

EXTENDED AFFINE LIE ALGEBRAS, VERTEX ALGEBRAS AND EQUIVARIANT ϕ -COORDINATED QUASI-MODULES

BY

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ABSTRACT

For any nullity 2 extended affine Lie algebra \mathcal{E} of maximal type and $\ell \in \mathbb{C}$, we prove that there exist a vertex algebra $V_{\mathcal{E}}(\ell)$ and an automorphism group G of $V_{\mathcal{E}}(\ell)$ equipped with a linear character χ , such that the category of restricted \mathcal{E} -modules of level ℓ is canonically isomorphic to the category of (G, χ) -equivariant ϕ -coordinated quasi $V_{\mathcal{E}}(\ell)$ -modules. Moreover, when ℓ is a nonnegative integer, there is a quotient vertex algebra $L_{\mathcal{E}}(\ell)$ of $V_{\mathcal{E}}(\ell)$ modulo by a G -stable ideal, and we prove that the integrable restricted \mathcal{E} -modules of level ℓ are exactly the (G, χ) -equivariant ϕ -coordinated quasi- $L_{\mathcal{E}}(\ell)$ -modules.

1. Introduction

Extended affine Lie algebra (EALA for short) was introduced by Hoegh-Krohn and Torresani in [H-KT] with applications to quantum gauge theory, and since then it has been studied extensively in the literature (see [N] and the references therein). An **EALA** by definition is a complex Lie algebra \mathcal{E} , together with a finite-dimensional ad-diagonalizable subalgebra and a nondegenerate invariant symmetric bilinear form, satisfying a list of natural axioms. The isotropic roots (i.e., roots of length 0) in \mathcal{E} generate a free abelian group of finite rank called the nullity of \mathcal{E} . EALAs of nullity 0 and 1 precisely coincide with the finite-dimensional simple Lie algebras and affine Kac–Moody algebras

Received October 16, 2021 and in revised form March 23, 2022

respectively [ABGP]. Meanwhile, the structure of EALAs with positive nullity are like affine Kac–Moody algebras in many ways [ABFP, BGK, CNPY]. It is well-known that affine Kac–Moody algebras through their restricted modules can be naturally associated with vertex algebras and (twisted) modules [FZ, FLM, Li1, Li2]. This association plays an important role in both affine Kac–Moody algebra theory and vertex (operator) algebra theory. It is a natural question to ask whether EALAs of nullity ≥ 2 can be associated with vertex algebras in a similar way as the affine Kac–Moody algebras.

The main goal of this paper is to associate the nullity 2 EALAs and their modules to vertex algebras. We would like to point out that the representation theory of nullity 2 EALAs is totally different from that for EALAs with nullity ≥ 3 (see [ESB, CLT2]), and the nullity 2 EALAs arose naturally in the work of Saito [Sa] and Slodowy [Sl] on simple elliptic singularities and can be connected with Ringel–Hall algebras [LPe]. The subalgebra of an EALA generated by its nonisotropic root vectors is called the core of the EALA, and the core modulo its center is often called the centerless core of the algebra. It is known that the classification of EALAs can be reduced to the classification of their centerless cores [N]. And the centerless cores of nullity 2 EALAs were classified by Allison–Berman–Pianzola in [ABP] (see also [GP]).

In this paper we deal with the nullity 2 EALAs of maximal type in the sense that their cores are centrally closed [BGK]. The representation theory of nullity 2 EALAs of maximal type has been extensively studied (see [G1, G2, G3, GZ, B2, CLT1, CLT2] for example). By applying the theory of equivariant ϕ -coordinated quasi modules for vertex algebras developed by Li (see [Li6, Li8]), we associate all nullity 2 EALAs of maximal type and their restricted modules to vertex algebras.

Li introduced the notion of (G, χ) -equivariant quasi-modules for vertex algebras in [Li3, Li4] to associate certain infinite-dimensional Lie algebras to vertex algebras, where G is a group and χ is a linear character of G . Li [Li6, Li8] also developed a theory of (G, χ) -equivariant ϕ -coordinated quasi-modules for non-local vertex algebras to associate quantum affine algebras with quantum vertex algebras (see also [JKLiT, CLTW]), where ϕ is an associate of the 1-dimensional additive formal group $F(z, w) = z + w$ [Li6]. In this paper we assume the associate $\phi = ze^w$, which is indeed the associate appearing in the quantum vertex algebra theory [Li6, Li8].

If \mathfrak{g} is a finite-dimensional simple Lie algebra with a diagram automorphism ν , we denote by $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$ the corresponding affine Kac–Moody algebra (i.e., nullity 1 EALA), and $\widehat{\mathcal{L}}(\mathfrak{g}, \nu) = [\tilde{\mathcal{L}}(\mathfrak{g}, \nu), \tilde{\mathcal{L}}(\mathfrak{g}, \nu)]$ the derived Lie subalgebra of $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$. We know that the restricted modules for the affine Lie algebra $\widehat{\mathcal{L}}(\mathfrak{g}, \nu)$ of level $\ell \in \mathbb{C}$ can be associated with the ν -twisted modules for the universal affine vertex algebra $V_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ of the untwisted affine Lie algebra $\widehat{\mathcal{L}}(\mathfrak{g}) = \widehat{\mathcal{L}}(\mathfrak{g}, \text{id})$ (see [FZ, Li1, FLM, Li2]). And the integrable restricted modules for $\widehat{\mathcal{L}}(\mathfrak{g}, \nu)$ of level $\ell \in \mathbb{N}$ can be associated with the ν -twisted modules for the corresponding simple affine vertex algebras $L_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ (see [FZ, Li1, Li2]).

Similar to the association of the affine Lie algebra $\widehat{\mathcal{L}}(\mathfrak{g}, \nu)$ and its restricted modules with the affine vertex algebras and their twisted modules, we can apply the equivariant ϕ -coordinated quasi-modules for the affine vertex algebras to associate with the restricted modules for the affine Kac–Moody algebra $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$. For this purpose, we first, by using results from [Z] and [Li5], investigate the natural connections among equivariant ϕ -coordinated quasi-modules, equivariant quasi modules and twisted modules for the general vertex operator algebras (see Proposition 3.4). And then we prove that the category of restricted (resp. integrable restricted) modules for the affine Kac–Moody algebra $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$ of level ℓ is isomorphic to the category of equivariant ϕ -coordinated quasi-modules for the universal (resp. simple) affine vertex algebra $V_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ (resp. $L_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$) (see Theorem 3.11).

Let \mathfrak{g} be the untwisted affine Kac–Moody algebra $\tilde{\mathcal{L}}(\mathfrak{g}, \text{id})$, and μ a nontransitive diagram automorphism of \mathfrak{g} . We can also define the twisted toroidal EALA $\tilde{\mathfrak{g}}[\mu]$ similar to the construction of twisted affine Kac–Moody algebra. Allison–Berman–Pianzola proved in [ABP] that a nullity 2 EALA of maximal type is either isomorphic to a twisted toroidal EALA $\tilde{\mathfrak{g}}[\mu]$, or to EALA $\tilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ of type A_{N-1} coordinated by an irrational quantum torus \mathbb{C}_q (see Section 5 for details). It was discovered by Billig in [B2] that an untwisted toroidal EALA is in general not a vertex Lie algebra in the sense of [DLM]. Thus one cannot associate the restricted $\tilde{\mathfrak{g}}[\mu]$ -modules, as in the affine Kac–Moody algebra case, to twisted modules of vertex algebras. It was also pointed out in [Li3] that, since the generating functions of $\tilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ are not “local” in general, the restricted modules of EALA $\tilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ cannot be directly associated to twisted modules of vertex algebras.

As the main result of this paper, we prove that every nullity 2 EALA of maximal type can be associated with a vertex algebra through equivariant ϕ -coordinated quasi-modules. More explicitly, for any nontransitive diagram automorphism μ of an untwisted affine Kac–Moody algebra \mathfrak{g} , we construct a vertex algebra $V_{\widehat{\mathfrak{g}}}(\ell, 0)$, a quotient vertex algebra $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ of $V_{\widehat{\mathfrak{g}}}(\ell, 0)$, an automorphism group G_{μ} of $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ and $L_{\widehat{\mathfrak{g}}}(\ell, 0)$, and a linear character χ_{ω} of G_{μ} . And then we establish a module category isomorphism from the category of restricted (resp. integrable restricted) $\widetilde{\mathfrak{g}}[\mu]$ -modules of level ℓ to the category of (G_{μ}, χ_{ω}) -equivariant ϕ -coordinated quasi-modules for $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ (resp. $L_{\widehat{\mathfrak{g}}}(\ell, 0)$) (see Theorem 6.8). Meanwhile, for any positive integer $N \geq 2$ and generic complex number q , we also construct a universal (resp. simple) affine vertex algebras $V_{\widehat{\mathfrak{sl}}_{\infty}}(\ell, 0)$ (resp. $L_{\widehat{\mathfrak{sl}}_{\infty}}(\ell, 0)$) associated to \mathfrak{sl}_{∞} , an automorphism group G_N of the affine vertex algebras, and a linear character χ_q of G_N . And then we prove that the category of restricted (resp. integrable restricted) $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ -modules of level ℓ is canonically isomorphic to the category of (G_N, χ_q) -equivariant ϕ -coordinated quasi-modules for $V_{\widehat{\mathfrak{sl}}_{\infty}}(\ell, 0)$ (resp. $L_{\widehat{\mathfrak{sl}}_{\infty}}(\ell, 0)$) (see Theorem 7.5).

The structure of the paper is given as follows. In Section 2 we recall the notion of (G, χ) -equivariant ϕ -coordinated quasi-module for a vertex algebra introduced in [Li7], and consider the (G, χ) -equivariant ϕ -coordinated quasi-modules for the universal enveloping vertex algebra of a conformal algebra. After giving natural connections among equivariant ϕ -coordinated quasi-modules, equivariant quasi-modules and twisted modules for vertex operator algebras and (general) universal affine vertex algebras, in Section 3 we prove the isomorphism between the categories of restricted modules for affine Kac–Moody algebras and equivariant ϕ -coordinated quasi-modules for affine vertex algebras. In Section 4, for any diagram automorphism μ of an untwisted affine Kac–Moody algebra \mathfrak{g} , we construct an automorphism $\widetilde{\mu}$ of the toroidal EALA $\widetilde{\mathfrak{g}}$ associated to \mathfrak{g} and study the Lie subalgebra $\widetilde{\mathfrak{g}}[\mu]$ of $\widetilde{\mathfrak{g}}$ fixed by the automorphism $\widetilde{\mu}$. We recall Allison–Berman–Pianzola’s classification result of nullity 2 EALAs of maximal type in Section 5. And then in Section 6, we define two vertex algebras $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ and $L_{\widehat{\mathfrak{g}}}(\ell, 0)$, and associate restricted (resp. integrable restricted) $\widetilde{\mathfrak{g}}[\mu]$ -modules of level ℓ with equivariant ϕ -coordinated quasi-modules for the vertex algebra $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ (resp. $L_{\widehat{\mathfrak{g}}}(\ell, 0)$). Finally, in Section 7, we associate

restricted (resp. integrable restricted) $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ -modules of level ℓ with equivariant ϕ -coordinated quasi-modules for universal (resp. simple) affine vertex algebras associated to \mathfrak{sl}_∞ .

In this paper we denote by $\mathbb{Z}, \mathbb{Z}^*, \mathbb{N}, \mathbb{C}$ and \mathbb{C}^* respectively the sets of integers, nonzero integers, nonnegative integers, complex numbers and nonzero complex numbers. And if \mathfrak{g} is a Lie algebra, we denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra.

2. Equivariant ϕ -coordinated quasi-modules for vertex algebras

In this section we recall the notion and some basics on equivariant ϕ -coordinated quasi-modules for vertex algebras (cf. [Li8, JKLiT, CLTW]).

2.1. DEFINITIONS AND BASIC PROPERTIES. Throughout this paper, let $z, w, z_0, z_1, z_2, \dots$ be mutually commuting independent formal variables. We use the standard notations and conventions as in [FHL, LeLi]. For example, for a vector space U , $U[[z_1, z_2, \dots, z_r]]$ is the space of formal (possibly doubly infinite) power series in z_1, z_2, \dots, z_r with coefficients in U , and $U((z_1, z_2, \dots, z_r))$ is the space of lower truncated Laurent power series in z_1, z_2, \dots, z_r with coefficients in U . Denote a vertex algebra by $V = (V, Y, \mathbf{1})$, where $\mathbf{1}$ is the vacuum vector and

$$Y(\cdot, z) : V \rightarrow \text{Hom}(V, V((z))),$$

$$v \mapsto \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$$

is the vertex operator. And the canonical derivation on V defined by $v \mapsto v_{-2}\mathbf{1}$ for $v \in V$ is denoted by \mathcal{D} .

For a subset Γ of \mathbb{C}^* , denote by $\mathbb{C}_\Gamma[z]$ the set of all polynomials in $\mathbb{C}[z]$ whose roots are contained in Γ . Let ϕ be the formal power series $\phi(z_2, z_0) = z_2 e^{z_0}$, which is a particular associate of the one-dimensional additive formal group $F(z, w) = z + w$ as defined in [Li3]. Now we recall the notions of equivariant ϕ -coordinated quasi-modules for a vertex algebra (see [Li8]).

Definition 2.1: Let $(V, Y, \mathbf{1})$ be a vertex algebra, G a group of automorphisms on V and $\chi : G \rightarrow \mathbb{C}^*$ a linear character of G . A (G, χ) -equivariant ϕ -coordinated quasi- V -module (W, Y_W^ϕ) is a vector space W equipped with a linear map

$$Y_W^\phi(\cdot, z) : V \rightarrow \text{Hom}(W, W((z))) \subset (\text{End}W)[[z, z^{-1}]]$$

satisfying the following three conditions:

- (i) $Y_W^\phi(\mathbf{1}, z) = 1_W$;
- (ii) $Y_W^\phi(gv, z) = Y_W^\phi(v, \chi(g)z)$ for $g \in G, v \in V$;
- (iii) for $u, v \in V$, there exists $f(z) \in \mathbb{C}_{\chi(G)}[z]$ such that

$$f(z_1/z_2)Y_W^\phi(u, z_1)Y_W^\phi(v, z_2) \in \text{Hom}(W, W((z_1, z_2))),$$

$$f(e^{z_0})Y_W^\phi(Y(u, z_0)v, z_2) = (f(z_1/z_2)Y_W^\phi(u, z_1)Y_W^\phi(v, z_2))|_{z_1=\phi(z_2, z_0)}.$$

Furthermore, a (G, χ) -equivariant ϕ -coordinated quasi- V -module (W, Y_W^ϕ, d) is a (G, χ) -equivariant ϕ -coordinated quasi- V -module (W, Y_W^ϕ) equipped with an endomorphism d of W such that $[d, Y_W^\phi(v, z)] = Y_W^\phi(Dv, z)$ for $v \in V$.

Now we fix a vertex algebra V , an automorphism group G of V , and a linear character χ of G . For a (G, χ) -equivariant ϕ -coordinated quasi- V -module (W, Y_W^ϕ) . It was proved in [Li6, Lemma 3.7] that

$$(2.1) \quad Y_W^\phi(Dv, z) = z \frac{d}{dz} Y_W^\phi(v, z), \quad \forall v \in V.$$

This immediately gives the following result.

LEMMA 2.2: For a (G, χ) -equivariant ϕ -coordinated quasi- V -module (W, Y_W^ϕ, d) , we have

$$[d, Y_W^\phi(v, z)] = z \frac{d}{dz} Y_W^\phi(v, z), \quad \forall v \in V.$$

Furthermore, we have

LEMMA 2.3: Let (W, Y_W^ϕ) be a (G, χ) -equivariant ϕ -coordinated quasi- V -module and d a linear transformation on W . Assume that there is a generating set A of V such that $[d, Y_W^\phi(v, z)] = z \frac{d}{dz} Y_W^\phi(v, z)$ for all $v \in A$. Then (W, Y_W^ϕ, d) is a (G, χ) -equivariant ϕ -coordinated quasi- V -module.

Proof. Set

$$V' = \{v \in V \mid [d, Y_W^\phi(v, z)] = Y_W^\phi(Dv, z)\}.$$

For $u, v \in V'$, take $f(z) \in \mathbb{C}_{\chi(G)}[z]$ such that

$$f(z_1/z_2)Y_W^\phi(u, z_1)Y_W^\phi(v, z_2) \in \text{Hom}(W, W((z_1, z_2))).$$

Then from (2.1) we have

$$f^2(z_1/z_2)Y_W^\phi(Du, z_1)Y_W^\phi(v, z_2), f^2(z_1/z_2)Y_W^\phi(u, z_1)Y_W^\phi(Dv, z_2)$$

$$\in \text{Hom}(W, W((z_1, z_2))),$$

which implies that

$$\begin{aligned}
 f^2(e^{z_0})Y_W^\phi(Y(\mathcal{D}u, z_0)v, z_2) &= (f^2(z_1/z_2)Y_W^\phi(\mathcal{D}u, z_1)Y_W^\phi(v, z_2))|_{z_1=\phi(z_2, z_0)}, \\
 f^2(e^{z_0})Y_W^\phi(Y(u, z_0)\mathcal{D}v, z_2) &= (f^2(z_1/z_2)Y_W^\phi(u, z_1)Y_W^\phi(\mathcal{D}v, z_2))|_{z_1=\phi(z_2, z_0)}.
 \end{aligned}$$

Since \mathcal{D} is a derivation on V and $u, v \in V'$, we have

$$\begin{aligned}
 f^2(e^{z_0})Y_W^\phi(\mathcal{D}Y(u, z_0)v, z_2) &= f^2(e^{z_0})Y_W^\phi(Y(\mathcal{D}u, z_0)v + Y(u, z_0)\mathcal{D}v, z_2) \\
 &= (f^2(z_1/z_2)(Y_W^\phi(\mathcal{D}u, z_1)Y_W^\phi(v, z_2) + Y_W^\phi(u, z_1)Y_W^\phi(\mathcal{D}v, z_2)))|_{z_1=\phi(z_2, z_0)} \\
 &= (f^2(z_1/z_2)([d, Y_W^\phi(u, z_1)]Y_W^\phi(v, z_2) + Y_W^\phi(u, z_1)[d, Y_W^\phi(v, z_2)])|_{z_1=\phi(z_2, z_0)} \\
 &= [d, (f^2(z_1/z_2)Y_W^\phi(u, z_1)Y_W^\phi(v, z_2))|_{z_1=\phi(z_2, z_0)}] \\
 &= [d, f^2(e^{z_0})Y_W^\phi(Y(u, z_0)v, z_2)].
 \end{aligned}$$

This implies that V' is a subalgebra of V . Since the generating set A lies in V' , we have $V' = V$, as required. ■

The following analogue of Borcherds commutator formula was proved in [CLTW, Proposition 5.2].

PROPOSITION 2.4: *Let (W, Y_W^ϕ) be a (G, χ) -equivariant ϕ -coordinated quasi V -module and $\psi : \chi(G) \rightarrow G$ be a section of χ . Then for $u, v \in V$,*

$$\begin{aligned}
 [Y_W^\phi(u, z_1), Y_W^\phi(v, z_2)] &= \text{Res}_{z_0} \sum_{g \in \psi(\chi(G))} Y_W^\phi(Y(gu, z_0)v, z_2) e^{z_0 z_2 \frac{\partial}{\partial z_2}} \left(\chi(g) \delta \left(\frac{\chi(g) z_2}{z_1} \right) \right).
 \end{aligned}$$

Using the above formula we have:

LEMMA 2.5: *Let (W, Y_W^ϕ) be a (G, χ) -equivariant ϕ -coordinated quasi V -module, and $u, v \in V$ be such that $u_n v = 0$ for $n \geq 0$. Then there exists a polynomial $q(z) \in \mathbb{C}_{\chi(G) \setminus \{1\}}[z]$ such that*

$$(2.2) \quad q(z_1/z_2)Y_W^\phi(u, z_1)Y_W^\phi(v, z_2) = q(z_1/z_2)Y_W^\phi(v, z_2)Y_W^\phi(u, z_1).$$

Furthermore, for any such polynomial $q(z)$ we have

$$(2.3) \quad Y_W^\phi(u_{-1}v, z_2) = (q(z_1/z_2)Y_W^\phi(u, z_1)Y_W^\phi(v, z_2))|_{z_1=z_2}.$$

Proof. Since $u_n v = 0$ for $n \geq 0$, from Proposition 2.4 it follows that there exist (possibly the same) $\lambda_1, \dots, \lambda_r \in \chi(G) \setminus \{1\}$ such that

$$(z_1/z_2 - \lambda_1) \cdots (z_1/z_2 - \lambda_r) [Y_W^\phi(u, z_1), Y_W^\phi(v, z_2)] = 0.$$

This proves (2.2) with $q(z) = (z - \lambda_1) \cdots (z - \lambda_r)$. Then by definition we have

$$q(e^{z_0}) Y_W^\phi(Y(u, z_0)v, z_2) = (q(z_1/z_2) Y_W^\phi(u, z_1) Y_W^\phi(v, z_2))|_{z_1=z_2 e^{z_0}}.$$

Note that $Y(u, z_0)v \in V[[z_0]]$; one can set $z_0 = 0$ in the above equality, which gives (2.3). ■

Furthermore, we have the following results (cf. [Li1, Proposition 2.3.6], [Li2, Proposition 2.10]).

PROPOSITION 2.6: *Let $u \in V$ such that $u_n u = 0$ for $n \geq 0$, ℓ a positive integer and (W, Y_W^ϕ) a (G, χ) -equivariant ϕ -coordinated quasi- V -module. Then there exists a polynomial $q(z) \in \mathbb{C}_{\chi(G) \setminus \{1\}}[z]$ such that*

$$(2.4) \quad \left(\prod_{1 \leq i < j \leq \ell+1} q(z_i/z_j) \right) Y_W^\phi(u, z_1) \cdots Y_W^\phi(u, z_{\ell+1}) \in \text{Hom}(W, W((z_1, \dots, z_{\ell+1}))).$$

Moreover, if $(u_{-1})^{\ell+1} \mathbf{1} = 0$, then

$$(2.5) \quad \left(\prod_{1 \leq i < j \leq \ell+1} q(z_i/z_j) \right) Y_W^\phi(u, z_1) \cdots Y_W^\phi(u, z_{\ell+1})|_{z_1=\dots=z_{\ell+1}} = 0.$$

And on the other hand, if W is faithful and (2.5) holds, then $(u_{-1})^{\ell+1} \mathbf{1} = 0$.

