right with respect to H_1 and left with respect to H_2, and*
- *can be partitioned into two spanning subhypergraphs H_1' and H_2' such that there exists an ordering of the vertices that is vanishing for both H_1' and H_2' and if e_1 is an edge of H_1' and e_2 is an edge of H_2' such that $|e_1 \cap e_2| = 2$, then the pair $e_1 \cap e_2$ is top with respect to H_1' and left with respect to H_2'.*

The uniform Turán density of H_0 is equal to $1/27$.

Proof. Fix an n-vertex 3-graph H_0 with the properties given in the statement of the lemma. Since H_0 has no vanishing ordering, its uniform Turán density is at least $1/27$ by Theorem 1 and Corollary 2. By Proposition 4, we need to show that for every $\delta > 0$, there exists N such that every N-partitioned hypergraph with density at least $1/27 + \delta$ embeds H_0. Apply Lemma 14 with n and δ to get N. Let H be an N-partitioned hypergraph with density at least $1/27 + \delta$ and let H' be an n-partitioned induced subhypergraph of H with one of the three properties given in Lemma 14. In each of the three cases given by which of the three properties holds, we use a partition given in the second or in the third case of the statement of the theorem, to embed H_1 using the edges formed by the vertices α_{ij}, β_{ij} and γ_{ij}, and to embed H_2 using the edges formed by the vertices β'_{ij}, γ'_{ij} and either β_{ij} or γ_{ij}.

If the first property given in Lemma 14 holds, we consider an ordering of the vertices of H_0 as described in the second bullet point in the statement of the theorem and choose α_{ij} for every pair i, j that is left with respect to H_1, choose β_{ij} for every pair i, j that is right with respect to H_1 or left with respect to H_2, choose γ_{ij} for every pair i, j that is top with respect to H_1, choose β'_{ij} for every pair i, j that is right with respect to H_2, and choose γ'_{ij} for every pair i, j that is top with respect to H_2. Hence, H' embeds H_0.

If the second property given in Lemma 14 holds, we consider an ordering of the vertices of H_0 as described in the third bullet point in the statement of the theorem and choose α_{ij} for every pair i, j that is left with respect to H_1, choose β_{ij} for every pair i, j that is right with respect to H_1, choose γ_{ij} for every pair i, j that is top with respect to H_1 or left with respect to H_2, choose β'_{ij} for every pair i, j that is right with respect to H_2, and choose γ'_{ij} for every pair i, j that is top with respect to H_2. Again, we conclude that H' embeds H_0.

The third property given in Lemma 14 is symmetric to the second (by reversing the order of the parts of H'), and the arguments as in the previous paragraph yield that H' embeds H_0. Since H' embeds H_0 regardless which of the three properties given in Lemma 14 holds, the uniform Turán density of H_0 is at most $1/27$ by Proposition 4. ∎

We next give examples of 3-graphs that satisfy the assumption of Theorem 15 and so their uniform Turán density is equal to $1/27$. We start with introducing a lemma, which will be useful to rule out the existence of a vanishing ordering of vertices of a 3-graph. We say that a directed graph is **simple** if every pair of its vertices is joined by at most one edge, i.e., there are no parallel edges or pairs of edges oriented in the opposite way.

Lemma 16: *A 3-graph H has a vanishing ordering if and only if there exists a simple directed graph G with the same vertex set as H such that each edge of H corresponds to a cyclically directed triangle with edges colored* 1, 2 *and* 3 *(in this order), and there exist distinct indices i and j, $i, j \in \{1, 2, 3\}$, such that the subgraph of G containing all edges colored with i and j is acyclic.*

Proof. We show that the existence of a directed graph G with properties as given in the statement of the lemma is equivalent to the existence of a vanishing ordering of H. First suppose that there exists a directed graph G with edges colored as described in the statement of the lemma. By symmetry, we may assume that the subgraph of G containing all edges colored with 1 and 2 is acyclic (otherwise, we cyclically rotate the colors to satisfy this). Consider a linear ordering of the vertices of H that is an extension of the partial order given by the existence of a directed path in G. We claim that this linear ordering is vanishing. Indeed, all left edges are colored with 1, all right edges with 2 and all top edges with 3.

Next suppose that there exists a vanishing ordering of the vertices of H and consider the following simple directed graph G: if $\{u, v, w\}$ is an edge of H such that uv is the left pair, vw is the right pair and uw is the top pair, include the edge uv directed from u to v and colored with 1, the edge vw directed from v to w and colored with 2, and the edge uw directed from w to u and colored with 3. The subgraph of G containing all edges colored with 1 and 2 satisfies that every edge is directed from a smaller vertex to a larger vertex, and so the subgraph is acyclic. Hence, G has the properties described in the statement of the lemma. ∎

As the first example of a 3-graph with uniform Turán density equal to $1/27$, we present a 3-graph with seven vertices, which is the smallest possible number of vertices. The 3-graph has a non-trivial group of automorphisms, which correspond to a vertical mirror symmetry in Figure 2 where the 3-graph is visualized.

