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ON POINTED HOPF ALGEBRAS OVER NILPOTENT GROUPS

BY

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ABSTRACT

We classify finite-dimensional Nichols algebras over finite nilpotent groups of odd order in group-theoretical terms. The main step is to show that the conjugacy classes of such finite groups are either abelian or of type C; this property also holds for finite conjugacy classes of finitely generated nilpotent groups whose torsion has odd order. To extend our approach to the setting of finite GK-dimension, we propose a new Conjecture on racks of type C. We also prove that the bosonization of a Nichols algebra of a Yetter–Drinfeld module over a group whose support is an infinite conjugacy class has infinite GK-dimension. We apply this to the study of the finite GK-dimensional pointed Hopf algebras over finitely generated torsion-free nilpotent groups.

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Introduction

0.1. The context. Let $\Bbbk$ be an algebraically closed field of characteristic 0. This paper contributes to the classification of Hopf algebras with finite Gelfand–Kirillov dimension, GK-dim for short. Despite recent interest on this question, see [21, 22] and references therein, the general structure of such Hopf algebras is still mysterious, so it is justified to focus on the class of pointed Hopf algebras; under this assumption we may follow the method from [11], first applied in the GK-dim context in [12], cf. also [38]. By a celebrated theorem of Gromov [27], a finitely generated group has finite growth if and only if it is nilpotent-by-finite. Thus, the first major goal within the method of [11] is to classify Nichols algebras with finite GK-dim over finitely generated nilpotent-by-finite groups. Towards this goal, the first natural question is to deal with the classification of Nichols algebras with finite GK-dim over abelian groups. We collect information on this question needed for the general case. There are various subclasses to consider:

- Nichols algebras of diagonal type, corresponding to semisimple Yetter–Drinfeld modules over finitely generated abelian groups. Those with finite dimension were classified in [29]. Conjecture 1.2 below from [6] reduces the classification of the finitely generated Nichols algebras of diagonal type with finite GK-dim to [29]. Also [5] deals with Nichols algebras of diagonal type with finite GK-dim which are not finitely generated.
- Next comes the class of Nichols algebras of blocks & points; here the classification of those with finite GK-dim was achieved in [6] for finite, and [5] for infinite, rank; both assume the validity of Conjecture 1.2.
- There are decomposable Yetter–Drinfeld modules over abelian groups that are not of the form blocks & points; they contain components known as pale blocks. The classification of those with finite GK-dim in rank 3 was also obtained in [6] while rank 4 is work in progress [7].

Summarizing, in order to have a classification of the finitely generated Nichols algebras with finite GK-dim over abelian groups it remains to conclude the classification of the blocks & pale blocks & points giving rise to Nichols algebras with finite GK-dim and to prove Conjecture 1.2; we believe that both objectives could be attained soon. We should also mention that the classification of finite-dimensional Nichols algebras over finite groups is far from complete notwithstanding intense activity in this direction. See [1, 23, 32] and references therein.

The focus of this article is on Nichols algebras over nilpotent groups whose bosonizations have finite GK-dim. The main results of this paper are:

- The description of the finite-dimensional Nichols algebras over a finite nilpotent group of odd order G up to the knowledge of the conjugacy classes and the representations of the centralizers of G, Theorem 3.11.
- The description of the bosonizations $\mathscr{B}(M)\#\Bbbk G$ with G a torsion-free nilpotent group and $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ semisimple having finite GK-dim up to knowledge of the irreducible representations of G. See Theorem 3.5.

We next describe how we achieve these results.

0.2. Finite nilpotent groups. Let us start with finite nilpotent groups of odd order. Our first basic result, Theorem 2.1, states that any conjugacy class of such a group is either abelian or of type C. The notion of rack of type C was introduced in [8], where it was shown that any Nichols algebra whose support is of type C has infinite dimension. Thus we are reduced to deal with abelian conjugacy classes which give rise to braided vector spaces of diagonal type. To give a more precise answer, we generalize a technique from [41], the only reference we know on Nichols algebras over nilpotent groups (beyond the abelian case). A similar analysis could be carried out in the context of finite GK-dim, but we need to assume the already mentioned Conjecture 1.2 and the new Conjecture 1.13 extending the criterium of type C from [8] to the setting of GK-dim.

0.3. Finitely generated nilpotent groups. Our second basic result, Theorem 2.6, shows that the bosonization $\mathscr{B}(M)\#\Bbbk G$ of the Nichols algebra of any Yetter–Drinfeld M module over any finitely generated group G, whose support is an infinite conjugacy class, has infinite GK-dim even if $\mathscr{B}(M)$ could have finite GK-dim. It is known that any conjugacy class of a finitely generated torsion-free nilpotent group G is either infinite or central, thus for such groups we just need to study Nichols algebras with central support. These arise also over abelian groups, discussed above in §0.1. For illustration we list those corresponding to M semisimple, see Theorem 3.5.

Finally let G be a finitely generated nilpotent group and assume that its torsion subgroup T has odd order. Then we show that any finite conjugacy class is either abelian or of type C, see Proposition 2.9, extending Theorem 2.1. To proceed further we need the validity of Conjectures 1.2 and 1.13. We also make a reduction when the order of T is coprime to 6.

The paper has four sections: in the first one we collect preliminary information on Nichols algebras. Section 2 deals with conjugacy classes including the basic theorems mentioned above. In Section 3 we establish the main results, stating some auxiliary lemmas in more generality for future applications and discussing a few examples. Comments on the open questions on Hopf algebras over nilpotent-by-finite groups are in the last Section 4.

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1. Preliminaries

1.1. Notations. We denote the cardinal of a set X by $|X|$. If $k < \ell$ are non-negative integers, then we set $\mathbb{I}_{k,\ell} = \{i \in \mathbb{N} : k \leq i \leq \ell\}$ and just $\mathbb{I}_\ell = \mathbb{I}_{1,\ell}$. Given a positive integer ℓ, we denote by $\mathbb{G}_\ell$ the group of ℓ-th roots of unity in $\Bbbk$, and by $\mathbb{G}'_\ell \subset \mathbb{G}_\ell$ the subset of those of order ℓ. The group of all roots of unity is denoted by $\mathbb{G}_\infty$ and $\mathbb{G}'_\infty := \mathbb{G}_\infty - \{1\}$.

Let G be a group. The identity, the group of characters and the center of G are denoted by e, $\widehat{G} = \operatorname{Hom}_{\text{groups}}(G, \Bbbk^\times)$ and $Z(G)$. The notations $F \leqslant G$, or $G \geqslant F$, mean that F is a subgroup of G, while $F \lhd G$, or $G \rhd F$, mean that $F \leqslant G$ is normal. We shall use the notations $x \triangleright y = xyx^{-1}$, $[x, y] = xyx^{-1}y^{-1}$ (the commutator), $|x|$ = the order of x, for $x, y \in G$. Given $x \in G$, let $\mathcal{O}_x$ be its conjugacy class and let G^x be its centralizer. If emphasis is needed, then we write $\mathcal{O}_x^G = \mathcal{O}_x$.

The symmetric and exterior algebras of a vector space V are denoted $S(V)$ and $\Lambda(V)$ respectively.

Let $\operatorname{Irr} \mathcal{C}$ be the set of isomorphism classes of simple objects in an abelian category $\mathcal{C}$. If A is an algebra and $\mathcal{C}$ is the category of A-modules, then $\operatorname{Irr} A := \operatorname{Irr} \mathcal{C}$; if $A = \Bbbk G$, then $\operatorname{Irr} G := \operatorname{Irr} A$. Also $\operatorname{Indec} \mathcal{C}$ denotes the set of isomorphism classes of indecomposable objects in $\mathcal{C}$ and corespondingly we have $\operatorname{Indec} A$, $\operatorname{Indec} G$.

1.2. Yetter–Drinfeld modules. A **braided vector space** is a pair (V, c) where V is a vector space and $c \in GL(V^{\otimes 2})$ satisfies the braid equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

A systematic way of producing braided vector spaces is through Yetter–Drinfeld modules over a Hopf algebra $\mathcal{H}$ (always assumed with bijective antipode); these are $\mathcal{H}$-modules and $\mathcal{H}$-comodules subject to a compatibility condition, see [36]. The category ${}^{\mathcal{H}}_{\mathcal{H}}\mathcal{YD}$ of Yetter–Drinfeld modules over $\mathcal{H}$ is a braided monoidal one. Hence the notion of Hopf algebras in ${}^{\mathcal{H}}_{\mathcal{H}}\mathcal{YD}$ is available. We refer to [1] for the concepts of Nichols algebra of a Yetter–Drinfeld module (a special kind of Hopf algebra in ${}^{\mathcal{H}}_{\mathcal{H}}\mathcal{YD}$) and Nichols algebra of a braided vector space (a non-categorical version of the former), central in the approach to the classification of pointed Hopf algebras pursued in this paper.

If $V \in {}^{\mathcal{H}}_{\mathcal{H}}\mathcal{YD}$, or if (V, c) is a braided vector space, then $\mathscr{B}(V)$ denotes its Nichols algebra and $\mathcal{J} = \mathcal{J}(V)$ its ideal of defining relations.