Proof. Note that the first part of the proposition follows from (2.2). For the second part, it suffices to prove the following identity:

$$(2.6) \quad \begin{aligned} & Y_W^\phi((u_{-1})^{\ell+1} \mathbf{1}, z_{\ell+1}) \\ &= \left(\prod_{1 \leq i < j \leq \ell+1} q(z_i/z_j) \right) Y_W^\phi(u, z_1) \cdots Y_W^\phi(u, z_{\ell+1})|_{z_1=\dots=z_{\ell+1}}. \end{aligned}$$

We prove this equation by induction on ℓ . When $\ell = 1$, the identity (2.6) follows from (2.3). Now we assume that $\ell > 1$ and set $v = (u_{-1})^\ell \mathbf{1}$. Then by induction

and (2.4), we obtain

$$\begin{aligned}
 & q(z_1/z_{\ell+1})^\ell Y_W^\phi(u, z_1) Y_W^\phi(v, z_{\ell+1}) \\
 &= q(z_1/z_{\ell+1})^\ell Y_W^\phi(u, z_1) \\
 &\quad \times \left(\prod_{2 \leq i < j \leq \ell+1} q(z_i/z_j) \right) Y_W^\phi(u, z_2) \cdots Y_W^\phi(u, z_{\ell+1})|_{z_2=\dots=z_{\ell+1}} \\
 &= \left(\prod_{1 \leq i < j \leq \ell+1} q(z_i/z_j) \right) Y_W^\phi(u, z_1) \cdots Y_W^\phi(u, z_{\ell+1})|_{z_2=\dots=z_{\ell+1}} \\
 &\in \text{Hom}(W, W((z_1, z_{\ell+1}))).
 \end{aligned}$$

Since $u_n u = 0$ for $n \geq 0$, we have $u_n v = 0$ for $n \geq 0$. Then it follows from (2.2) that

$$\begin{aligned}
 Y_W^\phi((u_{-1})^{\ell+1} \mathbf{1}, z_{\ell+1}) &= Y_W^\phi(u_{-1} v, z_{\ell+1}) \\
 &= (q(z_1/z_{\ell+1})^\ell Y_W^\phi(u, z_1) Y_W^\phi(v, z_{\ell+1}))|_{z_1=z_{\ell+1}} \\
 &= \left(\prod_{1 \leq i < j \leq \ell+1} q(z_i/z_j) \right) Y_W^\phi(u, z_1) \cdots Y_W^\phi(u, z_{\ell+1})|_{z_1=\dots=z_{\ell+1}},
 \end{aligned}$$

which proves (2.6) and hence completes the proof of the proposition. ■

2.2. (G, χ) -EQUIVARIANT ϕ -COORDINATED QUASI- $V_{\mathcal{C}}$ -MODULES. In this subsection we first recall the notion of conformal algebra, then study the equivariant ϕ -coordinated quasi-modules for the universal enveloping vertex algebra constructed from a conformal algebra [CLTW].

A **conformal algebra**, also known as a **vertex Lie algebra** (see [P, DLM]), is a vector space \mathcal{C} equipped with a linear operator ∂ and a linear map

$$\begin{aligned}
 & Y^- : \mathcal{C} \rightarrow \text{Hom}(\mathcal{C}, z^{-1}\mathcal{C}[z^{-1}]), \\
 (2.7) \quad & u \mapsto Y^-(u, z) = \sum_{n \geq 0} u_n z^{-n-1}
 \end{aligned}$$

such that for any $u, v \in \mathcal{C}$,

$$\begin{aligned}
 (2.8) \quad & [\partial, Y^-(u, z)] = Y^-(\partial u, z) = \frac{d}{dz} Y^-(u, z), \\
 & Y^-(u, z)v = \text{Sing}(e^{z\partial} Y^-(v, -z)u), \\
 & [Y^-(u, z), Y^-(v, w)] = \text{Sing}(Y^-(Y^-(u, z-w)v, w)),
 \end{aligned}$$

where Sing stands for the singular part.

It was proved in [P, Remark 4.2] that a conformal algebra structure on a vector space \mathcal{C} amounts to a Lie algebra structure on the following quotient space of $\mathbb{C}[t, t^{-1}] \otimes \mathcal{C}$:

$$\widehat{\mathcal{C}} = \mathbb{C}[t, t^{-1}] \otimes \mathcal{C} / \left(1 \otimes \partial + \frac{d}{dt} \otimes 1 \right) (\mathbb{C}[t, t^{-1}] \otimes \mathcal{C}).$$

LEMMA 2.7: *Let \mathcal{C} be a vector space equipped with a linear operator ∂ and a linear map Y^- as given in (2.7) such that (2.8) holds. Then \mathcal{C} is a conformal algebra if and only if there is a Lie algebra structure on $\widehat{\mathcal{C}}$ such that*

$$(2.9) \quad [u(m), v(n)] = \sum_{i \geq 0} \binom{m}{i} (u_i v)(m + n - i),$$

for $u, v \in \mathcal{C}$, $m, n \in \mathbb{Z}$, where $u(m)$ stands for the image of $t^m \otimes u$ in $\widehat{\mathcal{C}}$.

Let \mathcal{C} be a conformal algebra. Set

$$(2.10) \quad \begin{aligned} \widehat{\mathcal{C}}^- &= \text{Span}\{u(-m-1) \mid u \in \mathcal{C}, m \in \mathbb{N}\}, \\ \widehat{\mathcal{C}}^+ &= \text{Span}\{u(m) \mid u \in \mathcal{C}, m \in \mathbb{N}\}. \end{aligned}$$

Then $\widehat{\mathcal{C}} = \widehat{\mathcal{C}}^+ \oplus \widehat{\mathcal{C}}^-$ and both $\widehat{\mathcal{C}}^+$ and $\widehat{\mathcal{C}}^-$ are subalgebras of the Lie algebra $\widehat{\mathcal{C}}$. Moreover, the map

$$(2.11) \quad \mathcal{C} \rightarrow \widehat{\mathcal{C}}^-, \quad u \mapsto u(-1)$$

is an isomorphism of vector spaces [P, Theorem 4.6]. Consider the induced $\widehat{\mathcal{C}}$ -module

$$V_{\mathcal{C}} = \mathcal{U}(\widehat{\mathcal{C}}) \otimes_{\mathcal{U}(\widehat{\mathcal{C}}^+)} \mathbb{C},$$

where \mathbb{C} is the one-dimensional trivial $\widehat{\mathcal{C}}^+$ -module. Set $\mathbf{1} = 1 \otimes 1 \in V_{\mathcal{C}}$. Identify \mathcal{C} as a subspace of $V_{\mathcal{C}}$ through the linear map $u \mapsto u(-1)\mathbf{1}$. It was proved in [P] that there exists a unique vertex algebra structure on $V_{\mathcal{C}}$, called the **universal enveloping vertex algebra** of \mathcal{C} , with $\mathbf{1}$ as the vacuum vector and

$$Y(u, z) = u(z) = \sum_{n \in \mathbb{Z}} u(n)z^{-n-1}$$

for $u \in \mathcal{C}$. The map $-\frac{d}{dt} \otimes 1$ (or equivalently, $1 \otimes \partial$) on $\mathbb{C}[t, t^{-1}] \otimes \mathcal{C}$ induces a derivation \mathcal{D} on $\widehat{\mathcal{C}}$ such that

$$(2.12) \quad \mathcal{D}(u(m)) = -mu(m-1), \quad \forall u \in \mathcal{C}, m \in \mathbb{Z}.$$

Note that \mathcal{D} preserves the subalgebra $\widehat{\mathcal{C}}^-$. Thus it can be uniquely extended to a derivation on $\mathcal{U}(\widehat{\mathcal{C}}^-) \cong V_{\mathcal{C}}$, which coincides with the canonical derivation on the vertex algebra $V_{\mathcal{C}}$.

Recall that an **automorphism** φ of the conformal algebra \mathcal{C} is a linear automorphism such that $\varphi \circ \partial = \partial \circ \varphi$ and $\varphi(u_i v) = \varphi(u)_i \varphi(v)$ for $u, v \in \mathcal{C}$ and $i \in \mathbb{N}$ (see [K2]). We have

LEMMA 2.8: *Let φ be a linear map of the conformal algebra \mathcal{C} such that $\varphi \circ \partial = \partial \circ \varphi$. Then the linear map*

$$(2.13) \quad \begin{aligned} \hat{\varphi} : \widehat{\mathcal{C}} &\rightarrow \widehat{\mathcal{C}}, \\ u(m) &\mapsto \varphi(u)(m) \quad (u \in \mathcal{C}, m \in \mathbb{Z}) \end{aligned}$$

on the Lie algebra $\widehat{\mathcal{C}}$ is well-defined. Furthermore, φ is an automorphism of the conformal algebra \mathcal{C} if and only if $\hat{\varphi}$ is an automorphism of the Lie algebra $\widehat{\mathcal{C}}$.

Proof. The first assertion of the lemma is easy to see. For the second part, let $u, v \in \mathcal{C}$ and $m, n \in \mathbb{Z}$. Then by (2.9) we have

$$(2.14) \quad \begin{aligned} \hat{\varphi}([u(m), v(n)]) &= \sum_{i \geq 0} \binom{m}{i} \hat{\varphi}((u_i v)(m+n-i)) \\ &= \sum_{i \geq 0} \binom{m}{i} \varphi(u_i v)(m+n-i), \end{aligned}$$

and

$$(2.15) \quad \begin{aligned} \hat{\varphi}(u(m)), \hat{\varphi}(v(n))t &= [\varphi(u)(m), \varphi(v)(n)] \\ &= \sum_{i \geq 0} \binom{m}{i} ((\varphi u)_i(\varphi v))(m+n-i). \end{aligned}$$

This immediately implies that if φ is an automorphism of \mathcal{C} , then $\hat{\varphi}$ is a Lie algebra automorphism of $\widehat{\mathcal{C}}$. On the other hand, let $\hat{\varphi}$ be a Lie algebra automorphism of $\widehat{\mathcal{C}}$. Assume that φ is not an automorphism of \mathcal{C} . Then there exist $u, v \in \mathcal{C}$ such that $\varphi(u_i v) \neq (\varphi u)_i(\varphi v)$ for some $i \in \mathbb{N}$. Let i_0 be the maximal one among such integers. Take $m, n \in \mathbb{Z}$ such that $m+n = i_0 - 1$. Then we have

$$\sum_{i=0}^{i_0-1} \binom{m}{i} \varphi(u_i v)(m+n-i), \quad \sum_{i=0}^{i_0-1} \binom{m}{i} ((\varphi u)_i(\varphi v))(m+n-i) \in \widehat{\mathcal{C}}^+,$$

and

$$\sum_{i \geq i_0} \binom{m}{i} \varphi(u_i v)(m+n-i), \sum_{i \geq i_0} \binom{m}{i} ((\varphi u)_i (\varphi v))(m+n-i) \in \widehat{\mathcal{C}}^-.$$

Recall that $\widehat{\mathcal{C}}^+ \cap \widehat{\mathcal{C}}^- = \{0\}$ and $\hat{\varphi}$ is a Lie algebra automorphism. Then by (2.14), (2.15) and the maximality of i_0 we obtain that

$$\varphi(u_{i_0} v)(-1) = ((\varphi u)_{i_0} (\varphi v))(-1).$$

This together with (2.11) gives $\varphi(u_{i_0} v) = (\varphi u)_{i_0} (\varphi v)$, a contradiction. Therefore we have finished the proof of the lemma. ■

We define a multiplication on the loop space $\mathbb{C}[t, t^{-1}] \otimes \mathcal{C}$ by

$$(2.16) \quad \begin{aligned} &= \sum_{i \geq 0} \left(\frac{1}{i!} \left(t \frac{d}{dt} \right)^i f(t) \right) g(t) \otimes (u_i v) \\ & \qquad \qquad \qquad (f(t), g(t) \in \mathbb{C}[t, t^{-1}], u, v \in \mathcal{C}). \end{aligned}$$

By [K2, Remark 2.7d] this multiplication affords a Lie algebra structure on the quotient space

$$\overline{\mathcal{C}} = (\mathbb{C}[t, t^{-1}] \otimes \mathcal{C}) / \text{Im} \left(1 \otimes \partial + t \frac{d}{dt} \otimes 1 \right).$$

Denote by \mathbf{d} the derivation on $\overline{\mathcal{C}}$ induced from $-t \frac{d}{dt} \otimes 1 \in \text{End}(\mathbb{C}[t, t^{-1}] \otimes \mathcal{C})$. Form the semi-direct product Lie algebra

$$\widetilde{\mathcal{C}} = \overline{\mathcal{C}} \rtimes \mathbb{C}\mathbf{d}.$$

For $u \in \mathcal{C}$ and $m \in \mathbb{Z}$, denote by $u[m]$ the image of $t^m \otimes u$ in $\overline{\mathcal{C}}$. Then the Lie relations on $\widetilde{\mathcal{C}}$ are given by

$$(2.17) \quad [u[m], v[n]] = \sum_{i \geq 0} \frac{m^i}{i!} (u_i v)[m+n], \quad [\mathbf{d}, u[m]] = -mu[m]$$

for $u, v \in \mathcal{C}, m, n \in \mathbb{Z}$. The following notion was first introduced in [G-KK] (see also [Li3]).

Definition 2.9: A *G-conformal algebra* is a conformal algebra \mathcal{C} together with an automorphism group G of \mathcal{C} such that for $u \in \mathcal{C}, Y^-(gu, z) = 0$ for all but finitely many $g \in G$.

Let \mathcal{C} be a G -conformal algebra and $\chi : G \rightarrow \mathbb{C}^*$ be a linear character. For any $g \in G$, it is easy to check that (cf. [CLTW, Lemma 5.8]) the linear map

$$\bar{g} : \bar{\mathcal{C}} \rightarrow \bar{\mathcal{C}}, \quad u[m] \mapsto \chi(g)^m (gu)[m]$$

defines an automorphism of the Lie algebra $\bar{\mathcal{C}}$. Furthermore, \bar{g} can be extended to be an automorphism \tilde{g} of $\tilde{\mathcal{C}}$ by

$$(2.18) \quad \tilde{g}|_{\bar{\mathcal{C}}} = \bar{g} \quad \text{and} \quad \tilde{g}(\mathbf{d}) = \mathbf{d}.$$

Following [Li6, Section 4], we define a new operation on $\bar{\mathcal{C}}$ by

$$(a, b) \mapsto \sum_{g \in G} [\bar{g}a, b] \quad (a, b \in \bar{\mathcal{C}}).$$

It was proved therein that the quotient space

$$(2.19) \quad \bar{\mathcal{C}}[G] = \bar{\mathcal{C}} / \text{Span}\{\bar{g}a - a \mid a \in \bar{\mathcal{C}}, g \in G\}$$

is a Lie algebra under the operation. For $u \in \mathcal{C}$ and $n \in \mathbb{Z}$, we denote by $\overline{u[n]}$ the image of $u[n]$ under the quotient map $\bar{\mathcal{C}} \rightarrow \bar{\mathcal{C}}[G]$. Since $g \circ \mathbf{d} = \mathbf{d} \circ g$ for $g \in G$, \mathbf{d} descends to a derivation on $\bar{\mathcal{C}}[G]$ and hence we have the semi-direct product Lie algebra

$$(2.20) \quad \tilde{\mathcal{C}}[G] = \bar{\mathcal{C}}[G] \rtimes \mathbb{C}\mathbf{d}.$$

For $u \in \mathcal{C}$, we form the generating function $u[z] = \sum_{n \in \mathbb{Z}} \overline{u[n]} z^{-n}$, and we say that a module W for $\tilde{\mathcal{C}}[G]$ or $\bar{\mathcal{C}}[G]$ is **restricted** if $u[z] \in \text{Hom}(W, W((z)))$ for $u \in \mathcal{C}$. It is known from [P] that an automorphism of \mathcal{C} can be lifted uniquely to an automorphism of $V_{\mathcal{C}}$. In particular, the automorphism group G of \mathcal{C} can be viewed as an automorphism group of $V_{\mathcal{C}}$. Now we have the following result.

PROPOSITION 2.10: *Let \mathcal{C} be a G -conformal algebra and $\chi : G \rightarrow \mathbb{C}^*$ be an injective linear character. Then the (G, χ) -equivariant ϕ -coordinated quasi $V_{\mathcal{C}}$ -modules (W, Y_W^ϕ, d) are exactly the restricted $\tilde{\mathcal{C}}[G]$ -modules W with*

$$d = \mathbf{d} \quad \text{and} \quad Y_W^\phi(u, z) = u[z] \quad \text{for } u \in \mathcal{C}.$$

Proof. It was proved in [CLTW, Theorem 5.12] that the (G, χ) -equivariant ϕ -coordinated quasi- $V_{\mathcal{C}}$ -modules (W, Y_W^ϕ) are exactly the restricted $\bar{\mathcal{C}}[G]$ -modules W with $Y_W^\phi(u, z) = u[z]$ for $u \in \mathcal{C}$. Note that for $u \in \mathcal{C}$, we have

$$[\mathbf{d}, u[z]] = \sum_{n \in \mathbb{Z}} [\mathbf{d}, \overline{u[n]}] z^{-n} = \sum_{n \in \mathbb{Z}} -n \overline{u[n]} z^{-n} = z \frac{d}{dz} u[z].$$

This together with Lemma 2.2 and Lemma 2.3 proves the proposition. ■

Remark 2.11: Assume that $G = \langle g \rangle$ is a cyclic group of finite order T , and the linear character χ is injective. Then it is straightforward to see that $\overline{\mathcal{C}}[G]$ is isomorphic to the subalgebra of $\overline{\mathcal{C}}$ fixed by \bar{g} . And the isomorphism is given by

$$\overline{u[n]} \mapsto \sum_{p=0}^{T-1} (\bar{g})^p(u[n])$$

for $u \in \mathcal{C}$ and $n \in \mathbb{Z}$. Furthermore, this isomorphism can be extended to an isomorphism from $\widetilde{\mathcal{C}}[G]$ to the subalgebra of $\widetilde{\mathcal{C}}$ fixed by \tilde{g} such that $\mathbf{d} \mapsto \mathbf{d}$.

3. Equivariant ϕ -coordinated quasi-modules for affine vertex algebras

In this section, we study the connection between equivariant ϕ -coordinated quasi-modules for universal (resp. simple) affine vertex algebras and restricted (resp. integrable restricted) modules for affine Kac–Moody algebras.

3.1. EQUIVALENCE OF MODULE CATEGORIES FOR VERTEX OPERATOR ALGEBRAS. We first study connections among equivariant ϕ -coordinated quasi-modules, equivariant quasi-modules and twisted modules for vertex operator algebras (see [FLM, FHL]). Recall that for any finite order automorphism σ of a vertex algebra V , there is a (weak) σ -**twisted V -module** (W, Y_W^t) , where W is a vector space and

$$Y_W^t \in \text{Hom}(W, W((z^{1/N})))$$

with N the order of σ (cf. [Li2, FLM]). We also recall that a \mathbb{Z} -**graded vertex algebra** is a vertex algebra V equipped with a \mathbb{Z} -grading $V = \bigoplus_{n \in \mathbb{Z}} V_{(n)}$ such that

$$\mathbf{1} \in V_{(0)}$$

and

$$(3.1) \quad u_m V_{(n)} \subset V_{(n+k-m-1)} \quad \text{for } u \in V_{(k)}, m, n, k \in \mathbb{Z}.$$

Define the linear operator $L(0)$ on a \mathbb{Z} -graded vertex algebra V by $L(0)v = nv$ for $v \in V_{(n)}$ with $n \in \mathbb{Z}$. Then we have the following definition (cf. [Li3, Li4]):

Definition 3.1: Let $(V, Y, \mathbf{1})$ be a \mathbb{Z} -graded vertex algebra, G an automorphism group of V preserving the \mathbb{Z} -grading of V and $\chi : G \rightarrow \mathbb{C}^*$ a linear character. A (G, χ) -equivariant quasi- V -module (W, Y_W) is a vector space W equipped with a linear map $Y_W(\cdot, z) : V \rightarrow \text{Hom}(W, W((z)))$ satisfying the following conditions:

- (i) $Y_W(\mathbf{1}, z) = 1_W$;
- (ii) $Y_W(\chi(g)^{-L(0)}gv, z) = Y_W(v, \chi(g)z)$ for $g \in G, v \in V$;
- (iii) for $u, v \in V$, there exists $f(z) \in \mathbb{C}_{\chi(G)}[z]$ such that

$$f(z_1/z_2)Y_W(u, z_1)Y_W(v, z_2) \in \text{Hom}(W, W((z_1, z_2))),$$

$$f((z_2 + z_0)/z_2)Y_W(Y(u, z_0)v, z_2) = (f(z_1/z_2)Y_W(u, z_1)Y_W(v, z_2))|_{z_1=z_2+z_0}.$$

Remark 3.2: If the automorphism group $G = \{1\}$ is the trivial group in the definition, one has the usual module for vertex algebra. And if replacing the associate $\phi = z_2 + z_0$ in (iii) by $\phi = z_2e^{z_0}$, one gets the notion of a ϕ -coordinated V -module defined in Definition 2.1.

If $V = (V, Y(\cdot, z), \mathbf{1})$ is a \mathbb{Z} -graded vertex algebra, we define a linear map

$$Y[\cdot, z] : V \rightarrow \text{End}(V)[[z, z^{-1}]],$$

$$v \mapsto Y(e^{zL(0)}v, e^z - 1).$$

Then $(V, Y[\cdot, z], \mathbf{1})$ also carries a vertex algebra structure [Z, Li7]. Note that if G is an automorphism group of the vertex algebra V preserving the \mathbb{Z} -grading of V , then it is also an automorphism group of $(V, Y[\cdot, v], \mathbf{1})$. The following result is a generalization of [Li7, Proposition 5.8].