Theorem 17: *Let H be a 3-graph with seven vertices $a, \ldots, g$ and the following 9 edges: abc, ade, bcd, bcf, cde, def, abg, cdg and efg. The uniform Turán density of H is equal to $1/27$.*

Proof. Consider a directed graph G associated with the 3-graph H as described in the statement of Lemma 16. Note that if we fix an orientation and a coloring for the triple abc, then all the orientations and colors of the graph are fixed. Therefore G is unique up to cyclical shifts of the colors and a swap of all the orientations. Hence it suffices to consider the directed graph G depicted in Figure 2 together with the three subgraphs containing all edges of the colors 1 and 2, all edges of the colors 1 and 3, and all edges of the colors 2 and 3. Since neither of the three subgraphs is acyclic, the 3-graph H has no vanishing ordering by Lemma 16. Hence, the first condition in the statement of Theorem 15 holds.

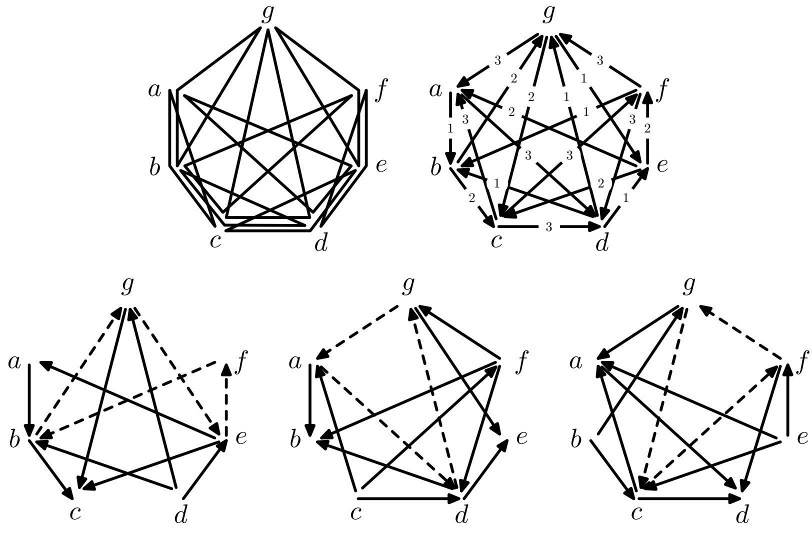

Figure 2. The 3-graph H described in the statement of Theorem 17 (the edges correspond to the drawn triangles), the unique (up to a symmetry) graph G associated with H as described in Lemma 16 and the three subgraphs containing all edges with distinct pairs of colors. Cycles witnessing that neither of the three subgraphs is acyclic are drawn dashed.

We next verify the second and third conditions in the statement of Theorem 15. We set H_1 and H_1' to be the 3-graphs with the same vertex set as H and the edge abg only, and H_2 and H_2' the 3-graphs obtained from H by removing the edge abg. We consider the ordering $egbdfac$ of the vertices of H_2 and the ordering $ebgdfac$ of the vertices of H_2'. The orderings are vanishing with respect to H_2 and H_2', respectively; this can be straightforwardly verified with the aid of Figure 3. Note that ab is the only pair shared by an edge of both H_1 and H_2, as well as the only pair shared by an edge of both H_1' and H_2'. The pair ab is left with respect to both H_2 and H_2'. Since the pair ab is right with respect to H_1 and top with respect to H_1' (the orderings are vanishing with respect to H_1 and H_1' as the 3-graph consists of a single edge), the second and third conditions in the statement of Theorem 15 hold. We conclude that the uniform Turán density of H is equal to $1/27$. ∎

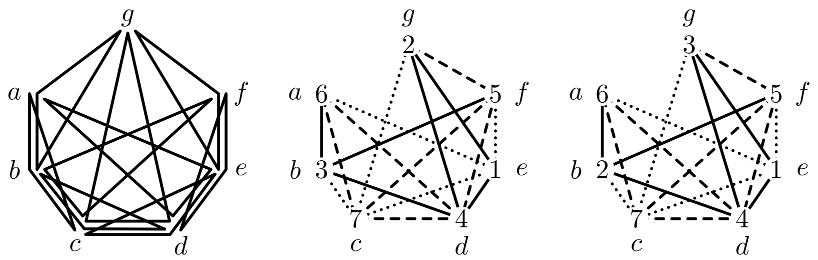

Figure 3. The 3-graph H described in the statement of Theorem 17 and the vanishing orders with respect to 3-graphs H_2 and H_2' as in the statement of Theorem 15; the 3-graphs H_2 and H_2' are obtained by removing the edge abg from H. The left pairs are drawn solid, the right pairs dashed and the top pairs dotted.