Example 1.1: Yetter–Drinfeld modules of dimension 1 are classified by YD-pairs over $\mathcal{H}$, that is pairs $(g,\chi) \in G(H) \times \operatorname{Hom}_{\mathrm{Alg}}(H, \Bbbk)$ satisfying

$$\chi(h)g = \chi(h_{(2)})h_{(1)}g\mathcal{S}(h_{(3)}), \quad h \in H.$$

Given a YD-pair (g,χ), we denote by $\Bbbk_g^\chi \in {}^{\mathcal{H}}_{\mathcal{H}}\mathcal{YD}$ the one-dimensional vector space where $\mathcal{H}$ acts by χ and co-acts by g. Let $q = \chi(g)$. It is well-known that

$$\mathscr{B}(\Bbbk_g^\chi) \simeq \begin{cases} \Bbbk[T]/T^N, & \text{if } q \in \mathbb{G}'_N, N > 1; \\ \Bbbk[T], & \text{otherwise,} \end{cases}$$

where T is an indeterminate.

1.3. Nichols algebras of diagonal type. Let $\theta \in \mathbb{N}$ and $\mathbb{I} := \mathbb{I}_\theta$. Given a matrix $\mathbf{q} = (q_{ij})_{i,j\in\mathbb{I}} \in (\Bbbk^\times)^{\mathbb{I}\times\mathbb{I}}$, we denote by $(V, c^{\mathbf{q}})$ the braided vector space of **diagonal type** associated to $\mathbf{q}$, where V has a basis $(x_i)_{i\in\mathbb{I}}$ and

$$c^{\mathbf{q}}(x_i \otimes x_j) = q_{ij}x_j \otimes x_i, \quad i, j \in \mathbb{I}. \tag{1.1}$$

In this case we set $\mathcal{J}_{\mathbf{q}} = \mathcal{J}(V)$, $\mathscr{B}_{\mathbf{q}} = \mathscr{B}(V)$, etc. The **Dynkin diagram** associated to $\mathbf{q}$ is the graph with θ vertices, where the vertex i is labelled by q_{ii}, and there is an edge between i and j labelled by $\widetilde{q}_{ij} := q_{ij}q_{ji}$. When $\widetilde{q}_{ij} = 1$, the edge is omitted except sometimes for the needs of the exposition.

Assume that a matrix $\mathbf{p} = (p_{ij})_{i,j\in\mathbb{I}}$ is **twist-equivalent** to $\mathbf{q}$, that is they have the same Dynkin diagram, i.e., $p_{ii} = q_{ii}$ and $\widetilde{p}_{ij} = \widetilde{q}_{ij}$ for all $i \neq j$. Then the Nichols algebras $\mathscr{B}_{\mathbf{p}}$ and $\mathscr{B}_{\mathbf{q}}$, which are not necessarily isomorphic, have the same Hilbert series, hence the same GK-dim by [34, Lemma 6.1]. We shall express this situation by $\mathscr{B}_{\mathbf{p}} \simeq_{\mathrm{tw}} \mathscr{B}_{\mathbf{q}}$.

We may refer to the connected components of the Dynkin diagram and a fortiori of $\mathbf{q}$. For many purposes we may assume that $\mathbf{q}$ is connected as $\mathscr{B}_{\mathbf{q}}$ is the twisted tensor product of the Nichols algebras of the connected components of $\mathbf{q}$. Under suitable hypotheses, a matrix $\mathbf{q}$ gives rise to a generalized root system [33]; if $\dim \mathscr{B}_{\mathbf{q}} < \infty$, then $\mathbf{q}$ has a finite root system. The classification of all $\mathbf{q}$ with finite root system and connected Dynkin diagram was given in [29]; this contains the classification of the finite-dimensional Nichols algebras of diagonal type. The Nichols algebras $\mathscr{B}_{\mathbf{q}}$ with $\mathbf{q}$ in the list of [29] have finite GK-dim. It was conjectured that these are all.

Conjecture 1.2 ([6, Conjecture 1.5]): *The root system of a Nichols algebra of diagonal type with finite GK-dimension is finite.*

The conjecture holds when $\theta \leq 3$, when $\mathbf{q}$ is of Cartan type, or when $\mathbf{q}$ is generic; see [4], [17] and references therein.

The defining relations of the Nichols algebras $\mathscr{B}_{\mathbf{q}}$ with $\mathbf{q}$ in the list of [29] appear in [14, 13]. See the survey [2].

We shall apply several times the following result. The first three items are well-known, the last follows from [4, Theorems 1, 2, 4.1]. The case $\theta = 1$ is covered by Example 1.1.

Lemma 1.3: *Assume that $\theta \geq 2$. Given $q \in \Bbbk^{\times}$, let V be a braided vector space of diagonal type with matrix $\mathbf{q} = (q_{ij})_{i,j\in\mathbb{I}}$ where $q_{ij} = q$ for all $i, j \in \mathbb{I}$; thus locally the Dynkin diagram is*

$$\overset{q}{\underset{i}{\circ}} \overset{q^2}{\text{------}} \overset{q}{\underset{j}{\circ}} \qquad \textit{for all } i \neq j.$$

Then:

(i) *If $q = 1$, then $\mathscr{B}(V) \simeq S(V)$.*
(ii) *If $q = -1$, then $\mathscr{B}(V) \simeq \Lambda(V)$.*
(iii) *If $q \in \mathbb{G}'_3$ and $\dim V = 2$, then $\mathscr{B}(V)$ is of Cartan type A_2 and has dimension 27.*
(iv) *Otherwise* $\operatorname{GK-dim} \mathscr{B}(V) = \infty$. ∎

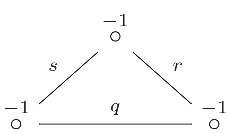
The next result will be useful too. For the first two items we assume Conjecture 1.2 and go through the list of [29]. Here by cycle we mean a closed path. For (iii), see [6, Lemma 2.8] inspired by [38].

Lemma 1.4: *Let V be a braided vector space of diagonal type such that its Dynkin diagram contains*

(i) *either an N-cycle, with $N > 3$,*
(ii) *or else a 3-cycle, with no vertex labelled by -1,*
(iii) *or else a sub-diagram of the form* $\overset{q}{\circ}\overset{r}{\rule[3pt]{40pt}{0.4pt}}\overset{1}{\circ}$, $r \neq 1$.

Then GK-dim $\mathscr{B}(V) = \infty$. ∎

Remark 1.5: The only 3-cycles with all vertices labelled by -1 in [29] are



where $q, r, s \neq 1$, $qrs = 1$.

These are of type $D(2,1;\alpha)$, [2, §5.4]. When $q = r = s \in \mathbb{G}'_3$, $\dim \mathscr{B}_{\mathbf{q}} = 2^4 3^3$. Furthermore, this diagram can not be embedded in a Dynkin diagram of rank 4 with finite root system; see [29, Table 4].

Remark 1.6: The only Dynkin diagrams of rank 2 in the list of [29] with both vertices labelled by the same $q \neq 1$ are

$\overset{q}{\circ}\overset{q^{-1}}{\rule[3pt]{60pt}{0.4pt}}\overset{q}{\circ}$, of type A_2;

$\overset{q}{\circ}\overset{\zeta}{\rule[3pt]{60pt}{0.4pt}}\overset{q}{\circ}$, where $\zeta \in \mathbb{G}'_{12}$ and $q = -\zeta^2$; of type $\mathfrak{ufo}(8)$, see [2, §10.8].

Definition 1.7: Let $(V, c^{\mathbf{q}})$ be a braided vector space of diagonal type. A **principal realization** of $(V, c^{\mathbf{q}})$ over a Hopf algebra $\mathcal{H}$ is a family $(g_i, \chi_i)_{i \in \mathbb{I}}$ of YD-pairs such that $q_{ij} = \chi_j(g_i)$ for all i, j. In this case $V = \bigoplus_i \Bbbk_{g_i}^{\chi_i} \in {}^{\mathcal{H}}_{\mathcal{H}}\mathcal{YD}$.

Let G be a finite group of odd order. By inspection of the list in [29], we see that a matrix $\mathbf{q}$ with finite root system and connected Dynkin diagram could have a principal realization over the the group algebra $\Bbbk G$ only when either it is of Cartan type, or else its Dynkin diagram is one of

$$\overset{\omega}{\circ}\overset{q^{-1}}{\rule[3pt]{40pt}{0.4pt}}\overset{q}{\circ}\,, \qquad \overset{\omega}{\circ}\overset{\omega^2 q}{\rule[3pt]{80pt}{0.4pt}}\overset{\omega q^{-1}}{\circ}\,, \tag{1.2}$$

$$\overset{\zeta}{\circ}\overset{\zeta^{-1}}{\rule[3pt]{40pt}{0.4pt}}\overset{\zeta}{\circ}\overset{\zeta^{-1}}{\rule[3pt]{40pt}{0.4pt}}\overset{\zeta^{-3}}{\circ} \qquad \overset{\zeta}{\circ}\overset{\zeta^{-1}}{\rule[3pt]{40pt}{0.4pt}}\overset{\zeta^{-4}}{\circ}\overset{\zeta^{4}}{\rule[3pt]{40pt}{0.4pt}}\overset{\zeta^{-3}}{\circ}\,, \tag{1.3}$$

where the order of q divides $|G|$ and is > 3, $\omega \in \mathbb{G}'_3$ and $\zeta \in \mathbb{G}'_9$. In particular, if 3 does not divide $|G|$, then $\mathbf{q}$ could admit a principal realization only if it is of Cartan type. See [2, §7.2, §7.3] for information on (1.2) of type $\mathbf{br}(2)$, resp. (1.3) of type $\mathbf{br}(3)$.

