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TOTALLY DISCONNECTED LOCALLY COMPACT GROUPS WITH JUST INFINITE LOCALLY NORMAL SUBGROUPS

BY

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ABSTRACT

We obtain some global features of totally disconnected locally compact (t.d.l.c.) groups G that are locally isomorphic to a just infinite profinite group, building on an earlier result of Barnea–Ershov–Weigel and also using tools developed by P.-E. Caprace, G. Willis and the author for studying local structure in t.d.l.c. groups. The approach uses the following property of just infinite profinite groups, essentially due to Wilson: given a locally normal subgroup K of G, then there is an open subgroup of K that is a direct factor of an open subgroup of G. This is a local property of t.d.l.c. groups and we obtain a characterization of the local isomorphism types of t.d.l.c. groups that have it.

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1. Introduction

1.1. Background. The background and motivation for this article comes primarily from three sources. First, we recall some of the structure theory of just infinite groups; see [9] and [19] for a detailed account.

Definition 1.1: A topological group G is **just infinite** if G is infinite and every nontrivial closed normal subgroup of G has finite index.

A **hereditarily just infinite** (h.j.i.) group is one for which every finite index open subgroup is just infinite.

A **branch group** is a compact or discrete group G acting faithfully by automorphisms on a locally finite rooted tree, with root vertex ε, such that for each sphere of vertices $S_n(\varepsilon)$ around the root, G acts transitively on $S_n(\varepsilon)$ and a finite index open subgroup of G splits as a direct product

$$\prod_{v \in S_n(\varepsilon)} R_v,$$

where R_v fixes all vertices not descended from v.

Theorem 1.1 ([18, Theorem 3], [19, Corollary 4.4]):[1] *Let G be a just infinite profinite or discrete group. Then every closed subnormal subgroup of G is a direct factor of a finite index open subgroup.*

Theorem 1.2 (See [9, Theorem 3]): *Let G be a just infinite profinite or discrete group that is not virtually abelian. Then exactly one of the following holds:*

(i) *There is a closed normal subgroup of G of finite index of the form $L_1 \times \cdots \times L_n$, where the L_i form a single conjugacy class of hereditarily just infinite subgroups of G;*
(ii) *G is a branch group.*

In the present article we will consider a property inspired by J. Wilson's Theorem 1.1, converted into a form expressible in the local structure of totally disconnected, locally compact (t.d.l.c.) groups.

Definition 1.2: Let G be a t.d.l.c. group. Say that G has **property (LD)** if for every closed subgroup K of G with open normalizer, then there is an open subgroup L of K that is a direct factor of an open subgroup of G.

[1] The cited results do not claim the result for a just infinite virtually abelian profinite group G, but this case can be verified by a similar argument to [18, Theorem 3A].

Theorem 1.1 shows that just infinite profinite groups have (LD). In fact, we will obtain a characterization of property (LD) that shows it is closely related to the just infinite property.

The second source of inspiration is the articles [1] and [3] of Barnea–Ershov–Weigel and Caprace–De Medts respectively, which introduce and develop the theory of local isomorphisms of t.d.l.c. groups. This theory is well-behaved for t.d.l.c. groups G where the **quasi-centre** $\mathrm{QZ}(G)$, that is, the group of elements with open centralizer in G, is trivial. In that case, there is a group $\mathscr{L}(G)$, the **group of germs of** G, which contains as an open subgroup every t.d.l.c. group that is locally isomorphic to G and has trivial quasi-centre. In [1], the closed normal subgroup structure of $G = \mathscr{L}(H)$ was considered where H is a h.j.i. profinite group that is not virtually abelian. Write $\mathrm{Res}(G)$ for the intersection of all open normal subgroups of G.

Theorem 1.3 ([1, Proposition 5.1 and Theorem 5.4]):[2] *Let H be a h.j.i. profinite group that is not virtually abelian and let G be a t.d.l.c. group containing H as an open subgroup. Then $\mathrm{QZ}(H) = \{1\}$ and exactly one of the following holds:*

Residually discrete type: $\mathrm{Res}(G/\mathrm{QZ}(G)) = \{1\}$;

Mysterious type: $\mathrm{Res}(G/\mathrm{QZ}(G)) \neq \{1\}$ *but* $\mathrm{Res}(\mathrm{Res}(G/\mathrm{QZ}(G))) = \{1\}$;

Simple type: $\mathrm{Res}(G/\mathrm{QZ}(G))$ *is open and topologically simple.*

In [1], the authors also comment that they are not aware of any groups of germs of h.j.i. profinite groups of mysterious type. As far as I am aware, it is still unknown whether there are t.d.l.c. groups of mysterious type (not just groups of germs) with trivial quasi-centre and a h.j.i. compact open subgroup.

In the present article we continue this analysis to other classes of t.d.l.c. groups G that have strong restrictions on their closed locally normal subgroups, including the case when G is locally isomorphic to a just infinite branch group.

The third source of inspiration is the series of articles [5], [6] and [7] of P.-E. Caprace, G. Willis and the present author, in which further methods were developed for using local properties of t.d.l.c. groups G to obtain restrictions on the global structure (particularly when G is nondiscrete, compactly generated and topologically simple). Particularly relevant here is the notion of the local decomposition lattice of a t.d.l.c. group G, which was also directly inspired by Wilson's approach to just infinite groups.

[2] The theorem was originally stated for $G = \mathscr{L}(H)$, but the argument only uses the more general hypothesis.

Definition 1.3: Let G be a t.d.l.c. group. A subgroup is **locally normal** if it has open normalizer. Two subgroups H and K are **locally equivalent** if $H \cap K$ is open in both H and K. The **structure lattice** $\mathcal{LN}(G)$ of G is the bounded lattice formed by the local equivalence classes of closed locally normal subgroups of G, ordered by inclusion of representatives; write 0 for the least element $[\{1\}]$ and ∞ for the greatest element $[G]$. When $\mathrm{QZ}(G)$ is discrete, the **local decomposition lattice** $\mathcal{LD}(G)$ consists of those elements α of $\mathcal{LN}(G)$ with a complement in $\mathcal{LN}(G)$, that is, $\beta := \alpha^{\perp}$ such that

$$\alpha \vee \beta = \infty \quad \text{and} \quad \alpha \wedge \beta = 0.$$

Note that for a t.d.l.c. group G such that $\mathrm{QZ}(G)$ is discrete, property (LD) is exactly the condition that $\mathcal{LN}(G) = \mathcal{LD}(G)$.

1.2. Local structure of groups with (LD). Our first main result is to characterize property (LD), establishing the connection with just infinite profinite groups.

Theorem 1.4 (See Section 3): *Let G be a t.d.l.c. group. The following are equivalent:*

(i) *G has (LD);*
(ii) *G is locally isomorphic to a profinite group of the form*

$$\prod_{i \in I} L_i,$$

such that finitely many factors L_i (possibly none) are just infinite profinite groups and the remaining factors are finite simple groups.

We also show that in a group with (LD), every closed subnormal subgroup has (LD) and has an open subgroup that is locally normal: see Lemma 3.1.

Given a t.d.l.c. group G with property (LD), then there is a compact open subgroup that decomposes into a direct product of monomial factors plus a leftover factor G_∞, which is either trivial or a direct product of infinitely many finite simple groups, such that each isomorphism type only appears finitely many times. Moreover, this factorization is stable under local isomorphisms. (See Propositions 4.4 and 4.5 for the exact statements.)

1.3. Global structure. For the rest of this introduction we will assume that $\mathrm{QZ}(G) = \{1\}$. In that case we have a decomposition into parts locally isomorphic to just infinite profinite groups, as follows.

Theorem 1.5 (See Theorem 4.6): *Let G be a t.d.l.c. group with property (LD) such that $\mathrm{QZ}(G) = \{1\}$. Then G is first-countable and has an open subgroup of the form*

$$M = M_1 \times M_2 \times \cdots \times M_n,$$

where for $1 \leq i \leq n$ the groups M_i have the following properties:

(i) *M_i is closed and characteristic in every closed locally normal subgroup of G containing M_i.*
(ii) *M_i is locally isomorphic to a just infinite profinite group.*

Thus to a large extent, the structure theory of groups with (LD) and trivial quasi-centre reduces to the case of groups locally isomorphic to a just infinite profinite group. If G is locally isomorphic to a h.j.i. profinite group, then the possibilities for the subnormal subgroup structure of G are accounted for by Theorem 1.3; see also Theorem 4.7. If G is locally isomorphic to a just infinite profinite branch group, we decompose $\mathrm{Res}(G)$ into noncompact, directly indecomposable parts, each of which is either residually discrete or topologically simple; see Theorem 4.11.

We conclude the introduction with a theorem concerning the relationship between topologically simple closed locally normal subgroups of G (which are necessarily nondiscrete and noncompact in the present setting) and contraction groups. In effect, we obtain restrictions on the mysterious type case of Theorem 1.3, and also on the analogous phenomenon for when G is locally isomorphic to a just infinite profinite branch group (that is, the possibility of a nontrivial direct factor of $\mathrm{Res}(G)$ that is residually discrete). A relative version for the action of compactly generated (not necessarily closed) subgroups of G is also given below, see Theorem 4.15.

Definition 1.4: Let G be a t.d.l.c. group. The **contraction group** of $g \in G$ is the group

$$\mathrm{con}(g) := \{x \in G \mid g^n x g^{-n} \to 1 \text{ as } n \to \infty\}.$$

The **Tits core** $G^\dagger$ of G is

$$G^\dagger = \langle \overline{\mathrm{con}(g)} \mid g \in G \rangle.$$

Theorem 1.6 (See Section 4.5): *Let G be a t.d.l.c. group with (LD) such that* $\mathrm{QZ}(G) = \{1\}$. *Then* $\overline{G^\dagger}$ *is a direct factor of* $\mathrm{Res}(G)$ *and is trivial or a direct product of finitely many topologically simple groups. If G is compactly generated, then* $\mathrm{Res}(G)/\overline{G^\dagger} \cong \mathrm{C}_{\mathrm{Res}(G)}(\overline{G^\dagger})$ *is locally isomorphic to a direct product of finitely many h.j.i. profinite groups.*

1.4. Acknowledgements. This article started as an offshoot of the project with Pierre-Emmanuel Caprace and George Willis that led to [5], [6] and [7], and was also partly developed during research visits to Alejandra Garrido and John Wilson at Oxford, and to Yiftach Barnea at RHUL. I thank all of them for their hospitality and for very helpful discussions. I also thank the referee, whose recommendations have led to a number of improvements to the presentation.

2. Preliminaries

2.1. The structure lattice and the decomposition lattice. We begin by setting some terminology.

Definition 2.1: Write $H \leq_o G$ to mean "H is an open subgroup of G". All groups in this article will be locally compact Hausdorff topological groups.

We distinguish between two groups H and K being **commensurable**, meaning there is an isomorphism from a finite index open subgroup of H to a finite index open subgroup of K, and **commensurate**, meaning they are both subgroups of some ambient group, such that $H \cap K$ is open and has finite index in both H and K. An automorphism α of some $G \geq H$ **commensurates** H if $\alpha(H)$ is commensurate with H.

Given t.d.l.c. groups H and K, a **local isomorphism** from H to K is a continuous injective open homomorphism $\theta : U \to K$, where $U \leq_o H$. Two t.d.l.c. groups are **locally isomorphic** if a local isomorphism between them exists.

Let G be a t.d.l.c. group. Given two subgroups H and K of G, we say H is **locally equivalent** to K, and write $H \sim_o K$, if $H \cap K$ is open in both H and K. Write $[H]$ for the local equivalence class of H. Say $H \leq G$ is **locally normal** if $\mathrm{N}_G(H)$ is open in G. The **structure lattice** of G is defined to be the set

$$\mathcal{LN}(G) := \{[H] \mid H \leq G \text{ closed}, \mathrm{N}_G(H) \leq_o G\},$$

partially ordered by setting $[H] \leq [K]$ if $H \cap K$ is open in H. We write 0 for the least element $[\{1\}]$ and ∞ for the greatest element $[G]$.

The **quasi-centre** $\mathrm{QZ}(G)$ of a t.d.l.c. group G is the set of all elements with open centralizer; we say G is **quasi-discrete** if $\mathrm{QZ}(G)$ is dense in G.

More generally, the **quasi-centralizer** $\mathrm{QC}_G(H)$ of $H \leq G$ is the set of all elements of G that centralize an open subgroup of H. Given a local equivalence class α of closed subgroups of G, we also define $\mathrm{QC}_G(\alpha)$ to be $\mathrm{QC}_G(H)$ where $H \in \alpha$ (note that the choice of representative is irrelevant). We also define $\mathrm{C}^2_G(H) := \mathrm{C}_G(\mathrm{C}_G(H))$ and $\mathrm{QC}^2_G(\alpha) := \mathrm{QC}_G(\mathrm{QC}_G(\alpha))$.