We next present an infinite family of 3-graphs with uniform Turán density equal to $1/27$; the smallest 3-graph in the family has eight vertices. The family enjoys three cyclic symmetries (by mapping the vertices c_i, d_i and e_i to each other in a cyclic way).

Theorem 18: *For a positive integer k, let H^k be the 3-graph with $5+3k$ vertices $a, b, c_0, \ldots, c_k, d_0, \ldots, d_k, e_0, \ldots e_k$ and the following $3(k+2)$ edges:*

$$abc_0,\ bc_0c_1,\ c_0c_1c_2, \ldots, c_{k-2}c_{k-1}c_k,\ c_{k-1}c_kd_k,$$

$$abd_0,\ bd_0d_1,\ d_0d_1d_2, \ldots, d_{k-2}d_{k-1}d_k,\ d_{k-1}d_ke_k,$$

$$abe_0,\ be_0e_1,\ e_0e_1e_2, \ldots, e_{k-2}e_{k-1}e_k,\ e_{k-1}e_kc_k.$$

The uniform Turán density of H^k is equal to $1/27$.

Proof. Fix a positive integer k. We first show using Lemma 16, Theorem 1 and Corollary 2 that the uniform Turán density of H^k is at least $1/27$. Consider the graph G as described in the statement of the lemma. By symmetry, we can assume that the edge ab is oriented from a to b and colored with 1 (see Figure 4 for $k = 1$). So, the edge bx_0 is oriented from b to x_0 and colored with 2 for each $x \in \{c, d, e\}$, and the edge x_0x_1 is oriented from x_0 to x_1 and colored with 3, etc. In particular, the edge $x_{k-1}x_k$ is oriented from x_{k-1} to x_k and colored with $k+2 \pmod 3$. It follows that the edges c_kd_k, d_ke_k and e_kc_k form a cyclically oriented triangle and each of the edges is colored with $k \pmod 3$, and the edges $c_{k-1}c_k$, c_ke_{k-1}, $e_{k-1}e_k$, e_kd_{k-1}, $d_{k-1}d_k$ and d_kc_{k-1} form an oriented cycle with edges colored with $k+2 \pmod 3$ and $k+1 \pmod 3$ in an alternating way. Hence, no pair of edge colors induces an acyclic subgraph, and so the 3-graph H^k has no vanishing ordering of the vertices by Lemma 16. We conclude that the uniform Turán density of H^k is at least $1/27$.

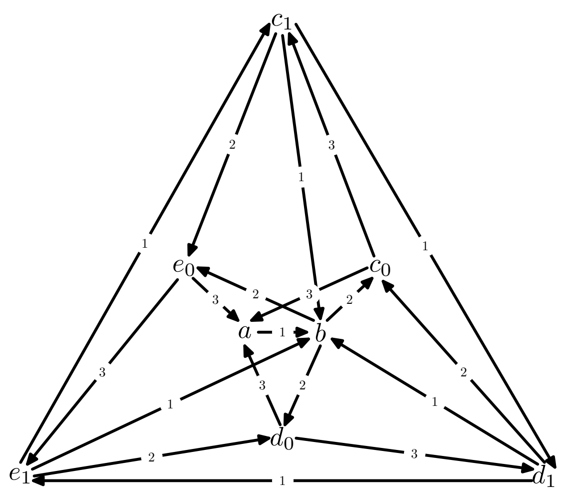

Figure 4. The graph G from the proof of Theorem 18 for $k = 1$.

We next verify the second and third conditions in the statement of Theorem 15. We set H_1 and H_1' to be the 3-graphs with the same vertex set as H^k and the edge $e_{k-1}e_kc_k$ only, and H_2 and H_2' the 3-graphs obtained from H^k by removing the edge $e_{k-1}e_kc_k$. Let A be the set containing all vertices x_i with $i \equiv k-1 \pmod 3$, $x \in \{c, d, e\}$, a if $k \equiv 2 \pmod 3$, and b if $k \equiv 0 \pmod 3$. Let B be the set containing all vertices x_i with $i \equiv k \pmod 3$, $x \in \{c, d, e\}$, except for c_k and d_k, a if $k \equiv 1 \pmod 3$, and b if $k \equiv 2 \pmod 3$. Finally, let C be the set containing all vertices x_i with $i \equiv k+1 \pmod 3$, $x \in \{c, d, e\}$, a if $k \equiv 0 \pmod 3$, and b if $k \equiv 1 \pmod 3$.