1.4. Yetter–Drinfeld modules over groups. Let G be a group. Recall that $\mathcal{O}_x$ denotes the conjugacy class and G^x the centralizer of $x \in G$. For any $y \in \mathcal{O}_x$ we fix $g_y \in G$ such that $g_y \triangleright x = y$. Then for $h \in G$ and $y \in \mathcal{O}_x$

$$t_{h,y} := g_{h \triangleright y}^{-1} h g_y \in G^x. \tag{1.4}$$

A Yetter–Drinfeld module $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is just a G-graded vector space $M = \bigoplus_{g \in G} M_g$ provided with a linear action of G such that

$$h \cdot M_g = M_{h \triangleright g}, \quad h, g \in G.$$

In such case, the support of M is $\operatorname{supp} M = \{g \in G : M_g \neq 0\}$, which is a disjoint union of conjugacy classes. If $v \in M_g$, then $\deg v := g$.

We next describe $\operatorname{Irr} {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ and $\operatorname{Indec} {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. First we consider $x \in G$ and a representation $\rho : G^x \to GL(W)$. We set

$$M(x, W) := \operatorname{Ind}_{G^x}^G W \simeq \Bbbk\mathcal{O} \otimes W \simeq \bigoplus_{y \in \mathcal{O}_x} g_y \otimes W.$$

As is known, $M(x, W)$ belongs to ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with action and grading

$$h \cdot (g_y \otimes w) = g_{h \triangleright y} \otimes t_{h,y} \cdot w, \quad \deg(g_y \otimes w) = y,$$

where $t_{h,y}$ is given by (1.4). We also use the notation $M(x, \rho) = M(x, W)$, and accordingly

$$\mathscr{B}(x, W) := \mathscr{B}(M(x, \rho)) =: \mathscr{B}(x, \rho).$$

For brevity, we set $g_y w = g_y \otimes w$. Then the braiding of $M(x, W)$ is given by

$$c(g_z u \otimes g_y w) = g_{z \triangleright y}(t_{z,y} \cdot w) \otimes g_z u, \quad z, y \in \mathcal{O}_x,\ u, w \in W. \tag{1.5}$$

Example 1.8: A YD-pair over $\Bbbk G$ is just a pair $(g, \chi) \in Z(G) \times \widehat{G}$; then

$$\Bbbk_g^\chi \simeq M(\mathcal{O}_g, \chi),$$

see Example 1.1. If, e.g., $g \in Z(G) \cap [G, G]$ and $\chi \in \widehat{G}$, then $\mathscr{B}(g, \chi) \simeq S(W)$ where $\dim W = 1$, thus $\operatorname{GK-dim} \mathscr{B}(g, \chi) = 1$. ∎

Proposition 1.9: $\operatorname{Irr}{}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ *is parametrized by pairs* $(\mathcal{O}, W)$ *where* $\mathcal{O}$ *is a conjugacy class and* $W \in \operatorname{Irr} G^x$ *for a fixed choice of* $x \in \mathcal{O}$. *Similarly* $\operatorname{Indec}{}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ *is parametrized by pairs* $(\mathcal{O}, W)$ *where now* $W \in \operatorname{Indec} G^x$.

We sketch a proof of this well-known result for completeness, as we have not found a reference for the case when G is infinite.

Proof. If M is indecomposable, then necessarily $\operatorname{supp} M = \mathcal{O}_x$ for some $x \in G$ that we fix. Thus $M = \bigoplus_{y \in \mathcal{O}_x} M_y$ and G^x acts on M_x. Let N_x be a G^x-submodule of M_x. For each $y \in \mathcal{O}_x$ choose $g_y \in G$ such that $g_y \triangleright x = y$. Then $g_y \cdot N_x \subset M_y$ and $N := \bigoplus_{y \in \mathcal{O}_x} g_y \cdot N_x$ is a Yetter–Drinfeld submodule of M. Thus, if M is indecomposable, respectively simple, then so is M_x as G^x-module, and $M \simeq M(x, M_x)$. The converse is proved similarly. ∎

Actually, there is an equivalence of categories between $\operatorname{Rep} G^x$ and the full subcategory of ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ whose objects have support on $\mathcal{O}_x$. See [30, Prop. 1.4.17]. I thank the referee for pointing out this fact.

1.5. Racks. Nichols algebras over groups are studied systematically through racks. We refer to [9] for an exposition on racks and Nichols algebras over groups and [1] for more recent results. Here we collect some material needed in this paper. A **rack** is a non-empty set X with a self-distributive operation $\triangleright : X \times X \to X$ such that $\varphi_x := x \triangleright \_$ is bijective for every $x \in X$. The main examples are subsets of groups stable under conjugation. All racks here are assumed to be subracks of groups. A rack X is **abelian** if $x \triangleright y = y$ for all $x, y \in X$. The inner group of a rack X is the subgroup $\operatorname{Inn} X$ of the group $\operatorname{Aut} X$ of rack automorphisms generated by φ_x for all $x \in X$.

Lemma 1.10 ([9, Lemma 1.8]): *A surjective morphism of racks* $\pi : X \to Y$ *extends to a surjective morphism of groups* $\operatorname{Inn} \pi : \operatorname{Inn} X \to \operatorname{Inn} Y$.

Proof. Given $x \in X$, define $\operatorname{Inn} \pi(\varphi_x) = \varphi_{\pi(x)}$. If $z \in X$ satisfies $\varphi_x = \varphi_z$, then $\varphi_{\pi(x)}(\pi(y)) = \pi(x) \triangleright \pi(y) = \pi(x \triangleright y) = \pi(z \triangleright y) = \varphi_{\pi(z)}(\pi(y))$. Since π is surjective, then $\varphi_{\pi(x)} = \varphi_{\pi(z)}$, i.e., $\operatorname{Inn} \pi$ is well-defined. Consider next

$$G = \{\sigma \in \mathbb{S}_X : \exists \nu \in \mathbb{S}_Y \text{ such that } \pi\sigma = \nu\pi\}.$$

Clearly such ν is unique, $G \leq \mathbb{S}_X$ and $\sigma \mapsto \nu$ is a morphism of groups. Thus $\operatorname{Inn} \pi$ is a morphism of groups which is surjective because π is so. ∎

The following statement is a consequence of [31, Theorem 2.1].

Theorem 1.11: *Let G be a finite non-abelian group and V and W be two simple Yetter–Drinfeld modules over G such that G is generated by the support of $V \oplus W$, $\dim V \leq \dim W$ and*

$$c^2{}_{|V\otimes W} \neq \mathrm{id}_{V\otimes W}\,. \tag{1.6}$$

If $\dim \mathscr{B}(V \oplus W) < \infty$, then $(\dim V, \dim W)$ belongs to

$$\{(1,3),(1,4),(2,2),(2,3),(2,4)\}. \tag{1.7}$$

When $\dim W = 3$ in (1.7), W is the braided vector space associated to the transpositions in $\mathbb{S}_3$ with the cocycle -1, which is not of diagonal type. ∎

Recall from [8, Definition 2.3] that a finite rack X is of **type C** when there are a decomposable subrack $Y = R \coprod S$ and elements $r \in R$, $s \in S$ such that

$$r \triangleright s \neq s \quad (\text{hence } s \triangleright r \neq r), \tag{1.8}$$

$$R = \mathcal{O}_r^{\operatorname{Inn} Y}, \quad S = \mathcal{O}_s^{\operatorname{Inn} Y}, \tag{1.9}$$

$$\min\{|R|,|S|\} > 2 \quad \text{or} \quad \max\{|R|,|S|\} > 4. \tag{1.10}$$

Theorem 1.12 ([8, Theorem 2.9]): *A finite rack of type C collapses, that is*

$$\dim \mathscr{B}(\mathcal{O}, \mathbf{q}) = \infty$$

for every finite faithful 2-cocycle $\mathbf{q}$. ∎

The proof of Theorem 1.12 relies on [31, Theorem 2.1], which in turn depends on the notion of Weyl groupoid [33]. For some of the arguments below we need the validity of the following conjecture; the adaptation of the proof of [31] does not appear to be straightforward.

Conjecture 1.13: *Let X be a finite rack of type C. Then*

$$\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, \mathbf{q}) = \infty$$

for every faithful 2-cocycle $\mathbf{q}$.

Example 1.14: Assume that $G = G_1 \times G_2$. If $x = (x_1, x_2) \in G$, then

$$\mathcal{O}_x = \mathcal{O}_{x_1} \times \mathcal{O}_{x_2}, \quad G^x = G^{x_1} \times G^{x_2}, \quad \operatorname{Irr} G^x \simeq \operatorname{Irr} G^{x_1} \times \operatorname{Irr} G^{x_2}. \tag{1.11}$$

Thus if $W \simeq W_1 \otimes W_2$ is a simple G^x-module, then

$$M(x, W) \simeq M(x_1, W_1) \otimes M(x_2, W_2). \tag{1.12}$$

Correspondingly, the tensor product of two braided vector spaces (V, c_V) (W, c_W) is $(V \otimes W, c_{V\otimes W})$ where $c_{V\otimes W} : V \otimes W \otimes V \otimes W \to V \otimes W \otimes V \otimes W$ is defined by

$$c_{V\otimes W} := (\mathrm{id} \otimes \tau_{V,W} \otimes \mathrm{id})(c_V \otimes c_W)(\mathrm{id} \otimes \tau_{W,V} \otimes \mathrm{id}).$$

Observe that there is no clear relation between the Nichols algebras $\mathscr{B}(x, W)$, $\mathscr{B}(x_1, W_1)$ and $\mathscr{B}(x_2, W_2)$. For instance, if all three modules in (1.12) have dimension one, then the braidings are given respectively by q, q_1 and q_2 with $q = q_1 q_2$ so that any of them could be 1 with the other two being non-trivial roots of 1. But the criterium of type C propagates in this setting.