In a profinite group, $\mathrm{QZ}(G)$ is just the union of all finite conjugacy classes of G. Let us note two useful consequences of this fact.

Lemma 2.1 (See [11, Theorems 5.1 and 5.2]): *Let G be a profinite group. Then every element of $\mathrm{QZ}(G)$ of finite order is contained in a finite normal subgroup of G. If $\mathrm{QZ}(G)$ is torsion-free, then it is abelian.*

The next lemma is clear from the fact that the conjugation action is continuous; in particular, we see that $\mathrm{QZ}(G)$ contains all discrete locally normal subgroups of a t.d.l.c. group G.

Lemma 2.2: *Let G be a t.d.l.c. group, let $O \leq_o G$ and let $g \in G$. Suppose that the set $\{ogo^{-1} \mid o \in O\}$ is discrete. Then $g \in \mathrm{QZ}(G)$.*

If we rule out quasi-central and abelian locally normal subgroups, we obtain a Boolean algebra canonically associated to the structure lattice.

Definition 2.2: A t.d.l.c. group G is **[A]-semisimple** if $\mathrm{QZ}(G) = \{1\}$ and there are no nontrivial abelian locally normal subgroups of G.

Definition 2.3: Given a bounded lattice $\mathcal{L}$, and $\alpha \in \mathcal{L}$, a **pseudocomplement** of α is an element $\alpha^\perp \in \mathcal{L}$ with the property

$$\forall \alpha, \beta \in \mathcal{L} : \alpha \wedge \beta = 0 \Leftrightarrow \beta \leq \alpha^\perp.$$

Pseudocomplements are unique if they exist, and if every element of $\mathcal{L}$ has a pseudocomplement, we say that $\mathcal{L}$ is **pseudocomplemented**, with associated **pseudocomplement map**

$$\perp : \mathcal{L} \to \mathcal{L}; \quad \alpha \mapsto \alpha^\perp.$$

The pseudocomplement $\alpha^\perp$ to α is a **complement** if in addition $\alpha \vee \alpha^\perp = \infty$; if all elements of $\mathcal{L}$ admit a complement, we say $\mathcal{L}$ is **complemented**. In particular, a **Boolean algebra** is a bounded distributive complemented lattice.

The set of pseudocomplements of any bounded lattice forms a Boolean algebra: see [8].

Theorem 2.3 (See [6, Theorems 3.19 and 5.2]): *Let G be an [A]-semisimple t.d.l.c. group. Then there is a well-defined pseudocomplement map $\perp$ on $\mathcal{LN}(G)$ given by $[K]^\perp = [\mathrm{C}_G(K)]$. Moreover, given closed locally normal subgroups K and L of G, then the following are equivalent:*

(i) *$K \cap L$ is discrete, that is, $[K] \wedge [L] = 0$;*
(ii) *$K \cap L$ is trivial;*
(iii) *$K \leq \mathrm{QC}_G(L)$ and $L \leq \mathrm{QC}_G(K)$;*
(iv) *$[K, L] = \{1\}$.*

Note that, since condition (i) in Theorem 2.3 only depends on the local equivalence classes of K and L, the same is true for the other three conditions.

The image $\perp(\mathcal{LN}(G))$ with induced partial order is the **centralizer lattice** $\mathcal{LC}(G)$. The operation $\perp$ is an involution on $\mathcal{LC}(G)$, that is, $\alpha^{\perp\perp\perp} = \alpha^\perp$ for all $\alpha \in \mathcal{LN}(G)$, and it serves as the complement for $\mathcal{LC}(G)$. The theorem also yields a global version of the centralizer lattice: we have a G-equivariant isomorphism of Boolean algebras

$$\mathcal{LC}(G) \to \mathrm{LC}(G); \quad \alpha \mapsto \mathrm{QC}^2_G(\alpha),$$

where $\mathrm{LC}(G)$ is the set of centralizers of closed locally normal subgroups of G, ordered by inclusion; the group $\mathrm{QC}^2_G(\alpha) = \mathrm{C}_G(\mathrm{QC}_G(\alpha))$ in fact contains every closed locally normal subgroup of G that represents α.

In this article, when we refer to direct products and factors, it is required that the topology is the product topology; in particular, the factors are closed subgroups. In an [A]-semisimple t.d.l.c. group, the elements α of $\mathcal{LN}(G)$ such that $\alpha \vee \alpha^\perp = \infty$ in $\mathcal{LN}(G)$ are exactly the elements represented by direct factors of open subgroups. These form another Boolean algebra $\mathcal{LD}(G)$, the **local decomposition lattice**, which is a sublattice of $\mathcal{LN}(G)$ and a subalgebra of $\mathcal{LC}(G)$. One can also define $\mathrm{LD}(G)$ as the subalgebra of $\mathrm{LC}(G)$ corresponding to $\mathcal{LD}(G)$; one sees that the elements of $\mathrm{LD}(G)$ are all direct factors of open subgroups of G. Moreover, any finite set of disjoint elements of $\mathrm{LD}(G)$ generates its direct product.

Lemma 2.4: *Let G be an [A]-semisimple t.d.l.c. group and let $K_1, \dots, K_n$ be direct factors of open subgroups of G. Then the group K generated by the K_i is closed in G and forms a direct product $K = K_1 \times \cdots \times K_n$.*

Proof. We proceed by induction on n; the case $n = 1$ is clear. Note that by Theorem 2.3, we have $[K_i, K_j] = \{1\}$ for $i \neq j$. Let $L = \langle K_2, \dots, K_n \rangle$; by the inductive hypothesis, L is closed and is the direct product of $K_2, \dots, K_n$. We now form the group $M = K_1 \mathrm{C}_G(K_1)$. Since K_1 is a direct factor of an open subgroup of G, given a compact open subgroup U of G then the product

$$V = (K_1 \cap U)\mathrm{C}_U(K_1)$$

is open in U, and hence M is open in G; consequently M is [A]-semisimple. By compactness, the natural map from the external direct product of $(K_1 \cap U)$ and $\mathrm{C}_U(K_1)$ to V is an isomorphism of topological groups; since V is an identity neighbourhood in M, it follows that as a topological group, M decomposes as a direct product $K_1 \times \mathrm{C}_G(K_1)$. Now $K = K_1 L$ is a subgroup of M, and given the direct decomposition of M, we see that K is closed and decomposes as $K_1 \times L$, and hence as $K_1 \times K_2 \times \cdots \times K_n$, as required. ∎

2.2. The Mel'nikov subgroup and just infinite groups.

Definition 2.4: A topological group H is **finitely decomposable** if $H = H_1 \times \cdots \times H_n$ for some $n \geq 1$, such that each H_i cannot be decomposed further as a direct product. A t.d.l.c. group G is **locally finitely decomposable** if every compact open subgroup of G is finitely decomposable.

Let G be a profinite group. The **Mel'nikov subgroup** $\mathrm{M}(G)$ of G is the intersection of all closed normal subgroups K of G such that G/K is simple. Say G is **Mel'nikov-finite** if $G/\mathrm{M}(G)$ is finite. A t.d.l.c. group L is **locally Mel'nikov-finite** if every compact open subgroup of L is Mel'nikov-finite.

We list here some useful facts about the Mel'nikov subgroup of a profinite group.

Proposition 2.5: *Let G be a profinite group.*

- (i) *We have $\mathrm{M}(G) < G$ unless G is the trivial group. In particular, if G is topologically characteristically simple, then $\mathrm{M}(G) = \{1\}$.*
- (ii) *Given a closed normal subgroup H of G, we have $\mathrm{M}(G/H) = \mathrm{M}(G)H/H$. In particular, $\mathrm{M}(G/\mathrm{M}(G)) = \{1\}$.*
- (iii) *The quotient $G/\mathrm{M}(G)$ is a direct product of finite simple groups. In particular, the torsion elements of $\mathrm{QZ}(G/\mathrm{M}(G))$ form a dense subgroup of $G/\mathrm{M}(G)$.*
- (iv) *If $G/\mathrm{M}(G)$ is finite, then G is finitely decomposable.*

(v) *Let H be a closed normal subgroup of G. Then $G = \mathrm{M}(G)H$ if and only if $G = H$.*

(vi) *Let H be a closed normal subgroup of G. Then $\mathrm{M}(H) \leq \mathrm{M}(G)$. In particular, it follows that G is locally Mel'nikov-finite if and only if there is a set of open normal Mel'nikov-finite subgroups of G forming a basis of identity neighbourhoods.*

(vii) *Define $M_0 = G$ and thereafter $M_{i+1} = \mathrm{M}(M_i)$. Then $\bigcap_{i \in \mathbb{N}} M_i = \{1\}$. Thus if G is locally Mel'nikov-finite, then it is first-countable, and indeed has a countable descending chain of open characteristic subgroups with trivial intersection.*

(viii) *Suppose that G is locally Mel'nikov-finite. Then for each natural number n, there are only finitely many open subgroups of G of index n.*

Proof. For (iii), see for instance [16, Lemma 8.2.2]. The other statements are straightforward exercises given the well-known properties of profinite groups. ∎

Definition 2.5: A profinite group G is **just infinite** if it is infinite, but every nontrivial closed normal subgroup of G is open (in other words, has finite index). It is **hereditarily just infinite** (h.j.i.) if every open subgroup of G is just infinite.

A profinite group G is **nearly just infinite** if G has an open subgroup in common with a just infinite profinite group. (Nearly) hereditarily just infinite groups and (nearly) just infinite branch groups are defined similarly.

There are few possibilities for the quasi-centre and Mel'nikov subgroup of a just infinite profinite group.

Lemma 2.6: *Let G be a just infinite profinite group.*

(i) *The quasi-centre $\mathrm{QZ}(G)$ is either trivial or the direct product of finitely many copies of $\mathbb{Z}_p$ for some fixed prime p; in particular, $\mathrm{QZ}(G)$ is closed.*

(ii) *We have $\mathrm{QZ}(B) \leq \mathrm{QZ}(G)$ for every closed locally normal subgroup B of G.*

(iii) *G is locally Mel'nikov-finite, and hence has only finitely many open subgroups of each index.*

Proof. Since G has no nontrivial finite normal subgroups, Lemma 2.1 implies that $\mathrm{QZ}(G)$ is torsion-free abelian; note also that G has no nontrivial finite locally normal subgroups. The closure Q of $\mathrm{QZ}(G)$ is an abelian closed normal subgroup, which is then either trivial or of finite index; in either case, we see that $Q = \mathrm{QZ}(G)$, so $\mathrm{QZ}(G)$ is closed. Using the just infinite property, it is now straightforward to see that if $Q \neq \{1\}$, then $Q \cong (\mathbb{Z}_p)^n$ for some prime p and natural number n, proving (i).

For (ii), consider first the case that G has no nontrivial abelian locally normal subgroup; we conclude from (i) that G is [A]-semisimple, and by Theorem 2.3 it follows that $\mathrm{QZ}(B) = \{1\}$ for every closed locally normal subgroup. Thus we may assume G has some nontrivial abelian locally normal subgroup C; note that C is then infinite. Let M be an open normal subgroup of G contained in $\mathrm{N}_G(C)$ and let

$$N = \langle g(C \cap M)g^{-1} \mid g \in G \rangle.$$

Then N is nilpotent by Fitting's theorem and is open in G by the just infinite property, so N has a nontrivial central subgroup $\mathrm{Z}(N)$, which is normal in G and contained in Q. Thus $Q > \{1\}$, so by part (i), Q has finite index in G and is torsion-free. Now consider an arbitrary closed locally normal subgroup B of G. Then $\mathrm{QZ}(B) \cap Q$ is torsion-free, from which it follows that the product of all finite normal subgroups of $\mathrm{QZ}(B)$ has order at most $|G : Q| < \infty$; since G has no nontrivial finite locally normal subgroups we deduce that $\mathrm{QZ}(B)$ is torsion-free, hence abelian. We now claim that $\mathrm{QZ}(B)$ commutes with the finite index subgroup $R = \mathrm{N}_Q(B)$ of G, so that $\mathrm{QZ}(B) \leq Q$. Letting $x \in \mathrm{QZ}(B)$ and $k \in \mathbb{N}$, we see that $[x^k, r] = [x, r]^k$ using the fact that $[\mathrm{QZ}(B), R]$ is central in $\mathrm{QZ}(B)$. For some $k > 0$ we have $x^k \in R$, so $[x^k, r] = 1$ and hence $[x, r]^k = 1$. Since $\mathrm{QZ}(B)$ is torsion-free it follows that $[x, r] = 1$, so indeed $\mathrm{QZ}(B) \leq Q$, completing the proof of (ii).