PROPOSITION 3.3: *Let $V = (V, Y(\cdot, z), \mathbf{1})$ be a \mathbb{Z} -graded vertex algebra, G an automorphism group of V preserving the \mathbb{Z} -grading of V and χ a linear character of G . Then the (G, χ) -equivariant quasi- V -modules (W, Y_W) are exactly the (G, χ) -equivariant ϕ -coordinated quasi-modules (W, Y_W^ϕ) for the vertex algebra $(V, Y[\cdot, z], \mathbf{1})$ with*

$$Y_W^\phi(v, z) = Y_W(z^{L(0)}v, z), \quad \forall v \in V.$$

Proof. Assume first that (W, Y_W) is a (G, χ) -equivariant quasi- V -module. Then for $v \in V$ and $g \in G$ we have

$$Y_W^\phi(v, \chi(g)z) = Y_W((\chi(g)z)^{L(0)}v, \chi(g)z) = Y_W(z^{L(0)}gv, z) = Y_W^\phi(gv, z).$$

This together with [Li7, Proposition 5.8] shows that (W, Y_W^ϕ) is a (G, χ) -equivariant ϕ -coordinated quasi module for $(V, Y[\cdot, z], \mathbf{1})$. Conversely, let (W, Y_W^ϕ) be a (G, χ) -equivariant ϕ -coordinated quasi-module for $(V, Y[\cdot, z], \mathbf{1})$. Then we have $Y_W(v, z) \in \text{Hom}(W, W((z)))$ for $v \in V$,

$$Y_W(\mathbf{1}, z) = Y_W^\phi(z^{-L(0)}\mathbf{1}, z) = Y_W^\phi(\mathbf{1}, z) = 1_W$$

and

$$\begin{aligned} Y_W(\chi(g)^{-L(0)}gv, z) &= Y_W^\phi((\chi(g)z)^{-L(0)}gv, z) \\ &= Y_W^\phi((\chi(g)z)^{-L(0)}v, \chi(g)z) = Y_W(v, \chi(g)z), \end{aligned}$$

for $v \in V$ and $g \in G$. For $u, v \in V$, let $f(z)$ be a nonzero polynomial in $\mathbb{C}_{\chi(G)}[z]$ such that Definition 2.1 (iii) holds. Then we have

$$\begin{aligned} f(z_1/z_2)Y_W(u, z_1)Y_W(v, z_2) \\ = f(z_1/z_2)Y_W^\phi(z_1^{-L(0)}u, z_1)Y_W^\phi(z_2^{-L(0)}v, z_2) \in \text{Hom}(W, W((z_1, z_2))). \end{aligned}$$

Furthermore, we have

$$f(z_1/z_2)Y_W^\phi((z_2 + w_0)^{-L(0)}u, z_1)Y_W(z_2^{-L(0)}v, z_2) \in \text{Hom}(W, W((z_1, z_2))[[w_0]])$$

and

$$\begin{aligned} f(z_1/z_2)Y_W^\phi((z_2 + w_0)^{-L(0)}u, z_1)Y_W^\phi(z_2^{-L(0)}v, z_2)|_{z_1=z_2e^{z_0}} \\ = f(e^{z_0})Y_W^\phi(Y[(z_2 + w_0)^{-L(0)}u, z_0]z_2^{-L(0)}v, z_2). \end{aligned}$$

By applying the substitution $z_0 = \log(1 + w_0/z_2)$ in the previous equation, we obtain from the left-hand side

$$\begin{aligned} (f(z_1/z_2)Y_W^\phi((z_2 + w_0)^{-L(0)}u, z_1)Y_W^\phi(z_2^{-L(0)}v, z_2)|_{z_1=z_2e^{z_0}})|_{z_0=\log(1+w_0/z_2)} \\ = f(z_1/z_2)Y_W^\phi(z_1^{-L(0)}u, z_1)Y_W^\phi(z_2^{-L(0)}v, z_2)|_{z_1=z_2+w_0} \\ = f(z_1/z_2)Y_W(u, z_1)Y_W(v, z_2)|_{z_1=z_2+w_0}, \end{aligned}$$

while by using the fact that $z_2^{L(0)}Y(u, z_0)z_2^{-L(0)} = Y(z_2^{L(0)}u, z_0z)$ (cf. [FHL]), we obtain from the right-hand side

$$\begin{aligned} (f(e^{z_0})Y_W^\phi(Y[(z_2 + w_0)^{-L(0)}u, z_0]z_2^{-L(0)}v, z_2))|_{z_0=\log(1+w_0/z_2)} \\ = (f(e^{z_0})Y_W(z_2^{L(0)}Y(e^{z_0L(0)}(z_2 + w_0)^{-L(0)}u, e^{z_0}-1)z_2^{-L(0)}v, z_2))|_{z_0=\log(1+w_0/z_2)} \\ = (f(e^{z_0})Y_W(Y(z_2^{L(0)}e^{z_0L(0)}(z_2 + w_0)^{-L(0)}u, (e^{z_0}-1)z_2)v, z_2))|_{z_0=\log(1+w_0/z_2)} \\ = f(z_2 + w_0/z_2)Y_W(Y(u, w_0)v, z_2). \end{aligned}$$

This proves the proposition. ■

Let V be a vertex operator algebra with the Virasoro element ω . We say $v \in V$ is a **primary vector** if $L(n)v = 0$ for all $n \geq 1$, where

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}.$$

PROPOSITION 3.4: *Assume that V is a vertex operator algebra, G is a group of automorphisms of V and χ a linear character of G . Then the following two categories are isomorphic:*

- (i) *the category of (G, χ) -equivariant ϕ -coordinated quasi- V -modules (W, Y_W^ϕ) ;*
- (ii) *the category of (G, χ) -equivariant quasi- V -modules (W, Y_W) .*

Moreover, if $G = \langle \sigma \rangle$ is a cyclic group of finite order N , and $\chi(\sigma) = e^{-\frac{2\pi\sqrt{-1}}{N}}$, then these two categories are also isomorphic to the following category

- (iii) *the category of (weak) σ -twisted V -modules (W, Y_W^t) .*

And furthermore, if $v \in V$ is a primary vector, then we have the following identities:

$$(3.2) \quad Y_W^\phi(z^{-L(0)}v, z) = Y_W(v, z) = Y_W^t((Nz^{N-1})^{L(0)}v, z^N).$$

Proof. Let $(V, Y(\cdot, z), \mathbf{1}, \omega)$ be a vertex operator algebra of central charge ℓ . Set $\tilde{\omega} = \omega - \frac{\ell}{24}\mathbf{1}$. It was proved in [Z] that $(V, Y[\cdot, z], \mathbf{1}, \tilde{\omega})$ carries a vertex operator algebra structure. And, an explicit isomorphism f from $(V, Y(\cdot, z), \mathbf{1}, \omega)$ to $(V, Y[\cdot, z], \mathbf{1}, \tilde{\omega})$ was constructed therein. This together with Proposition 3.3 implies that the (G, χ) -equivariant ϕ -coordinated quasi- V -modules (W, Y_W^ϕ) are exactly the (G, χ) -equivariant quasi- V -modules (W, Y_W) with

$$Y_W^\phi(v, z) = Y_W(z^{L(0)}f(v), z), \quad \text{for } v \in V.$$

In particular, if v is a primary vector, then from [Z] we have $f(v) = v$ and hence

$$Y_W^\phi(z^{-L(0)}v, z) = Y_W(v, z).$$

Finally, if $G = \langle \sigma \rangle$ is a finite cyclic group of order N and $\chi(\sigma) = e^{-\frac{2\pi\sqrt{-1}}{N}}$, then it was proved in [Li5] that the (G, χ) -equivariant quasi- V -modules (W, Y_W) are exactly the (weak) σ -twisted V -modules (W, Y_W^t) with

$$Y_W(v, z) = Y_W(\Phi(z)v, z^N), \quad \text{for } v \in V,$$

where $\Phi(z)$ is an operator in $\text{Hom}(V, V((z)))$ such that $\Phi(z)v = (Nz^{N-1})^{L(0)}v$ whenever v is primary. This completes the proof of the proposition. ■

3.2. EQUIVALENCE OF MODULE CATEGORIES FOR UNIVERSAL AFFINE VERTEX ALGEBRAS. In this subsection, we study the equivalence of certain module categories for universal affine vertex algebras. Let \mathfrak{b} be a (possibly infinite-dimensional) Lie algebra equipped with a nondegenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$. Then we have the affine Lie algebra

$$\tilde{\mathcal{L}}(\mathfrak{b}) = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{b}) \oplus \mathbb{C}k \oplus \mathbb{C}d,$$

where k is a central element and

$$(3.3) \quad [t^m \otimes a, t^n \otimes b] = t^{m+n} \otimes [a, b] + \langle a, b \rangle \delta_{m+n, 0} k, \quad [d, t^m \otimes a] = m t^m \otimes a,$$

for $m, n \in \mathbb{Z}$ and $a, b \in \mathfrak{b}$. Equip $\tilde{\mathcal{L}}(\mathfrak{b})$ with a \mathbb{Z} -grading structure with respect to the adjoint action of $-d$, and form the following \mathbb{Z} -graded Lie subalgebra of $\tilde{\mathcal{L}}(\mathfrak{b})$:

$$\widehat{\mathcal{L}}(\mathfrak{b}) = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{b}) \oplus \mathbb{C}k.$$

Let ℓ be a fixed complex number. View \mathbb{C} as a $(\mathbb{C}[t] \otimes \mathfrak{b} \oplus \mathbb{C}k)$ -module with $\mathbb{C}[t] \otimes \mathfrak{b}$ acting trivially and k acting as scalar ℓ . Then we have the induced $\widehat{\mathcal{L}}(\mathfrak{b})$ -module

$$(3.4) \quad V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0) = \mathcal{U}(\widehat{\mathcal{L}}(\mathfrak{b})) \otimes_{\mathcal{U}(\mathbb{C}[t] \otimes \mathfrak{b} \oplus \mathbb{C}k)} \mathbb{C},$$

which is naturally \mathbb{N} -graded by defining $\deg \mathbb{C} = 0$. Set $\mathbf{1} = 1 \otimes 1 \in V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ and identify \mathfrak{b} as the degree-one subspace of $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ through the linear map

$$a \mapsto (t^{-1} \otimes a)\mathbf{1} \in V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0),$$

for $a \in \mathfrak{b}$. It is known (cf. [FZ, Li1]) that there exists a vertex algebra structure on $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$, which is uniquely determined by the condition that $\mathbf{1}$ is the vacuum vector and

$$Y(a, z) = a(z) = \sum_{m \in \mathbb{Z}} (t^m \otimes a) z^{-m-1},$$

for $a \in \mathfrak{b}$. The vertex algebra $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ is often called the **universal affine vertex algebra** associated to \mathfrak{b} . Denote by $J_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ the unique maximal graded $\widehat{\mathcal{L}}(\mathfrak{b})$ -submodule of $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$. Then $J_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ is an ideal of the vertex algebra $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$. Define

$$L_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0) = V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0) / J_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0),$$

which is a simple \mathbb{Z} -graded vertex algebra.

Assume that G is an automorphism group of \mathfrak{b} preserving the bilinear form. It is easy to see that G can be uniquely lifted to an automorphism group of $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ that preserves the \mathbb{Z} -grading. Moreover, as $J_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ is G -stable, G also induces an automorphism group of $L_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$. Assume further that for $a, b \in \mathfrak{b}$,

$$(3.5) \quad [ga, b] = 0, \quad \langle ga, b \rangle = 0$$

for all but finitely many $g \in G$. For any linear character $\chi : G \rightarrow \mathbb{C}^*$, we define a quotient space

$$(3.6) \quad \widehat{\mathcal{L}}(\mathfrak{b}, G) = \widehat{\mathcal{L}}(\mathfrak{b}) / \text{Span}\{\chi(g)^m(t^m \otimes ga) - t^m \otimes a \mid g \in G, m \in \mathbb{Z}, a \in \mathfrak{b}\}.$$

It was proved in [Li4] that $\widehat{\mathcal{L}}(\mathfrak{b}, G)$ carries a Lie algebra structure with Lie bracket given by

$$[\overline{t^m \otimes a}, \overline{t^n \otimes b}] = \sum_{g \in G} \chi(g)^m \overline{t^{m+n} \otimes [ga, b]} + \delta_{m+n,0} \langle ga, b \rangle \bar{k}$$

for $a, b \in \mathfrak{b}$ and $m, n \in \mathbb{Z}$, where $\overline{t^m \otimes a}$ and \bar{k} stand for the images of $t^m \otimes a$ and k in $\widehat{\mathcal{L}}(\mathfrak{b}, G)$ respectively.

We say that an $\widehat{\mathcal{L}}(\mathfrak{b}, G)$ -module W is **restricted** if for any $a \in \mathfrak{b}$ and $w \in W$,

$$\overline{t^n \otimes a} \cdot w = 0$$

for n sufficiently large, and is of **level** ℓ if \bar{k} acts as the scalar ℓ . From [CLTW, Proposition 6.4] we have the following result.

PROPOSITION 3.5: *Let G, χ be as above. The following categories are isomorphic to each other:*

- (i) *the category of (G, χ) -equivariant ϕ -coordinated quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ -modules (W, Y_W^ϕ) ;*
- (ii) *the category of (G, χ) -equivariant quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ -modules (W, Y_W) ;*
- (iii) *the category of restricted $\widehat{\mathcal{L}}(\mathfrak{b}, G)$ -modules W of level ℓ .*

And, the isomorphisms are determined by

$$Y_W^\phi(a, z) = zY_W(a, z) = \sum_{n \in \mathbb{Z}} \overline{t^n \otimes a} z^{-n},$$

for $a \in \mathfrak{b}$.

We assume now that σ is an automorphism of \mathfrak{b} with finite order N and preserving the bilinear form, and set $\omega_N = e^{\frac{2\pi\sqrt{-1}}{N}}$. For $a \in \mathfrak{b}$ and $m \in \mathbb{Z}$, set $a_{(m)} = \sum_{p=0}^{N-1} \omega_N^{-mp} \sigma^p(a)$ and $\mathfrak{b}_{(m)} = \{a_{(m)} \mid a \in \mathfrak{b}\}$ (the σ -eigenspace of \mathfrak{b} with eigenvalue ω_N^m). Then we have the following Lie subalgebras of $\tilde{\mathcal{L}}(\mathfrak{b})$, called σ -twisted affine Lie algebras (cf. [K1]):

$$\begin{aligned} \tilde{\mathcal{L}}(\mathfrak{b}, \sigma) &= \left(\sum_{m \in \mathbb{Z}} \mathbb{C}t^m \otimes \mathfrak{b}_{(m)} \right) \oplus \mathbb{C}k \oplus \mathbb{C}d, \\ \widehat{\mathcal{L}}(\mathfrak{b}, \sigma) &= \left(\sum_{m \in \mathbb{Z}} \mathbb{C}t^m \otimes \mathfrak{b}_{(m)} \right) \oplus \mathbb{C}k. \end{aligned}$$

Let χ_{ω_N} be the linear character of the cyclic group $\langle \sigma \rangle$ such that $\chi_{\omega_N}(\sigma) = \omega_N^{-1}$. We take $G = \langle \sigma \rangle$ and $\chi = \chi_{\omega_N}$ in (3.6). Then it is easy to check that the linear map given by

$$(3.7) \quad \overline{t^m \otimes a} \mapsto t^m \otimes a_{(m)}, \quad \bar{k} \mapsto Nk, \quad \text{for } a \in \mathfrak{b}, m \in \mathbb{Z},$$

is a Lie algebra isomorphism from $\widehat{\mathcal{L}}(\mathfrak{b}, \langle \sigma \rangle)$ to $\widehat{\mathcal{L}}(\mathfrak{b}, \sigma)$.

Definition 3.6: Let W be a module of $\tilde{\mathcal{L}}(\mathfrak{b}, \sigma)$ or $\widehat{\mathcal{L}}(\mathfrak{b}, \sigma)$. We say that W is **restricted** if for any $a \in \mathfrak{b}$ and $w \in W$,

$$(t^n \otimes a_{(n)}) \cdot w = 0$$

for n sufficiently large, and is of **level** ℓ if k acts as scalar ℓ/N .

From Proposition 3.5 and the isomorphism (3.7), one immediately obtains the following result.

PROPOSITION 3.7: *The following categories are isomorphic to each other:*

- (i) *the category of $(\langle \sigma \rangle, \chi_{\omega_N})$ -equivariant ϕ -coordinated quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ -modules (W, Y_W^ϕ) ;*
- (ii) *the category of $(\langle \sigma \rangle, \chi_{\omega_N})$ -equivariant quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ -modules (W, Y_W) ;*
- (iii) *the category of restricted $\widehat{\mathcal{L}}(\mathfrak{b}, \sigma)$ -modules W of level ℓ .*

And, the isomorphisms are determined by

$$Y_W^\phi(a, z) = zY_W(a, z) = a[z] := \sum_{n \in \mathbb{Z}} (t^n \otimes a_{(n)})z^{-n},$$

for $a \in \mathfrak{b}$.

Furthermore, by Proposition 3.7, Lemma 2.2, Lemma 2.3 and the fact that $[d, a[z]] = -z \frac{d}{dz} a[z]$ for $a \in \mathfrak{b}$, we have

PROPOSITION 3.8: *For any complex number ℓ , the restricted $\widetilde{\mathcal{L}}(\mathfrak{b}, \sigma)$ -modules W of level ℓ are exactly the $(\langle \sigma \rangle, \chi_{\omega_N})$ -equivariant ϕ -coordinated quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ -modules (W, Y_W^ϕ, d) with*

$$d = -d, \quad a[z] = Y_W^\phi(a, z),$$

for $a \in \mathfrak{b}$.

Remark 3.9: We also have the following variant of the twisted affine Lie algebra $\widehat{\mathcal{L}}(\mathfrak{b}, \sigma)$ (cf. [FLM]):

$$\widehat{\mathcal{L}}[\mathfrak{b}, \sigma] = \left(\sum_{m \in \frac{1}{N}\mathbb{Z}} \mathbb{C}t^m \otimes \mathfrak{b}_{(Nm)} \right) \oplus \mathbb{C}\mathfrak{k},$$

where the Lie bracket is given by (3.3) with $m, n \in \frac{1}{N}\mathbb{Z}$. Note that $\widehat{\mathcal{L}}(\mathfrak{b}, \sigma)$ is isomorphic to $\widehat{\mathcal{L}}[\mathfrak{b}, \sigma]$ with the mapping

$$(3.8) \quad t^m \otimes a_{(m)} \mapsto t^{m/N} \otimes a_{(m)}, \quad N\mathfrak{k} \mapsto \mathfrak{k},$$

for $a \in \mathfrak{b}, m \in \mathbb{Z}$. It is well-known that the σ -twisted $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ -modules (W, Y_W^t) are exactly the restricted $\widehat{\mathcal{L}}[\mathfrak{b}, \sigma]$ -modules W of level ℓ with

$$Y_W^t(a, z) = \sum_{m \in \mathbb{Z}} \left(t^{\frac{m}{N}} \otimes \frac{a_{(m)}}{N} \right) z^{-\frac{m}{N}-1}$$

for $a \in \mathfrak{b}$ (cf. [Li2]).

Remark 3.10: One can also define the notion of a σ -twisted $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ -module (W, Y_W^t, d) such that $[d, Y_W^t(v, z)] = Y_W^t(\mathcal{D}v, z)$ for $v \in V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ (see [Li2] for example). Then such σ -twisted $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ -modules are precisely the restricted modules of level ℓ for the algebra

$$\widehat{\mathcal{L}}[\mathfrak{b}, \sigma] \rtimes \mathbb{C} \left(\frac{d}{dt} \otimes 1 \right),$$

as $[d, Y_W^t(v, z)] = \frac{d}{dz} Y_W^t(v, z)$ for $v \in V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ (cf. [Li2]). And a similar result also holds for the equivariant quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ -modules (cf. [Li3]).

3.3. ASSOCIATING AFFINE KAC–MOODY ALGEBRAS WITH VERTEX ALGEBRAS. In this subsection, we associate the nullity 1 EALAs (i.e., affine Kac–Moody algebras with derivations) to vertex algebras via equivariant ϕ -coordinated quasi-modules. As a by-product, we obtain a characterization of integrable restricted modules for affine Kac–Moody algebras, which will be used later to associate the nullity 2 EALAs with vertex algebras.

Let \mathfrak{g} be a finite-dimensional simple Lie algebra over the complex field, \mathfrak{h} a Cartan subalgebra of \mathfrak{g} , and $\dot{\Delta}$ the root system of \mathfrak{g} with respect to \mathfrak{h} . Let $\langle \cdot, \cdot \rangle$ be the nondegenerate invariant symmetric bilinear form on \mathfrak{g} which is normalized so that the square length of long roots is equal to 2. For $\dot{\alpha} \in \dot{\Delta}$, choose nonzero root vectors $x_{\dot{\alpha}} \in \mathfrak{g}_{\dot{\alpha}}$ such that $\{x_{\dot{\alpha}}, \dot{\alpha}^\vee, x_{-\dot{\alpha}}\}$ form an \mathfrak{sl}_2 -triple, where $\dot{\alpha}^\vee \in \mathfrak{h}$ denotes the coroot of $\dot{\alpha}$.

Fix a simple root system $\dot{\Pi} = \{\dot{\alpha}_1, \dots, \dot{\alpha}_l\}$ of $\dot{\Delta}$, where $l = \dim \mathfrak{h}$ is the rank of \mathfrak{g} . Let ν be a diagram automorphism of \mathfrak{g} of order N ($N = 1, 2$ or 3). By definition, there exists a permutation $\dot{\nu}$ on the set $\dot{I} = \{1, 2, \dots, l\}$ such that $\nu(x_{\pm\dot{\alpha}_i}) = x_{\pm\dot{\alpha}_{\nu(i)}}$ for $i \in \dot{I}$. Then the Lie algebra $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$ as defined in Section 3.2 is a Kac–Moody algebra of affine type, and any affine Kac–Moody algebra has such a form [K1]. We say that an $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$ -module (or an $\widehat{\mathcal{L}}(\mathfrak{g}, \nu)$ -module) is **integrable** if all real root vectors $t^m \otimes x_{\dot{\alpha}(m)}$ for $\dot{\alpha} \in \dot{\Delta}, m \in \mathbb{Z}$ act locally nilpotent on it [K1]. Now we give the main result of this section.

THEOREM 3.11: *Let ℓ be any complex number. For any restricted $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$ -module W of level ℓ , there is a $(\langle \dot{\nu} \rangle, \chi_{\omega_N})$ -equivariant ϕ -coordinated quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ -module structure (Y_W^ϕ, d) on W , which is uniquely determined by*

$$d = -d, \quad Y_W^\phi(a, z) = a[z] \left(= \sum_{n \in \mathbb{Z}} (t^n \otimes a_{(n)}) z^{-n} \right),$$

for $a \in \mathfrak{g}$. On the other hand, for any $(\langle \dot{\nu} \rangle, \chi_{\omega_N})$ -equivariant ϕ -coordinated quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ -module (W, Y_W^ϕ, d) , W is a restricted $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$ -module of level ℓ with action given by

$$\begin{aligned} d &= -d, \\ a[z] &= Y_W^\phi(a, z), \end{aligned}$$

for $a \in \mathfrak{g}$. Furthermore, when ℓ is a nonnegative integer, the integrable restricted $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$ -modules of level ℓ exactly correspond to the $(\langle \dot{\nu} \rangle, \chi_{\omega_N})$ -equivariant ϕ -coordinated quasi- $L_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ -modules.