Namely, let $X = X_1 \times X_2$ be a direct product of racks. If either X_1 or X_2 is of type C, then so is X. Indeed, let $Y_1 = R_1 \coprod S_1$ be a subrack of X_1 and elements $r_1 \in R_1$, $s_1 \in S_1$ satisfying (1.8), (1.9) and (1.10). Pick any $x_2 \in X_2$ and set $Y = Y_1 \times \{x_2\}$, $R = R_1 \times \{x_2\}$, $S = S_1 \times \{x_2\}$, $r = (r_1, x_2)$, $s = (s_1, x_2)$. Then (1.8), (1.9) and (1.10) hold for them.

2. Hopf algebras and conjugacy classes

Recall that the upper central series of a group G is the sequence of subgroups $e = Z_0 \lhd Z_1 \lhd \cdots \lhd Z_i \lhd \cdots$, where

$$Z_{n+1} = Z_{n+1}(G) = \{x \in G : [x, G] \leq Z_n\};$$

G is nilpotent iff the upper centralizer series stabilizes in G [24, Th. 2.2].

2.1. Conjugacy classes in finite nilpotent groups. Here is our first basic result on Nichols algebras over finite nilpotent groups.

Theorem 2.1: *Let $\mathcal{O}$ be a conjugacy class in a finite nilpotent group G of odd order. Then $\mathcal{O}$ is either of type C or else an abelian rack.*

There are examples of conjugacy classes of finite nilpotent groups that are of type C; see Subsection 3.8.3.

Proof. It is well-known that a finite group is nilpotent if and only if it is isomorphic to the product of its Sylow subgroups; see, e.g., [24, Theorem 2.13]. Hence, by Example 1.14, we may assume that G is a p-group with p an odd prime. Let us assume that $\mathcal{O}$ is not abelian. That is, there exist $r, s \in \mathcal{O}$ such that $r \triangleright s \neq s$ (and then $s \triangleright r \neq r$). Let $H = \langle r, s\rangle \leq G$.

If $R := \mathcal{O}_r^H \neq S := \mathcal{O}_s^H$, then $Y := R \coprod S$ is a decomposable subrack of $\mathcal{O}$ that satisfies (1.9), because $H = \langle Y \rangle$ so $R = \mathcal{O}_r^H = \mathcal{O}_r^{\operatorname{Inn} Y}$ and $\mathcal{O}_s^H = \mathcal{O}_s^{\operatorname{Inn} Y}$. Now (1.8) holds by assumption. Evidently p divides both $|R|$ and $|S|$. Since $p \geq 3$, (1.10) holds and $\mathcal{O}$ is of type C.

Next suppose that $s \in \mathcal{O}_r^H = Y$. Then $H = \langle Y \rangle$, hence $Y = \mathcal{O}_r^H = \mathcal{O}_r^{\operatorname{Inn} Y}$ is indecomposable by [9, Lemma 1.15]. Also $\operatorname{Inn} Y \simeq H/Z(H)$ by [9, Lemma 1.9], hence $\operatorname{Inn} Y$ is a p-group. Now by a routine recursive argument, there exists a surjective morphism of racks $\pi : Y \to Z$ where Z is simple. Considering the surjective morphism of groups $\operatorname{Inn} \pi : \operatorname{Inn} Y \to \operatorname{Inn} Z$ given by Lemma 1.10, we see that $\operatorname{Inn} Z$ is a p-group, so in particular $|Z|$ should be a power of p. But then $\operatorname{Inn} Z$ could not be a p-group being a semidirect product of a p-group with a group of order not divisible by p, see [9, Proposition 3.2 and Theorem 3.7], and also the discussion at the end of page 204 and the beginning of [9, p. 205]. ∎

Note that this is not a statement on racks with p^n elements, p an odd prime but on conjugacy classes of p-groups.

When G is a 2-group, Theorem 2.1 is no longer true. Indeed, the group $\mathbb{D}_4 = \langle x, y | x^2 = e = y^4, xyx = y^3 \rangle$ admits a a finite-dimensional Nichols algebra $\mathscr{B}(V)$ where

$$\operatorname{supp} V = Y = \mathcal{O}_x \coprod \mathcal{O}_{xy}$$

which is neither abelian nor of type C. The rack Y can be realized as conjugacy class in the group $D_4 \rtimes \mathbb{Z}/4$ determined by the automorphism $\sigma : \mathbb{D}_4 \to \mathbb{D}_4$,

$$\sigma(x) = xy, \quad \sigma(y) = y.$$

2.2. Hopf algebras and infinite conjugacy classes. Let $\mathcal{H}$ be a Hopf algebra. If R is a Hopf algebra in ${}^{\mathcal{H}}_{\mathcal{H}}\mathcal{YD}$, then the **bosonization** (or biproduct) $R\#\mathcal{H}$ is the Hopf algebra with underlying vector space $R \otimes \mathcal{H}$ and structure as in [36, Section 11.6].

Remark 2.2: Recall that an affine algebra is a finitely generated one. Let $\mathcal{O}$ be a conjugacy class in a finitely generated group G. Let $x \in \mathcal{O}$ and let $W \in \operatorname{Rep} G^x$ be a finitely generated module. Set $M = M(\mathcal{O}, W)$. Then $T(M)\#\Bbbk G$ is affine, hence so is $\mathscr{B}(M)\#\Bbbk G$.

Proof. Let $(g_i)_{i \in I}$ be a family of generators of G and $(w_j)_{j \in J}$ be a family of generators of W. Then the g_i's together with the w_j's generate the algebra $T(M)\#\Bbbk G$. ∎

Let G be a finitely generated group and let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ be such that the action of G is locally finite. By [6, Lemma 2.3.1], we have

$$\text{GK-dim}\,\mathscr{B}(M)\#\Bbbk G \leq \text{GK-dim}\,\mathscr{B}(M) + \text{GK-dim}\,\Bbbk G. \tag{2.1}$$

Furthermore, if $\dim M < \infty$, then the equality holds in (2.1). Our second basic result, Theorem 2.6 (inspired by [6, Example 2.3.3]), roughly states that $\text{GK-dim}\,\mathscr{B}(M)\#\Bbbk G = \infty$ if the support of M is an infinite conjugacy class, even if $\text{GK-dim}\,\mathscr{B}(M)$ is finite, in sharp contrast with (2.1). We start by a theorem of Malcev needed for our approach.

Theorem 2.3 (Malcev [24, Theorems 2.23, 2.24]): *Let G be a finitely generated nilpotent group and let $H \leq G$. Assume that there exists a finite set X of generators such that for any $g \in X$ there exists a positive integer n such that $g^n \in H$. Then the index of H in G is finite.*

Furthermore, if for any $g \in X$ the integer n is a power of a fixed prime p, then $[G : H]$ is also a power of p. ∎

Actually, the last claim holds more generally if n is a ϖ-number, where ϖ is a fixed set of primes.

Corollary 2.4: *Let G be a finitely generated nilpotent-by-finite group and let $H \leq G$. Assume that for every $g \in G$ there exists a positive integer n such that $g^n \in H$. Then the index of H in G is finite.*

Proof. Let $N \leq G$ be nilpotent of finite index; N is finitely generated by [37, 1.6.11]. If $g \in N$, then there exists $n \in \mathbb{N}$ such that $g^n \in H \cap N$; thus $[N : H \cap N]$ is finite by Theorem 2.3 and so is $[G : H \cap N] = [G : N][N : H \cap N]$. But $[G : H \cap N] = [G : H][H : H \cap N]$, so $[G : H]$ is finite. ∎

Corollary 2.5: *Let G be a finitely generated nilpotent-by-finite group, let $\mathcal{O} \subset G$ be an infinite conjugacy class and pick $x \in \mathcal{O}$. Then there exists $g \in G$ such that $g^n \triangleright x \neq x$ for all $n \in \mathbb{N}$.*

Proof. If for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g^n \triangleright x = x$, then $|\mathcal{O}| = [G : G^x]$ is finite by Corollary 2.4. ∎

Theorem 2.6: *Let G be a finitely generated group and $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ such that $\mathcal{O} = \operatorname{supp} M$ is an infinite conjugacy class. Then*

$$\text{GK-dim}\,\mathscr{B}(M)\#\Bbbk G = \infty.$$

Proof. By Gromov's Theorem we may assume that G is nilpotent-by-finite. Let $x \in \mathcal{O}$. Since M_x is the union of its finitely generated G^x-submodules, we may assume that it is finitely generated. Let S be a finite set of generators of the group G and let F be a finite set of generators of the G^x-module M_x. Pick $g \in G$ such that $x_n = g^n \triangleright x \neq x$ for all $n \in \mathbb{N}$, which exists by Corollary 2.5, and $m_0 \in F\backslash 0$. Let $V = \langle 1, S, g^{\pm 1}, F\rangle$, a set of generators of $\mathscr{B}(M)\#\Bbbk G$. For $n \in \mathbb{N}$ we set $m_n = g^n m_0 g^{-n} \in V^{2n+1}$. Given $s \in \mathbb{I}_{0,n}$, let

$$\Lambda_{n,s} = \{m_{i_1} \cdots m_{i_s} : 1 \leq i_1 < \cdots < i_s \leq n\}, \quad \Lambda_n = \bigcup_{s \in \mathbb{I}_{0,n}} \Lambda_{n,s}.$$

We claim that (i) $|\Lambda_n| = 2^n$ (exercise), (ii) $\Lambda_n \subset V^{(n+1)^2}$, and (iii) Λ_n is a linearly independent set. For (ii), just observe that

$$\Lambda_1 = \{1, m_1\} \subset V^3 \subset V^4, \quad \Lambda_n \subset \Lambda_{n-1}\{1, m_n\} \subset V^{n^2} V^{2n+1} \subset V^{(n+1)^2}.$$

For (iii), it is enough to prove that $\Lambda_{n,s} \subset \mathscr{B}^s(M)$ is a linearly independent set. Clearly, $m_n \in M_{x_n}$ for all $n \in \mathbb{N}_0$. The claim (iii) is a particular case of

Claim (iv): *Given $\widetilde{m}_i \in M_{x_i}\backslash 0$, $i \in \mathbb{I}_n$, the subset*

$$\widetilde{\Lambda}_{n,s} = \{\widetilde{m}_{i_1} \cdots \widetilde{m}_{i_s} : 1 \leq i_1 < \cdots < i_s \leq n\}$$

of $\mathscr{B}(M)$ is linearly independent.