If $Q > \{1\}$ then G is virtually isomorphic to $\mathbb{Z}_p^n$ for some n and p by part (i), and (iii) is clear. Otherwise, for each open normal subgroup N of G, then $\mathrm{QZ}(N) = \{1\}$ by part (ii), and hence $\mathrm{M}(N) > \{1\}$ by Proposition 2.5(iii); by the just infinite property of G it follows that $\mathrm{M}(N)$ is open in N. Part (iii) now follows by parts (vi) and (viii) of Proposition 2.5. ∎

2.3. Contraction in t.d.l.c. groups. We recall some definitions relating to contraction groups in t.d.l.c. groups.

Definition 2.6: The **contraction group** of an automorphism ϕ of a group G is the group

$$\mathrm{con}(\phi) := \{x \in G \mid \phi^n(x) \to 1 \text{ as } n \to \infty\}.$$

Given $g \in G$, we let g act on G by left conjugation. Let G be a t.d.l.c. group and let $H \leq \mathrm{Aut}(G)$. We define the **relative Tits core** $G^\dagger_H$ of H acting on G as

$$G^\dagger_H = \langle \overline{\mathrm{con}(h)} \mid h \in H \rangle.$$

Given a closed subgroup K of G and $H \leq \mathrm{N}_G(K)$, we define $K^\dagger_H$ to be $K^\dagger_{\phi(H)}$, where ϕ is the left conjugation action of H on K. In particular, we define the **Tits core** $G^\dagger := G^\dagger_G$. For a single automorphism α, we write $G^\dagger_\alpha := G^\dagger_{\langle\alpha\rangle}$.

The Tits core is not sensitive to passing to an open subgroup of finite index, as follows from the following basic observation.

LEMMA 2.7: *Let G be a t.d.l.c. group, let $\alpha \in \mathrm{Aut}(G)$ and let n be a positive integer. Then*

$$\mathrm{con}(\alpha) = \mathrm{con}(\alpha^n).$$

Proof. We have $\mathrm{con}(\alpha^n) \geq \mathrm{con}(\alpha)$, since $(\alpha^{ni})_{i\in\mathbb{N}}$ is a subsequence of $(\alpha^i)_{i\in\mathbb{N}}$. Let $H = \overline{\mathrm{con}(\alpha)}$, let $x \in \mathrm{con}(\alpha^n)$ and set $x_i = \alpha^i(x)$. Then the sequence $(x_{ni})_{i\in\mathbb{N}}$ converges to the identity. Since α is a continuous automorphism and $\alpha^j(x_k) = x_{j+k}$ for all $j, k \in \mathbb{Z}$, it follows that the sequence $(x_{j+ni})_{i\in\mathbb{N}}$ converges to $\alpha^j(1) = 1$. Hence $(x_i)_{i\in\mathbb{N}}$ converges to the identity, since it can be partitioned into finitely many subsequences $(x_{j+ni})_{i\in\mathbb{N}}$ for $0 \leq j < n$, each of which converges to the identity. In other words, $x \in \mathrm{con}(\alpha)$. ∎

For individual automorphisms, having trivial relative Tits core is equivalent to having small invariant neighbourhoods, as follows.

LEMMA 2.8: *Let G be a t.d.l.c. group and let α be an automorphism of G. Then $G^\dagger_\alpha = \{1\}$ if and only if the compact open subgroups U of G such that $\alpha(U) = U$ form a base of neighbourhoods of the identity.*

Proof. If $G^\dagger_\alpha = \{1\}$, then by [2, Proposition 3.24], α normalizes a compact open subgroup U, and by [2, Theorem 3.32], U can be made arbitrarily small.[3] Conversely, if the compact open subgroups U of G such that $\alpha(U) = U$ form a base of neighbourhoods of the identity, then clearly $\mathrm{con}(\alpha^n) = \{1\}$ for all $n \in \mathbb{Z}$, so $G^\dagger_\alpha = \{1\}$. ∎

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[3] Some of the results in [2, §3] assume G is metrizable, but the assumption is only used to prove [2, Theorem 3.8]. The latter theorem was later proved without the assumption of metrizability by Jaworski [10], and consequently the subsequent results in [2, §3] are in fact valid for all t.d.l.c. groups.

Given a group G and a subgroup H, write $\mathrm{Res}_G(H)$ for the intersection of all open H-invariant subgroups of H.

Lemma 2.9 (See [15, Theorem B]): *Let G be a t.d.l.c. group and let H be a compactly generated (not necessarily closed) subgroup of G. Then there is a compactly generated open subgroup E of G such that $H \leq E$ and $\mathrm{Res}_G(H) = \mathrm{Res}(E)$.*

3. Groups in which every closed locally normal subgroup is locally a direct factor

3.1. First observations. The goal of this section is to prove Theorem 1.4. We start by establishing some general features of the property (LD).

Lemma 3.1: *Let G be a t.d.l.c. group with (LD) and let K be a closed subnormal subgroup of an open subgroup of G. Then some open subgroup of K is a direct factor of an open subgroup of G; moreover, K has (LD).*

Proof. Let $U \leq_o G$ such that K is subnormal in U and let

$$K = U_0 \trianglelefteq U_1 \trianglelefteq \cdots \trianglelefteq U_n = U$$

be a subnormal series from K to U of shortest possible length. By induction on n, there is an open subgroup V of U_1 that is a direct factor of an open subgroup of U, so there is $M \leq U$ such that $\langle V, M\rangle \cong V \times M \leq_o U$. Now $K \cap V$ is normal in $\langle V, M\rangle$, hence locally normal in G, so by (LD), G has an open subgroup $K_0 \times N$ where $K_0 \leq_o K \cap V$. In particular, K_0 is an open subgroup of K that is a direct factor of an open subgroup of G.

Every locally normal subgroup R of K_0 is also locally normal in G, and so there is $R_2 \leq_o R$ and $L_2 \leq G$ such that $R_2 \times L_2 \leq_o G$. We now see that

$$R_2 \times (L_2 \cap K_0) \leq_o K_0,$$

so R_2 is a direct factor of an open subgroup of K_0. Thus K_0 has (LD); since K_0 is open in K, it follows that K has (LD). ∎

When the quasi-centre is trivial, property (LD) can be characterized in terms of quasi-centralizers.

Lemma 3.2: *Let G be a t.d.l.c. group with (LD) and let K be a closed locally normal subgroup of G such that $\mathrm{QZ}(G) \cap K = \{1\}$. Then $\mathrm{QZ}(K) = \{1\}$.*

Proof. Without loss of generality, K is normal in G. Let K_0 be an open subgroup of K that is a direct factor of an open subgroup of G. Then every element of $\mathrm{QZ}(K_0)$ has open centralizer in G, so we have

$$\mathrm{QZ}(K) \cap K_0 = \mathrm{QZ}(K_0) \leq \mathrm{QZ}(G) \cap K = \{1\}.$$

Thus $\mathrm{QZ}(K)$ is a discrete normal subgroup of G; hence $\mathrm{QZ}(K) \leq \mathrm{QZ}(G)$ by Lemma 2.2, so in fact $\mathrm{QZ}(K) = \{1\}$. ∎

Lemma 3.3: *Let G be a t.d.l.c. group with $\mathrm{QZ}(G) = \{1\}$. The following are equivalent:*

(i) *G has (LD);*

(ii) *G is [A]-semisimple and for every compact locally normal subgroup K of G that is not open, then $\mathrm{QC}_G(K)$ is nondiscrete.*

Proof. Let K be a compact locally normal subgroup of G that is not open.

Suppose G has (LD). By Lemma 3.2, every closed locally normal subgroup of G has trivial quasi-centre, so G is [A]-semisimple. After replacing K with an open subgroup we find that $K \times \mathrm{QC}_G(K)$ is open in G, so certainly $\mathrm{QC}_G(K)$ is not discrete. Thus (ii) holds.

Conversely, suppose (ii) holds. Given a compact open subgroup U containing K, we have a compact locally normal subgroup of the form

$$L = K \times \mathrm{QC}_U(K)$$

Because $[\mathrm{QC}_U(K)]$ is the pseudocomplement of $[K]$ in $\mathcal{LN}(G)$ (see Section 2.1), we see that $[\mathrm{QC}_U(L)] = 0$, so by (ii), L must be open. Thus K is a direct factor of an open subgroup, showing that G has (LD). ∎

We have the following restriction on direct factors of open subgroups.

Lemma 3.4: *Let G be a t.d.l.c. group with (LD) such that $\mathrm{QZ}(G) = \{1\}$. Then $K/\overline{[K,K]}$ and $K/\mathrm{M}(K)$ are both finite for every compact locally normal subgroup K of G. In particular, G is locally finitely decomposable and first-countable.*

Proof. Let K be a compact locally normal subgroup of G. Then K has (LD) by Lemma 3.1 and trivial quasi-centre by Lemma 3.2.

By property (LD), K has an open subgroup of the form $L_1 \times L_2$ where $L_1 \leq_o \overline{[K,K]}$. Since L_1 has finite index in $\overline{[K,K]}$, L_2 is virtually abelian, and indeed L_2 can be chosen to be abelian. But then $L_2 \leq \mathrm{QZ}(K) = \{1\}$, so in fact L_1 is open in K, and hence $\overline{[K,K]}$ is open in K. Similarly, K has an open subgroup of the form $M_1 \times M_2$ where $M_1 \leq_o \mathrm{M}(K)$. We see that M_2 is commensurable with a profinite group R such that $\mathrm{M}(R) = \{1\}$. Then $\mathrm{QZ}(R)$ is dense in R by Proposition 2.5(iii). But then $\overline{\mathrm{QZ}(M_2)}$ must be open in M_2; since $\mathrm{QZ}(M_2) = \{1\}$, we conclude that M_2 is finite and M_1 is open in K, so $\mathrm{M}(K)$ is open in K.

Applying parts (iv) and (vii) of Proposition 2.5 to a compact open subgroup, we see that G is locally finitely decomposable and first-countable. ∎

The next two lemmas will allow us to split the proof of Theorem 1.4 into the quasi-discrete case and the trivial quasi-centre case.

Lemma 3.5: *Let G be a t.d.l.c. group with (LD). Then G has an open subgroup of the form $Q \times R$, with the following properties:*

(i) *Q and R are closed subgroups with property (LD);*
(ii) *Q is characteristic in G, the quotient G/Q has (LD) with $\mathrm{QZ}(G/Q)=\{1\}$, and $\overline{\mathrm{QZ}(Q)}$ is open in Q;*
(iii) *R is a compact locally normal subgroup of G such that $\mathrm{QZ}(R) = \{1\}$.*

Proof. Suppose G has (LD). Let $K = \overline{\mathrm{QZ}(G)}$ and let $Q/K = \mathrm{QZ}(G/K)$; by construction, Q is characteristic in G. Then G has a compact open subgroup of the form $Q_1 \times R$ where Q_1 is open in K; note that R is compact and locally normal in G. Since Q_1 already accounts for an open subgroup of K, the intersection $K \cap R$ is discrete, and by passing to an open subgroup of R we may ensure that $K \cap R = \{1\}$, so $\mathrm{QZ}(R) = \{1\}$ by Lemma 3.2. In the quotient G/K, we have a copy of R embedded as an open subgroup; since R has trivial quasi-centre we see that $Q/K = \mathrm{QZ}(G/K)$ is discrete, and in particular closed, so that Q is closed in G and K is open in Q. The quotient G/Q then has trivial quasi-centre. In addition, $Q \cap KR = K$, so $Q \cap R = \{1\}$. Now G has an open subgroup QR, such that the factors Q and R normalize each other and have trivial intersection, and R is compact; consequently

$$QR \cong Q \times R$$

as topological groups. By Lemma 3.1, Q and R have (LD). In turn, G/Q has (LD) since it is locally isomorphic to R. ∎

Lemma 3.6: *Let G be a t.d.l.c. group with an open subgroup $Q \times R$, where Q and R have (LD) and $\mathrm{QZ}(R) = \{1\}$. Then G has (LD).*

Proof. Since property (LD) is a local property we may assume that $G = Q \times R$ and that G is compact. Let K be a closed locally normal subgroup of G; we must show that an open subgroup of K is a direct factor of an open subgroup of G. Let K_Q and K_R be the projections of K onto Q and R respectively. By passing to an open subgroup, we may assume K_Q is a direct factor of Q and K_R is a direct factor of R. Note also that K_Q and K_R have (LD) by Lemma 3.1, and $\mathrm{QZ}(K_R) = \{1\}$ by Lemma 3.2. So it suffices to consider the case when $K_R = R$.

By Lemma 3.4, $\overline{[R,R]}$ is open in R. Moreover, since $R \leq KQ$ and K is normal in G, we have $\overline{[R,R]} = \overline{[K,R]} \leq K$, so indeed $K \cap R$ is open in R. It follows that the subgroup $(K \cap Q) \times (K \cap R)$ is open in K. Using property (LD) in Q, we obtain an open subgroup of G of the form $Q_2 \times K_2 \times (K \cap R)$ where $Q_2 \leq Q$ and $K_2 \leq_o K \cap Q$. Thus G has (LD) as required. ∎

3.2. [A]-semisimple groups with property (LD). Lemma 3.4 is a strong restriction on closed locally normal subgroups that will lead to a characterization of property (LD) in the [A]-semisimple case.