First consider any ordering of the vertices of H^k that contains first all vertices of A except for e_{k-1}, then c_k, then e_{k-1}, then d_k, then all vertices of B, and then all vertices of C. Observe that this ordering is a vanishing ordering with respect to H_1' and the pair $e_{k-1}e_k$ is left in this ordering. Indeed, each edge of H_1' except for those containing the vertex c_k or the vertex d_k, i.e., except for $c_{k-2}c_{k-1}c_k$, $d_{k-2}d_{k-1}d_k$, $c_{k-1}c_kd_k$ and $d_{k-1}d_ke_k$, contains exactly one vertex from A, one from B and one from C, and so the pairs involving a vertex from A and a vertex from B except the pair $d_{k-1}e_k$ are left, the pairs involving a vertex from A and a vertex from C are top, and the pairs involving a vertex from B and a vertex from C are right; in addition, the pairs $c_{k-1}c_k$ and $d_{k-1}d_k$ are left, the pairs c_kc_{k-2}, d_kd_{k-2} and d_ke_k are right, and the pairs $c_{k-1}d_k$ and $d_{k-1}e_k$ are top, which is in line with the ordering of the four exceptional edges. Since H_1 contains a single edge (the edge $e_{k-1}e_kc_k$), the ordering is also a vanishing ordering with respect to H_1. Furthermore, $e_{k-1}e_k$ is the only pair shared by H_1 and H_1' and the pair right with respect to this ordering in H_1 and left in H_1'. Hence, the second condition in the statement of Theorem 15 holds.

Next consider any ordering of the vertices of H^k that contains first all vertices of A, then c_k, then d_k, then all vertices of B, and then all vertices of C. Observe that this ordering is a vanishing ordering with respect to H_2' and the pair $e_{k-1}e_k$ is left in this ordering (the argument is analogous to the previous case). Since H_2 contains a single edge, the ordering is also a vanishing ordering with respect to H_2 and the pair $e_{k-1}e_k$ is top. We conclude that the third condition in the statement of Theorem 15 also holds, and so the uniform Turán density of H is equal to $1/27$. ∎

6. Conclusion

The 7-vertex 3-graph with uniform Turán density $1/27$ described in Theorem 17 has the smallest possible number of vertices but it is not the unique 7-vertex 3-graph with uniform Turán density equal to $1/27$. Using a computer, we have generated all minimal 7-vertex 3-graphs with uniform Turán density equal to $1/27$ and we include their list below (three of them have one fewer edge than the 3-graph from Theorem 17, however, they enjoy less symmetries than the presented 3-graph and so we preferred analyzing a more symmetric 3-graph with a larger number of edges). The vertices are denoted by $a, \dots, g$ and each line below is the edge set of one of them; the first line contains the 3-graph described in Theorem 17 (with vertices renamed).

$$abc, abd, abe, acf, acg, bdf, bdg, cef, deg$$
$$abc, abd, abe, acf, acg, bdf, cdg, cef, efg$$
$$abc, abd, abe, acf, adg, bdf, cef, efg$$
$$abc, abd, abe, acf, aeg, bdf, bfg, cde, cdg, cef$$
$$abc, abd, abe, acf, bcg, bdf, cde, ceg, efg$$
$$abc, abd, ace, adg, bcf, bde, bfg, cdf, ceg$$
$$abc, abd, ace, aef, afg, bcf, bde, beg, cdf, cdg$$
$$abc, abd, ace, afg, bcf, bde, bfg, def$$
$$abc, abd, ace, bde, bfg, cdf, ceg, cfg$$

For each of the remaining 15 minimal 7-vertex 3-graphs H with positive uniform Turán density (out of which 6 have isolated vertices), for every $\varepsilon > 0$, there exist arbitrarily large $(4/27, \varepsilon)$-dense 3-graphs that avoid H. Each of these 3-graphs H is avoided by one of the following two constructions of random n-vertex 3-graphs: Order the n vertices randomly and color the pairs of vertices randomly red and blue with probability $2/3$ and $1/3$, respectively. In the first construction, we include an edge if the left and right pairs are red and the top pair is blue, and in the second construction, we include an edge if the left and top pairs are red and the right pair is blue.

Theorem 15 gives a sufficient condition on a 3-graph to have the uniform Turán density equal to $1/27$. We believe that this condition is not necessary, however, we do not have an example of a 3-graph with uniform Turán density $1/27$ that does not satisfy the condition and do not also have a conjecture for a possible classification of 3-graphs with uniform Turán density $1/27$.

Problem 1: Characterize the 3-graphs with uniform Turán density equal to $1/27$.

In view of Corollary 2, it is natural to ask whether a similar phenomenon appears for the uniform Turán density of $1/27$, in particular, all 3-graphs that we know to fail to satisfy the conditions of Theorems 1 and 15 have uniform Turán density at least $4/27$.

Problem 2: Does there exist $\delta > 0$ such that the uniform Turán density of every 3-graph is either at most $1/27$ or at least $1/27 + \delta$?

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