Proof of Claim (iv). By induction on s and n. The elements x_i are all different by our choice of g, thus the case $s = 1$ follows. After completing appropriately the family $(\widetilde{m}_i)$ to a homogeneous basis of M, we know that there exist skew-derivations $\partial_j : \mathscr{B}(M) \to \mathscr{B}(M)$, $j \in \mathbb{I}_n$, such that

$$\partial_j(uv) = \partial_j(u)(x_j \cdot v) + u\,\partial_j(v), \quad \partial_j(\widetilde{m}_i) = \delta_{i,j}.$$

Set $\mathbf{I}_s = \{\mathbf{i} = (i_1, \dots, i_s) : 1 \leq i_1 < \cdots < i_s \leq n\}$, $\widetilde{m}_{\mathbf{i}} = \widetilde{m}_{i_1} \cdots \widetilde{m}_{i_s}$. Then

$$\partial_j(\widetilde{m}_{\mathbf{i}}) = \begin{cases} 0, & \text{if } j \notin \{i_1, \dots, i_s\}, \\ \widetilde{m}_{i_1} \cdots \widetilde{m}_{i_{h-1}}(x_j \cdot \widetilde{m}_{i_{h+1}}) \cdots (x_j \cdot \widetilde{m}_{i_s}) & \text{if } j = i_h,\ h \in \mathbb{I}_s. \end{cases}$$

Thus we consider the map ψ_j from $\mathbf{I}_{s;j} := \{\mathbf{i} = (i_1, \dots, i_s) \in \mathbf{I}_s : \exists h, i_h = j\}$ to $\mathbf{I}_{s-1;\neg j} := \{\mathbf{i} = (i_1, \dots, i_{s-1}) \in \mathbf{I}_{s-1} : \nexists k, i_k = j\}$ given by

$$\mathbf{i} \mapsto (i_1, \dots, i_{h-1}, i_{h+1}, \dots, i_s).$$

It is easy to see that this map ψ_j is bijective. Let now $\lambda_{\mathbf{i}}$, $\mathbf{i} \in \mathbf{I}_s$, be a family of scalars such that $\sum_{\mathbf{i}\in\mathbf{I}_s} \lambda_{\mathbf{i}} \widetilde{m}_{\mathbf{i}} = 0$. Then for any $j \in \mathbb{I}_n$

$$0 = \partial_j\left(\sum_{\mathbf{i}\in\mathbf{I}_s} \lambda_{\mathbf{i}}\widetilde{m}_{\mathbf{i}}\right) = \sum_{\mathbf{i}\in\mathbf{I}_{s;j}} \lambda_{\mathbf{i}}\widetilde{m}_{i_1}\cdots\widetilde{m}_{i_{h-1}}(x_j\cdot\widetilde{m}_{i_{h+1}})\cdots(x_j\cdot\widetilde{m}_{i_s}).$$

If $j = n$, then

$$0 = \sum_{\mathbf{i}\in\mathbf{I}_{s;n}} \lambda_{\mathbf{i}}\widetilde{m}_{i_1}\cdots\widetilde{m}_{i_{s-1}} \overset{\star}{\Longrightarrow} \lambda_{\mathbf{i}} = 0\ \forall \mathbf{i}\in\mathbf{I}_{s;n}$$
$$\Longrightarrow 0 = \sum_{\mathbf{i}\in\mathbf{I}_{s;\neg n}} \lambda_{\mathbf{i}}\widetilde{m}_{\mathbf{i}} = \sum_{\mathbf{i}\in\mathbf{I}_{s;n-1}} \lambda_{\mathbf{i}}\widetilde{m}_{\mathbf{i}} \overset{\diamond}{\Longrightarrow} \lambda_{\mathbf{i}} = 0\ \forall \mathbf{i}\in\mathbf{I}_{s;\neg n},$$

where $\star$ is by the inductive hypothesis on s and $\diamond$ is by the inductive hypothesis on n. Claim (iv) is proved. ∎

By (i), (ii) and (iii), $2^n \leq \dim V^{(n+1)^2}$ for all n; the Theorem follows. ∎

2.3. Finitely generated torsion-free nilpotent groups. Recall that the FC-center $FC(G)$ of a group G is the union of all finite conjugacy classes of G; $FC(G)$ is a characteristic subgroup of G containing $Z(G)$ [18]. The following result is folklore; see, for instance, [25] for a different proof.

Lemma 2.7: *If G is a finitely generated torsion-free nilpotent group, then every non-central conjugacy class is infinite.*

Proof. Since G is finitely generated nilpotent, so is $FC(G)$. By [39, Theorem 1.6], $[FC(G), FC(G)]$ is finite. Since G is torsion-free,

$$[FC(G), FC(G)] = e,$$

i.e., $FC(G) = Z(FC(G)) = Z(G)$, the last equality by [25, Lemma 2.2]. ∎

2.4. Finite conjugacy classes in nilpotent groups. We generalize Theorem 2.1 to nilpotent groups whose torsion has odd order, using a well-known result of Gruenberg.

Theorem 2.8 ([28]): *Let G be a finitely generated nilpotent group with torsion $T \neq e$ and $e \neq g \in G$. Then there exists a prime p that divides $|T|$, a finite p-group P and a morphism $\pi : G \to P$ such that $\pi(g) \neq e$.*

Proposition 2.9: *Let G be a finitely generated nilpotent group whose torsion subgroup $T \neq e$ is non-trivial and has odd order. Then a finite conjugacy class $\mathcal{O}$ of G is either abelian or else of type C.*

Proof. Suppose that $\mathcal{O}$ is not abelian. Pick $r, s \in \mathcal{O}$ such that $r \triangleright s \neq s$, i.e., $[r, s] \neq e$. By Theorem 2.8 there exist an odd prime p and an epimorphism π from G to a finite p-group P such that $[\pi(r), \pi(s)] = \pi([r, s]) \neq e$. Let

$$H = \langle \pi(r), \pi(s) \rangle \leq P, \quad \widetilde{R} := \mathcal{O}^P_{\pi(r)}, \quad \widetilde{S} := \mathcal{O}_\pi(r)^P.$$

Arguing as in the proof of Theorem 2.1, we see that $\widetilde{Y} := \widetilde{R} \coprod \widetilde{S}$ is a decomposable subrack of P. Let $R := \pi^{-1}(\widetilde{R}) \cap \mathcal{O}$, $S := \pi^{-1}(\widetilde{R}) \cap \mathcal{O}$. Then $Y := R \coprod S$ is a decomposable subrack of $\mathcal{O}$. Now the elements $\pi(s)$, $\pi(r) \triangleright \pi(s)$, $\pi(r)^2 \triangleright \pi(s)$ of $\widetilde{S}$, respectively $\pi(r)$, $\pi(s) \triangleright \pi(r)$, $\pi(s)^2 \triangleright \pi(r)$ of $\widetilde{R}$, are different. Hence $s, r \triangleright s, r^2 \triangleright s \in S$, respectively $r, s \triangleright r, s^2 \triangleright r \in R$, are different, (1.10) holds and $\mathcal{O}$ is of type C. ∎

Theorem 2.6 and Proposition 2.9 show that the classification of pointed Hopf algebras with finite GK-dim over a finitely generated nilpotent group whose torsion has odd order goes through Nichols algebras over abelian groups. For instance, a Hopf algebra H like this is co-Frobenius if and only if $\operatorname{gr} H \simeq \mathscr{B}(V)$, where V is of diagonal type and $\dim \mathscr{B}(V) < \infty$, as follows from the preceding results and [6, Theorem 1.4.2]; see loc. cit. for details. This last claim does not assume Conjecture 1.13.

We finish this Section with a result needed later, see Lemma 3.13.

Lemma 2.10: *Let G be a finitely generated nilpotent group with torsion T and let $\mathcal{O} = \mathcal{O}_x$ be a finite conjugacy class in G. Then $|\mathcal{O}|$ divides $|T|$.*

Proof. As G/T is torsion-free, $FC(G/T) = Z(G/T)$ by Lemma 2.7. Hence the image of x is central in G/T; i.e., $[G, x] \leq T$. Let $\phi : G \to T$ be given by $g \mapsto [g, x]$, $g \in G$ and let $S = \phi^{-1}(T \cap Z(G))$. If $g \in G$ and $h \in S$, then

$$\phi(gh) = [gh, x] = ghxh^{-1}g^{-1}x^{-1} = g[h, x]xg^{-1}x^{-1} = \phi(g)\phi(h). \tag{2.2}$$

Thus the restriction $\phi : S \to T \cap Z(G)$ is a homomorphism; clearly, $G^x \leq S$ and S/G^x embeds into $T \cap Z(G)$. Hence $|\mathcal{O}^S_x|$ divides $|T \cap Z(G)|$. Let now

$$k = \min\{k \in \mathbb{N}_0 : T \leq Z_k(G)\}.$$

We argue by induction on k and $|\mathcal{O}|$. If $k = 0$, then G is torsion-free and Lemma 2.7 applies. If $k = 1$, then $T \leq Z(G)$, $G = S$ and the claim follows.