Lemma 3.7: *Let G be an [A]-semisimple profinite group such that every closed normal subgroup is finitely decomposable. Then $\mathcal{LD}(G)^G$ is finite.*

Proof. Suppose $\mathcal{LD}(G)^G$ is infinite and let U be an open normal subgroup of G. Then we can find an infinite set $\mathcal{L}$ of pairwise disjoint elements of the Boolean algebra $\mathcal{LD}(G)^G$, as follows. In any finite subset $\mathcal{F}$ of $\mathcal{LD}(G)^G$ with infinitely many elements below the join of $\mathcal{F}$, there is some element of $\mathcal{F}$ lying above infinitely many elements of $\mathcal{LD}(G)^G$. By the axiom of choice there is therefore an infinite properly descending chain $(\alpha_i)_{i \geq 0}$ in $\mathcal{LD}(G)^G$, where $\alpha_0 = \infty$ and thereafter we take $0 < \beta_{i+1} < \alpha_i$ and then set α_{i+1} to be β_{i+1} or $\alpha_i \wedge \beta_{i+1}^\perp$, whichever lies above infinitely many elements. Now set

$$\mathcal{L} = \{\alpha_i \wedge \alpha_{i+1}^\perp \mid i \geq 0\}.$$

The corresponding direct product

$$\prod_{\alpha \in \mathcal{L}} \mathrm{QC}^2_U(\alpha)$$

is then a closed normal subgroup of G that is not finitely decomposable. ∎

An **atom** of a Boolean algebra is a minimal nonzero element.

Lemma 3.8: *Let G be an [A]-semisimple profinite group. Suppose that N is a closed normal subgroup of G, such that $[N]$ is an atom of $\mathcal{LN}(G)^G$ and such that $N\mathrm{C}_G(N)$ is open in G. Then N has (LD) and is commensurable with a just infinite profinite group.*

Proof. To show N has (LD), by Lemma 3.3 it suffices to show that every closed locally normal subgroup of N of infinite index has nondiscrete (quasi-)centralizer in N. Let K be a closed locally normal subgroup of N of infinite index. By replacing N and K with $N \cap U$ and $K \cap U$ respectively for a sufficiently small open normal subgroup U of G, we may assume K is normal in U and $N \leq U$. Let $K_1, \dots, K_n$ be the conjugates of K in G, with $K_1 = K$. By the minimality of $[N]$, we have $\bigcap_{i=1}^n K_i = \{1\}$. Let I be a subset of $\{1, \dots, n\}$ of largest possible size such that

$$L := \bigcap_{i \in I} K_i > \{1\};$$

since I is a proper subset of $\{1, \dots, n\}$, we may assume $1 \notin I$. Then L is a normal subgroup of N and $K \cap L = \{1\}$, so $\mathrm{QC}_N(K) \geq L > \{1\}$. Indeed $\mathrm{QC}_N(K)$ must be nondiscrete, since $\overline{\mathrm{QC}_N(K)}$ is a nontrivial locally normal subgroup of G. Thus N has (LD) as required.

Since $N\mathrm{C}_G(N)$ is open in G, we can identify $\mathcal{LN}(N)$ with the set $\mathcal{I}$ of elements α of $\mathcal{LN}(G)$ such that $\alpha \leq [N]$; we also note that

$$N \cap \mathrm{C}_G(N) \leq \mathrm{QZ}(G) = \{1\}.$$

Let $H = G/\mathrm{C}_G(N)$; we see that there is a G-equivariant isomorphism between $\mathcal{LN}(H)$ and $\mathcal{I}$, and H is commensurable with N. If $x \in G$ is such that $x\mathrm{C}_G(N)$ centralizes an open subgroup of $G/\mathrm{C}_G(N)$, then $x \in \mathrm{QC}_G(N) = \mathrm{C}_G(N)$, so $\mathrm{QZ}(H) = \{1\}$; thus H has no nontrivial finite normal subgroups. Moreover, given the minimality of $[N]$ and the way H arises as a quotient of G, the action of H on $\mathcal{LN}(H)$ has no nontrivial fixed points. Thus H is just infinite. ∎

Corollary 3.9: *Let G be an [A]-semisimple profinite group with (LD). Then $\mathcal{LN}(G)^G$ is finite. Moreover, there is an open characteristic subgroup U of G that decomposes as a finite direct product, where each factor is commensurable with a just infinite group.*

Proof. Given a closed locally normal subgroup H of G, then by Lemmas 3.1 and 3.2, H has (LD) with $\mathrm{QZ}(H) = \{1\}$, so $\mathcal{LN}(H) = \mathcal{LD}(H)$ and H is finitely decomposable by Lemma 3.4. It follows by Lemma 3.7 and property (LD) that $\mathcal{LN}(G)^G$ is finite.

Let $\alpha_1, \dots, \alpha_n$ be the atoms of $\mathcal{LN}(G)^G$. By property (LD),

$$\mathcal{LN}(G) = \mathcal{LD}(G),$$

and hence G has an open characteristic subgroup $U = \prod_{i=1}^n H_i$ where

$$H_i = \mathrm{QC}^2_G(\alpha_i).$$

For $1 \leq i \leq n$, we see that H_i is commensurable with a just infinite group by Lemma 3.8. ∎

3.3. The quasi-discrete case. Let us now focus on the case of a profinite group G such that $\mathrm{QZ}(G)$ is dense in G. Write $\mathbb{P}$ for the set of all prime numbers.

Lemma 3.10: *Let G be a profinite group that is elementary abelian, that is, G is abelian and there is some $p \in \mathbb{P}$ such that $x^p = 1$ for all $x \in G$. Then every closed subgroup of G is a direct factor, with a closed direct complement. In particular, G has (LD).*

Proof. The Pontryagin dual $\widehat{G} := \mathrm{Hom}(G, \mathbb{T})$ is an elementary abelian discrete group, which we may regard as a vector space over the field of p elements. Let K be a closed subgroup of G. Then the annihilator M of K in $\widehat{G}$ is a subspace of $\widehat{G}$, so we have $\widehat{G} = M \oplus N$ for some other subspace N of $\widehat{G}$ (for instance, we could take a basis A of M, extend to a basis B of $\widehat{G}$, and set N to be the span of $B \setminus A$). Now the annihilator R of N in G, that is, the set of elements $g \in G$ such that $g \in \ker \phi$ for all $\phi \in \widehat{G}$, is a closed subgroup of G, and we have $G = K \times R$. ∎

We can now prove Theorem 1.4 for profinite groups with dense quasi-centre.

Proposition 3.11: *Let G be a quasi-discrete profinite group. The following are equivalent:*

(i) *G has (LD).*
(ii) *G has an open subgroup of the form*

$$\prod_{i=1}^k (\mathbb{Z}_{p_i})^{n_i} \times E,$$

where $k \geq 0$, $\{p_1, \dots, p_k\}$ is a set of primes, with associated positive integers $(n_1, \dots, n_k)$, and E is a direct product of finite simple groups.

Proof. Suppose that G has (LD). Let K be the closed subgroup of G topologically generated by all minimal finite normal subgroups of G. Every minimal finite normal subgroup of G is a product of copies of a finite simple group; we see from this that K/K_0 is a direct product of simple groups for every open normal subgroup K_0 of K, and hence $\mathrm{M}(K) = \{1\}$. Thus by Proposition 2.5(iii), $K = \prod_{i \in I} S_i$ for some finite simple groups S_i. By property (LD), there is an open subgroup of G of the form $G_2 = K_2 \times L$ where $K_2 \leq_o K$ and L intersects K trivially; we see that $\mathrm{QZ}(G_2/K_2)$ is dense in G_2/K_2, so $\mathrm{QZ}(L)$ is dense in L. We can also take

$$K_2 = \prod_{i \in I'} S_i,$$

where I' is a cofinite subset of I.

Let $x \in \mathrm{QZ}(L)$ and suppose x has finite order. Then $x \in \mathrm{QZ}(G)$, so by Lemma 2.1, x is contained in a finite G-invariant subgroup F of L. Now F is a finite normal subgroup of N that intersects trivially with K. From the definition of K, this forces $F = \{1\}$, so $x = 1$. We have thus shown that $\mathrm{QZ}(L)$ is torsion-free. By Lemma 2.1, $\mathrm{QZ}(L)$ is abelian, hence $L = \overline{\mathrm{QZ}(L)}$ is abelian; in particular, $L = \mathrm{QZ}(L)$. Now L has (LD) by Lemma 3.1. There is therefore an open subgroup of L of the form $L_1 \times L_2$, where $L_1 \leq_o \mathrm{M}(L)$. We see that $L/\mathrm{M}(L)$ is a product of elementary abelian groups, so by passing to an open subgroup, we can make L_2 a product of elementary abelian groups. But L is torsion-free, so then $L_2 = \{1\}$. We conclude that $|L : \mathrm{M}(L)| < \infty$; since L is torsion-free abelian, it follows that

$$L = \prod_{i=1}^{k} (\mathbb{Z}_{p_i})^{n_i}$$

for some distinct primes $p_1, \dots, p_k$ ($k \geq 0$) and positive integers $n_1, \dots, n_k$. Thus (ii) holds.

Conversely, suppose that (ii) holds. Without loss of generality, let us suppose that

$$G = T \times A \times E,$$

where $T = \prod_{i=1}^{k} (\mathbb{Z}_{p_i})^{n_i}$, $A = \prod_{p \in \mathbb{P}} A_p$ where A_p is an elementary abelian pro-p group and E is a direct product of nonabelian finite simple groups. Let K be a closed locally normal subgroup of G. On passing to an open subgroup of G, we may in fact assume that K is normal in G. In particular, $K_E = K \cap E$ is normal in E, from which we may conclude that $\mathrm{Z}(K_E) = \{1\}$ and also

$$K = \mathrm{C}_K(K_E)K_E$$

(since G only induces inner automorphisms on the normal subgroups of E by conjugation), so in fact $K = \mathrm{C}_K(K_E)K_E$. So to show that G has (LD), it suffices to show that $T \times A$ has (LD), and thus we may assume $E = \{1\}$. In this case, K is abelian, so admits a canonical direct decomposition

$$K := \prod_{p \in \mathbb{P}} K_p$$

by Sylow's theorem, where K_p is the p-Sylow subgroup of K. Note that T is a pro-π group for a finite set of primes π. We can construct a subgroup $L = \prod_{p \in \mathbb{P}} L_p$ that is almost a direct complement of K in G (in the sense that an open subgroup of G is of the form $K_2 \times L$, with $K_2 \leq_o K$) as follows:

For all primes $p \notin \pi$, we have $K_p \leq A_p$, and we may choose L_p to be a direct complement of K_p in A_p using Lemma 3.10.

Given $p \in \pi$, choose a direct complement R_p to $K_p \cap A = K_p \cap A_p$ in A_p, noting that $K_p \cap A$ is the torsion subgroup of K_p (since A contains the torsion subgroup of G). Let P be the p-Sylow subgroup of G. Then

$$P/A_p \cong (\mathbb{Z}_p)^n$$

for some integer n. The group $(\mathbb{Z}_p)^n$ has (LD) for the following reason: given a closed subgroup P of $\mathbb{Z}_p^n$, we embed $\mathbb{Z}_p^n$ in the vector space $\mathbb{Q}_p^n$, obtain a $\mathbb{Q}_p$-linear complement Q to the span of P, and observe that $P \times (Q \cap \mathbb{Z}_p^n)$ spans $\mathbb{Q}_p^n$ and is therefore an open subgroup of $\mathbb{Z}_p^n$. Thus P/A_p has (LD). Note that there is a continuous injective homomorphism from P/A to $P \cap T$ given by $\phi : x \mapsto x^p$. Let Q/A_p be a closed subgroup such that $Q/A_p \times K_pA_p/A_p$ is open in P/A_p, and let $S_p = \phi(Q)$. Finally, set $L_p = R_p \times S_p$. We see that P has an open subgroup of the form

$$S_p \times \phi(K_p) \times R_p \times (K_p \cap A_p),$$

and $\phi(K_p) \times (K_p \cap A_p)$ is an open subgroup of K_p.