Proof. The one-to-one correspondence between the category of restricted $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$ -modules of level ℓ and the category of $(\langle \nu \rangle, \chi_{\omega_N})$ -equivariant ϕ -coordinated quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ -modules follows from Proposition 3.8.

For the last assertion of the theorem, we note that $L_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ is a rational vertex operator algebra, and each $a \in \mathfrak{g}$ is a primary vector of weight 1 in $L_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$. Thus, by Proposition 3.4, it is easy to see that the $(\langle \nu \rangle, \chi_{\omega_N})$ -equivariant ϕ -coordinated quasi- $L_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ -modules (W, Y_W^ϕ) are exactly the ν -twisted $L_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ -modules (W, Y_W^t) with

$$Y_W^\phi(a, z) = Nz^N Y_W^t(a, z),$$

for $a \in \mathfrak{g}$. Recall the algebra $\widehat{\mathcal{L}}[\mathfrak{g}, \nu]$ defined in Remark 3.9. It is known (cf. [Li2]) that the integrable restricted $\widehat{\mathcal{L}}[\mathfrak{g}, \nu]$ -modules of level ℓ are exactly the ν -twisted $L_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ -modules with

$$Y_W^t(a, z) = \sum_{m \in \mathbb{Z}} \left(t^{\frac{m}{N}} \otimes \frac{a(m)}{N} \right) z^{-\frac{m}{N}-1},$$

for $a \in \mathfrak{g}$. Via the isomorphism (3.8), we obtain that the integrable restricted $\widehat{\mathcal{L}}(\mathfrak{g}, \nu)$ -modules of level ℓ are exactly the $(\langle \nu \rangle, \chi_{\omega_N})$ -equivariant ϕ -coordinated quasi- $L_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ -modules (W, Y_W^ϕ) with

$$Y_W^\phi(a, z) = Nz^N Y_W^t(a, z) = \sum_{m \in \mathbb{Z}} (t^{\frac{m}{N}} \otimes a(m)) z^{-m} = \sum_{m \in \mathbb{Z}} (t^m \otimes a(m)) z^{-m} = a[z],$$

for $a \in \mathfrak{g}$. Thus, the last assertion of the theorem follows from this and Lemmas 2.2, 2.3. ■

We write $\dot{\Delta} = \dot{\Delta}_1 \cup \dot{\Delta}_2$, where

$$\dot{\Delta}_2 = \{ \dot{\alpha} \in \dot{\Delta} \mid \langle \dot{\alpha}, \nu(\dot{\alpha}) \rangle = -1 \} \quad \text{and} \quad \dot{\Delta}_1 = \dot{\Delta} \setminus \dot{\Delta}_2.$$

For each $\dot{\alpha} \in \dot{\Delta}$, we set $\epsilon_{\dot{\alpha}} = \frac{2}{\langle \dot{\alpha}, \dot{\alpha} \rangle}$ ($= 1, 2$ or 3), and $p_{\dot{\alpha}}(z) = \frac{1-z^s}{1-z}$ if $\dot{\alpha} \in \dot{\Delta}_s$ for $s = 1, 2$.

Note that if $\dot{\alpha} \in \dot{\Delta}_2$ (resp. $\dot{\Delta}_1$), then the Dynkin diagram associated to the ν -orbit $\{ \nu^p(\dot{\alpha}) \mid p = 0, \dots, N-1 \}$ of $\dot{\alpha}$ is of type A_2 (resp. a direct sum of type A_1). Using this, one can check that

$$(3.9) \quad p_{\dot{\alpha}}(z/w) \cdot [x_{\dot{\alpha}}[z], x_{\dot{\alpha}}[w]] = 0,$$

for $\dot{\alpha} \in \dot{\Delta}$. As a by-product of Theorem 3.11, we obtain a characterization of the integrable restricted $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$ -modules.

PROPOSITION 3.12: *Let W be a restricted $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$ -module of level ℓ . Then W is integrable if and only if ℓ is a nonnegative integer, and for each $\dot{\alpha} \in \dot{\Delta}$,*

$$(3.10) \quad \left(\prod_{1 \leq i < j \leq \epsilon_{\dot{\alpha}} \ell + 1} p_{\dot{\alpha}}(z_i / z_j) \right) x_{\dot{\alpha}}[z_1] x_{\dot{\alpha}}[z_2] \cdots x_{\dot{\alpha}}[z_{\epsilon_{\dot{\alpha}} \ell + 1}] |_{z_1 = z_2 = \cdots = z_{\epsilon_{\dot{\alpha}} \ell + 1}} = 0$$

on W .

Proof. Note that if W is integrable, then ℓ is a nonnegative integer and the ideal $J_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0)$ is generated by the elements $((x_{\dot{\alpha}})_{-1})^{\epsilon_{\dot{\alpha}} \ell + 1} \mathbf{1}$ for $\dot{\alpha} \in \dot{\Delta}$ [K1]. Let W be a restricted $\tilde{\mathcal{L}}(\mathfrak{g}, \nu)$ -module of level ℓ , and hence by Theorem 3.11 there is a $(\langle \dot{\nu} \rangle, \chi_{\omega_N})$ -equivariant ϕ -coordinated quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ -module (W, Y_W^ϕ, d) . Furthermore, if W is also integrable, then (W, Y_W^ϕ, d) becomes a $(\langle \dot{\nu} \rangle, \chi_{\omega_N})$ -equivariant ϕ -coordinated quasi- $L_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ -module. This then implies that

$$((x_{\dot{\alpha}})_{-1})^{\epsilon_{\dot{\alpha}} \ell + 1} \mathbf{1} = 0$$

on W . Note that $(x_{\dot{\alpha}})_n (x_{\dot{\alpha}}) = 0$ for $n \geq 0$ in $V_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$. Thus by Proposition 2.6 and (3.9), we see that (3.10) holds.

Conversely, if (3.10) holds, again by Proposition 2.6 and (3.9), we see that $((x_{\dot{\alpha}})_{-1})^{\epsilon_{\dot{\alpha}} \ell + 1} \mathbf{1} = 0$ on W , where W is now viewed as a faithful $(\langle \dot{\nu} \rangle, \chi_{\omega_N})$ -equivariant ϕ -coordinated quasi-module for $V_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0) / \ker Y_W^\phi$. This implies that $J_{\widehat{\mathcal{L}}(\mathfrak{b})}(\ell, 0) \subset \ker Y_W^\phi$, and hence (W, Y_W^ϕ, d) becomes a $(\langle \dot{\nu} \rangle, \chi_{\omega_N})$ -equivariant ϕ -coordinated quasi- $L_{\widehat{\mathcal{L}}(\mathfrak{g})}(\ell, 0)$ -module. Finally, the integrability of W follows from Theorem 3.11. ■

Remark 3.13: When $\nu = \text{id}$, the identity (3.10) is known as the Lepowsky–Prime integrability relations [LePr].

4. Twisted toroidal extended affine Lie algebras

In this section, we recall the construction of the nullity 2 twisted toroidal extended affine Lie algebras arising from diagram automorphisms of affine Kac–Moody algebras.

4.1. TOROIDAL EALA $\tilde{\mathfrak{g}}$. We first recall the definition of nullity 2 toroidal extended affine Lie algebras in this subsection [BGK, B2]. Let $\mathcal{R} = \mathbb{C}[t_0, t_0^{-1}, t_1, t_1^{-1}]$ be a Laurent polynomial ring in commuting variables t_0 and t_1 . Denote by

$$\Omega_{\mathcal{R}}^1 = \mathcal{R}dt_0 \oplus \mathcal{R}dt_1$$

the space of Kähler differentials on \mathcal{R} . Define the 1-forms

$$k_0 = t_0^{-1}dt_0, \quad k_1 = t_1^{-1}dt_1.$$

Then $\{k_0, k_1\}$ forms a \mathcal{R} -basis of $\Omega_{\mathcal{R}}^1$. Let

$$d(\mathcal{R}) = \left\{ df = \frac{\partial f}{\partial t_0} dt_0 + \frac{\partial f}{\partial t_1} dt_1 \mid f \in \mathcal{R} \right\}$$

be the space of exact 1-forms in $\Omega_{\mathcal{R}}^1$, and set

$$\mathcal{K} = \Omega_{\mathcal{R}}^1 / d(\mathcal{R}),$$

and

$$(4.1) \quad k_{m_0, m_1} = \begin{cases} \frac{1}{m_1} t_0^{m_0} t_1^{m_1} k_0 & \text{if } m_1 \neq 0, \\ -\frac{1}{m_0} t_0^{m_0} k_1 & \text{if } m_0 \neq 0, m_1 = 0, \\ 0 & \text{if } m_0 \neq 0, m_1 = 0, \end{cases}$$

for $m_0, m_1 \in \mathbb{Z}$. Then the set

$$(4.2) \quad \mathbb{B}_{\mathcal{K}} = \{k_0, k_1\} \cup \{k_{m_0, m_1} \mid (m_0, m_1) \in \mathbb{Z}^2 \setminus \{(0, 0)\}\}$$

forms a basis of \mathcal{K} . The set $\mathbb{B}_{\mathcal{K}}$ can also be expressed as follows:

$$(4.3) \quad \mathbb{B}_{\mathcal{K}} = \{k_0\} \cup \{t_0^{m_0} k_1, k_{m_0, m_1} \mid m_0 \in \mathbb{Z}, m_1 \in \mathbb{Z}^*\},$$

which will be used later on. Recall that \mathfrak{g} is a finite-dimensional simple Lie algebra and $\langle \cdot, \cdot \rangle$ is the normalized bilinear form on \mathfrak{g} . Let

$$\mathfrak{t}(\mathfrak{g}) = (\mathcal{R} \otimes \mathfrak{g}) \oplus \mathcal{K}$$

be a central extension of the double-loop algebra $\mathcal{R} \otimes \mathfrak{g}$ by \mathcal{K} with Lie product

$$(4.4) \quad \begin{aligned} & [t_0^{m_0} t_1^{m_1} \otimes x, t_0^{n_0} t_1^{n_1} \otimes y] \\ & = t_0^{m_0+n_0} t_1^{m_1+n_1} \otimes [x, y] + \langle x, y \rangle \sum_{i=0,1} m_i t_0^{m_0+n_0} t_1^{m_1+n_1} k_i, \end{aligned}$$

for $x, y \in \mathfrak{g}$ and $m_0, m_1, n_0, n_1 \in \mathbb{Z}$. It is proved in [MRY] that the Lie algebra $\mathfrak{t}(\mathfrak{g})$ is the universal central extension of $\mathcal{R} \otimes \mathfrak{g}$, and it is often called the **nullity 2 toroidal Lie algebra**.

Let

$$\text{Der}(\mathcal{R}) = \mathcal{R} \frac{\partial}{\partial t_0} \oplus \mathcal{R} \frac{\partial}{\partial t_1}$$

be the space of derivations of \mathcal{R} . Set

$$d_0 = t_0 \frac{\partial}{\partial t_0}, \quad d_1 = t_1 \frac{\partial}{\partial t_1}.$$

Then $\{d_0, d_1\}$ form a \mathcal{R} -basis of $\text{Der}(\mathcal{R})$, and the derivation Lie algebra $\text{Der}(\mathcal{R})$ also acts on $\mathcal{R} \otimes \mathfrak{g}$ with

$$\psi(f \otimes x) = \psi(f) \otimes x,$$

for $\psi \in \text{Der}(\mathcal{R}), f \in \mathcal{R}, x \in \mathfrak{g}$. One notes that the $\text{Der}(\mathcal{R})$ -action on $\mathcal{R} \otimes \mathfrak{g}$ can be uniquely extended to an action on the center \mathcal{K} of the toroidal Lie algebra $\mathfrak{t}(\mathfrak{g})$ with

$$\psi(fdg) = \psi(f)dg + fd\psi(g),$$

for $\psi \in \text{Der}(\mathcal{R}), g \in \mathcal{R}$. Now, we form the semi-direct product Lie algebra

$$\mathcal{T}(\mathfrak{g}) = \mathfrak{t}(\mathfrak{g}) \rtimes \text{Der}(\mathcal{R}) = \mathcal{R} \otimes \mathfrak{g} \oplus \mathcal{K} \oplus \text{Der}(\mathcal{R}),$$

which is often called the **full toroidal Lie algebra** [B1]. Note that in $\mathcal{T}(\mathfrak{g})$ we have

$$(4.5) \quad [t_0^{m_0} t_1^{m_1} d_i, t_0^{n_0} t_1^{n_1} \otimes x] = n_i (t_0^{m_0+n_0} t_1^{m_1+n_1} \otimes x),$$

$$(4.6) \quad [t_0^{m_0} t_1^{m_1} d_i, t_0^{n_0} t_1^{n_1} k_j] = n_i t_0^{m_0+n_0} t_1^{m_1+n_1} k_j + \delta_{i,j} \sum_{r=0,1} m_r t_0^{m_0+n_0} t_1^{m_1+n_1} k_r,$$

$$(4.7) \quad [t_0^{m_0} t_1^{m_1} d_i, t_0^{n_0} t_1^{n_1} d_j] = n_i t_0^{m_0+n_0} t_1^{m_1+n_1} d_j - m_j t_0^{m_0+n_0} t_1^{m_1+n_1} d_i,$$

for $x \in \mathfrak{g}, m_0, n_0, m_1, n_1 \in \mathbb{Z}$, and $i, j \in \{0, 1\}$.

We define a Lie subalgebra \mathcal{S} of $\text{Der}(\mathcal{R})$ as follows:

$$\mathcal{S} = \{f_0 d_0 + f_1 d_1 \in \text{Der}(\mathcal{R}) \mid f_0, f_1 \in \mathcal{R}, d_0(f_0) + d_1(f_1) = 0\}.$$

The elements in \mathcal{S} are often called **skew derivations** over \mathcal{R} (cf. [BGK, N]), and also known as **divergence-zero derivations** (cf. [B2]). It is easy to see that, for $m_0, m_1 \in \mathbb{Z}$,

$$\tilde{d}_{m_0, m_1} = m_0 t_0^{m_0} t_1^{m_1} d_1 - m_1 t_0^{m_0} t_1^{m_1} d_0$$

are elements of \mathcal{S} , and the set

$$\mathbb{B}_{\mathcal{S}} = \{d_0, d_1\} \cup \{\tilde{d}_{m_0, m_1} \mid (m_0, m_1) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}\}$$

forms a \mathbb{C} -basis of \mathcal{S} . Note that $\tilde{d}_{m,0} = mt_0^m d_1$ for $m \in \mathbb{Z}$, we can rewrite $\mathbb{B}_{\mathcal{S}}$ as follows:

$$(4.8) \quad \mathbb{B}_{\mathcal{S}} = \{d_0\} \cup \{t_0^{m_0} d_1, \tilde{d}_{m_0, m_1} \mid m_0 \in \mathbb{Z}, m_1 \in \mathbb{Z}^*\},$$

and the Lie product relations are given as follows

$$(4.9) \quad [d_i, \tilde{d}_{m_0, m_1}] = m_i \tilde{d}_{m_0, m_1}, \quad [\tilde{d}_{m_0, m_1}, \tilde{d}_{n_0, n_1}] = (m_0 n_1 - m_1 n_0) \tilde{d}_{m_0+n_0, m_1+n_1},$$

for $i \in \{0, 1\}$, and $m_0, m_1, n_0, n_1 \in \mathbb{Z}$.

Form the following subalgebra of $\mathcal{T}(\tilde{\mathfrak{g}})$:

$$(4.10) \quad \tilde{\mathfrak{g}} = \mathfrak{t}(\tilde{\mathfrak{g}}) \rtimes \mathcal{S} = \mathcal{R} \otimes \tilde{\mathfrak{g}} \oplus \mathcal{K} \oplus \mathcal{S},$$

which is often called the **nullity 2 toroidal extended affine Lie algebra** [BGK, B2].

From (4.5) and (4.6), one has the following Lie product relations in $\tilde{\mathfrak{g}}$:

$$(4.11) \quad [\tilde{d}_{m_0, m_1}, t_0^{n_0} t_1^{n_1} \otimes x] = (m_0 n_1 - m_1 n_0) t_0^{m_0+n_0} t_1^{m_1+n_1} \otimes x,$$

$$(4.12) \quad [\tilde{d}_{m_0, m_1}, k_{n_0, n_1}] = (m_0 n_1 - m_1 n_0) k_{m_0+n_0, m_1+n_1} + \delta_{m_0+n_0, 0} \delta_{m_1+n_1, 0} (m_0 k_0 + m_1 k_0),$$

for $m_0, n_0, m_1, n_1 \in \mathbb{Z}$, and $x \in \tilde{\mathfrak{g}}$. And the Lie subalgebra

$$\mathfrak{g} := (\mathbb{C}[t_1, t_1^{-1}] \otimes \tilde{\mathfrak{g}}) \oplus \mathbb{C}k_1 \oplus \mathbb{C}d_1$$

of $\tilde{\mathfrak{g}}$ is isomorphic to the affine Kac–Moody algebra $\tilde{\mathcal{L}}(\tilde{\mathfrak{g}})$. We extend the normalized bilinear form $\langle \cdot, \cdot \rangle$ of $\tilde{\mathfrak{g}}$ to a nondegenerate invariant symmetric bilinear form on \mathfrak{g} by defining

$$(t_1^m \otimes x + a k_1 + b d_1, t_1^n \otimes y + a' k_1 + b' d_1) = \delta_{m+n, 0} \langle x, y \rangle + ab' + ba',$$

where $m, n \in \mathbb{Z}$, $x, y \in \tilde{\mathfrak{g}}$, and $a, b, a', b' \in \mathbb{C}$. It is easy to see from (4.3) and (4.8) that the Lie algebra $\tilde{\mathfrak{g}}$ is linearly spanned by the set

$$(4.13) \quad \{t_0^m u, k_{m, n}, \tilde{d}_{m, n}, d_0, k_0 \mid u \in \mathfrak{g}, m \in \mathbb{Z}, n \in \mathbb{Z}^*\},$$

with a nondegenerate and invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$(4.14) \quad \langle t_0^m x, t_0^{-m} y \rangle = \langle x, y \rangle, \quad \langle k_0, d_0 \rangle = \langle \tilde{d}_{m, n}, k_{-m, -n} \rangle = 1,$$

for $x, y \in \mathfrak{g}$, $m \in \mathbb{Z}$, $n \in \mathbb{Z}^*$, and a self-centralizing ad-diagonalizable subalgebra

$$\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}k_0 \oplus \mathbb{C}k_1 \oplus \mathbb{C}d_0 \oplus \mathbb{C}d_1.$$

4.2. DIAGRAM AUTOMORPHISMS OF AFFINE KAC–MOODY ALGEBRAS. For any diagram automorphism μ of the affine Kac–Moody algebra \mathfrak{g} , we define an automorphism $\hat{\mu}$ for the full toroidal Lie algebra $\mathcal{T}(\dot{\mathfrak{g}})$, which will be used to construct twisted toroidal extended affine Lie algebras in the next subsection.

Denote by

$$\mathfrak{h} = \dot{\mathfrak{h}} \oplus \mathbb{C}k_1 \oplus \mathbb{C}d_1$$

the Cartan subalgebra of the affine Kac–Moody algebra \mathfrak{g} , and \mathfrak{h}^* the dual space of \mathfrak{h} . We identify $\dot{\mathfrak{h}}^*$ as a subspace of \mathfrak{h}^* such that $\dot{\alpha}(k_1) = \dot{\alpha}(d_1) = 0$ for $\dot{\alpha} \in \dot{\mathfrak{h}}^*$, and define the null root $\delta_1 \in \mathfrak{h}^*$ by setting

$$\delta_1(\dot{\mathfrak{h}}) = \delta(k_1) = 0, \quad \delta_1(d_1) = 1.$$

Then we have the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where

$$\Delta = \{ \dot{\alpha} + m\delta_1, n\delta_1 \mid \dot{\alpha} \in \dot{\Delta}, m \in \mathbb{Z}, n \in \mathbb{Z}^* \}.$$

Denote by $\Delta^\times = \{ \dot{\alpha} + m\delta_1 \mid \dot{\alpha} \in \dot{\Delta}, m \in \mathbb{Z} \}$ the set of real roots in Δ . Recall that for each $\dot{\alpha} \in \dot{\Delta}$ and $\alpha = \dot{\alpha} + m\delta_1 \in \Delta^\times$, there are \mathfrak{sl}_2 -triples $\{x_\alpha, \alpha^\vee, x_{-\alpha}\}$ in $\dot{\mathfrak{g}}$, and $\{x_\alpha, \alpha^\vee, x_{-\alpha}\}$ in \mathfrak{g} , where

$$\alpha^\vee = \dot{\alpha}^\vee + \frac{2m}{\langle \dot{\alpha}, \dot{\alpha} \rangle} k_1 \quad x_\alpha = t_1^m \otimes x_{\dot{\alpha}} \in \mathfrak{g}_\alpha.$$

Let $\dot{\theta}$ be the highest root in $\dot{\Delta}$ with respect to the simple root system $\dot{\Pi} = \{\dot{\alpha}_1, \dots, \dot{\alpha}_l\}$ of $\dot{\Delta}$. Then $\Pi = \{\alpha_0, \dots, \alpha_l\}$ forms a simple root system of Δ with $\alpha_0 = \delta_1 - \dot{\theta}$ and $\alpha_i = \dot{\alpha}_i$ for $1 \leq i \leq l$. Denote by

$$A = (a_{ij})_{i,j \in I} = \left(2 \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} \right)_{i,j \in I}$$

the generalized Cartan matrix of the affine Kac–Moody algebra \mathfrak{g} , where $I = \{0, 1, \dots, l\}$. Let μ be an automorphism of the generalized Cartan matrix A , which by definition is a permutation of I such that $a_{ij} = a_{\mu(i)\mu(j)}$ for $i, j \in I$. It is known (cf. [KW], [FSS]) that there is a unique automorphism of \mathfrak{g} , still denoted by μ , such that

$$(4.15) \quad \mu(x_{\pm\alpha_i}) = x_{\pm\alpha_{\mu(i)}}, \quad \langle \mu(x), \mu(y) \rangle = \langle x, y \rangle$$

for $i \in I$ and $x, y \in \mathfrak{g}$. Moreover, μ stabilizes \mathfrak{h} and has the same order as it is viewed as a permutation of I .