Assume that $k > 1$. Let $H = G/(T \cap Z(G))$ and let $\pi : G \to H$ be the natural projection. Given $y \in \mathcal{O}^H_{\pi(x)}$, we fix $z_y \in \mathcal{O}^G_x = \mathcal{O}$ such that $\pi(z_y) = y$ and $g_y \in G$ such that $z_y = g_y \triangleright x$. For $t \in \mathcal{O}^S_x$, fix $g_t \in S$ such that $t = g_t \triangleright x$. We claim that the map

$$\begin{aligned} &\sigma : \mathcal{O}^H_{\pi(x)} \times \mathcal{O}^S_x \to \mathcal{O}, \\ &\sigma(y,t) = z_y t x^{-1} = z_t[g_t, x], \qquad y \in \mathcal{O}^H_{\pi(x)}, t \in \mathcal{O}^S_x, \end{aligned}$$

is a well-defined bijection. First, since $\phi(g_t) \in Z(G)$,

$$z_y t x^{-1} = [g_y, x]x[g_t, x] = \phi(g_y)\phi(g_t)x \overset{(2.2)}{=} \phi(g_y g_t)x = g_y g_t \triangleright x \in \mathcal{O};$$

we have

$$\sigma(y,t) = \sigma(w,s) \implies \pi(z_y\phi(g_t)) = \pi(z_w\phi(g_s)) \implies y = w \implies t = s.$$

Finally, let $z \in \mathcal{O}$ and $y = \pi(z)$; then $z = z_y u$ where $u \in T \cap Z(G)$. Now

$$z = z_y u = (g_y \triangleright x)(g_y \triangleright u) = g_y \triangleright (xu) \implies xu \in \mathcal{O};$$

pick $g \in G$ such that $ux = xu = g \triangleright x$ and set $t = g \triangleright x$; since $u = (g \triangleright x)x^{-1} = \phi(g)$, we conclude that $g \in S$, $t \in \mathcal{O}^S_x$ and $\sigma(y,t) = z$. The claim is proved.

The torsion of H is $T_1 = T/T \cap Z(G)$. Then

$$Z_j(H) \cap T_1 = \pi(Z_{j+1}(G) \cap T), \quad j \in \mathbb{N}_0,$$

and so $Z_{k-1}(H) \cap T_1 = T_1$. By the inductive hypothesis on $k-1$, $|\mathcal{O}^H_{\pi(x)}|$ divides $|T_1|$; thus $|\mathcal{O}| = |\mathcal{O}^H_{\pi(x)}||\mathcal{O}^S_x|$ divides $|T_1||T \cap Z(G)| = |T|$. ∎

3. Nichols algebras

3.1. First remarks: $\dim W \geq 2$. In this subsection, we fix

- ◦ A group G, a finite conjugacy class $\mathcal{O}$ in G, $x \in \mathcal{O}$.
- ◦ $W \in \operatorname{Irr} G^x$ with representation $\rho : G^x \to GL(W)$. We assume that $\dim W$ is countable; this is the case if G is finitely generated. By the Schur Lemma, aka Dixmier's Lemma [40, 0.5.2], there exists $\eta \in \widehat{Z(G^x)}$ implementing the action of $Z(G^x)$ on W.
- ◦ $\chi \in \widehat{G^x}$; set $q := \chi(x)$.

Remark 3.1: Pick $y \in \mathcal{O}$ and $g_y \in G$ such that $g_y \triangleright x = y$. Then $G^y = g_y \triangleright G^x$ and

$$\rho^y : G^y \to GL(W),$$
$$\rho^y(g) = \rho(g_y^{-1} \triangleright g)$$

is an irreducible representation of G^y. Hence

$$M(\mathcal{O}, \rho) \simeq M(\mathcal{O}, \rho^y) \quad \text{and} \quad \eta(x) = \eta^y(y).$$

Similarly,

$$M(\mathcal{O}, \chi) \simeq M(\mathcal{O}, \chi^y) \quad \text{and} \quad q = \chi^y(y).$$

We start with an argument going back to [26, 3.1], based on Lemma 1.3.

Lemma 3.2: *Let $W \in \operatorname{Irr} G^x$ be as above. Assume that $\dim W \geq 2$. Let $\mathtt{M} = \Bbbk x \otimes W$, a braided subspace of $M(\mathcal{O}, W)$ with $\dim \mathtt{M} = \dim W$. Then $\mathscr{B}(\mathtt{M})$ is a braided Hopf subalgebra of $\mathscr{B}(\mathcal{O}, W)$ and we have*

- (i) *If $\eta(x) = 1$, then $\mathscr{B}(\mathtt{M}) \simeq S(\mathtt{M})$. Thus $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, W) = \infty$ implies that $\dim W < \infty$.*
- (ii) *If $\eta(x) = -1$, then $\mathscr{B}(\mathtt{M}) \simeq \Lambda(\mathtt{M})$.*
- (iii) *If $\eta(x) \in \mathbb{G}_3'$ and $\dim W = 2$, then $\mathscr{B}(\mathtt{M})$ is of Cartan type A_2 and has dimension 27.*
- (iv) *In any other case, $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, W) = \infty$.*

Proof. Choose $g_x = x$, thus $t_{x,x} = x$, cf. (1.4). Fix a basis $(w_i)_{i \in I}$ of W, so that the symbols xw_i form a basis of $\mathtt{M}$ and its braiding is given by

$$c(xw_i \otimes xw_j) = \eta(x)\, xw_j \otimes xw_i, \quad i, j \in I.$$

Then $\operatorname{GK-dim} \mathscr{B}(\mathtt{M})$ can be read off from Lemma 1.3. ∎

We next generalize [41, 3.5].

Lemma 3.3: *Let G be a finite group of odd order and let $W \in \operatorname{Irr} G^x$ as above.*

- (a) *If $\dim W \geq 2$ and $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, W)$ is finite, then $\eta(x) = 1$ and consequently $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, W) > 0$.*
- (b) *If $x \in Z(G)$ and $\eta(x) = 1$, then $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, W) = \dim W$.*
- (c) *If $\dim \mathscr{B}(x, W) < \infty$, necessarily $\dim W = 1$.*

Proof. (a) and (b) follow from Lemma 3.2: as $|G|$ is odd $\eta(x) \neq -1$, and $\dim W$, a divisor of $|G^x|$, could not be 2. In turn (a) implies (c). ∎

3.2. Nichols algebras with central support. Let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with central support. Then the braided vector space M can be realized in ${}^{\Bbbk Z(G)}_{\Bbbk Z(G)}\mathcal{YD}$, hence it fits into the theory of Nichols algebras over abelian groups sketched at the Introduction. For illustration we describe the semisimple $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of finite length and central support such that $\operatorname{GK-dim} \mathscr{B}(M) < \infty$, up to Conjecture 1.2. See [6, 3] for Nichols algebras of indecomposable M.

Proposition 3.4: *Let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ be semisimple of the form*

$$M \simeq M_1 \oplus \cdots \oplus M_t \oplus M_{t+1} \oplus \cdots \oplus M_\theta, \tag{3.1}$$

where $M_i \simeq M(g_i, W_i)$ with $g_i \in Z(G)$, $W_i \in \operatorname{Irr} G$ and

- *if $i \in \mathbb{I}_t$, then W_i has dimension ≥ 2 and central character η_i;*
- *if $i \in \mathbb{I}_{t+1,\theta}$, then W_i has dimension 1 and action given by $\chi_i \in \widehat{G}$.*

Let $\mathbf{q} = (q_{ij})_{i,j \in \mathbb{I}_{t+1,\theta}}$, where $q_{ij} = \chi_j(g_i)$. Then

$$\operatorname{GK-dim} \mathscr{B}(M) < \infty$$

if and only if the following conditions hold:

(a) *The connected components of the diagram of $\mathbf{q}$ are either points labelled by 1 or else belong to the list of* [29].
(b) *If $i \in \mathbb{I}_t$, then $\eta_i(g_i) \in \mathbb{G}_2 \cup \mathbb{G}_3$.*
(c) *If $i \in \mathbb{I}_t$ and $\eta_i(g_i) = 1$, then $\eta_i(g_j)\eta_j(g_i) = 1$ for all $i \neq j \in \mathbb{I}_t$ and $\eta_i(g_k)\chi_k(g_i) = 1$ for all $k \in \mathbb{I}_{t+1,\theta}$.*
(d) *If $i \neq j \in \mathbb{I}_t$ and $\eta_i(g_i) \neq 1 \neq \eta_j(g_j)$, then $\eta_i(g_j)\eta_j(g_i) = 1$.*
(e) *If $i \in \mathbb{I}_t$, $k \in \mathbb{I}_{t+1,\theta}$ and $\eta_i(g_i) = \omega \in \mathbb{G}'_3$, then $\eta_i(g_k)\chi_k(g_i) = 1$ unless $\{k\}$ is a connected component of $\mathbf{q}$ labelled by -1 and $\eta_i(g_k)\chi_k(g_i) = \omega^2$.*
(f) *If $i \in \mathbb{I}_t$, $k \in \mathbb{I}_{t+1,\theta}$ and $\eta_i(g_i) = -1$, then $\eta_i(g_k)\chi_k(g_i) = 1$ except when $\dim W = 2$ or 3 and the points from $g_i \otimes W$ together with the connected component of k appear in one of the following: rows 1, 8, 15 in Table 2, rows 5, 18 in Table 3, or row 8 in Table 4 from* [29].