Finally, set

$$K_2 = \prod_{p \in \pi} (\phi(K_p) \times (K_p \cap A_p)) \times \prod_{p \notin \pi} K_p$$

and $L = \prod_{p \in \mathbb{P}} L_p$. Then $K_2 \times L$ is an open subgroup of G with K_2 open in K, as required. ∎

3.4. Characterization of property (LD). We now complete the characterization of property (LD).

Proof of Theorem 1.4. Suppose G has (LD). Then by Lemma 3.5, G has an open subgroup of the form $Q \times R$ where Q is quasi-discrete and $\mathrm{QZ}(R) = \{1\}$. By Proposition 3.11, Q has an open subgroup that is a direct product of finite simple groups (possibly including cyclic groups of prime order) together with finitely many groups $\mathbb{Z}_{p_i}$, the latter being just infinite profinite groups. By Corollary 3.9, R has an open subgroup that is commensurable with a finite direct product of just infinite groups. Thus G has an open subgroup in common with a direct product

$$\prod_{i \in I} L_i,$$

such that finitely many factors L_i (possibly none) are just infinite profinite groups and the remaining factors are finite simple groups, as required.

Conversely, suppose G has an open subgroup in common with a direct product

$$\prod_{i \in I} L_i,$$

such that finitely many factors L_i (possibly none) are just infinite profinite groups and the remaining factors are finite simple groups. Without loss of generality we may assume actually $G = \prod_{i \in I} L_i$. Recalling from Lemma 2.6(i) the structure of just infinite groups with nontrivial quasi-centre, we can write an open subgroup of G as a direct product

$$\prod_{i=1}^{k} (\mathbb{Z}_{p_i})^{n_i} \times E \times J_1 \times \cdots \times J_n,$$

where E is a direct product of finite simple groups and each of the groups J_i is just infinite with trivial quasi-centre. Appealing to Proposition 3.11, the group

$$Q = \prod_{i=1}^{k} (\mathbb{Z}_{p_i})^{n_i} \times E$$

has (LD), whilst J_i has (LD) for all i by Lemma 3.8. Using Lemma 3.6, we conclude that G has (LD), completing the proof of the theorem. ∎

4. Global structure

We now turn to the study of t.d.l.c. groups with (LD), using what we know about the compact open subgroups to deduce global properties.

4.1. Similarity classes. At this point it is useful to introduce a notion of homogeneity of closed locally normal subgroups, which will capture many examples of groups with (LD).

Definition 4.1: Let G_1 and G_2 be t.d.l.c. groups. Say G_1 and G_2 are **commensurable** if there is an isomorphism from a finite index open subgroup of G_1 to a finite index open subgroup of G_2. Say G_1 and G_2 are **similar** if there are natural numbers a_1 and a_2 such that $G_1^{a_1}$ is commensurable with $G_2^{a_2}$. Say G_1 and G_2 are **locally similar** if they have compact open subgroups U_1 and U_2 respectively such that U_1 is similar to U_2.

A t.d.l.c. group G is (**locally**) **monomial** if it is nondiscrete and (locally) similar to every infinite closed locally normal subgroup of G^a for all natural numbers a.

It is easily seen that similarity and local similarity are equivalence relations and that the local similarity type is stable under passage to an open subgroup. It is also easy to see that among profinite groups, the monomial property is a similarity invariant. We can characterize the locally monomial t.d.l.c. groups as follows. In the next theorem "LS" is for "locally semisimple", the "J" stands for "just infinite", and then the letter after indicates the relevant type of just infinite profinite group (either (virtually) abelian, (similar to) hereditarily just infinite, or (nearly) branch).

Theorem 4.1: *Let G be a nondiscrete t.d.l.c. group. Then the following are equivalent:*

(i) *G is locally monomial and has (LD).*

(ii) *One of the following holds:*

(LS) *Some open subgroup of G is a direct product of $\aleph_0$ copies of a finite simple group;*

(JA) *G has a compact open subgroup isomorphic to $\mathbb{Z}_p^d$ for some natural number d and prime p;*

(JH) $\mathrm{QZ}(G)$ *is discrete,* $\mathcal{LN}(G)$ *is finite and an open subgroup of* G *is a direct product of* n *copies of a h.j.i. profinite group, where* $2^n = |\mathcal{LN}(G)|$;

(JB) $\mathrm{QZ}(G)$ *is discrete,* $\mathcal{LN}(G)$ *is infinite,* $G/\mathrm{QZ}(G)$ *acts faithfully on* $\mathcal{LN}(G)$ *and* G *is locally isomorphic to a just infinite profinite branch group.*

We begin the proof with two lemmas.

Lemma 4.2: *Every nearly just infinite profinite group is monomial.*

Proof. Since the monomial property is a commensurability invariant, we only need to consider just infinite profinite groups. Let J be a just infinite profinite group and let H be an infinite closed locally normal subgroup of J^a for some $a \geq 1$; we must show that H is similar to J. If $\mathrm{QZ}(J) > \{1\}$, then by Lemma 2.6(i), J is commensurable with $\mathbb{Z}_p^m$ for some prime p and natural number m, and then H is commensurable with $\mathbb{Z}_p^n$ for some $1 \leq n \leq am$; hence H is similar to J. Thus we may assume $\mathrm{QZ}(J) = \{1\}$. In this case, J^a is a finite index open subgroup of the just infinite group $J' = J \wr \mathrm{Sym}(a)$, which contains H as a locally normal subgroup. Let U be an open normal subgroup of J' that normalizes H; without loss of generality we may replace H with $U \cap H$ and assume that H is normal in U. Then $\mathcal{LN}(U)^U$ is finite by Corollary 3.9, so it is generated by its atoms. We can then take a closed normal subgroup K of U representing an atom of $\mathcal{LN}(U)^U$ such that the distinct J'-conjugates $K_1, \dots, K_n$ of K represent distinct atoms of $\mathcal{LN}(U)^U$. Since J' is just infinite,

$$\mathcal{LN}(U)^{J'} = \{0, \infty\}.$$

and so U has an open subgroup of the form $K_1 \times \cdots \times K_n$; we then see that every closed normal subgroup of U is commensurate with the product of some subset of $K_1, \dots, K_n$ and hence is similar to K. In particular, H is similar to K; clearly also K is similar to J' and J' is similar to J. ∎

Lemma 4.3: *Let* $G = J_1 \times \cdots \times J_n$, *where* $n \geq 1$ *and* $J_1, \dots, J_n$ *are nearly just infinite profinite groups. Then the following are equivalent:*

(i) G *is monomial;*

(ii) $J_1, \dots, J_n$ *are similar;*

(iii) G *is nearly just infinite.*

Proof. Given Lemma 4.2, the only nontrivial implication is that (ii) implies (iii), so let us suppose (ii) holds. If $\mathrm{QZ}(J_i) > \{1\}$ for some i, then $\mathrm{QZ}(J_j) > \{1\}$ for $1 \leq j \leq n$, and hence by Lemma 2.6(i) and similarity, G is commensurable with $\mathbb{Z}_p^m$ (for some prime p and natural number m), which in turn is commensurable with a just infinite virtually abelian group of the form $\mathbb{Z}_p^m \rtimes F$ for some finite subgroup F of $\mathrm{GL}_m(\mathbb{Z}_p)$ acting irreducibly over $\mathbb{Q}_p$, implying (iii). Thus we may assume $\mathrm{QZ}(J_i) = \{1\}$ for $1 \leq i \leq n$; hence also $\mathrm{QZ}(G) = \{1\}$.

There is a profinite group H and natural numbers $k_1, \dots, k_n$ such that $J_i^{k_i}$ is commensurable with H for $1 \leq i \leq n$. We can clearly take H to be just infinite, for instance $H = J_1 \wr \mathrm{Sym}(k_1)$. Taking $k = \prod_{i=1}^n k_i$, we see that G^k is commensurable with a power H^l of H. In turn, H^l is commensurable with the just infinite profinite group $L = H \wr \mathrm{Sym}(l)$.

We now argue that G itself is commensurable with a just infinite profinite group. We have an isomorphism θ from V^k, where V is open in G, to an open subgroup U of L. In particular, $\theta(V)$ represents an element of $\mathcal{LN}(U)^U$. Similar to the proof of Lemma 4.2, we find that V is commensurable with a direct power K^m of K, where K represents an atom of $\mathcal{LN}(U)^U$. It follows by Lemma 3.8 that K is commensurable with a just infinite profinite group K_2, and thus G is commensurable with the just infinite profinite group $K_2 \wr \mathrm{Sym}(m)$. Thus (ii) implies (iii), completing the cycle of implications. ∎

Proof of Theorem 4.1. Let us first consider the closed locally normal subgroups of an infinite direct product $E = \prod_{i \in I} S_i$, where S_i is a finite simple group. We see that every closed locally normal subgroup of E is commensurate with the subgroup $\prod_{i \in I'} S_i$ for some subset I' of I, where I' is determined up to adding or removing finitely many elements. From this description, we see that if E is monomial, then all but finitely many S_i belong to a single isomorphism class; moreover, the fact that E is similar to $\prod_{i \in I'} S_i$ where $|I'| = \aleph_0$ ensures that $|I| = \aleph_0$. Conversely, if all but finitely many S_i belong to a single isomorphism class and I is countably infinite, it is clear that E is monomial.

Suppose G is locally monomial and has (LD). Then given Theorem 1.4, G is locally isomorphic a direct product

$$J \times E,$$

where J is a direct product of finitely many (possibly no) just infinite profinite groups and E is a direct product of finite simple groups. It follows that $J \times E$ is monomial; clearly, for this to be the case we must have either J trivial or E finite.

If J is trivial, then E is monomial, and then (LS) follows by the first paragraph. So let us suppose that E is finite. Then by Lemma 4.3 we know that J, and hence G, is locally isomorphic to a just infinite group J'. If J' is virtually abelian then (JA) follows. If J' is not virtually abelian, then $\mathrm{QZ}(J') = \{1\}$, from which it follows that $\mathrm{QZ}(G)$ is discrete. There are then two possibilities as in Theorem 1.2: if $\mathcal{LN}(G)$ is finite, then J' is commensurable with a direct product of n copies of a hereditarily just infinite profinite group, and we see that $2^n = |\mathcal{LN}(G)|$; after passing to an open subgroup, we obtain such a direct product that also occurs as a compact open subgroup of G, and hence (JH) holds. If instead $\mathcal{LN}(G)$ is infinite, then J' is a just infinite branch group, which means that J' acts faithfully on $\mathcal{LN}(J')$ and hence the kernel of the action of G on $\mathcal{LN}(G)$ is discrete. At the same time, $\mathrm{QZ}(G)$ acts trivially on $\mathcal{LN}(G)$ and is the largest discrete normal subgroup of G; thus we have a faithful action of $G/\mathrm{QZ}(G)$ on $\mathcal{LN}(G)$, and (JB) follows. Thus (i) implies (ii).

Conversely, suppose that (ii) holds. In case (LS), it follows from the first paragraph that G is locally monomial, and from Theorem 1.4 that G has (LD). For (JA), (JH) and (JB), we see that G is locally isomorphic to a just infinite profinite group, and the same conclusions follow by Lemma 4.3 and Theorem 1.4. Thus (ii) implies (i). ∎

Moving away from the locally monomial case, we now obtain a canonical factorization of an open subgroup.

Definition 4.2: Given a profinite group H with (LD), a **monomial factorization** of H is a factorization of H as a direct product

$$H = H_1 \times H_2 \times \cdots \times H_n \times \prod_{S \in \mathcal{C}} H_S \times H_\infty$$

with the following properties:

(i) H_i is nearly just infinite for $1 \leq i \leq n$, and for $i \neq j$ then H_i and H_j are not similar;

(ii) $\mathcal{C}$ is a collection of isomorphism types of finite simple groups (possibly empty), and for each $S \in \mathcal{C}$ then H_S is the direct product of $\aleph_0$ copies of S;

(iii) H_∞ is either trivial or a direct product of infinitely many finite simple groups not belonging to $\mathcal{C}$, such that each isomorphism type only appears finitely many times.

Note in particular that the factors of the monomial factorization are monomial, with the exception of the leftover part H_∞.

Proposition 4.4: *Let G be a first-countable t.d.l.c. group with (LD).*

(i) *G has an open subgroup H admitting a monomial factorization*

$$H = H_1 \times H_2 \times \cdots \times H_n \times \prod_{S \in \mathcal{C}} H_S \times H_\infty.$$

(ii) *Let N be an infinite compact monomial locally normal subgroup of G. Then N is virtually contained in one of the factors of the monomial factorization of H.*

(iii) *Let K be a first-countable profinite group admitting a monomial factorization*

$$K = K_1 \times K_2 \times \cdots \times K_m \times \prod_{S \in \mathcal{C}'} H_S \times K_\infty,$$

such that there is a local isomorphism $\phi : U \to H$ from K to H. Then the monomial factorizations are equivalent in the following sense: We have $m = n$ and $\mathcal{C} = \mathcal{C}'$; there is a permutation π of $\{1, \dots, m\}$ such that

$$[\phi(K_i \cap U)] = [H_{\pi(i)}];$$
$$[\phi(K_S)] = [H_S] \qquad \textit{for all } S \in \mathcal{C};$$
$$[\phi(K_\infty)] = [H_\infty].$$

Proof. (i) By Theorem 1.4, after passing to an open subgroup we can take

$$G = \prod_{i \in I} L_i,$$

where finitely many factors, say $L_1, \dots, L_{n'}$, are nearly just infinite profinite groups and the remaining factors are finite simple groups. Since G is first-countable, we can take the indexing set I to be countable. This immediately yields a factorization

$$G = L_1 \times \cdots \times L_{n'} \times \prod_{S \in \mathcal{C}} H_S \times H_\infty,$$

where $\mathcal{C}$ is the set of isomorphism types of finite simple factors of G that occur infinitely many times, and H_∞ is the product of the finite simple factors of G whose isomorphism type only occurs finitely many times. We can then group the factors $L_1, \dots, L_{n'}$ into similarity classes; by Lemma 4.3, the product of the factors in a given similarity class is nearly just infinite. This yields the desired factorization of some open subgroup H of G.