Remark 4.1: The automorphism μ of the affine Kac–Moody algebra \mathfrak{g} determined by (4.15) is called a **diagram automorphism** associated to an automorphism μ of the generalized Cartan matrix A .

From now on, we fix a diagram automorphism μ of \mathfrak{g} with order T . Recall that $\mu(\mathfrak{h}) = \mathfrak{h}$. Then there is an action of μ on \mathfrak{h}^* defined by

$$\mu(\alpha)(h) = \alpha(\mu^{-1}(h))$$

for $\alpha \in \mathfrak{h}^*, h \in \mathfrak{h}$. Let

$$Q = \mathbb{Z}\Pi = \mathbb{Z}\alpha_0 \oplus \cdots \oplus \mathbb{Z}\alpha_l$$

be the root lattice of \mathfrak{g} , and $\dot{Q} = \mathbb{Z}\dot{\Pi}$ be the root lattice of $\dot{\mathfrak{g}}$. Note that

$$\mu(\alpha_i) = \alpha_{\mu(i)} \quad \text{for } i \in I$$

and hence $\mu(Q) = Q$. Furthermore, it is known that $\mu(\delta_1) = \delta_1$ (cf. [FSS]). Since $Q = \dot{Q} \oplus \mathbb{Z}\delta_1$, for each $\dot{\alpha} \in \dot{Q}$, we can write

$$\mu(\dot{\alpha}) = \dot{\mu}(\dot{\alpha}) + \rho_\mu(\dot{\alpha})\delta_1,$$

where $\dot{\mu}(\dot{\alpha}) \in \dot{Q}$ and $\rho_\mu(\dot{\alpha}) \in \mathbb{Z}$. One can easily check that

$$\dot{\mu} : \dot{Q} \rightarrow \dot{Q}, \quad \dot{\alpha} \mapsto \dot{\mu}(\dot{\alpha})$$

defines an automorphism of \dot{Q} , and the map

$$\rho_\mu : \dot{Q} \rightarrow \mathbb{Z}, \quad \dot{\alpha} \mapsto \rho_\mu(\dot{\alpha})$$

is a homomorphism of abelian groups.

Now identify $\dot{\mathfrak{h}}$ with $\dot{\mathfrak{h}}^*$ via $\langle \cdot, \cdot \rangle$ so that the coroot $\dot{\alpha}^\vee = \frac{2\dot{\alpha}}{\langle \dot{\alpha}, \dot{\alpha} \rangle}$ for $\dot{\alpha} \in \dot{\Delta}$. Then by \mathbb{C} -linearity we obtain the two linear maps $\dot{\mu} : \dot{\mathfrak{h}} \rightarrow \dot{\mathfrak{h}}$ and $\rho_\mu : \dot{\mathfrak{h}} \rightarrow \mathbb{C}$ such that

$$(4.16) \quad \mu(h) = \dot{\mu}(h) + \rho_\mu(h)k_1$$

for $h \in \dot{\mathfrak{h}}$. Furthermore, we can extend $\dot{\mu}$ to an automorphism of the Lie algebra $\dot{\mathfrak{g}}$ by the following rule (cf. [CJKT]):

$$(4.17) \quad \mu(x_{\dot{\alpha}}) = t_1^{\rho_\mu(\dot{\alpha})} \otimes \dot{\mu}(x_{\dot{\alpha}}) \in \mathfrak{g}_{\dot{\mu}(\dot{\alpha}) + \rho_\mu(\dot{\alpha})\delta_1}$$

for $\dot{\alpha} \in \dot{\Delta}$ (recall that $\mu(\dot{\alpha}) = \dot{\mu}(\dot{\alpha}) + \rho_\mu(\dot{\alpha})\delta_1$). Then we have the following results:

LEMMA 4.2: For $\dot{\alpha} \in \dot{\Delta}$ and $n \in \mathbb{Z}$, we have

$$(4.18) \quad \mu(t_1^n \otimes x_{\dot{\alpha}}) = t_1^{\rho_\mu(\dot{\alpha})+n} \otimes \dot{\mu}(x_{\dot{\alpha}}), \quad \mu(k_1) = k_1,$$

$$(4.19) \quad \mu(t_1^n \otimes \dot{\alpha}^\vee) = t_1^n \otimes \dot{\mu}(\dot{\alpha}^\vee) + \delta_{n,0} \rho_\mu(\dot{\alpha}^\vee) k_1,$$

$$(4.20) \quad \mu(d_1) = d_1 + \mathbf{h} - \frac{\langle \mathbf{h}, \mathbf{h} \rangle}{2} k_1,$$

where $\mathbf{h} \in \dot{\mathfrak{h}}$ is determined by

$$(4.21) \quad \dot{\mu}(\dot{\gamma})(\mathbf{h}) = -\rho_\mu(\dot{\gamma}) \quad \text{for } \dot{\gamma} \in \dot{Q}.$$

Furthermore, for any $h \in \dot{\mathfrak{h}}$ we have

$$(4.22) \quad \begin{aligned} \dot{\mu}^T(h) &= h, \\ \sum_{p=0}^{T-1} \rho_\mu(\dot{\mu}^p(h)) &= 0, \\ \sum_{p=0}^{T-1} \dot{\mu}^p(\mathbf{h}) &= 0 = \sum_{p=1}^{T-1} (T-p) \langle \dot{\mu}^p(\mathbf{h}), \mathbf{h} \rangle + \frac{T \langle \mathbf{h}, \mathbf{h} \rangle}{2}. \end{aligned}$$

Proof. The equalities (4.18) and (4.19) were proved in [CJKT, Proposition 2.2]. Assume that $\mu(d_1) = \mathbf{h} + ak_1 + bd_1$ with $\mathbf{h} \in \dot{\mathfrak{h}}$ and $a, b \in \mathbb{C}$. Since

$$1 = \langle d_1, k_1 \rangle = \langle \mu(d_1), \mu(k_1) \rangle = \langle \mathbf{h} + ak_1 + bd_1, k_1 \rangle = b,$$

we obtain $b = 1$. Then for any $\dot{\alpha} \in \dot{Q}$,

$$0 = \langle \mu(d_1), \mu(\dot{\alpha}) \rangle = \langle \mathbf{h} + ak_1 + d_1, \dot{\mu}(\dot{\alpha}) + \rho_\mu(\dot{\alpha})k_1 \rangle = \langle \mathbf{h}, \dot{\mu}(\dot{\alpha}) \rangle + \rho_\mu(\dot{\alpha}),$$

and hence \mathbf{h} is determined by the condition (4.21). And by using the fact that $\langle \mu(d_1), \mu(d_1) \rangle = 0$, one obtains

$$a = -\frac{\langle \mathbf{h}, \mathbf{h} \rangle}{2},$$

which implies (4.20). Finally, (4.22) follows from the equalities (4.19), (4.20) and the fact that μ has order T . ■

Now we are ready to define an automorphism of the full toroidal Lie algebra $\mathcal{T}(\dot{\mathfrak{g}})$ from the diagram automorphism μ of $\dot{\mathfrak{g}}$.

PROPOSITION 4.3: *The following assignment, for $\dot{\alpha} \in \dot{\Delta}$, $m_0, m_1 \in \mathbb{Z}$ and $i = 0, 1$,*

$$\begin{aligned} t_0^{m_0} t_1^{m_1} \otimes x_{\dot{\alpha}} &\mapsto t_0^{m_0} t_1^{m_1 + \rho_\mu(\dot{\alpha})} \otimes \dot{\mu}(x_{\dot{\alpha}}), \\ t_0^{m_0} t_1^{m_1} \otimes \dot{\alpha}^\vee &\mapsto t_0^{m_0} t_1^{m_1} \otimes \dot{\mu}(\dot{\alpha}^\vee) + \rho_\mu(\dot{\alpha}^\vee) t_0^{m_0} t_1^{m_1} k_1, \\ t_0^{m_0} t_1^{m_1} k_i &\mapsto t_0^{m_0} t_1^{m_1} k_i, \\ t_0^{m_0} t_1^{m_1} d_0 &\mapsto t_0^{m_0} t_1^{m_1} d_0, \\ t_0^{m_0} t_1^{m_1} d_1 &\mapsto t_0^{m_0} t_1^{m_1} d_1 + t_0^{m_0} t_1^{m_1} \otimes \mathbf{h} - \frac{\langle \mathbf{h}, \mathbf{h} \rangle}{2} t_0^{m_0} t_1^{m_1} k_1, \end{aligned}$$

defines an automorphism, denoted by $\hat{\mu}$, of the full toroidal Lie algebra $\mathcal{T}(\mathfrak{g})$ with order T .

Proof. For $m_0, m_1, n_0, n_1 \in \mathbb{Z}$, by applying (4.4)–(4.7), we have

$$\begin{aligned} &[\hat{\mu}(t_0^{m_0} t_1^{m_1} d_1), \hat{\mu}(t_0^{n_0} t_1^{n_1} d_1)] \\ &= [t_0^{m_0} t_1^{m_1} d_1, t_0^{n_0} t_1^{n_1} d_1] - \frac{\langle \mathbf{h}, \mathbf{h} \rangle}{2} ([t_0^{m_0} t_1^{m_1} d_1, t_0^{n_0} t_1^{n_1} k_1] + [t_0^{m_0} t_1^{m_1} k_1, t_0^{n_0} t_1^{n_1} d_1]) \\ &\quad + [t_0^{m_0} t_1^{m_1} d_1, t_0^{n_0} t_1^{n_1} \otimes \mathbf{h}] + [t_0^{m_0} t_1^{m_1} \otimes \mathbf{h}, t_0^{n_0} t_1^{n_1} d_1] + [t_0^{m_0} t_1^{m_1} \otimes \mathbf{h}, t_0^{n_0} t_1^{n_1} \otimes \mathbf{h}] \\ &= (n_1 - m_1) \left(t_0^{m_0+n_0} t_1^{m_1+n_1} d_1 + t_0^{m_0+n_0} t_1^{m_1+n_1} \otimes \mathbf{h} - \frac{\langle \mathbf{h}, \mathbf{h} \rangle}{2} t_0^{m_0+n_0} t_1^{m_1+n_1} k_1 \right) \\ &\quad - \frac{\langle \mathbf{h}, \mathbf{h} \rangle}{2} \left(\sum_{i=1,2} m_i t_0^{m_0+n_0} t_1^{m_1+n_1} k_i - \sum_{i=1,2} n_i t_0^{m_0+n_0} t_1^{m_1+n_1} k_i \right) \\ &\quad + \langle \mathbf{h}, \mathbf{h} \rangle \sum_{i=1,2} m_i t_0^{m_0+n_0} t_1^{m_1+n_1} k_i \\ &= \hat{\mu}([t_0^{m_0} t_1^{m_1} d_1, t_0^{n_0} t_1^{n_1} d_1]). \end{aligned}$$

One can check that $\hat{\mu}$ preserves all other relations by a similar argument as above. And it follows from (4.22) that the order of the automorphism $\hat{\mu}$ is equal to T . ■

4.3. SUBALGEBRA OF $\tilde{\mathfrak{g}}$ FIXED BY THE AUTOMORPHISM $\tilde{\mu}$. From the definition of the automorphism $\hat{\mu}$ of the full toroidal Lie algebra $\mathcal{T}(\mathfrak{g})$ given in the previous subsection, it is easy to see that $\hat{\mu}(\tilde{\mathfrak{g}}) = \tilde{\mathfrak{g}}$. Fix a \mathbb{Z} -grading $\tilde{\mathfrak{g}} = \bigoplus_{m \in \mathbb{Z}} \tilde{\mathfrak{g}}(m)$ by the adjoint action of $-d_0$, i.e., $\tilde{\mathfrak{g}}(m) = \{x \in \tilde{\mathfrak{g}} \mid [d_0, x] = -mx\}$. Set

$$\omega = \omega_T = e^{2\pi\sqrt{-1}/T}$$

and let ω^{-d_0} be the automorphism of $\tilde{\mathfrak{g}}$ defined by $\omega^{-d_0}(a) = \omega^m a$ for $a \in \tilde{\mathfrak{g}}(m)$. Therefore,

$$\tilde{\mu} = \omega^{-d_0} \circ \hat{\mu}|_{\tilde{\mathfrak{g}}}$$

defines an automorphism for the toroidal EALA $\tilde{\mathfrak{g}}$. Now we investigate the Lie subalgebra $\tilde{\mathfrak{g}}[\mu]$ of $\tilde{\mathfrak{g}}$ fixed by the automorphism $\tilde{\mu}$.

Firstly, for $x \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta^\times \cup \{0\}$, $h \in \mathfrak{h}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$, we have

$$\begin{aligned}
 \tilde{\mu}(t_0^m x) &= \omega^{-m} t_0^m \mu(x), \\
 \tilde{\mu}(t_0^m t_1^n \otimes h) &= \omega^{-m} (t_0^m t_1^n \otimes \dot{\mu}(h) - \rho_\mu(h) m k_{m,n}), \\
 \tilde{\mu}(k_0) &= k_0, \\
 \tilde{\mu}(d_0) &= d_0, \\
 \tilde{\mu}(k_{m,n}) &= \omega^{-m} k_{m,n}, \\
 \tilde{\mu}(\tilde{d}_{m,n}) &= \omega^{-m} \left(\tilde{d}_{m,n} + m t_0^m t_1^n \otimes \mathbf{h} + \frac{\langle \mathbf{h}, \mathbf{h} \rangle}{2} m^2 k_{m,n} \right).
 \end{aligned}
 \tag{4.23}$$

Moreover, we recall that the diagram automorphism μ preserves the Cartan subalgebra \mathfrak{h} and the bilinear form $\langle \cdot, \cdot \rangle$. Therefore, by (4.14), (4.21) and (4.23), we have the following results:

LEMMA 4.4: $\tilde{\mu}(\tilde{\mathfrak{h}}) = \tilde{\mathfrak{h}}$, and $\langle \tilde{\mu}(x), \tilde{\mu}(y) \rangle = \langle x, y \rangle$ for $x, y \in \tilde{\mathfrak{g}}$.

Denote by $\tilde{\mathfrak{g}}[\mu]$ and $\tilde{\mathfrak{h}}[\mu]$ the Lie subalgebras of $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{h}}$ respectively fixed by $\tilde{\mu}$. We consider the root space decomposition of $\tilde{\mathfrak{g}}[\mu]$ with respect to its abelian subalgebra $\tilde{\mathfrak{h}}[\mu]$. Denote by $(\mathfrak{h}^*)^\mu$ the subspace of \mathfrak{h}^* that is fixed by μ and denote by

$$\pi_\mu : \mathfrak{h}^* \rightarrow (\mathfrak{h}^*)^\mu, \quad \alpha \mapsto \tilde{\alpha} = \frac{1}{T} \sum_{p=0}^{T-1} \mu^p(\alpha)$$

the canonical projection. Since $\tilde{\alpha}(h - \mu(h)) = 0$ for $\alpha \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$, we may identify $(\mathfrak{h}^\mu)^*$ with $(\mathfrak{h}^*)^\mu$, where \mathfrak{h}^μ is the subspace of \mathfrak{h} fixed by μ . Note that

$$\tilde{\mathfrak{h}}[\mu] = \mathfrak{h}^\mu \oplus \mathbb{C}k_0 \oplus \mathbb{C}d_0.$$

We view $(\mathfrak{h}^\mu)^*$ as a subspace of $\tilde{\mathfrak{h}}[\mu]^*$ such that

$$\alpha(k_0) = \alpha(d_0) = 0$$

for $\alpha \in (\mathfrak{h}^\mu)^*$, and define $\delta_0 \in \tilde{\mathfrak{h}}[\mu]^*$ by

$$\delta_0(\mathfrak{h}^\mu) = \delta_0(k_0) = 0, \quad \delta_0(d_0) = 1.
 \tag{4.24}$$

In particular, for each $\alpha \in \Delta$, $\tilde{\alpha} \in (\mathfrak{h}^*)^\mu = (\mathfrak{h}^\mu)^*$ is an element in $\tilde{\mathfrak{h}}[\mu]^*$.

For $\alpha \in \tilde{\mathfrak{h}}[\mu]^*$, set

$$\tilde{\mathfrak{g}}[\mu]_\alpha = \{x \in \tilde{\mathfrak{g}}[\mu] \mid [h, x] = \alpha(h)x, h \in \tilde{\mathfrak{h}}[\mu]\}.$$

Then we have the following root space decomposition:

$$\tilde{\mathfrak{g}}[\mu] = \tilde{\mathfrak{g}}[\mu]_0 \oplus \sum_{\alpha \in \tilde{\Delta}_\mu} \tilde{\mathfrak{g}}[\mu]_\alpha,$$

where $\tilde{\Delta}_\mu = \{\alpha \in \tilde{\mathfrak{h}}[\mu]^* \setminus \{0\} \mid \tilde{\mathfrak{g}}[\mu]_\alpha \neq 0\}$. Let

$$(4.25) \quad \eta_\mu : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}[\mu], \quad x \mapsto \sum_{p=0}^{T-1} \tilde{\mu}^p(x)$$

be a projection from $\tilde{\mathfrak{g}}$ to $\tilde{\mathfrak{g}}[\mu]$. It can be readily seen that

$$(4.26) \quad \eta_\mu(t_0^m u) \in \tilde{\mathfrak{g}}[\mu]_{\tilde{\alpha} + m\delta_0}, \eta_\mu(k_{m,n}), \eta_\mu(\tilde{d}_{m,n}) \in \tilde{\mathfrak{g}}[\mu]_{m\delta_0 + n\delta_1}, k_0, d_0 \in \tilde{\mathfrak{g}}[\mu]_0$$

for $u \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta \cup \{0\}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$. Then we have

LEMMA 4.5: *The root system $\tilde{\Delta}_\mu \subset (\pi_\mu(\Delta) + \mathbb{Z}\delta_0) \cup \mathbb{Z}\delta_0$ and $\tilde{\mathfrak{g}}[\mu]_0 = \tilde{\mathfrak{h}}[\mu]$.*

Proof. The first assertion follows from (4.13) and (4.26). For the second one, note that if $\alpha = \sum_{i \in I} a_i \alpha_i$ is a root in Δ , then

$$\tilde{\alpha} = \frac{1}{T} \sum_{i \in I} \sum_{p=0}^{T-1} a_i \alpha_{i\mu^p(i)}$$

is clearly nonzero. This together with (4.26) implies that

$$\tilde{\mathfrak{g}}[\mu]_0 = \eta_\mu(\mathfrak{h}) \oplus \mathbb{C}k_0 \oplus \mathbb{C}d_0 = \tilde{\mathfrak{h}}[\mu]. \quad \blacksquare$$

Denote by $\mathfrak{t}(\tilde{\mathfrak{g}}, \mu)$ the subalgebra of $\mathfrak{t}(\tilde{\mathfrak{g}})$ fixed by $\tilde{\mu}$ (noting that $\tilde{\mu}(\mathfrak{t}(\tilde{\mathfrak{g}})) = \mathfrak{t}(\tilde{\mathfrak{g}})$). We have the following result from [CJKT].

PROPOSITION 4.6: *The Lie algebra $\mathfrak{t}(\tilde{\mathfrak{g}}, \mu)$ is centrally closed.*

In the rest of this subsection, we recall a characterization for the subset of the root system $\tilde{\Delta}_\mu$ given in [CJKT]:

$$\tilde{\Delta}_\mu^\times = \{\tilde{\alpha} + m\delta_0 \in \tilde{\Delta}_\mu \mid \alpha \in \Delta, m \in \mathbb{Z} \text{ and } \langle \tilde{\alpha}, \tilde{\alpha} \rangle \neq 0\}.$$

For every $i \in I = \{0, 1, \dots, l\}$, denote by $\mathcal{O}(i)$ the orbit containing i under the action of the group $\langle \mu \rangle$. We say μ is **transitive** if $\mathcal{O}(i) = I$ for each $i \in I$. Observe that a diagram automorphism on \mathfrak{g} is transitive if and only if \mathfrak{g} is of type $A_l^{(1)}$, and it has order $l + 1$. Note that in this case we have $\pi_\mu(\alpha_i) = \delta_1$ for any $i \in I$, and hence $\tilde{\Delta}_\mu^\times = \emptyset$.

From now on we assume that μ is nontransitive, and in this case, it is known that the folded matrix

$$\check{A} = \left(2 \frac{\langle \check{\alpha}_i, \check{\alpha}_j \rangle}{\langle \check{\alpha}_i, \check{\alpha}_i \rangle} \right)_{i,j \in \check{I}}$$

of A associated to μ is also an affine generalized Cartan matrix (cf. [FSS, ABP]), where $\check{I} = \{i \in I \mid \mu^k(i) \geq i \text{ for } k \in \mathbb{Z}\}$ is a set of representative elements in I . Denote by $\check{\Delta}$ and \check{W} respectively the root system and the Weyl group associated to the folded matrix \check{A} . Then $\{\check{\alpha}_i\}_{i \in \check{I}}$ is a simple root base of $\check{\Delta}$. Furthermore, we have the following result from [ABP, Lemma 12.15].

LEMMA 4.7: *For each $i \in I$, one and only one of the following statement holds:*

- (a) *The elements α_p , for $p \in \mathcal{O}(i)$, are pairwise orthogonal;*
- (b) *$\mathcal{O}(i) = \{i, \mu(i)\}$, and $a_{i\mu(i)} = -1 = a_{\mu(i)i}$.*

Recall that T is the order of the diagram automorphism μ . For every $i \in I$, we set

$$(4.27) \quad T_i = T/|\mathcal{O}(i)|, \quad \text{and} \quad s_i = \begin{cases} 1, & \text{if (a) holds in Lemma 4.7;} \\ 2, & \text{if (b) holds in Lemma 4.7.} \end{cases}$$

The following result was proved in [CJKT, Proposition 5.1].

PROPOSITION 4.8: *If μ is nontransitive, then*

$$\begin{aligned} \check{\Delta}_\mu^\times = & \{ \check{w}\check{\alpha}_i + T_i m \delta_0 \mid \check{w} \in \check{W}, i \in \check{I}, m \in \mathbb{Z} \} \\ & \cup \left\{ 2\check{w}\check{\alpha}_i + \left(\frac{T}{2} + mT \right) \delta_0 \mid \check{w} \in \check{W}, i \in \check{I} \text{ with } s_i = 2, m \in \mathbb{Z} \right\}. \end{aligned}$$

Remark 4.9: If μ is nontransitive, then $\check{\Delta}_\mu^\times$ is a nullity 2 reduced extended affine root system introduced by Saito (cf. [Sa]). Conversely, any nullity 2 reduced extended affine root system is of such a form (cf. [ABP]). Furthermore, we will see in the next section that the triple $(\check{\mathfrak{g}}[\mu], \check{\mathfrak{h}}[\mu], \langle \cdot, \cdot \rangle)$ is a nullity 2 EALA, and which we call a **nullity 2 twisted toroidal extended affine Lie algebra**.