Proof. If $\operatorname{GK-dim} \mathscr{B}(M) < \infty$, then (a) follows from [29] assuming Conjecture 1.2; (b), from Lemma 3.2; (c), from Lemma 1.4(iii); (d), from Lemma 1.4(i). Now (e) and (f) follow by inspecting the list in [29]; the exception in (e) is from row 15, Table 2 in [29]. The proof of the converse implication is standard. ∎

3.3. Hopf algebras over torsion-free nilpotent groups. Combining Lemma 2.7 with Theorem 2.6 and Proposition 3.4 we get:

Theorem 3.5: *Let G be a finitely generated torsion-free nilpotent group. Let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ be semisimple of finite length. Then*

$$\operatorname{GK-dim} \mathscr{B}(M)\#\Bbbk G < \infty$$

if and only if

$$\operatorname{supp} M \subset Z(G)$$

and M is as in Proposition 3.4. ∎

As already mentioned, the classification of all $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of finite dimension with $\operatorname{GK-dim} \mathscr{B}(M)\#\Bbbk G < \infty$ would follow from the abelian case once this is settled completely.

3.4. Abelian non-central conjugacy classes. We next study Nichols algebras over finite conjugacy classes that are abelian as racks. We keep the notation from §3.1 and assume from now on that

$$\mathcal{O} \text{ is abelian but not central.}$$

The braided vector space $M(\mathcal{O}, W)$ can be realized in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ where $\Gamma = \langle \mathcal{O} \rangle$ is an abelian subgroup of G, hence the theory of braided vector spaces over abelian groups applies again. In order to give more precisions, we start with some general reductions.

Let $y, z \in \mathcal{O}$ and $g_y, g_z \in G$ such that $g_y \triangleright x = y$, $g_z \triangleright x = z$. Since $\mathcal{O}$ is abelian, we see as in (1.4) that $t_{z,y} = g_y^{-1} z g_y = (g_y^{-1} g_z) \triangleright x$. Now (1.5) says

$$\begin{aligned} c(g_z u \otimes g_y w) &= g_y(t_{z,y} \cdot w) \otimes g_z u, \\ c(g_y w \otimes g_z u) &= g_z(t_{y,z} \cdot u) \otimes g_y w, \end{aligned} \qquad u, w \in W. \tag{3.2}$$

We first consider the case when $\dim W \geq 2$; we elaborate on Lemma 3.2 using a result on pale blocks from [6, §8].

Lemma 3.6: *Let $W \in \operatorname{Irr} G^x$ such that $\dim W \geq 2$ and $\eta(x) \neq -1$. Then $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, W) < \infty$ if and only if $\dim W < \infty$, $t_{z,y}$ acts on W by a scalar $\tau_{z,y}$ such that $\tau_{z,y} = \tau_{y,z}^{-1}$, for any $y \neq z \in \mathcal{O}$, and either*

(i) *$\eta(x) = 1$; then $\mathscr{B}(\mathcal{O}, W) \simeq_{tw} S(W^{|\mathcal{O}|})$, $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, W) = \dim W |\mathcal{O}|$;*

(ii) *or else $\eta(x) \in \mathbb{G}_3'$, $\dim W = 2$; in this case $\dim \mathscr{B}(\mathcal{O}, W) = 27^{|\mathcal{O}|}$.*

Proof. Fix $y \neq z \in \mathcal{O}$. Assume that w is an eigenvector for $t_{z,y}$ with eigenvalue λ and that $u \in W$ is an eigenvector for $t_{y,z}$ with eigenvalue μ.

(a) By (3.2), the 2-dimensional braided vector space spanned by $g_z u$, $g_y w$ is of diagonal type with Dynkin diagram $\overset{\eta(x)}{\circ}\overset{\lambda\mu}{\frac{\qquad\quad}{}}\overset{\eta(x)}{\circ}$.

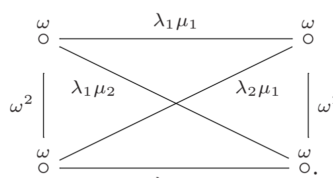
(b) Assume that $\widetilde{u} \in W$ satisfies $t_{y,z} \cdot \widetilde{u} = \mu(\widetilde{u} + u)$. Let $v_1 = g_z u$, $v_2 = g_z \widetilde{u}$ and $v_3 = g_y w$. Then the braided vector space V with basis $(v_i)_{i \in \mathbb{I}_3}$ has braiding given by

$$(c(v_i \otimes v_j))_{i,j \in \mathbb{I}_3} = \begin{pmatrix} \eta(x) v_1 \otimes v_1 & \eta(x) v_2 \otimes v_1 & \lambda v_3 \otimes v_1 \\ \eta(x) v_1 \otimes v_2 & \eta(x) v_2 \otimes v_2 & \lambda v_3 \otimes v_2 \\ \mu v_1 \otimes v_3 & \mu(v_2 + v_1) \otimes v_3 & \eta(x) v_3 \otimes v_3 \end{pmatrix}.$$

Now [6, Theorem 8.1.3] says that $\operatorname{GK-dim} \mathscr{B}(V) < \infty$ if and only if

$$\eta(x) = -1 \quad \text{and} \lambda\mu = \begin{cases} 1, & \text{thus } \operatorname{GK-dim} \mathscr{B}(V) = 1; \text{ or} \\ -1, & \text{thus } \operatorname{GK-dim} \mathscr{B}(V) = 2. \end{cases} \tag{3.3}$$

First we assume that $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, W) < \infty$. Since one of our assumptions is that $\eta(x) \neq -1$, both $t_{y,z}$ and $t_{z,y}$ act diagonally on W by (b).
(i): If $\eta(x) = 1$, then $\dim W < \infty$ by Lemma 3.2. Pick eigenvalues λ, μ as above. Then $\lambda\mu = 1$ by Lemma 1.4 (iii); thus μ and $\lambda = \mu^{-1}$ are uniquely determined, i.e., $t_{z,y}$ and $t_{y,z}$ act by inverse scalars.
(ii): Assume that $\eta(x) =: \omega \in \mathbb{G}_3'$ and $\dim W = 2$. Let λ_1 and λ_2 be the eigenvalues of $t_{y,z}$ acting on W, respectively μ_1 and μ_2 the eigenvalues of $t_{z,y}$. Then the Dynkin diagram of $V := g_z W \oplus g_y W$ has the form



If at least two of the $\lambda_i \mu_j$ are different from 1, then this diagram has either a 3-cycle or a 4-cycle with all vertices equal to ω, so $\operatorname{GK-dim} \mathscr{B}(V) = \infty$, while if three of them are 1, then the fourth also is 1. The converse is clear. ∎

The case when $\dim W \geq 2$ and $\eta(x) = -1$ is still open, see §4.2.

Next we treat the case when $\dim W = 1$, i.e., given by a character χ.

Lemma 3.7: *Assume that $\mathcal{O}$ is an abelian rack and that*

$$\chi((g^{-1} \triangleright x)(g \triangleright x)) = 1 \quad \text{for every } g \in G \backslash G^x. \tag{3.4}$$

Then $M(\mathcal{O}, \chi)$ is a braided vector space of diagonal type whose Dynkin diagram is totally disconnected with all vertices labelled by $q = \chi(x)$. Hence

$$\begin{aligned} &\dim \mathscr{B}(\mathcal{O}, \chi) = N^{|\mathcal{O}|}, \ \text{GK-dim}\, \mathscr{B}(\mathcal{O}, \chi) = 0, && \text{if } q \in \mathbb{G}'_N, \ N \geq 2; \\ &\text{GK-dim}\, \mathscr{B}(\mathcal{O}, \chi) = |\mathcal{O}|, && \text{if } q = 1 \text{ or } q \in \Bbbk \backslash \mathbb{G}_\infty. \end{aligned} \tag{3.5}$$

Proof. By (3.2), we have $c(g_z \otimes g_y) = \chi((g_y^{-1} g_z) \triangleright x) g_y \otimes g_z$, therefore

$$c^2(g_z \otimes g_y) = \chi((g_z^{-1} g_y) \triangleright x) \chi((g_y^{-1} g_z) \triangleright x) g_z \otimes g_y.$$

By (3.4) the Dynkin diagram is totally disconnected and (3.5) follows. ∎

If $\dim W = 1$ but (3.4) does not hold, then we apply an argument generalizing [41, Lemma 3.7]. Given $g \in G$, as $\mathcal{O}_x$ is finite, the set

$$\mathcal{Z}_g^x := \{z_i := g^i \triangleright x : i \in \mathbb{Z}\} \subset \mathcal{O}_x \tag{3.6}$$

is finite; $|\mathcal{Z}_g^x| = 1$ iff $g \in G^x$ and $|\mathcal{Z}_g^x|$ divides $|g|$ when this last is finite.

Lemma 3.8: *Fix $g \in G \backslash G^x$. Assume that*

$$\zeta := \chi(z_{-1} z_1) = \chi((g^{-1} \triangleright x)(g \triangleright x)) \neq 1. \tag{3.7}$$

If GK-dim $\mathscr{B}(\mathcal{O}, \chi) < \infty$, *then*

$$q = \chi(x) \neq 1, \tag{3.8}$$

$$n := |\mathcal{Z}_g^x| \in \{2, 3\}. \tag{3.9}$$

Furthermore:

(i) *If $n = 2$, then either $\zeta = q^{-1}$ or else $\zeta \in \mathbb{G}'_{12}$ and $q = -\zeta^2$.*
(ii) *If $n = 3$, then $q = -1$ and $\zeta \in \mathbb{G}'_3$.*

For this Lemma, we just need $\mathcal{Z}_g^x$ to be an abelian subrack of $\mathcal{O}$.