(ii) By Theorem 4.1, after replacing N with a finite index open subgroup, we may assume either that N is a direct product of $\aleph_0$ copies of a finite simple group S or N is nearly just infinite.

If N is a direct product of $\aleph_0$ copies of a finite simple group S, let $\mathcal{I}_S$ be the set of elements of $\mathcal{LN}(G)$ with representatives locally isomorphic to N. We see that $S \in \mathcal{C}$, $[N] \in \mathcal{I}_S$, and $[H_S]$ is the unique largest element of $\mathcal{I}_S$: we see the last asssertion, for example, by observing that H/H_S has no closed locally normal subgroup locally isomorphic to H_S. Thus $[N] \leq [H_S]$, that is, N is virtually contained in H_S. If instead N is nearly just infinite, one sees similarly that the set $\mathcal{J}$ of elements of $\mathcal{LN}(G)$ with representatives similar to N has a unique largest element, which is represented by H_i for $1 \leq i \leq n$, and hence N is virtually contained in H_i.

(iii) Writing $\theta([L]) = [\phi(L \cap U)]$, we have an order isomorphism θ from $\mathcal{LN}(K)$ to $\mathcal{LN}(H)$, which also preserves commensurability of the representatives. In particular, applying (ii), we see that if $S \in \mathcal{C}'$, then $S \in \mathcal{C}$ and $\theta([K_S]) \leq [H_S]$; after applying the same argument to θ^{-1}, we see that $\mathcal{C} = \mathcal{C}'$ and $\theta([K_S]) = [H_S]$ for every $S \in \mathcal{C}'$. Similarly, given $1 \leq i \leq m$ we see that $\theta([K_i]) \leq [H_j]$ for some $1 \leq j \leq n$, ensuring that H_j is similar to K_i, and conversely given $1 \leq j \leq n$ then $\theta^{-1}([H_j]) \leq [K_i]$ for some $1 \leq i \leq n$ such that K_i is similar to H_i. Thus in fact there is a permutation π of $\{1, \dots, m\}$ such that $\theta([K_i]) = [H_{\pi(i)}]$, where π is uniquely determined by the correspondence between the similarity classes of the nearly just infinite factors in the monomial factorizations of K and H. Finally, we can characterize $[K_\infty]$ in $\mathcal{LN}(K)$ as follows, and similarly for $[H_\infty]$ in $\mathcal{LN}(H)$: Given a closed locally normal subgroup N of K such that the finite simple normal subgroups of N that do not belong to $\mathcal{C}$ generate a dense subgroup of N, then N is virtually contained in K_∞, and $[K_\infty]$ is the smallest element of $\mathcal{LN}(K)$ with this property. Thus $\theta([K_\infty]) = [H_\infty]$. ∎

Remark 4.1: In Proposition 4.4 we restricted to the first-countable case to avoid some technicalities, and because dropping this condition does not generalize the class of groups under consideration in any interesting way. There are only countably many isomorphism types of finite group and every just infinite profinite group is first-countable, so the only difference between first-countable and general profinite groups with (LD) in the context of Theorem 1.4 is that in the latter case, the factorization $\prod_{i \in I} L_i$ can include uncountably many copies of the same finite simple group.

4.2. Monomial constituents. Given a t.d.l.c. group G, a similarity class $\mathcal{S}$ of profinite groups is a **monomial constituent** of G if groups in $\mathcal{S}$ are monomial and there exists $H \in \mathcal{S}$ such that H is isomorphic to a closed locally normal subgroup of G. Proposition 4.4 shows that if G has (LD), then any monomial factorization of a profinite group locally isomorphic to G will yield all the monomial constituents of G.

Let us prove some analogues to Proposition 4.4 describing closed normal subgroups of G as a whole. In particular, it will follow that the structure theory of t.d.l.c. groups with (LD) can to some extent be reduced to the locally monomial case.

Proposition 4.5: *Let G be a first-countable t.d.l.c. group with (LD) and let $\mathcal{S}$ be a monomial constituent of G. Then there is a continuous injective homomorphism $\phi : G_{\mathcal{S}} \rightarrow G$, where $G_{\mathcal{S}}$ is a first-countable t.d.l.c. group, with the following properties:*

(i) *$G_{\mathcal{S}}$ is generated by its compact open subgroups, all of which belong to $\mathcal{S}$.*
(ii) *Given a compact locally normal subgroup N of G such that $N \in \mathcal{S}$, then ϕ restricts to a homeomorphism from $\phi^{-1}(N)$ to N.*
(iii) *$\phi(G_{\mathcal{S}})$ is characteristic in any open subgroup of G that contains it.*

Moreover, given an open subgroup K of G containing $\phi(G_{\mathcal{S}})$, then the map $G_{\mathcal{S}} \rightarrow K$ has the same properties with respect to K in place of G.

Proof. Given a compact open subgroup U of G, by Proposition 4.4 there is an open subgroup of U with a monomial factorization, such that exactly one of the factors $U_{\mathcal{S}}$ belongs to the similarity class $\mathcal{S}$.

Now let $\mathcal{L}$ be the set of all compact locally normal subgroups of G that belong to the similarity class $\mathcal{S}$. Given $N \in \mathcal{L}$, we see by Proposition 4.4(ii) that N is virtually contained in $U_{\mathcal{S}}$, and by Proposition 4.4(iii), N commensurates $U_{\mathcal{S}}$. It follows that the subgroup $\langle \mathcal{L} \rangle$ can be equipped with a t.d.l.c. group topology such that $U_{\mathcal{S}}$ is embedded as an open subgroup; let $G_{\mathcal{S}}$ be $\langle \mathcal{L} \rangle$ with this topology and let ϕ be the inclusion map into G. Note that $U_{\mathcal{S}} \in \mathcal{L}$, and up to finite index, $U_{\mathcal{S}}$ does not depend on the choice of U, so $G_{\mathcal{S}}$ does not depend on U.

Since G is first-countable, so is $U_{\mathcal{S}}$, and hence $G_{\mathcal{S}}$ is first-countable. It is clear from the construction that $\phi(G_{\mathcal{S}})$ is characteristic in G and that (ii) holds; in particular, since $U_{\mathcal{S}}$ is open in $G_{\mathcal{S}}$, we see from (ii) that ϕ is continuous at the identity. Since ϕ is evidently also a group homomorphism, it is continuous everywhere. The profinite group $U_{\mathcal{S}}$, and hence all profinite groups commensurable with it, belong to $\mathcal{S}$; thus (i) holds.

Finally, if K is an open subgroup of G containing $\phi(G_{\mathcal{S}})$, then we see that the set of monomial factors in $\mathcal{S}$ of compact locally normal subgroups of K belonging to $\mathcal{S}$ is the same as for G. Thus we can identify $G_{\mathcal{S}}$ with $K_{\mathcal{S}}$ in the obvious way, ensuring that (iii) holds and the map $G_{\mathcal{S}} \to K$ has the same properties with respect to K in place of G. ∎

Remark 4.2: The map ϕ is not closed in general, even when G is compact. For example, if $G = \mathbb{Z}_p \times \prod_{\aleph_0} \mathbb{Z}/p\mathbb{Z}$ and $\mathcal{S}$ is the monomial constituent containing $\mathbb{Z}_p$, then ϕ is bijective since G is generated by copies of $\mathbb{Z}_p$, but $G_{\mathcal{S}}$ is equipped with the topology in which $\mathbb{Z}_p$ is a compact open subgroup, so for instance there is a subgroup

$$\mathbb{Z}_p \times \bigoplus_{\aleph_0} \mathbb{Z}/p\mathbb{Z}$$

that is closed in $G_{\mathcal{S}}$ but not in G. This complication only arises when $\mathrm{QZ}(G) > \{1\}$, as we will see from Theorem 4.6 below.

We now obtain an extended version of Theorem 1.5.

Theorem 4.6: *Let G be a t.d.l.c. group with property (LD) such that $\mathrm{QZ}(G) = \{1\}$. Then G is first-countable and has an open subgroup of the form*

$$M = M_1 \times M_2 \times \cdots \times M_n,$$

where for $1 \le i \le n$ the groups M_i have the following properties:

(i) *M_i is closed and characteristic in every closed locally normal subgroup of G containing M_i.*
(ii) *M_i is locally isomorphic to a just infinite profinite group J_i, so in particular M_i is locally monomial; additionally M_i contains every closed locally normal subgroup of G that is locally similar to J_i.*
(iii) *In the case that G contains a compact just infinite representative J_i of $[M_i]$, then $J_i \le M_i$ and every nontrivial closed normal subgroup of M_i is open in M_i.*
(iv) *If $G = \mathscr{L}(G)$, then $G = M$, $\mathscr{L}(M_i) = M_i$ for $1 \le i \le n$, and M_i has a just infinite compact open subgroup for all $1 \le i \le n$.*

Proof. By Lemma 3.3, G is [A]-semisimple; by Lemma 3.4, G is first-countable. By Proposition 4.4, there is a compact open subgroup U of G with a factorization

$$U = U_1 \times U_2 \times \cdots \times U_n$$

such that the factors $U_1, \dots, U_n$ are each nearly just infinite, with trivial quasi-centre, and no two factors are similar. Let α_i be the element of $\mathcal{LN}(G) = \mathcal{LD}(G)$ represented by U_i and let

$$M_i = \mathrm{C}^2_G(U_i);$$

in particular, M_i is closed and is the largest representative of α_i. Since $\{\alpha_1, \dots, \alpha_n\}$ forms a partition of ∞ in $\mathcal{LD}(G)$, we have an open subgroup

$$M := \langle M_1, M_2, \dots, M_n \rangle = M_1 \times M_2 \times \cdots \times M_n,$$

where the direct product decomposition holds by Lemma 2.4. Let J_i be a compact just infinite representative of $[M_i]$, if one exists; otherwise let J_i be a just infinite profinite group commensurable with U_i. Let N be a closed locally normal subgroup of G that is locally similar to J_i. By Proposition 4.4(ii), $U_i \cap N$ is open in N, so also $M_i \cap N$ is open in N. By Theorem 2.3 it follows that M_j centralizes N for all $j \neq i$, and hence $N \leq M_i$. Thus (ii) holds.

Consider a closed locally normal subgroup K of G and let β_i be the element of $\mathcal{LD}(K)$ represented by $K \cap U_i$; note that the product $\prod_{i=1}^n (K \cap U_i)$ is a compact open subgroup of K, and also that K has (LD) and trivial quasi-centre by Lemmas 3.1 and 3.2. For $1 \leq i \leq n$, since the compact representatives of α_i are monomial, they are similar to the representatives of β_i. Since the representatives of α_i lie in different similarity classes for $1 \leq i \leq n$, we deduce that $\prod_{i=1}^n (K \cap U_i)$ is a monomial factorization of an open subgroup of K, and by Proposition 4.4(iii), it follows that every automorphism of K will preserve each of the elements $\beta_1, \dots, \beta_n$ of $\mathcal{LD}(K)$. In particular, if $M_i \leq K$ for some i, we see that $\alpha_i = \beta_i$ and that M_i is the largest representative of β_i, so M_i is characteristic in K. This proves (i).

Suppose G contains a compact just infinite representative J_i of $[M_i]$. Then $J_i \leq M_i$ by (ii), and hence

$$\mathcal{LN}(M_i)^{M_i} = \{0, \infty\},$$

ensuring that every closed normal subgroup of M_i is discrete or open in M_i. The former is ruled out by the fact that $\mathrm{QZ}(M_i) = \{1\}$; thus (iii) holds.