5. Nullity 2 EALA of maximal type

In this section, we first recall the definition of extended affine Lie algebra (cf. [AABGP, N]), and the classification of nullity 2 extended affine Lie algebras of maximal type based on Allison–Berman–Pianzola’s work [ABP].

Let \mathcal{E} be a Lie algebra equipped with a nontrivial finite-dimensional self-centralizing ad-diagonalizable subalgebra \mathcal{H} and a nondegenerate invariant symmetric bilinear form $(\cdot|\cdot)$. Let

$$\mathcal{E} = \mathcal{H} \oplus \sum_{\alpha \in \Phi} \mathcal{E}_\alpha$$

be the root space decomposition with respect to \mathcal{H} , where Φ is the corresponding root system. The form $(\cdot|\cdot)$ is also nondegenerate when it is restricted to $\mathcal{H} = \mathcal{E}_0$. Hence it induces a nondegenerate symmetric bilinear form on \mathcal{H}^* . Set

$$\begin{aligned} \Phi^\times &= \{\alpha \in \Phi \mid (\alpha|\alpha) \neq 0\}, \\ \Phi^0 &= \{\alpha \in \Phi \mid (\alpha|\alpha) = 0\}. \end{aligned}$$

Let \mathcal{E}_c be the subalgebra of \mathcal{E} generated by the root spaces $\mathcal{E}_\alpha, \alpha \in \Phi^\times$, which is called the **core** of \mathcal{E} .

Definition 5.1: $\mathcal{E} := (\mathcal{E}, \mathcal{H}, (\cdot|\cdot))$ is called an **extended affine Lie algebra** (EALA for short) if

- (1) $\text{ad}(x)$ is locally nilpotent for $x \in \mathcal{E}_\alpha, \alpha \in \Phi^\times$;
- (2) Φ^\times cannot be decomposed as a union of two orthogonal nonempty subsets;
- (3) the centralizer of \mathcal{E}_c in \mathcal{E} is contained in \mathcal{E}_c ;
- (4) Φ is a discrete subset of \mathcal{H}^* with respect to its Euclidean topology.

The axiom (4) implies that the subgroup $\langle \Phi^0 \rangle$ of \mathcal{H}^* generated by Φ^0 is a free abelian group of finite rank, and this rank is called the **nullity** of \mathcal{E} . Indeed, the nullity 0 EALAs are exactly the finite dimensional simple Lie algebras, while nullity 1 EALAs are exactly the affine Kac–Moody algebras (cf. [ABGP]). For the purpose of classifying EALAs, the following notion was introduced in [BGK].

Definition 5.2: An EALA is of **maximal type** if its core is centrally closed.

Note that the nullity 0 and nullity 1 EALAs are all of maximal type. In general, an EALA may not be of maximal type. However, the maximal EALAs appear to be the most interesting ones as from them one can know the structure of other EALAs (see [BGK, Remark 3.73] and [N] for details). In what follows, we give two classes of nullity 2 EALAs of maximal type.

PROPOSITION 5.3: *Let μ be a nontransitive diagram automorphism of an untwisted affine Kac–Moody algebra \mathfrak{g} . Then the triple $(\tilde{\mathfrak{g}}[\mu], \tilde{\mathfrak{h}}[\mu], \langle \cdot, \cdot \rangle)$, defined in the previous section, is a nullity 2 EALA of maximal type, and $\mathfrak{t}(\tilde{\mathfrak{g}}, \mu)$ is the core of $\tilde{\mathfrak{g}}[\mu]$.*

Proof. Recall from Lemma 4.5 that the ad-diagonalizable subalgebra $\tilde{\mathfrak{h}}[\mu]$ of $\tilde{\mathfrak{g}}[\mu]$ is self-centralized, and it follows from Lemma 4.4 that the invariant form $\langle \cdot, \cdot \rangle$ restricted to $\tilde{\mathfrak{g}}[\mu]$ is still nondegenerate. One can easily check that $\mathfrak{t}(\tilde{\mathfrak{g}}, \mu)$ is generated by the elements $\eta_\mu(t_0^m x_{\pm\alpha_i})$ for $m \in \mathbb{Z}$ and $i \in I$. This together with Proposition 4.8 implies that the core of $\tilde{\mathfrak{g}}[\mu]$ is $\mathfrak{t}(\tilde{\mathfrak{g}}, \mu)$. Now we check the axioms (1)–(4) in Definition 5.1. The axiom (1) follows from Proposition 4.8, while the axiom (2) is implied by Remark 4.9 as the root system defined by Saito is irreducible (see [Sa, Definition 1]). The axioms (3) and (4) are obvious. Finally, Proposition 4.6 implies the maximality of $\tilde{\mathfrak{g}}[\mu]$. ■

For the second class of nullity 2 EALAs of maximal type, we let $q \in \mathbb{C}^*$ be generic, i.e, q is not a root of unity, and $\mathbb{C}_q := \mathbb{C}_q[t_0^{\pm 1}, t_1^{\pm 1}]$ the quantum torus associated to q such that $t_0 t_1 = q t_1 t_0$. For any positive integer $N \geq 2$, $\mathfrak{gl}_N(\mathbb{C}_q)$ denotes the general linear Lie algebra over \mathbb{C}_q , and $\mathfrak{sl}_N(\mathbb{C}_q)$ the derived subalgebra of $\mathfrak{gl}_N(\mathbb{C}_q)$. For $1 \leq i, j \leq N$, $a \in \mathbb{C}_q$, we write $E_{i,j} a$ for the matrix whose only possible nonzero entry is the (i, j) -entry which is a . We consider the central extension of the Lie algebra $\mathfrak{gl}_N(\mathbb{C}_q)$:

$$\widehat{\mathfrak{gl}}_N(\mathbb{C}_q) := \mathfrak{gl}_N(\mathbb{C}_q) \oplus \mathbb{C}k_0 \oplus \mathbb{C}k_1,$$

where k_0, k_1 are central elements, and

$$\begin{aligned} [E_{i,j} t_0^{m_0} t_1^{m_1}, E_{k,l} t_0^{n_0} t_1^{n_1}] = & \delta_{j,k} q^{m_1 n_0} E_{i,l} t_0^{m_0+n_0} t_1^{m_1+n_1} \\ & - \delta_{i,l} q^{n_1 m_0} E_{k,j} t_0^{m_0+n_0} t_1^{m_1+n_1} \\ & + \delta_{j,k} \delta_{i,l} \delta_{m_0+n_0,0} \delta_{m_1+n_1,0} q^{m_1 n_0} (m_0 k_0 + m_1 k_1), \end{aligned} \tag{5.1}$$

for $1 \leq i, j, k, l \leq N$ and $m_0, m_1, n_0, n_1 \in \mathbb{Z}$. Moreover, we define two derivations d_0, d_1 acting on $\widehat{\mathfrak{gl}}_N(\mathbb{C}_q)$ by

$$[d_r, E_{i,j} t_0^{m_0} t_1^{m_1}] = m_r E_{i,j} t_0^{m_0} t_1^{m_1}, \quad [d_r, k_s] = 0 = [d_r, d_s],$$

for $r, s \in \{0, 1\}$ and for $1 \leq i, j \leq N, m_0, m_1 \in \mathbb{Z}$. Therefore, we obtain a Lie algebra

$$\tilde{\mathfrak{gl}}_N(\mathbb{C}_q) := \widehat{\mathfrak{gl}}_N(\mathbb{C}_q) \oplus \mathbb{C}d_0 \oplus \mathbb{C}d_1.$$

Set

$$\widehat{\mathfrak{sl}}_N(\mathbb{C}_q) = [\widehat{\mathfrak{gl}}_N(\mathbb{C}_q), \widehat{\mathfrak{gl}}_N(\mathbb{C}_q)] = \mathfrak{sl}_N(\mathbb{C}_q) \oplus \mathbb{C}k_0 \oplus \mathbb{C}k_1,$$

the derived subalgebra of $\widehat{\mathfrak{gl}}_N(\mathbb{C}_q)$. It is known that $\widehat{\mathfrak{sl}}_N(\mathbb{C}_q)$ is centrally closed and $\widehat{\mathfrak{gl}}_N(\mathbb{C}_q) = \widehat{\mathfrak{sl}}_N(\mathbb{C}_q) \oplus \mathbb{C}I_N$ (cf. [BGK]), where I_N is the identity matrix. Furthermore, we define

$$\begin{aligned} \widetilde{\mathfrak{sl}}_N(\mathbb{C}_q) &:= \widehat{\mathfrak{sl}}_N(\mathbb{C}_q) \oplus \mathbb{C}d_0 \oplus \mathbb{C}d_1, \\ \widetilde{\mathcal{H}} &:= \sum_{i=1}^{N-1} \mathbb{C}(E_{i,i} - E_{i+1,i+1}) \oplus \mathbb{C}k_0 \oplus \mathbb{C}k_1 \oplus \mathbb{C}d_0 \oplus \mathbb{C}d_1, \end{aligned}$$

and a bilinear form $\langle \cdot, \cdot \rangle$ of $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ such that

$$\langle E_{i,j}t_0^{m_0}t_1^{m_1}, E_{j,i}t_0^{-m_0}t_1^{-m_1} \rangle = 1 = \langle d_r, k_r \rangle,$$

for $1 \leq i, j \leq N, m_0, m_1 \in \mathbb{Z}, r \in \{0, 1\}$, and trivial for others. The following result is from [BGK]:

PROPOSITION 5.4: *Let $N \geq 2$ be a positive integer and $q \in \mathbb{C}^*$ be generic. Then the triple $(\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q), \widetilde{\mathcal{H}}, \langle \cdot, \cdot \rangle)$ is a nullity 2 EALA of maximal type, and $\widehat{\mathfrak{sl}}_N(\mathbb{C}_q)$ is the core of $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$.*

For any given EALA \mathcal{E} , we call $\mathcal{E}_{cc} := \mathcal{E}_c/Z(\mathcal{E}_c)$ the **centerless core** of \mathcal{E} , where $Z(\mathcal{E}_c)$ is the center of the core \mathcal{E}_c . One notes that the centerless core of the affine Kac–Moody algebra \mathfrak{g} is isomorphic to the loop algebra $\overline{\mathfrak{g}} = \mathbb{C}[t_1, t_1^{-1}] \otimes \mathfrak{g}$. Denote by $\bar{\mu}$ the automorphism of $\overline{\mathfrak{g}}$ induced by μ . Then the centerless core of $\widetilde{\mathfrak{g}}[\mu]$ is isomorphic to the $\bar{\mu}$ -twisted loop algebra

$$\mathcal{L}(\overline{\mathfrak{g}}, \bar{\mu}) = \sum_{n \in \mathbb{Z}} t_0^n \otimes \overline{\mathfrak{g}}_{(m)} \subset \mathbb{C}[t_0, t_0^{-1}] \otimes \overline{\mathfrak{g}} = \mathcal{R} \otimes \mathfrak{g},$$

where $\overline{\mathfrak{g}}_{(m)} = \{x \in \overline{\mathfrak{g}} \mid \bar{\mu}(x) = \omega^m x\}$.

It is clear that the centerless core of $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ with q generic is isomorphic to $\mathfrak{sl}_N(\mathbb{C}_q)$. The following classification of centerless cores of nullity 2 EALAs was given in [ABP]:

PROPOSITION 5.5: *The centerless core of a nullity 2 EALA is either isomorphic to $\mathfrak{sl}_N(\mathbb{C}_q)$ for some positive integer $N \geq 2$ and generic $q \in \mathbb{C}^*$, or to $\mathcal{L}(\overline{\mathfrak{g}}, \bar{\mu})$ for some nontransitive diagram automorphism μ of an untwisted affine Kac–Moody algebra \mathfrak{g} .*

Two EALAs are said to be **equivalent** if their cores are isomorphic. Then we have the following result.

THEOREM 5.6: *Up to equivalence, a nullity 2 extended affine Lie algebra of maximal type either has the form $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ for some positive integer $N \geq 2$ and generic $q \in \mathbb{C}^*$, or has the form $\widetilde{\mathfrak{g}}[\mu]$ for some nontransitive diagram automorphism μ of an untwisted affine Kac–Moody algebra \mathfrak{g} .*

Proof. Let \mathcal{E} be a nullity 2 EALA of maximal type. Then by Proposition 5.5, either $\mathcal{E}_{cc} \cong \mathfrak{sl}_N(\mathbb{C}_q)$, or $\mathcal{E}_{cc} \cong \mathcal{L}(\widetilde{\mathfrak{g}}, \bar{\mu})$. Then the maximality of \mathcal{E} forces that either $\mathcal{E}_c \cong \widehat{\mathfrak{sl}}_N(\mathbb{C}_q)$ or $\mathcal{E}_c \cong \mathfrak{t}(\widehat{\mathfrak{g}}, \mu)$. Thus it follows from Propositions 5.3 and 5.4 that \mathcal{E} is either equivalent to $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ or to $\widetilde{\mathfrak{g}}[\mu]$. ■

Remark 5.7: Following [BGK, N], a \mathcal{K} -valued affine cocycle τ on \mathcal{S} is an abelian 2-cocycle $\tau : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{K}$ such that $\tau(\mathcal{S}, d_i) = 0$ for $i = 0, 1$, and $\langle \tau(s_1, s_2), s_3 \rangle = \langle s_1, \tau(s_2, s_3) \rangle$ for $s_1, s_2, s_3 \in \mathcal{S}$. For any affine cocycle τ , one can define a new Lie multiplication $[\cdot, \cdot]_\tau$ on $\widetilde{\mathfrak{g}}$ defined by

$$[x_1 + s_1, x_2 + s_2]_\tau = [x_1 + s_1, x_2 + s_2] + \tau(s_1, s_2)$$

for $x_1, x_2 \in \mathfrak{t}(\widehat{\mathfrak{g}})$ and $s_1, s_2 \in \mathcal{S}$. We denote the resulting Lie algebra by $\widetilde{\mathfrak{g}}_\tau$. It is easy to see that the action (4.23) also defines an automorphism $\tilde{\mu}$ of $\widetilde{\mathfrak{g}}_\tau$, denote by $\widetilde{\mathfrak{g}}[\mu]_\tau$ the subalgebra of $\widetilde{\mathfrak{g}}_\tau$ fixed by $\tilde{\mu}$. Similar to the proof of Proposition 5.3, one can check that $(\widetilde{\mathfrak{g}}[\mu]_\tau, \widehat{\mathfrak{h}}[\mu], \langle \cdot, \cdot \rangle)$ is a nullity 2 EALA of maximal type if μ is nontransitive. By the explicit construction of EALAs (of maximal type) given by Neher in [N], one can prove that if an EALA is equivalent to $\widetilde{\mathfrak{g}}[\mu]$, then it is isomorphic to $\widetilde{\mathfrak{g}}[\mu]_\tau$ for some affine cocycle τ . And, if an EALA is equivalent to $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$, then it must be isomorphic to $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$.

Remark 5.8: It was shown in [MRY] that there exist nontrivial affine cocycles. For example, for any complex number a , the bilinear map $\tau_a : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{K}$ defined by

$$\begin{aligned} \tau_a(\mathcal{S}, d_r) &= 0, \\ \tau_a(d_{m_0, m_1}, d_{n_0, n_1}) &= a(m_0 n_1 - n_0 m_1)^3 k_{m_0 + n_0, m_1 + n_1} \end{aligned}$$

for $m_0, m_1, n_0, n_1 \in \mathbb{Z}$ and $r = 0, 1$, is an affine cocycle. Then we have the Lie algebras $\widetilde{\mathfrak{g}}[\mu]_{\tau_a}$ with $\widetilde{\mathfrak{g}}[\mu]_{\tau_0} = \widetilde{\mathfrak{g}}[\mu]$. It is conjectured that any \mathcal{K} -valued affine cocycle on \mathcal{S} has the form τ_a for some $a \in \mathbb{C}$. This will imply that $\widetilde{\mathfrak{g}}[\mu]_{\tau_a}$ and $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ exhaust all nullity 2 EALAs of maximal type up to isomorphism.

6. Associating $\tilde{\mathfrak{g}}[\mu]$ with vertex algebras

Let μ be a fixed diagram automorphism of the untwisted affine Kac–Moody algebra \mathfrak{g} . In this section, we associate the twisted toroidal EALA $\tilde{\mathfrak{g}}[\mu]$ with vertex algebras through equivariant ϕ -coordinated quasi-modules.

6.1. VERTEX ALGEBRAS $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ AND $L_{\widehat{\mathfrak{g}}}(\ell, 0)$. We recall the variant of the skew derivation algebra \mathcal{S} introduced in [CLiT]:

$$\widehat{\mathcal{S}} = \left\{ f_0 \frac{\partial}{\partial t_0} + f_1 d_1 \mid f_0, f_1 \in \mathcal{R}, \frac{\partial}{\partial t_0}(f_0) + d_1(f_1) = 0 \right\} \subset \text{Der}(\mathcal{R}).$$

For $m, n \in \mathbb{Z}$, set

$$\widehat{d}_{n,m} = (n + 1)t_0^n t_1^m d_1 - m t_0^n t_1^m d_0,$$

It is easy to see that the set

$$\begin{aligned} \mathbb{B}_{\widehat{\mathcal{S}}} &= \{t_0^{-1}d_0, t_0^{-1}d_1\} \cup \{\widehat{d}_{n,m} \mid (n, m) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(-1, 0)\}\} \\ &= \{t_0^{-1}d_0\} \cup \{t_0^n d_1, \widehat{d}_{n,m} \mid n \in \mathbb{Z}, m \in \mathbb{Z}^*\} \end{aligned}$$

forms a \mathbb{C} -basis of $\widehat{\mathcal{S}}$, and subject to the following relations:

$$(6.1) \quad [t_0^{-1}d_0, \widehat{d}_{n,m}] = (n + 1)\widehat{d}_{n-1,m}, \quad [t_0^{-1}d_1, \widehat{d}_{n,m}] = m\widehat{d}_{n-1,m},$$

$$(6.2) \quad [\widehat{d}_{n,m}, \widehat{d}_{n_1,m_1}] = ((n + 1)m_1 - m(n_1 + 1))\widehat{d}_{n+n_1,m+m_1},$$

for $n, m, n_1, m_1 \in \mathbb{Z}$. In view of this, we have the following subalgebra of the full toroidal Lie algebra $\mathcal{T}(\widehat{\mathfrak{g}})$:

$$\widehat{\mathfrak{g}} = \mathfrak{t}(\widehat{\mathfrak{g}}) + [\widehat{\mathcal{S}}, \widehat{\mathcal{S}}] + \mathbb{C}t_0^{-1}d_1.$$

Note that $\widehat{\mathfrak{g}}$ is linearly spanned by the set

$$(6.3) \quad \{t_0^n u, k_{n,m}, k_0, \widehat{d}_{n,m} \mid u \in \mathfrak{g}, n \in \mathbb{Z}, m \in \mathbb{Z}^*\}.$$

From (4.5) and (4.6), we have

$$(6.4) \quad [\widehat{d}_{i,m}, t_0^j t_1^n \otimes x] = ((i + 1)n - mj)t_0^{i+j} t_1^{m+n} \otimes x,$$

$$(6.5) \quad \begin{aligned} [\widehat{d}_{i,m}, k_{j,n}] &= ((i + 1)(m + n) - m(i + j))k_{i+j,m+n} \\ &\quad + \delta_{m+n,0} \delta_{i+j,0} ((i + 1)k_0 + mk_1), \end{aligned}$$

for $i, j, m, n \in \mathbb{Z}$ and $x \in \widehat{\mathfrak{g}}$. Set

$$(6.6) \quad \widehat{\mathfrak{g}}_+ = \text{Span}\{t_0^n u, k_{n+1,m}, \widehat{d}_{n-1,m} \mid u \in \mathfrak{g}, n \in \mathbb{N}, m \in \mathbb{Z}\},$$

$$(6.7) \quad \widehat{\mathfrak{g}}_- = \text{Span}\{t_0^{-n-1}u, k_{-n,m}, \widehat{d}_{-n-2,m} \mid u \in \mathfrak{g}, n \in \mathbb{N}, m \in \mathbb{Z}\}.$$

Then both $\widehat{\mathfrak{g}}_+$ and $\widehat{\mathfrak{g}}_-$ are subalgebras of $\widehat{\mathfrak{g}}$. Furthermore, we have the decomposition:

$$(6.8) \quad \widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \oplus \mathbb{C}k_0 \oplus \widehat{\mathfrak{g}}_-.$$

Let ℓ be a complex number. View \mathbb{C} as a $(\widehat{\mathfrak{g}}_+ + \mathbb{C}k_0)$ -module with $\widehat{\mathfrak{g}}_+$ acting trivially and with k_0 acting as scalar ℓ . We form the induced $\widehat{\mathfrak{g}}$ -module

$$(6.9) \quad V_{\widehat{\mathfrak{g}}}(\ell, 0) = \mathcal{U}(\widehat{\mathfrak{g}}) \otimes_{\mathcal{U}(\widehat{\mathfrak{g}}_+ + \mathbb{C}k_0)} \mathbb{C}.$$

Let \mathcal{A} be a vector space with a basis $\{K_n, D_n \mid n \in \mathbb{Z}^*\}$, and set

$$\mathcal{A}_{\mathfrak{g}} = \mathfrak{g} \oplus \mathcal{A}.$$

Form the generating functions $a(z)$, $a \in \mathcal{A}_{\mathfrak{g}}$ in $\widehat{\mathfrak{g}}[[z, z^{-1}]]$ as follows:

$$u(z) = \sum_{n \in \mathbb{Z}} (t_0^n u) z^{-n-1}, \quad D_m(z) = \sum_{n \in \mathbb{Z}} \widehat{d}_{n,m} z^{-n-2}, \quad K_m(z) = \sum_{n \in \mathbb{Z}} k_{n,m} z^{-n},$$

for $u \in \mathfrak{g}$ and $m \in \mathbb{Z}^*$. Set $\mathbf{1} = 1 \otimes 1 \in V_{\widehat{\mathfrak{g}}}(\ell, 0)$. Identify $\mathcal{A}_{\mathfrak{g}}$ as a subspace of $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ through the linear map

$$(6.10) \quad u \mapsto (t_0^{-1}u)\mathbf{1}, \quad K_n \mapsto k_{0,n}\mathbf{1}, \quad D_n \mapsto \widehat{d}_{-2,n}\mathbf{1},$$

for $u \in \mathfrak{g}$, $n \in \mathbb{Z}^*$. It is proved in [CLiT] that there is a unique vertex algebra structure on $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ with $Y(a, z) = a(z)$ for $a \in \mathcal{A}_{\mathfrak{g}}$, and $\mathbf{1}$ the vacuum vector.