Finally, let us suppose that $G = \mathscr{L}(G)$. Given $g \in G$, then conjugation by g induces a local automorphism of U; using Proposition 4.4(iii), we can restrict this local automorphism to an isomorphism

$$V_1 \times V_2 \times \cdots \times V_n \to gV_1g^{-1} \times gV_2g^{-1} \times \cdots \times gV_ng^{-1},$$

where V_i and gV_ig^{-1} are open subgroups of U_i. We then see that there are elements $g_1, \dots, g_n$ of G such that $g_ivg_i^{-1} = gvg^{-1}$ for all $v \in V_i$, but $g_i \in \mathrm{C}_G(V_j)$ for all $j \neq i$. We then have $g_i \in M_i$ and $g = g_1g_2 \dots g_n$. This proves that $G = M$. On the other hand, we see that every local automorphism of M_i extends to a local automorphism of G, and hence can be realized by conjugation in G; thus $M_i = \mathscr{L}(M_i)$ for $1 \leq i \leq n$. Finally, it is now clear that J_i appears as a compact open subgroup of M_i, completing the proof of (iv). ∎

Remark 4.3: In the situation of Theorem 4.6, one sees that factors $M_1, \dots, M_n$ correspond to the monomial constituents $\mathcal{S}_1, \dots, \mathcal{S}_n$ of G from Proposition 4.5, and for $1 \leq i \leq n$ the map $\phi : G_{\mathcal{S}_i} \to G$ constructed in Proposition 4.5 restricts to an open embedding (in particular, a closed map) from $G_{\mathcal{S}_i}$ to M_i.

4.3. Groups of type (JH). We now give the analogue of Theorem 1.3 for groups of type (JH).

Theorem 4.7: *Let G be a t.d.l.c. group of type (JH), with $\mathrm{QZ}(G) = \{1\}$. Then there is a hereditarily just infinite profinite group J and an open subgroup*

$$P = P_1 \times P_2 \times \cdots \times P_n$$

of G, where the factors $P_1, \dots, P_n$ (which we call the ***atomic factors*** *of G) have the following properties:*

(i) *Every automorphism of G permutes the factors $\{P_1, \dots, P_n\}$ (possibly trivially); in particular, P is characteristic in G.*
(ii) *For $1 \leq i \leq n$, $\mathrm{QZ}(P_i) = \{1\}$ and P_i has a compact open subgroup isomorphic to J.*
(iii) *Given $1 \leq i \leq n$, exactly one of the following holds:*
 Reducible type: *$P_i \cap \mathrm{Res}(G) = \{1\}$;*
 Mysterious type: *$P_i \cap \mathrm{Res}(G)$ is open, but $P_i \cap \mathrm{Res}(\mathrm{Res}(G)) = \{1\}$;*
 Simple type: *$P_i \cap \mathrm{Res}(G) = \mathrm{Res}(P_i)$ is open in P_i and topologically simple.*

Proof. By Theorem 4.1, $\mathcal{LN}(G)$ is finite and there is an open subgroup J' of G that is a direct product

$$J' = J_1 \times J_2 \times \cdots \times J_n,$$

where $J_i \cong J_j$ for all i and j and $J := J_1$ is hereditarily just infinite. Note that the elements $[J_1], \dots, [J_n]$ of $\mathcal{LN}(G)$ are distinct and are exactly the atoms of $\mathcal{LN}(G)$. We set $P_i = \mathrm{QC}^2_G([J_i])$; then we have an open subgroup

$$P = P_1 \times P_2 \times \cdots \times P_n.$$

Since the factors $P_1, \dots, P_n$ are obtained from $\mathcal{LN}(G)$ in a canonical way, they form a characteristic class of subgroups; thus (i) is satisfied. We also note that

$$\mathrm{QZ}(P_i) \le \mathrm{QZ}(G) = \{1\} \quad \text{for all } 1 \le i \le n,$$

and clearly $J_i \le_o P_i$; this proves (ii).

It remains to divide the atomic factors into three types as in (iii). Without loss of generality, we can replace G with the finite index open subgroup

$$\bigcap_{i=1}^{n} \mathrm{N}_G(P_i);$$

note that this does not change $\mathrm{Res}(G)$. Clearly the three listed types are mutually exclusive; note also that because of (ii), any closed locally normal subgroup of P_i is either trivial or open. Fix $1 \le i \le n$ and suppose P_i is not of reducible or mysterious type. Then we see that $\mathrm{Res}(\mathrm{Res}(G)) \cap P_i$ must be open in P_i; since P is open and characteristic in G, we see that $\mathrm{Res}(G) \le P$ and hence

$$P_i \cap \mathrm{Res}(\mathrm{Res}(G)) = \mathrm{Res}(R),$$

where $R = P_i \cap \mathrm{Res}(G)$. In particular, the product of $\mathrm{Res}(R)$ with P_j for $j \neq i$ is an open normal subgroup O of G; we then have

$$\mathrm{Res}(R) = P_i \cap O \ge P_i \cap \mathrm{Res}(G) = R,$$

thus $\mathrm{Res}(R) = R$. Since in addition, every nontrivial closed normal subgroup of R is open, we conclude that R is topologically simple. We then have

$$\mathrm{Res}(R) \le \mathrm{Res}(P_i) \le R,$$

so $R = \mathrm{Res}(P_i)$. Thus P_i is of simple type, completing the proof. ∎

We do not resolve the question of Barnea–Ershov–Weigel of whether locally h.j.i. groups of mysterious type actually exist, but we can put some restrictions on when they occur.

Theorem 4.8: *Let G be a t.d.l.c. group with $\mathrm{QZ}(G) = \{1\}$, such that G has a h.j.i. compact open subgroup and G is compactly generated. Suppose that $R = \mathrm{Res}(G)$ is open but $\mathrm{Res}(R)$ is trivial. Then the following hold:*

(i) *There is a noncompact open normal subgroup K of R such that, given any nontrivial closed normal subgroup Q of R, then Q contains a G-conjugate of K.*
(ii) *There is a G-conjugate of K that properly contains K.*
(iii) *Given a compactly generated subgroup H of G such that $R \leq H$, then $\mathrm{Res}_R(H)$ is open in R. In particular, R is not compactly generated.*
(iv) *Let L be a subgroup of G containing R such that L/R is virtually polycyclic. Then every open normal subgroup of R contains an open normal subgroup of L; in particular, L is residually discrete.*

Part (iv) is derived from a more general fact, which we prove separately in a lemma.

Lemma 4.9 ([12, Corollary 1.4]): *Let U be a profinite group such that U has only finitely many open subgroups of each index, and such that U is isomorphic to a proper open subgroup of itself. Then U has an infinite abelian normal subgroup.*

Lemma 4.10: *Let G be a t.d.l.c. group. Make the following assumptions:*

(i) *We have $G = \langle K, x \rangle$, where K is a residually discrete open normal subgroup of G.*
(ii) *The action of G on $\mathcal{LN}(G)$ has no nontrivial fixed points.*
(iii) *K has no infinite abelian locally normal subgroups.*
(iv) *Given a compact open subgroup U of K, then U has only finitely many open subgroups of each index.*

Then G is residually discrete; indeed, every open normal subgroup of K contains an open normal subgroup of G.

Proof. Let H be an open normal subgroup of K and let U be a compact open subgroup of H. Define the following subgroups:

$$H_+ = \bigcap_{i=0}^{\infty} x^i H x^{-i}; \quad H_- = \bigcap_{i=0}^{\infty} x^{-i} H x^i;$$

$$U_+ = \bigcap_{i=0}^{\infty} x^i U x^{-i}; \quad U_- = \bigcap_{i=0}^{\infty} x^{-i} U x^i.$$

Note that all conjugates of H and their intersections are closed and normal in K; furthermore, xH_+x^{-1} and $x^{-1}H_+x$ are both locally equivalent to H_+. Hence $\alpha_+ = [H_+]$ is an element of $\mathcal{LN}(G)$ that is fixed by G. It follows that $\alpha_+ \in \{0, \infty\}$. By [17, Lemma 1], the set U_+U_- is a neighbourhood of the identity in H. Suppose that $\alpha_+ = 0$; then H_+ is discrete, so U_+ is finite, and hence U_- is open in K. By construction, xU_-x^{-1} is a proper open subgroup of U_- that is isomorphic to U_-. Thus K has an infinite abelian locally normal subgroup by Lemma 4.9, a contradiction. Thus $\alpha_+ = \infty$, in other words, H_+ is open in K. By the same argument, H_- is also open in K, so $H_+ \cap H_-$ is open in K; by construction, $H_+ \cap H_-$ is normal in G and contained in H. Hence every open normal subgroup of K contains an open normal subgroup of G. ∎

Proof of Theorem 4.8. Let U be a proper compact open subgroup of R and note that the G-conjugates of U have trivial intersection. By [14, Theorem 1.2], there is some $x \in R \setminus \{1\}$ such that the G-conjugacy class of x accumulates at the identity. In particular, every open subgroup of R contains gxg^{-1} for some $g \in G$.

Now let

$$K = \overline{\langle rxr^{-1} \mid r \in R \rangle}$$

and consider a nontrivial closed normal subgroup Q of R. Then K and Q are both open, because R is locally h.j.i. and has trivial quasi-centre. We then see that Q contains gxg^{-1} for some $g \in G$; since Q is normal in R, it follows that Q contains $rgxg^{-1}r^{-1}$ for every $r \in R$. Since R is normal in G, in fact Q contains $grxr^{-1}g^{-1}$ for every $r \in R$; since Q is a closed subgroup of R, it follows that

$$gKg^{-1} \leq Q.$$

We have now proved (i) except for the fact that K is not compact.

Consider the case $Q = K \cap gKg^{-1}$, where $g \in G$ is such that $K \not\leq gKg^{-1}$. Then Q contains hKh^{-1} for some $h \in G$, and then we see that $hKh^{-1} < K$. This proves (ii). However, by Lemma 4.9, no compact open subgroup of R is isomorphic to a proper open subgroup of itself. Thus K is not compact, completing the proof of (i).

We note by (i) that R has no compact open normal subgroups. It follows that if H is a compactly generated subgroup containing R, then H also has no compact open normal subgroups. By [4, Corollary 4.1] it follows that

$$\mathrm{Res}(H) \neq \{1\},$$

and hence $\mathrm{Res}(H)$ is open. Since R is open and normal in H, we have

$$\mathrm{Res}(H) = \mathrm{Res}_R(H).$$

Since $\mathrm{Res}(R) = \{1\}$, we see that R cannot be compactly generated, proving (iii).

Finally, consider a subgroup L of G containing R, such that L/R is virtually polycyclic; then there is $R \leq L_0 \leq L$ such that L_0/R is polycyclic and L_0 is a normal subgroup of L of finite index. By repeated application of Lemma 4.10 we see that every open normal subgroup of R contains an open normal subgroup of L_0. In turn, every open normal subgroup M of L_0 contains the open normal subgroup

$$M' = \bigcap_{y \in Y} yMy^{-1}$$

of L, where Y is a finite set of coset representatives for L_0 in L. This proves (iv). ∎

4.4. Groups locally isomorphic to just infinite profinite branch groups. We now turn to groups G of type (JB). In this case there is no atomic decomposition coming from the structure lattice, since $\mathcal{LN}(G)$ is atomless. Instead, we take the approach of decomposing $\mathrm{Res}(G)$ into directly indecomposable parts. Other than that, we obtain a statement similar to Theorem 4.7.

Theorem 4.11: *Let G be a t.d.l.c. group of type (JB), with $\mathrm{QZ}(G) = \{1\}$. Then $\mathrm{Res}(G)$ is a direct factor of an open characteristic subgroup and admits a decomposition into finitely many directly indecomposable direct factors*

$$\mathrm{Res}(G) = P_1 \times P_2 \times \cdots \times P_n,$$

which we call the ***components*** *of G, with the following properties:*

(i) *Every direct factor of* $\mathrm{Res}(G)$ *is a direct product of a subset of the components.*

(ii) *Each component* P *is noncompact, but locally isomorphic to a just infinite profinite branch group, and we have* $\mathcal{LN}(P)^P = \{0, \infty\}$. *Moreover,* G *does not normalize any proper nontrivial closed subgroup of* P.