Remark 6.1: Note that $\mathfrak{t}(\widehat{\mathfrak{g}}) \oplus \widehat{\mathcal{S}} = \widehat{\mathfrak{g}} \oplus \mathbb{C}t_0^{-1}d_0$ with

$$[-t_0^{-1}d_0, a(z)] = \frac{d}{dz}a(z), \quad a \in \mathcal{A}_{\mathfrak{g}}.$$

This implies that $V_{\widehat{\mathfrak{g}}}(\ell)$ is a $\mathfrak{t}(\widehat{\mathfrak{g}}) \oplus \widehat{\mathcal{S}}$ -module with $-t_0^{-1}d_0$ acting as the canonical derivation \mathcal{D} . In particular, from [CLiT] it follows that an ideal of vertex algebra $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ is the same as a $\mathfrak{t}(\widehat{\mathfrak{g}}) \oplus \widehat{\mathcal{S}}$ -submodule.

When ℓ is a nonnegative integer, denote by $J_{\widehat{\mathfrak{g}}}(\ell, 0)$ the $\widehat{\mathfrak{g}}$ -submodule of $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ generated by the vectors

$$(6.11) \quad (t_0^{-1}x_{\pm\alpha_i})^{\epsilon_i \ell + 1} \mathbf{1}, \quad i \in I,$$

where $\epsilon_i = \frac{2}{\langle \alpha_i, \alpha_i \rangle}$. It is straightforward to check that $J_{\widehat{\mathfrak{g}}}(\ell, 0)$ is $t_0^{-1}d_0$ -stable (cf. [CLiT, Lemma 3.13]), and hence by Remark 6.1 it is an ideal of the vertex algebra $V_{\widehat{\mathfrak{g}}}(\ell, 0)$. Let

$$(6.12) \quad L_{\widehat{\mathfrak{g}}}(\ell, 0) = V_{\widehat{\mathfrak{g}}}(\ell, 0) / J_{\widehat{\mathfrak{g}}}(\ell, 0)$$

be a quotient vertex algebra of $V_{\widehat{\mathfrak{g}}}(\ell, 0)$.

6.2. CONFORMAL ALGEBRA $\mathcal{C}_{\mathfrak{g}}$. In order to associate the twisted toroidal EALA $\widetilde{\mathfrak{g}}[\mu]$ with the vertex algebras $V_{\widetilde{\mathfrak{g}}}(\ell, 0)$ and $L_{\widetilde{\mathfrak{g}}}(\ell, 0)$, we define a G_{μ} -conformal algebra $\mathcal{C}_{\mathfrak{g}}$ such that $\widehat{\mathcal{C}}_{\mathfrak{g}} \cong \widehat{\mathfrak{g}}$, and $\widetilde{\mathcal{C}}_{\mathfrak{g}}[G_{\mu}] \cong \widetilde{\mathfrak{g}}[\mu]$. As a vector space, we set

$$\mathcal{C}_{\mathfrak{g}} = (\mathbb{C}[\partial] \otimes \mathcal{A}_{\mathfrak{g}}) \oplus \mathbb{C}k_0,$$

and define ∂ to be a linear transformation on $\mathcal{C}_{\mathfrak{g}}$ such that

$$\partial(\partial^m \otimes x) = \partial^{m+1} \otimes x, \quad \partial(k_0) = 0$$

for $m \in \mathbb{N}$, $x \in \mathcal{A}_{\mathfrak{g}}$. Let

$$Y^- : \mathcal{C}_{\mathfrak{g}} \rightarrow \text{Hom}(\mathcal{C}_{\mathfrak{g}}, z^{-1}\mathcal{C}_{\mathfrak{g}}[z^{-1}]), \quad a \mapsto \sum_{i \in \mathbb{N}} a_i z^{-i-1}$$

be the unique linear map such that the property (2.8) holds, and the nontrivial i -products on $\mathcal{A}_{\mathfrak{g}} \oplus \mathbb{C}k_0$ are as follows:

$$\begin{aligned} (t_1^m \otimes u)_0(t_1^n \otimes v) &= t_1^{m+n} \otimes [u, v] + \langle u, v \rangle m(\partial \otimes K_{m+n}) + \delta_{m+n,0} \langle u, v \rangle m k_1, \\ (t_1^m \otimes u)_1(t_1^n \otimes v) &= (m+n) \langle u, v \rangle K_{m+n} + \delta_{m+n,0} \langle u, v \rangle k_0, \\ (D_r)_0(t_1^m \otimes u) &= m(\partial \otimes (t_1^{r+m} \otimes u)), \\ (t_1^m \otimes u)_0(D_r) &= r(\partial \otimes (t_1^{r+m} \otimes u)), \\ (D_r)_1(t_1^m \otimes u) &= (t_1^m \otimes u)_1(D_r) = (r+m)t_1^{r+m} \otimes u, \\ (D_r)_0(K_s) &= r(\partial \otimes K_{r+s}) + \delta_{r+s,0} r k_1, \\ (K_s)_0(D_r) &= s(\partial \otimes K_{r+s}) + \delta_{r+s,0} (-r k_1 + \partial \otimes k_0), \\ (D_r)_1(K_s) &= (K_s)_1(D_r) = (r+s)K_{r+s} + \delta_{r+s,0} k_0, \\ (d_1)_0(t_1^m \otimes u) &= -(t_1^m \otimes u)_0(d_1) = m t_1^m \otimes u, \\ (d_1)_1(k_1) &= (k_1)_1(d_1) = k_0, \\ (d_1)_0(K_r) &= -(K_r)_0(d_1) = r K_r, \\ (d_1)_0(D_r) &= -(D_r)_0(d_1) = D_r, \\ (D_r)_0(D_s) &= r \partial \otimes D_{r+s} + \delta_{r+s,0} (-r \partial^2 \otimes d_1), \\ (D_r)_1(D_s) &= (r+s)D_{r+s}, \end{aligned}$$

where $u, v \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, $r, s \in \mathbb{Z}^*$, and we have used the convention

$$K_0 = 0 = D_0.$$

PROPOSITION 6.2: *The vector space $\mathcal{C}_{\mathfrak{g}}$, together with the linear maps ∂ and Y^- , as defined above, is a conformal algebra, and the linear map $\hat{i}_{\mathfrak{g}} : \widehat{\mathcal{C}}_{\mathfrak{g}} \rightarrow \widehat{\mathfrak{g}}$ defined by*

$$u(m) \mapsto t_0^m u, \quad K_n(m) \mapsto k_{m+1,n}, \quad D_n(m) \mapsto \widehat{d}_{m-1,n}, \quad k_0(m) \mapsto \delta_{m,-1} k_0,$$

for $u \in \mathfrak{g}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$, is an isomorphism of Lie algebras. Furthermore, the linear map $\tilde{i}_{\mathfrak{g}} : \widetilde{\mathcal{C}}_{\mathfrak{g}} \rightarrow \widetilde{\mathfrak{g}}$ defined by

$$(6.13) \quad \begin{aligned} u[m] &\mapsto t_0^m u, & K_n[m] &\mapsto k_{m,n}, & D_n[m] &\mapsto \widehat{d}_{m,n}, \\ k_0[m] &\mapsto \delta_{m,0} k_0, & \mathbf{d} &\mapsto -d_0, \end{aligned}$$

for $u \in \mathfrak{g}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$, is also an isomorphism of Lie algebras.

Proof. By definition, $\widehat{\mathcal{C}}_{\mathfrak{g}}$ is linearly spanned by the elements $a(m), k_0(m)$ for $a \in \mathcal{A}_{\mathfrak{g}}$, $m \in \mathbb{Z}$. Note that $k_0(m) = 0$ if $m \neq -1$. Using these with (6.3), it is easy to see that the map $\hat{i}_{\mathfrak{g}}$ is an isomorphism of vector spaces. Then $\widehat{\mathcal{C}}_{\mathfrak{g}}$ admits a Lie algebra structure transferring from $\widehat{\mathfrak{g}}$ via the linear isomorphism $\hat{i}_{\mathfrak{g}}$. By using the relations (4.4), (6.1), (6.2), (6.4), (6.5) and the facts that $\widehat{d}_{n,0} = (n+1)t_0^n d_1$ and $k_{m,0} = -\frac{1}{m}t_0^m k_1$ for $n \in \mathbb{Z}$ and $m \in \mathbb{Z}^*$, one can check that the Lie brackets on $\widehat{\mathcal{C}}_{\mathfrak{g}}$ coincide with that in (2.9) with the i -products defined above. Thus, by Lemma 2.7, $(\mathcal{C}_{\mathfrak{g}}, \partial, Y_-)$ is a conformal algebra with $\hat{i}_{\mathfrak{g}}$ a Lie algebra isomorphism.

For the second assertion of the proposition, we note that $\widetilde{\mathcal{C}}_{\mathfrak{g}}$ is linearly spanned by the elements $a[m], k_0[m], \mathbf{d}$ for $a \in \mathcal{A}_{\mathfrak{g}}$ and $m \in \mathbb{Z}$. Also note that $k_0[n] = 0$ if $n \neq 0$. These together with (4.13) give that $\tilde{i}_{\mathfrak{g}}$ is a linear isomorphism. Furthermore, by comparing the Lie relation (2.17) in $\widetilde{\mathcal{C}}_{\mathfrak{g}}$ and the Lie relations (4.4), (4.9), (4.11), and (4.12) in $\widetilde{\mathfrak{g}}$, it is straightforward to check that $\tilde{i}_{\mathfrak{g}}$ is a Lie algebra homomorphism, as required. ■

Now we define an automorphism group G_{μ} on $\mathcal{C}_{\mathfrak{g}}$ so that $\widetilde{\mathcal{C}}_{\mathfrak{g}}[G_{\mu}] \cong \widetilde{\mathfrak{g}}[\mu]$. We first define a linear transformation R_{μ} on $\mathcal{C}_{\mathfrak{g}}$ by

$$\begin{aligned} R_{\mu}(\partial^m \otimes x) &= \partial^m \otimes \mu(x), \\ R_{\mu}(k_0) &= k_0, \\ R_{\mu}(\partial^m \otimes K_n) &= \partial^m \otimes K_n, \\ R_{\mu}(\partial^m \otimes (t_1^n \otimes h)) &= \partial^m \otimes (t_1^n \otimes \dot{\mu}(h)) + \rho_{\mu}(h) \partial^{m+1} \otimes K_n, \\ R_{\mu}(\partial^m \otimes D_n) &= \partial^m \otimes D_n - \partial^{m+1} \otimes (t_1^n \otimes \mathbf{h}) + \frac{\langle \mathbf{h}, \mathbf{h} \rangle}{2} \partial^{m+2} \otimes K_n, \end{aligned}$$

for $x \in \mathfrak{g}_{\alpha}$ with $\alpha \in \Delta^{\times} \cup \{0\}$, $h \in \mathfrak{h}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$.

LEMMA 6.3: *The linear transformation R_μ , as defined above, is an automorphism of $\mathcal{C}_\mathfrak{g}$ with order T .*

Proof. Note that $R_\mu \circ \partial = \partial \circ R_\mu$ on $\mathcal{C}_\mathfrak{g}$. Then we have a linear map $\widehat{R}_\mu : \widehat{\mathcal{C}}_\mathfrak{g} \rightarrow \widehat{\mathcal{C}}_\mathfrak{g}$ determined by (2.13). Via the isomorphism $\widehat{i}_\mathfrak{g} : \widehat{\mathcal{C}}_\mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$ given in Proposition 6.2, \widehat{R}_μ induces a linear map on $\widehat{\mathfrak{g}}$ determined by

$$\begin{aligned}
 \widehat{R}_\mu(t_0^m x) &= t_0^m \mu(x), \\
 \widehat{R}_\mu(t_0^m t_1^m \otimes h) &= t_0^m t_1^m \otimes \widehat{\mu}(h) - \rho_\mu(h) m k_{m,n}, \\
 \widehat{R}_\mu(k_{m,n}) &= k_{m,n}, \\
 \widehat{R}_\mu(d_{m,n}) &= d_{m,n} + (m + 1) t_0^m t_1^n \otimes \mathbf{h} + \frac{\langle \mathbf{h}, \mathbf{h} \rangle}{2} (m + 1) m k_{m,n},
 \end{aligned}
 \tag{6.14}$$

for $x \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta^\times \cup \{0\}$, $h \in \mathfrak{h}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$. Moreover, from Proposition 4.3 we have a Lie automorphism $\widehat{\mu}$ of $\mathcal{T}(\mathfrak{g})$, which preserves $\widehat{\mathfrak{g}}$. And one can check that $\widehat{\mu} = \widehat{R}_\mu$ on $\widehat{\mathfrak{g}}$. Thus \widehat{R}_μ is an automorphism of the Lie algebra $\widehat{\mathcal{C}}_\mathfrak{g}$ with order T . The assertion of the lemma then follows from Lemma 2.8. ■

Set $G_\mu = \langle R_\mu \rangle$, an automorphism group of $\mathcal{C}_\mathfrak{g}$, and let χ_ω be the linear character of G_μ defined by $\chi_\omega(R_\mu) = \omega^{-1}$. Recall that for any G -conformal algebra \mathcal{C} and any linear character χ of G , there is a Lie algebra $\widetilde{\mathcal{C}}[G]$ defined in (2.20). By specializing $G = G_\mu$, $\mathcal{C} = \mathcal{C}_\mathfrak{g}$ and $\chi = \chi_\omega$, we obtain a Lie algebra $\widetilde{\mathcal{C}}_\mathfrak{g}[G_\mu]$. Recall also the surjective map $\eta_\mu : \widetilde{\mathfrak{g}} \rightarrow \widetilde{\mathfrak{g}}[\mu]$ defined in (4.25); we have

PROPOSITION 6.4: *The following assignment:*

$$\begin{aligned}
 \overline{u[m]} &\mapsto \eta_\mu(t_0^m u), & \overline{K_n[m]} &\mapsto \eta_\mu(k_{m,n}), & \overline{D_n[m]} &\mapsto \eta_\mu(\widetilde{d}_{m,n}), \\
 \overline{k_0[m]} &\mapsto \delta_{m,0} T k_0, & \mathbf{d} &\mapsto -d_0
 \end{aligned}
 \tag{6.15}$$

for $u \in \mathfrak{g}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$, determines an isomorphism from the Lie algebra $\widetilde{\mathcal{C}}_\mathfrak{g}[G_\mu]$ to the Lie algebra $\widetilde{\mathfrak{g}}[\mu]$.

Proof. Corresponding to the automorphism R_μ of $\mathcal{C}_\mathfrak{g}$, there is an automorphism \widetilde{R}_μ of $\widetilde{\mathcal{C}}_\mathfrak{g}$ (see (2.18)). Via the isomorphism $\widetilde{i}_\mathfrak{g} : \widetilde{\mathcal{C}}_\mathfrak{g} \rightarrow \widetilde{\mathfrak{g}}$ given in Proposition 6.2, \widetilde{R}_μ induces an automorphism of $\widetilde{\mathfrak{g}}$. It is straightforward to check that this automorphism of $\widetilde{\mathfrak{g}}$ coincides with $\widetilde{\mu}$ (see (4.23)). Since G_μ is a cyclic group of order T , it follows from Remark 2.11 that $\widetilde{\mathcal{C}}_\mathfrak{g}[G_\mu]$ is isomorphic to the subalgebra of $\widetilde{\mathcal{C}}_\mathfrak{g}$ fixed by \widetilde{R}_μ . This implies that $\widetilde{\mathcal{C}}_\mathfrak{g}[G_\mu]$ is isomorphic to $\widetilde{\mathfrak{g}}[\mu]$ with the isomorphism given in the proposition. ■

6.3. THE CORRESPONDENCE THEOREM FOR $\tilde{\mathfrak{g}}[\mu]$. In this subsection we generalize the correspondence theorem (Theorem 3.11) for affine Kac–Moody algebras to the nullity 2 twisted toroidal extended affine Lie algebras $\tilde{\mathfrak{g}}[\mu]$.

Form the following generating functions $a^\mu[z]$, $a \in \mathcal{A}_{\mathfrak{g}}$ in $\tilde{\mathfrak{g}}[\mu][[z, z^{-1}]]$:

$$u^\mu[z] = \sum_{n \in \mathbb{Z}} \eta_\mu(t_0^n u) z^{-n}, \quad D_m^\mu[z] = \sum_{n \in \mathbb{Z}} \eta_\mu(\tilde{d}_{n,m}) z^{-n},$$

$$K_m^\mu[z] = \sum_{n \in \mathbb{Z}} \eta_\mu(k_{n,m}) z^{-n},$$

for $u \in \mathfrak{g}$ and $m \in \mathbb{Z}^*$. Note that all the components of these generating functions together with k_0, d_0 span the algebra $\tilde{\mathfrak{g}}[\mu]$. As in the affine Kac–Moody algebra case, we formulate the following definition.

Definition 6.5: We say that a $\tilde{\mathfrak{g}}[\mu]$ -module W is **restricted** if for any $a \in \mathcal{A}_{\mathfrak{g}}$, $a^\mu[z] \in \text{Hom}(W, W((z)))$. And W is said to be of **level** $\ell \in \mathbb{C}$ if the central element k_0 acts as the scalar ℓ/T . Furthermore, if μ is nontransitive, we say that W is **integrable** if for any $\alpha \in \tilde{\Delta}_\mu^\times$, $\tilde{\mathfrak{g}}[\mu]_\alpha$ acts locally nilpotently on W .

For each $i \in I = \{0, 1, \dots, l\}$, one recalls the positive integers T_i, s_i defined in (4.27), and sets

$$p_i(z) = \frac{1 - z^{s_i T_i}}{1 - z^{T_i}}.$$

Then we have the following analogue of Proposition 3.12.

PROPOSITION 6.6: *Assume that μ is nontransitive. Then for any $i \in I$,*

$$(6.16) \quad p_i(z_1/z_2)[x_{\pm\alpha_i}^\mu[z_1], x_{\pm\alpha_i}^\mu[z_2]] = 0.$$

Furthermore, if W is a restricted $\tilde{\mathfrak{g}}[\mu]$ -module of level ℓ , then W is integrable if and only if ℓ is a nonnegative integer and for any $i \in I$,

$$(6.17) \quad \left(\prod_{1 \leq i < j \leq \epsilon_i \ell + 1} p_i(z_i/z_j) \right) x_{\pm\alpha_i}^\mu[z_1] x_{\pm\alpha_i}^\mu[z_2] \cdots x_{\pm\alpha_i}^\mu[z_{\epsilon_i \ell + 1}]|_{z_1 = z_2 = \cdots = z_{\epsilon_i \ell + 1}} = 0$$

on W .

Proof. For each $i \in I$, denote by $\tilde{\mathfrak{g}}[\mu]_i$ the subalgebra of $\tilde{\mathfrak{g}}[\mu]$ generated by the elements $t_0^m x_{\pm\alpha_i}, d_0$ for $m \in \mathbb{Z}$. We first show that $\tilde{\mathfrak{g}}[\mu]_i$ is isomorphic to the affine Kac–Moody algebra of type $A_{s_i}^{(s_i)}$. For $k = 1, 2$, we denote by θ_k the order k diagram automorphism of the simple Lie algebra \mathfrak{sl}_{k+1} . Then for each $i \in I$, we have an affine Kac–Moody algebra $\tilde{\mathcal{L}}(\mathfrak{sl}_{s_i+1}, \theta_{s_i})$ of type $A_{s_i}^{(s_i)}$ (see

Section 3.2). Recall that $\eta_\mu(t_0^m x_{\pm\alpha_i}) = \sum_{p=0}^{T-1} t_0^m x_{\pm\alpha_{\mu(i)}}$ for $i \in I$ and $m \in \mathbb{Z}$. In particular, we have $\eta_\mu(t_0^m x_{\pm\alpha_i}) \neq 0$ if and only if $m \in T_i\mathbb{Z}$. By Lemma 4.7, it is straightforward to see that the assignment ($m \in \mathbb{Z}$ and β a fixed simple root of \mathfrak{sl}_{s_i+1})

$$\eta_\mu(t_0^{T_i m} x_{\pm\alpha_i}) \mapsto t^m \otimes (x_{\pm\beta})_{(m)}, \quad \frac{T}{s_i}k_0 \mapsto k, \quad d_0 \mapsto T_i d$$

determines an isomorphism from the Lie algebra $\widetilde{\mathfrak{g}}[\mu]_i$ to the Lie algebra $\widetilde{\mathcal{L}}(\mathfrak{sl}_{s_i+1}, \theta_{s_i})$. This together with (3.9) implies the first assertion of the proposition.

For the second assertion, let W be a restricted $\widetilde{\mathfrak{g}}[\mu]$ -module of level ℓ . For each $i \in I$, via the isomorphism $\widetilde{\mathfrak{g}}[\mu]_i \cong \widetilde{\mathcal{L}}(\mathfrak{sl}_{s_i+1}, \theta_{s_i})$, W becomes a restricted $\widetilde{\mathcal{L}}(\mathfrak{sl}_{s_i+1}, \theta_i)$ -module of level ℓ . Note that the $\widetilde{\mathfrak{g}}[\mu]$ -module W is integrable if and only if the elements $t_0^m x_{\pm\alpha_i}$, $i \in I, m \in \mathbb{Z}$ act locally nilpotently (as they generate the core $\mathfrak{t}(\widehat{\mathfrak{g}}, \mu)$). Thus, W is integrable if and only if W is integrable as an $\widetilde{\mathcal{L}}(\mathfrak{sl}_{s_i+1}, \theta_i)$ -module for all $i \in I$. Thus the assertion follows from Proposition 3.12. ■

Note that k_0 is a central element in $\mathcal{C}_{\mathfrak{g}}$. Thus, for any $\ell \in \mathbb{C}$, the $\widehat{\mathcal{C}}_{\mathfrak{g}}$ -submodule $\langle k_0 - \ell \rangle$ of $V_{\mathcal{C}_{\mathfrak{g}}}$ generated by $k_0 - \ell$ is an ideal of $V_{\mathcal{C}_{\mathfrak{g}}}$ (as a vertex algebra). Recall the isomorphism $\widehat{i}_{\mathfrak{g}} : \widehat{\mathcal{C}}_{\mathfrak{g}} \rightarrow \widehat{\mathfrak{g}}$ given in Proposition 6.2. One can readily check that $\widehat{i}_{\mathfrak{g}}(\widehat{\mathcal{C}}_{\mathfrak{g}}^+) = \widehat{\mathfrak{g}}_+$ and $\widehat{i}_{\mathfrak{g}}(\widehat{\mathcal{C}}_{\mathfrak{g}}^-) = \widehat{\mathfrak{g}}_- \oplus \mathbb{C}k_0$ (see (2.10) and (6.6)). This implies that $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ is isomorphic to the quotient vertex algebra $V_{\mathcal{C}_{\mathfrak{g}}}/\langle k_0 - \ell \rangle$. Recall also that the automorphism group $G_\mu = \langle R_\mu \rangle$ of $\mathcal{C}_{\mathfrak{g}}$ can be uniquely lifted to an automorphism group of its universal enveloping vertex algebra $V_{\mathcal{C}_{\mathfrak{g}}}$. As $R_\mu(k_0) = k_0$, G_μ is naturally an automorphism group of $V_{\widehat{\mathfrak{g}}}(\ell, 0)$. Furthermore, we have

LEMMA 6.7: For each nonnegative integer ℓ , $J_{\widehat{\mathfrak{g}}}(\ell, 0)$ is a R_μ -stable ideal of $V_{\widehat{\mathfrak{g}}}(\ell, 0)$.

Proof. The assertion follows from the fact that

$$R_\mu((t_0^{-1} x_{\pm\alpha_i})^{\epsilon_i \ell + 1} \mathbf{1}) = (t_0^{-1} \mu(x_{\pm\alpha_i}))^{\epsilon_i \ell + 1} \mathbf{1} = (t_0^{-1} x_{\pm\alpha_{\mu(i)}})^{\epsilon_{\mu(i)} \ell + 1} \mathbf{1} \in J(\ell),$$

where $\epsilon_{\mu(i)} = \epsilon_i$ for $i \in I$. ■

In view of the above lemma, G_μ is also an automorphism group of the vertex algebra $L_{\widehat{\mathfrak{g}}}(\ell, 0)$. Now we state one of the main results of the paper.

THEOREM 6.8: *Let ℓ be a complex number. For any restricted $\tilde{\mathfrak{g}}[\mu]$ -module W of level ℓ , there is a (G_μ, χ_ω) -equivariant ϕ -coordinated quasi- $V_{\tilde{\mathfrak{g}}}(\ell, 0)$ -module structure (Y_W^ϕ, d) on W , which is uniquely determined by*

$$d = -d_0, \quad Y_W^\phi(a, z) = a^\mu[z],$$

for $a \in \mathcal{A}_{\tilde{\mathfrak{g}}}$. On the other hand, for any (G_μ, χ_ω) -equivariant ϕ -coordinated quasi- $V_{\tilde{\mathfrak{g}}}(\ell, 0)$ -module (W, Y_W^ϕ, d) , W is a restricted $\tilde{\mathfrak{g}}[\mu]$ -module of level ℓ with action given by

$$d_0 = -d, \quad a^\mu[z] = Y_W^\phi(a, z),$$

for $a \in \mathcal{A}_{\tilde{\mathfrak{g}}}$. Furthermore, if ℓ is a nonnegative integer and μ is nontransitive, then the integrable restricted $\tilde{\mathfrak{g}}[\mu]$ -modules of level ℓ are exactly the (G_μ, χ_ω) -equivariant ϕ -coordinated quasi- $L_{\tilde{\mathfrak{g}}}(\ell, 0)$ -modules (W, Y_W^ϕ, d) .

Proof. By Propositions 2.6, 2.10 and 6.4, the restricted $\tilde{\mathfrak{g}}[\mu]$ -modules are exactly the (G_μ, χ_ω) -equivariant ϕ -coordinated quasi- $V_{\tilde{\mathfrak{C}}_{\tilde{\mathfrak{g}}}}$ -modules. Thus the fact that $V_{\tilde{\mathfrak{g}}}(\ell, 0) \cong V_{\tilde{\mathfrak{C}}_{\tilde{\mathfrak{g}}}}/\langle k_0 - \ell \rangle$ implies the first assertion of the theorem.

For the second part of the theorem, we assume that ℓ is a nonnegative integer, μ is nontransitive and W is an integrable restricted $\tilde{\mathfrak{g}}[\mu]$ -module of level ℓ . Note that for each i , we have $[x_{\pm\alpha_i}(z_1), x_{\pm\alpha_i}(z_2)] = 0$ on $\tilde{\mathfrak{g}}$. This implies that for any $n \in \mathbb{N}$ and $i \in I$,

$$(6.18) \quad (x_{\pm\alpha_i})_n(x_{\pm\alpha_i}) = 0 \quad \text{on } V_{\tilde{\mathfrak{g}}}(\ell, 0).$$

Viewing W as a faithful (G_μ, χ_ω) -equivariant ϕ -coordinated quasi- $V_{\tilde{\mathfrak{g}}}(\ell, 0)/\ker Y_W^\phi$ -module, by Proposition 6.6, we have

$$(6.19) \quad \left(\prod_{1 \leq i < j \leq \epsilon_i \ell + 1} p_i(z_i/z_j) \right) \times Y_W^\phi(x_{\pm\alpha_i}, z_1) Y_W^\phi(x_{\pm\alpha_i}, z_2) \cdots Y_W^\phi(x_{\pm\alpha_i}, z_{\epsilon_i \ell + 1})|_{z_1 = z_2 = \cdots = z_{\epsilon_i \ell + 1}} = 0 \quad \text{on } W$$

for all $i \in I$. This together with (6.18) and Proposition 2.6 proves that $((x_{\pm\alpha_i})_{-1})^{\epsilon_i \ell + 1} \mathbf{1} \in \ker Y_W^\phi$ for $i \in I$. Thus, we have $J_{\tilde{\mathfrak{g}}}(\ell, 0) \subset \ker Y_W^\phi$ and W becomes a (G_μ, χ_ω) -equivariant ϕ -coordinated quasi- $L_{\tilde{\mathfrak{g}}}(\ell, 0)$ -module.

Conversely, let (W, Y_W^ϕ, d) be a (G_μ, χ_ω) -equivariant ϕ -coordinated quasi- $L_{\tilde{\mathfrak{g}}}(\ell, 0)$ -module. Then it is also a (G_μ, χ_ω) -equivariant ϕ -coordinated quasi- $V_{\tilde{\mathfrak{g}}}(\ell, 0)$ -module, and such that for any $i \in I$, $((x_{\pm\alpha_i})_{-1})^{\epsilon_i \ell + 1} \mathbf{1}$ acts trivially

on W . Recall from the first part of the theorem that W is then a $\tilde{\mathfrak{g}}[\mu]$ -module with $a^\mu[z] = Y_W^\phi(a, z)$ for $a \in \mathcal{A}_{\mathfrak{g}}$. Combining this with (6.16), we obtain

$$\left(\prod_{1 \leq i < j \leq \epsilon_i \ell + 1} p_i(z_i/z_j) \right) Y_W^\phi(x_{\pm\alpha_i}, z_1) Y_W^\phi(x_{\pm\alpha_i}, z_2) \cdots Y_W^\phi(x_{\pm\alpha_i}, z_{\epsilon_i \ell + 1}) \in \text{Hom}(W, W((z_1, \dots, z_{\epsilon_i \ell + 1}))).$$

Then again by (6.18) and Proposition 2.6, we see that (6.19) holds. This implies, viewing W as a restricted $\tilde{\mathfrak{g}}[\mu]$ -module of level ℓ , that (6.17) holds. Thus, by Proposition 6.6, W is an integrable $\tilde{\mathfrak{g}}[\mu]$ -module. This completes the proof of the theorem. ■

Remark 6.9: For any complex number a , two vertex algebras $V_{\widehat{\mathfrak{t}}(\mathfrak{g}, a)^\circ}(\ell)$ and $L_{\widehat{\mathfrak{t}}(\mathfrak{g}, a)^\circ}(\ell)$ were constructed in [CLiT], and such that $V_{\widehat{\mathfrak{t}}(\mathfrak{g}, 0)^\circ}(\ell) = V_{\widehat{\mathfrak{g}}}(\ell, 0)$ and $L_{\widehat{\mathfrak{t}}(\mathfrak{g}, 0)^\circ}(\ell) = L_{\widehat{\mathfrak{g}}}(\ell, 0)$. By a similar argument as above, one can prove that the Lie algebra $\tilde{\mathfrak{g}}[\mu]_{\tau_a}$ can be associated with the vertex algebras $V_{\widehat{\mathfrak{t}}(\mathfrak{g}, a)^\circ}(\ell)$ and $L_{\widehat{\mathfrak{t}}(\mathfrak{g}, a)^\circ}(\ell)$ via their equivariant ϕ -coordinated quasi-modules, where $\tilde{\mathfrak{g}}[\mu]_{\tau_a}$ and the affine cocycle τ_a are defined in Remark 5.8.

7. Associating $\tilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ with vertex algebras

Let $N \geq 2$ be a positive integer and $q \in \mathbb{C}^*$ a generic complex number. In this section we prove an analog of Theorem 6.8 for the extended affine Lie algebra $\tilde{\mathfrak{sl}}_N(\mathbb{C}_q)$.

First, we define the following generating functions in $\tilde{\mathfrak{sl}}_N(\mathbb{C}_q)[[z, z^{-1}]]$:

$$\begin{aligned} (E_{i,j}t_1^m)[z] &= \sum_{n \in \mathbb{Z}} (E_{i,j}t_0^n t_1^m) z^{-n}, \\ H_k[z] &= \sum_{n \in \mathbb{Z}} (E_{k,k} - E_{k+1,k+1}) t_0^n z^{-n}, \\ H_N[z] &= \sum_{n \in \mathbb{Z}} (E_{N,N}t_0^n - q^{-n} E_{1,1}t_0^n + \delta_{n,0} k_1) z^{-n}, \end{aligned}$$

where $1 \leq i, j \leq N$, $m \in \mathbb{Z}$ with $(i - j, m) \neq (0, 0)$ and $1 \leq k \leq N - 1$. Note that all the coefficients of these generating functions, together with k_0, d_0 and d_1 , form a basis of $\tilde{\mathfrak{sl}}_N(\mathbb{C}_q)$.

Definition 7.1: We say that an $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ -module W is **restricted** if

$$(E_{i,j}t_1^m)[z], H_k[z] \in \text{Hom}(W, W((z)))$$

for $1 \leq i, j, k \leq N$, $m \in \mathbb{Z}$ with $(i - j, m) \neq (0, 0)$. Furthermore, W is said to be of **level** $\ell \in \mathbb{C}$ if the central element k_0 acts as the scalar ℓ . Furthermore, we say that W is **integrable** if $E_{i,j}t_0^n t_1^m$ acts locally nilpotent on W for $1 \leq i \neq j \leq N$, $m, n \in \mathbb{Z}$.

The following result is from [CLiTW, Proposition 3.13].

PROPOSITION 7.2: *Let W be a restricted $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ -module of level ℓ . Then W is integrable if and only if ℓ is a nonnegative integer and*

$$(E_{i,j}t_1^m)[z]^{\ell+1} = 0 \quad \text{on } W$$

for $1 \leq i \neq j \leq N$ and $m \in \mathbb{Z}$.

Let \mathfrak{gl}_∞ be the algebra of all doubly infinite complex matrices with only finitely many nonzero entries. For $m, n \in \mathbb{Z}$, let $E_{m,n}$ denote the unit matrix whose only nonzero entry is the (m, n) -entry which is equal to 1. Equip \mathfrak{gl}_∞ with a nondegenerate, invariant and symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle E_{i,j}, E_{k,l} \rangle = \delta_{j,k} \delta_{i,l},$$

for $i, j, k, l \in \mathbb{Z}$. Let $\mathfrak{sl}_\infty = [\mathfrak{gl}_\infty, \mathfrak{gl}_\infty]$ be the derived subalgebra of \mathfrak{gl}_∞ . Then $\langle \cdot, \cdot \rangle$ is also nondegenerate on \mathfrak{sl}_∞ . And associated to the pair $(\mathfrak{sl}_\infty, \langle \cdot, \cdot \rangle)$, we have the corresponding affine Lie algebra $\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)$, the universal affine vertex algebra $V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$, and the simple affine vertex algebra

$$L_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0) = V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0) / J_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$$

(see Subsection 3.2). The following result is given in [CLiTW, Lemma 3.11].

LEMMA 7.3: *If ℓ is a nonnegative integer, then $J_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$, as $\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)$ -module, is generated by the set of vectors*

$$(7.1) \quad \{(t^{-1} \otimes E_{mN+i, nN+j})^{\ell+1} \mathbf{1} \mid \text{for } 1 \leq i \neq j \leq N, m, n \in \mathbb{Z}\}.$$

Let σ_N be the automorphism of the algebra \mathfrak{gl}_∞ defined by

$$(7.2) \quad \sigma_N(E_{m,n}) = E_{m+N, n+N},$$

for $m, n \in \mathbb{Z}$. Restricting σ_N to the subalgebra \mathfrak{sl}_∞ , we see that σ_N is also an automorphism of \mathfrak{sl}_∞ that preserves the bilinear form $\langle \cdot, \cdot \rangle$. Denote by $G_N = \langle \sigma_N \rangle$

the automorphism group of \mathfrak{sl}_∞ generated by σ_N . As pointed out in Section 3.2, G_N can be extended uniquely to an automorphism group of the vertex algebras $V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$ and $L_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$. Let $\chi_q : G_N \rightarrow \mathbb{C}^\times$ be the linear character defined by $\chi_q(\sigma_N^n) = q^n$ for $n \in \mathbb{Z}$.

Define a \mathbb{Z} -grading $\mathfrak{g}^\ell_\infty = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}^{\ell(n)}_\infty$ on \mathfrak{g}^ℓ_∞ by assigning

$$(7.3) \quad \deg E_{mN+i, nN+j} = m - n,$$

for $m, n \in \mathbb{Z}$, $1 \leq i, j \leq N$. Note that \mathfrak{sl}_∞ is a graded subalgebra of \mathfrak{g}^ℓ_∞ . Denote by \mathcal{P} the derivation of \mathfrak{sl}_∞ defined by $\mathcal{P}(a) = na$ if $a \in \mathfrak{sl}_\infty$ with $\deg a = n$. Note that for $a, b \in \mathfrak{sl}_\infty$, one has $\langle \mathcal{P}a, b \rangle + \langle a, \mathcal{P}b \rangle = 0$. This allows us to lift the derivation \mathcal{P} of \mathfrak{sl}_∞ to be a derivation of the affine Lie algebra $\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)$ with

$$(7.4) \quad \mathcal{P}(k) = 0, \quad \mathcal{P}(t^n \otimes a) = t^n \otimes \mathcal{P}(a),$$

for $n \in \mathbb{Z}$, $a \in \mathfrak{sl}_\infty$. As $\mathcal{P}(t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{sl}_\infty) \subset t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{sl}_\infty$, \mathcal{P} is also a derivation of the associative algebra $\mathcal{U}(t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{sl}_\infty)$. Via the isomorphism

$$\mathcal{U}(t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{sl}_\infty) \cong V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0),$$

\mathcal{P} becomes a derivation of $V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$ (as a vertex algebra). Furthermore, if ℓ is a nonnegative integer, then by Lemma 7.3, we see that \mathcal{P} also preserves the submodule $J_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$. Therefore, it descends to a derivation of $L_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$.

Definition 7.4: A (G_N, χ_q) -equivariant ϕ -coordinated quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$ -module (W, Y_W^ϕ, d, p) is a (G_N, χ_q) -equivariant ϕ -coordinated quasi-module (W, Y_W^ϕ, d) equipped with a linear transformation p on W such that

$$[p, Y_W^\phi(v, z)] = Y_W^\phi(\mathcal{P}v, z),$$

for $v \in V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$. Similarly, when ℓ is a nonnegative integer, we can define the notion of (G_N, χ_q) -equivariant ϕ -coordinated quasi- $L_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$ -module (W, Y_W^ϕ, d, p) .

THEOREM 7.5: *Let ℓ be a complex number. If W is a restricted $\widetilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ -module of level ℓ , then there is a (G_N, χ_q) -equivariant ϕ -coordinated quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$ -module structure (Y_W^ϕ, d, p) on W uniquely determined by*

$$p = -d_1, \quad d = -d_0,$$

$$Y_W^\phi(E_{i, mN+j}, z) = (E_{i, j} t_1^m)[z], \quad Y_W^\phi(E_{k, k} - E_{k+1, k+1}, z) = H_k[z]$$

for $1 \leq i, j, k \leq N$, $m \in \mathbb{Z}$ with $(i - j, m) \neq (0, 0)$. On the other hand, if (W, Y_W^ϕ, d, p) is a (G_N, χ_q) -equivariant ϕ -coordinated quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$ -module, then W is a restricted $\widehat{\mathfrak{sl}}_N(\mathbb{C}_q)$ -module of level ℓ with action given by

$$d_1 = -p, \quad d_0 = -d,$$

$$(E_{i,j}t_1^m)[z] = Y_W^\phi(E_{i,mN+j}, z), \quad H_k[z] = Y_W^\phi(E_{k,k} - E_{k+1,k+1}, z)$$

for $1 \leq i, j, k \leq N$, $m \in \mathbb{Z}$ with $(i - j, m) \neq (0, 0)$.

Furthermore, if ℓ is a nonnegative integer, then the integrable restricted $\widehat{\mathfrak{sl}}_N(\mathbb{C}_q)$ -modules of level ℓ are exactly the (G_N, χ_q) -equivariant ϕ -coordinated quasi- $L_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$ -modules (W, Y_W^ϕ, d, p) .

Proof. Recall that $\widehat{\mathfrak{sl}}_N(\mathbb{C}_q) = \widehat{\mathfrak{sl}}_N(\mathbb{C}_q) \oplus \mathbb{C}d_0 \oplus \mathbb{C}d_1$. It was proved in [CLiTW, Proposition 3.8] that the restricted $\widehat{\mathfrak{sl}}_N(\mathbb{C}_q)$ -modules W of level ℓ are exactly the (G_N, χ_q) -equivariant quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$ -modules (W, Y_W) with

$$z^{-1}(E_{i,j}t_1^m)[z] = Y_W(E_{i,mN+j}, z), \quad z^{-1}H_k[z] = Y_W(E_{k,k} - E_{k+1,k+1}, z)$$

for $1 \leq i, j, k \leq N$, $m \in \mathbb{Z}$ with $(i - j, m) \neq (0, 0)$. It then follows from Proposition 3.5 that the restricted $\widehat{\mathfrak{sl}}_N(\mathbb{C}_q)$ -modules W of level ℓ are exactly the (G_N, χ_q) -equivariant ϕ -coordinated quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$ -modules (W, Y_W^ϕ) with

$$(E_{i,j}t_1^m)[z] = Y_W^\phi(E_{i,mN+j}, z), \quad H_k[z] = Y_W^\phi(E_{k,k} - E_{k+1,k+1}, z).$$

Furthermore, by (7.3) and Lemma 2.2 we have

$$\begin{aligned} [-d_1, (E_{i,j}t_1^m)[z]] &= -m(E_{i,j}t_1^m)[z] = -mY_W^\phi(E_{i,mN+j}, z) \\ &= Y_W^\phi(\mathcal{P}(E_{i,mN+j}), z), \\ [-d_1, H_k[z]] &= 0 = Y_W^\phi(\mathcal{P}(E_{k,k} - E_{k+1,k+1}), z), \\ [-d_0, (E_{i,j}t_1^m)[z]] &= z \frac{d}{dz}(E_{i,j}t_1^m)[z] \\ &= z \frac{d}{dz}Y_W^\phi(E_{i,mN+j}, z) = Y_W^\phi(\mathcal{D}(E_{i,mN+j}), z), \\ [-d_0, H_k[z]] &= z \frac{d}{dz}H_k[z] = z \frac{d}{dz}Y_W^\phi(E_{k,k} - E_{k+1,k+1}, z) \\ &= Y_W^\phi(\mathcal{D}(E_{k,k} - E_{k+1,k+1}), z). \end{aligned}$$

Therefore, we have finished the proof for the first part of the theorem.

To prove the second part of the theorem, we suppose that ℓ is a nonnegative integer and W an integrable restricted $\tilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ -module of level ℓ . Then for $1 \leq i \neq j \leq N$ and $m, n \in \mathbb{Z}$, we have $(E_{nN+i, mN+j})_r (E_{nN+i, nmN+j}) = 0$ for $r \geq 0$ in $V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$. And as a (G_N, χ_q) -equivariant ϕ -coordinated quasi- $V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$ -module we have

$$\begin{aligned} Y_W^\phi(E_{nN+i, mN+j}, z) &= Y_W^\phi(\sigma_N^n(E_{i, (m-n)N+j}), z) \\ &= Y_W^\phi(E_{i, (m-n)N+j}, \chi_q(\sigma_N)^n z) = (E_{i,j} t_1^{m-n})(q^n z). \end{aligned}$$

This implies that

$$\begin{aligned} [Y_W^\phi(E_{nN+i, mN+j}, z_1), Y_W^\phi(E_{nN+i, mN+j}, z_2)] \\ = [(E_{i,j} t_1^{m-n})(q^n z_1), (E_{i,j} t_1^{m-n})(q^n z_2)] = 0. \end{aligned}$$

Thus, by Propositions 2.6 and 7.2, we have $((E_{nN+i, mN+j})_{-1})^{\ell+1} \mathbf{1} = 0$ in $V_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0) / \ker Y_W^\phi$. This together with Lemma 7.3 implies that

$$J_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0) \subset \ker Y_W^\phi,$$

and hence (W, Y_W^ϕ, d, p) is a (G_N, χ_q) -equivariant ϕ -coordinated quasi- $L_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$ -module. Conversely, let (W, Y_W^ϕ, d, p) be a (G_N, χ_q) -equivariant ϕ -coordinated quasi- $L_{\widehat{\mathcal{L}}(\mathfrak{sl}_\infty)}(\ell, 0)$ -module. Then it is a restricted $\tilde{\mathfrak{sl}}_N(\mathbb{C}_q)$ -module of level ℓ . Again by Lemma 7.3, Proposition 2.6 and Proposition 7.2, one deduces that W is integrable as required. ■

ACKNOWLEDGEMENT. This work was supported by the National Natural Science Foundation of China (Nos. 11971396, 11971397, 12131018, 12161141001), and the Fundamental Research Funds for the Central Universities (No. 20720200067). The authors would like to thank the Institute for Advanced Study in Mathematics, Zhejiang. Part of this work was carried out while the first and third authors were visiting the institute.

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