(iii) *Given* $1 \leq i \leq n$, *exactly one of the following holds:*

Mysterious type: P_i *is residually discrete;*

Simple type: P_i *is topologically simple.*

Proof. We see that $R = \mathrm{Res}(G)$ is itself of type (JB) with trivial quasi-centre. In particular, we have $\mathcal{LN}(R) = \mathcal{LD}(R)$, R is [A]-semisimple, and R acts faithfully on $\mathcal{LN}(R)$ with finitely many fixed points. Let $\alpha_1, \dots, \alpha_n$ be the atoms of $\mathcal{LN}(R)^R$ and set $P_i = \mathrm{QC}^2_R(\alpha_i)$ for $1 \leq i \leq n$. Then we obtain an open subgroup of R

$$O = P_1 \times P_2 \times \cdots \times P_n.$$

From the construction we see that O is characteristic in R, hence normal in G; from there we see that $\mathrm{C}_G(R) \times O$ is an open normal subgroup of G. Thus

$$R \leq \mathrm{C}_G(R) \times O$$

and in fact $R = O$, so R is a direct factor of the open characteristic subgroup $\mathrm{C}_G(R) \times R$. It is now clear that

$$\mathcal{LN}(P_i)^{P_i} = \{0, \infty\} \quad \text{for } 1 \leq i \leq n;$$

in particular, P_i is directly indecomposable, and every nontrivial closed normal subgroup of P_i is open. Moreover, P_i is of type (JB), so it is locally isomorphic to a just infinite profinite branch group. Given a nontrivial closed subgroup Q of P_i that is normalized by G, we see that Q is open in P_i, and hence the product Q' of Q with the components other than P_i is an open normal subgroup of G. But then $R \leq Q'$, so $P_i \leq Q$. To complete the proof of (ii), suppose for a contradiction P_i is compact. Then P_i is a just infinite profinite group; in particular, $\mathrm{M}(P_i)$ is open in P_i by Lemma 2.6(iii). But then $\mathrm{M}(P_i)$ is characteristic, hence G-invariant, and we have a contradiction, since we have just shown that G preserves no proper nontrivial closed subgroup of P_i. Thus P_i is noncompact as claimed.

Consider now a direct factor D of $\mathrm{Res}(G)$. Since R is [A]-semisimple, D has a unique direct complement $\mathrm{C}_R(D)$, and then $D = \mathrm{C}^2_R(D)$, so D is closed and is the largest representative of an element $\alpha \in \mathcal{LN}(R)$. Since $R = D \times \mathrm{C}_R(D)$, in fact $\alpha \in \mathcal{LN}(R)^R$, so we can write α as the join of $\{\alpha_i \mid i \in I\}$ for some subset I of $\{1, \dots, n\}$. Given the direct decomposition of R into components, we see that in fact D is generated by the components $\{P_i \mid i \in I\}$. This proves (i).

It remains to divide the components into two types. Fix $1 \leq i \leq n$. If $\mathrm{Res}(P_i) > \{1\}$, then by (ii), we must have $\mathrm{Res}(P_i) = P_i$. Since every nontrivial closed normal subgroup of P_i is open, it follows that P_i is topologically simple. Otherwise, clearly P_i is of mysterious type. ∎

Similar to type (JH), there are no known examples of mysterious components; however, unlike for type (JH), we can rule out the mysterious components when G is compactly generated. In fact, in the present situation we can prove a result about $\mathrm{Res}_G(H)$, where H is any compactly generated subgroup of G. We first recall a sufficient condition for a t.d.l.c. group to have a nontrivial contraction group.

Lemma 4.12 (See [7, Proposition 6.14 and Theorem 6.19]): *Let G be a compactly generated t.d.l.c. group that is [A]-semisimple and let $\mathcal{A}$ be a subalgebra of $\mathcal{LC}(G)$ on which G acts faithfully.*

(i) *Suppose that G has a compact open subgroup U such that*

$$\bigcap_{g \in G} gUg^{-1} = \{1\}.$$

Then there is a finite subset $\{\alpha_1, \dots, \alpha_n\}$ of $\mathcal{A}$ such that for all $\beta \in \mathcal{A} \setminus \{0\}$, there is some $g \in G$ and $i \in \{1, \dots, n\}$ such that $g\alpha_i < \beta$.

(ii) *Let V be a compact open subgroup of G, and suppose there is $g \in G$ and $\alpha \in \mathcal{A}$ such that $g\alpha < \alpha$. Then there is a natural number n_0 such that*

$$\mathrm{QC}^2_U(g^{n_0}\alpha \setminus g^{n_0+1}\alpha) \leq \mathrm{con}(g).$$

Lemma 4.13: *Let G be a t.d.l.c. group of type (JB), with $\mathrm{QZ}(G) = \{1\}$, and let H be a compactly generated (not necessarily closed) subgroup of G. Then $\mathrm{Res}_G(H)$ is the direct product of finitely many (possibly none) topologically simple groups, each of which is locally normal in G. Moreover, if H is closed and $\mathrm{Res}_G(H) \leq H$, then*

$$\mathrm{Res}_G(H) = \overline{G^\dagger_H} = \overline{H^\dagger}.$$

Proof. By Lemma 2.9 and replacing G with an open subgroup, we may assume that $\mathrm{Res}_G(H) = \mathrm{Res}(G)$. Since $\mathrm{Res}(G)$ is a direct factor of an open subgroup, there is a natural isomorphism of topological groups

$$\mathrm{Res}(G) \to \mathrm{Res}(G)\mathrm{C}_G(\mathrm{Res}(G))/\mathrm{C}_G(\mathrm{Res}(G)) = \mathrm{Res}(G/\mathrm{C}_G(\mathrm{Res}(G))).$$

If $\mathrm{Res}(G) = \{1\}$ there is nothing more to prove, so without loss of generality, $\mathrm{Res}(G) > \{1\}$. In that case $\mathrm{Res}(G)$ is locally similar to G, since G is locally monomial by Theorem 4.1. We can now replace G with $G/\mathrm{C}_G(\mathrm{Res}(G))$ and hence assume

$$\mathrm{C}_G(\mathrm{Res}(G)) = \{1\}.$$

As a result, $\mathrm{Res}(G)$ is open in G.

By Theorem 4.11, we have

$$\mathrm{Res}(G) = P_1 \times P_2 \times \cdots \times P_n,$$

where $P_1, \dots, P_n$ are the components of G; after replacing G with a finite index open subgroup G_0 and H with $H \cap G_0$, we may assume that $P_1, \dots, P_n$ are normal in G. (Note that by Lemma 2.7, this does not change the (relative) Tits core of H, nor its discrete residual on G.) Under this assumption we see that the elements $\alpha_i = [P_i]$ of $\mathcal{LD}(G)$ are exactly the atoms of $\mathcal{LD}(G)^G$, while $P_1, \dots, P_n$ are exactly the minimal nontrivial closed normal subgroups of G. Applying Lemma 4.12, for $1 \le i \le n$ there is $g_i \in G$ and $0 < \beta_i < \alpha_i$ such that $g_i\beta_i < \beta_i$, and then the contraction group $\mathrm{con}_{P_i}(g_i)$ of g_i acting on P_i is nontrivial. In particular, the semidirect product $R_i = P_i \rtimes \langle g_i \rangle$ (equipped with the product topology) is not residually discrete.

Suppose that P_i is residually discrete. Then R_i satisfies the hypotheses of Lemma 4.10. Hypothesis (i) is clear; hypothesis (ii) follows from Theorem 4.11, ensuring that P_i acts on

$$\mathcal{LD}(P_i) \cong \mathcal{LN}(R_i)$$

with no nontrivial fixed points. For hypotheses (iii) and (iv), we note that P_i has trivial quasi-centre and is locally isomorphic to a just infinite profinite branch group, and then appeal to Lemma 2.6. But then R_i is residually discrete, a contradiction. Thus P_i is not residually discrete, so by Theorem 4.11 it is topologically simple.

In the case that H is closed and $\mathrm{Res}_G(H) \le H$, it is clear that $\overline{G^\dagger_H} = \overline{H^\dagger}$, and the existence of the elements g_i makes it clear that $\overline{H^\dagger} = \mathrm{Res}_G(H)$. ∎

4.5. Noncompact topologically simple direct factors of open subgroups. We can now prove Theorem 1.6 from the introduction.

Proof of Theorem 1.6. We see that $\overline{G^\dagger}$ is a closed G-invariant subgroup of $\mathrm{Res}(G)$, and by Theorem 4.6, $\mathrm{Res}(G)$ splits into monomial parts. Given Theorems 4.7 and 4.11, we can write

$$\mathrm{Res}(G) = P_1 \times P_2 \times \cdots \times P_n,$$

where $\{P_1, \ldots, P_n\}$ is a characteristic class of subgroups of G, and each of the factors P_i is one of the following:

(a) the intersection of an atomic factor of mysterious type with $\mathrm{Res}(G)$;
(b) a locally branch component of mysterious type;
(c) a topologically simple group that is locally isomorphic to a just infinite profinite group.

Note that in all cases,

$$\mathcal{LN}(P_i)^{P_i} = \{0, \infty\}.$$

After passing to a finite index open subgroup (which changes neither $G^\dagger$ nor $\mathrm{Res}(G)$), we may assume P_i is normal in G for $1 \leq i \leq n$. We then see that

$$\overline{G^\dagger} = Q_1 \times Q_2 \times \cdots \times Q_n,$$

where $Q_i = \overline{(P_i)^\dagger_G}$.

Fix $1 \leq i \leq n$ and suppose $Q_i \neq \{1\}$; that is, there is some $g \in G$ with a nontrivial contraction group on P_i. In particular, the group $P_i \rtimes \langle g \rangle$ is not residually discrete. By Lemma 4.10, it follows that P_i is not residually discrete, so we are in case (c), that is, P_i is topologically simple. In particular, $P_i = Q_i$.

Now suppose G is compactly generated. The group $\mathrm{Res}(G)/\overline{G^\dagger}$ is isomorphic to the direct product of those P_i such that $Q_i = \{1\}$. By Lemma 4.13, the factors P_i of type (b) are ruled out, and for type (c), if P_i is locally branch we have $P_i = Q_i$. Thus $\mathrm{Res}(G)/\overline{G^\dagger}$ is a direct product of finitely many groups of type (JH); in particular, it is locally isomorphic to a finite direct product of h.j.i. profinite groups. ∎

Theorem 1.6 has the following corollary, which illustrates a significant connection between topologically simple t.d.l.c. groups and the internal structure of just infinite profinite groups.

Corollary 4.14: *Let U be a just infinite profinite group that is not virtually abelian. Let $V \leq_o U$ and let $\theta : V \to U$ be a continuous injective open homomorphism such that $\theta(W) \neq W$ for all $W \leq_o V$. Then there is a compactly generated t.d.l.c. group G with an open normal topologically simple subgroup S, such that S is locally isomorphic to U (in the case that U is hereditarily just infinite) or locally similar to U (in the case that U is a branch group). Consequently, the composition factors of U are of bounded order.*

Proof. As explained in [1], U naturally embeds in $L = \mathscr{L}(U)$ as a compact open subgroup, and then θ is the restriction of an inner automorphism of L, induced by the element x say of L. By the construction of the group of germs, L is locally just infinite and $\mathrm{QZ}(L) = \{1\}$. By the assumptions on θ and Lemma 2.8, we see that $L^\dagger_x \neq \{1\}$. It follows that $H^\dagger \neq \{1\}$, where $H = \langle U, x\rangle$. By Theorem 1.6, H has a topologically simple locally normal subgroup S. If U is hereditarily just infinite then S is open in H, hence locally isomorphic to U; otherwise, U is monomial by Lemma 4.2, so S is locally similar to U. Finally, we form the group $G = \mathrm{N}_H(S)/\mathrm{C}_H(S)$, which has S as an open normal subgroup.

By [7, Proposition 4.6], taking a compact open subgroup V of S, then the composition factors of V are of bounded order. Since U is similar to V, it follows that the composition factors of U are also of bounded order. ∎

We note also the following result concerning relative discrete residuals. Note that if H has a cocompact polycyclic subgroup, then $\mathrm{Res}_G(H) = \overline{G^\dagger_H}$: see [13, Corollary 1.12].

Theorem 4.15: *Let G be a t.d.l.c. group with (LD) such that $\mathrm{QZ}(G) = \{1\}$ and let H be a compactly generated (not necessarily closed) subgroup of G. Suppose that at least one of the following holds:*

(i) *$H\mathrm{Res}_G(H)/\mathrm{Res}_G(H)$ is virtually polycyclic;*
(ii) *G has no hereditarily just infinite compact locally normal subgroup.*

Then $\mathrm{Res}_G(H)$ is trivial or a direct product of finitely many topologically simple groups, each of which is locally normal in G.

Proof. By Lemma 2.9 and replacing G with an open subgroup, we may assume that $\mathrm{Res}_G(H) = \mathrm{Res}(G)$. We first take the monomial factorization of an open normal subgroup of G,

$$M = M_1 \times M_2 \times \cdots \times M_m,$$

as in Theorem 4.6; it is then clear that $\mathrm{Res}_G(H)$ is the direct product of $\mathrm{Res}_{M_i}(H)$ for $1 \leq i \leq m$. Thus we reduce to the case when G is locally monomial, and hence of type (JH) or (JB).

If G is of type (JH), we are in the case that $H\mathrm{Res}_G(H)/\mathrm{Res}_G(H)$ is virtually polycyclic. We take the atomic factorization of G as in Theorem 4.7, and see by Theorem 4.8(iv) that only the topologically simple atomic factors of G can contribute to $\mathrm{Res}_G(H)$, and hence there are no atomic factors of mysterious type; in other words, $\mathrm{Res}_G(H)$ is a finite direct product of topologically simple groups.

If G is of type (JB), then the conclusions follow from Lemma 4.13. ∎

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