Proof. Let $\mathbb{J} = \mathbb{I}_{0,n-1}$. In the notation (1.4), choosing $g_{z_i} = g^i$ we have that

$$t_{z_i, z_j} = g_{z_j}^{-1} z_i g_{z_j} = g^{-j} z_i g^j = z_{i-j}, \quad i, j \in \mathbb{J},$$

since $\mathcal{Z}_g^x$ is abelian. Set $v_i = g^i 1 \in M(\mathcal{O}, \chi)$, $V = \langle v_i : i \in \mathbb{J} \rangle$. Then

$$c(v_i \otimes v_j) = \chi(z_{i-j}) v_j \otimes v_i, \quad i, j \in \mathbb{J}. \tag{3.10}$$

Thus the Dynkin diagram of V has locally the form

$$\cdots \underset{i}{\overset{q}{\circ}} \xrightarrow{\quad \chi(z_{-1}z_1) \quad} \underset{i+1}{\overset{q}{\circ}} \cdots$$

By Lemma 1.4 (iii), $q \neq 1$. If $n = 2$, then Remark 1.6 applies. If $2 < n < \infty$, then this is an n-cycle by (3.7). Thus Lemma 1.4 and Remark 1.5 imply that $n = 3$ and (ii). ∎

Assume that G is finite; hence $n = |\mathcal{Z}_g^x|$ divides $|G|$. If G has odd order, then (i) and (ii) could not happen and we have the following consequence.

Lemma 3.9: *Assume that G is a finite group of odd order and that $\mathcal{O}$ is an abelian rack. Then* $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, \chi) < \infty$ *if and only if* (3.4) *holds. When this happens,* $\dim \mathscr{B}(\mathcal{O}, \chi) < \infty$ *if and only if* $q \neq 1$.

Proof. If (3.4) does not hold, then $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, \chi) = \infty$ by Lemma 3.8. If (3.4) holds, then $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, \chi) < \infty$ by Lemma 3.7. The last claim follows from (3.5) since G is finite. ∎

Example 3.10: If $\mathcal{O}$ is an abelian rack, then $\mathcal{O} \subset G^x$. In this case, the extra assumption $\mathcal{O} \subset \ker \chi$ implies (3.4) and $q = 1$. Thus $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, \chi) = |\mathcal{O}|$.

3.5. Nilpotent groups of odd order. We are now ready to determine the finite-dimensional Nichols algebras over a finite nilpotent group of odd order in terms of its group structure and representation theory.

Theorem 3.11: *Let G be a finite nilpotent group of odd order. Given a finite-dimensional $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, we have that* $\dim \mathscr{B}(M) < \infty$ *if and only if* $M \simeq M_0 \oplus M_1 \oplus \cdots \oplus M_t$ *where:*

(i) $\operatorname{supp} M_0 \subseteq Z(G)$, *hence M_0 is given by a family of YD-pairs $(g_i, \chi_i)_{i \in J}$ such that the connected components of the matrix $\mathbf{q} = (q_{ij})_{i,j \in J}$ belong to the list in* [29].

(ii) *For $j \in \mathbb{I}_t$, $M_j \simeq M(\mathcal{O}_j, \chi_j)$ where $\mathcal{O}_j$ is not central and abelian as rack; $\chi_j \in \widehat{G^{x_j}}$ for a fixed $x_j \in \mathcal{O}_j$ that satisfies* (3.4)*; and $q_j := \chi_j(x_j)$ has order $2 < N_j < \infty$. Also* $\dim \mathscr{B}(\mathcal{O}_j, \chi_j) = N_j^{|\mathcal{O}_j|}$.

Furthermore,

$$c^2_{|M_i \otimes M_j} = \mathrm{id}_{|M_i \otimes M_j}, \quad i \neq j \in \mathbb{I}_{0,t}. \tag{3.11}$$

Proof. Since ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is semisimple, we may decompose $M = \bigoplus_{i\in I} M_i$ where the M_i's are simple. Let

$$J = \{i \in I : \operatorname{supp} M_i \subseteq Z(G)\}$$

and $M_0 = \bigoplus_{i\in J} M_i$. If $\operatorname{supp} M_i$ is central, then $M_i \simeq M(\{g_i\}, W)$ where $g_i \in Z(G)$; by Lemma 3.3(c), $\dim W = 1$. Thus there is a family of YD-pairs (g_i, χ_i) such that $M_i \simeq \Bbbk_{g_i}^{\chi_i}$ for all $i \in J$. By [29], $\dim \mathscr{B}(M_0) < \infty$ if and only if (i) holds.

Assume that $M \simeq M(\mathcal{O}, W)$ for some non-central conjugacy class $\mathcal{O}$. By Theorem 2.1 $\mathcal{O}$ is either of type C, or is an abelian rack. In the first case $\dim \mathscr{B}(M) = \infty$ by Theorem 1.12. Assume that $\mathcal{O}$ is abelian and $\dim \mathscr{B}(M) < \infty$; fix $x \in \mathcal{O}$. By Lemma 3.3, $\dim W = 1$, so W is given by $\chi \in \widehat{G^x}$. By Lemma 3.9, (3.4) should hold. Conversely, if $\mathcal{O}$ is abelian, W is given by $\chi \in \widehat{G^x}$ and (3.4) holds, then $\dim \mathscr{B}(M) < \infty$ by Lemma 3.9. Up to renumbering $I \backslash J$, we have (ii).

Finally, assume that $M', M'' \in \operatorname{Irr} {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ satisfy $\dim \mathscr{B}(M' \oplus M'') < \infty$, where $M' \simeq M(\mathcal{O}, \chi)$ and $\mathcal{O}$ is a non-central conjugacy class. By (ii), M' is of diagonal type. We may replace G by the subgroup generated by $\operatorname{supp} M' \cup \operatorname{supp} M''$. Clearly $\dim M'$ is odd and > 1. If (3.11) does not hold, then $\dim M'$ should be 3 by Theorem 1.11, but then M' is not of diagonal type, a contradiction. ∎

Algorithm 3.12: Let G be a finite nilpotent group of odd order. By Theorem 3.11, to list all Nichols algebras in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ we should do the following.

(i) Compute $Z(G)$, $[G,G]$ and $\widehat{G} = \widehat{[G,G]}$. Thus we have all YD-pairs $(g,\chi) \in Z(G) \times \widehat{G}$.

(ii) For any braided vector space $(V, c^{\mathbf{q}})$ either of Cartan type or of diagonal type (1.2) or else (1.3), compute all principal realizations over G; see Definition 1.7 and the subsequent discussion.

(iii) Compute the set of abelian conjugacy classes

$$\operatorname{Cl}_{\mathrm{ab}}(G) = \{\mathcal{O} \text{ conjugacy class of } G : [\mathcal{O},\mathcal{O}] = e,\ \mathcal{O} \not\subset Z(G)\}. \tag{3.12}$$

Given $\mathcal{O} \in \operatorname{Cl}_{\mathrm{ab}}(G)$, pick $x \in \mathcal{O}$ and compute G^x and $\widehat{G^x}$. Thus the set of pairs (with an evident abuse of notation)

$$\{(\mathcal{O},\chi) : \mathcal{O} \in \operatorname{Cl}_{\mathrm{ab}}(G), \chi \in \widehat{G^x} \text{ satisfies (3.4) and } \chi(x) \neq 1\} \tag{3.13}$$

parametrizes the finite-dimensional Nichols algebras in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of irreducible objects with abelian non-central support.

(iv) Compute all pairs $(\mathcal{O},\chi),(\mathcal{O}',\chi')$ as in (3.13) such that

$$[\mathcal{O},\mathcal{O}'] = e,$$
$$\chi((g_z')^{-1} \triangleright y)\chi'((g_y)^{-1} \triangleright z) = 1, \qquad \forall y \in \mathcal{O}, z \in \mathcal{O}',$$

where $x \in \mathcal{O}$, $x' \in \mathcal{O}'$ and $g_y, g_z' \in G$ satisfy

$$g_y \triangleright x = y, \quad g_z' \triangleright x' = z.$$

(v) Similarly for $(\mathcal{O},\chi)$ as in (3.13) and $(g',\chi') \in Z(G) \times \widehat{G}$.

3.6. Nilpotent groups of odd order, II. Let G be a finite nilpotent group of odd order. We extend the discussion in the previous subsection to determine all $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ such that $\operatorname{GK-dim} \mathscr{B}(M) < \infty$. To have a complete picture we still need Conjectures 1.2 and 1.13, and an analogue of Theorem 1.11. Here are the necessary steps:

1: $\operatorname{supp} M \subset Z(G)$.

Since every object in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is semisimple, Proposition 3.4 gives a complete picture, up to Conjecture 1.2.

2: $M \simeq M(\mathcal{O}, W)$ *where* $\mathcal{O}$ *is not central.*

By Theorem 2.1, $\mathcal{O}$ is either abelian or of type C. To discard type C, we need the validity of Conjecture 1.13.

Assume that $\mathcal{O}$ is abelian; fix $x \in \mathcal{O}$. If $\dim W > 1$, then $\eta(x) = 1$ and $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, W) = \dim W|\mathcal{O}|$ by Lemma 3.6. If $\dim W = 1$, then (3.4) should hold and then $\operatorname{GK-dim} \mathscr{B}(M) < \infty$, given by (3.4); see Lemma 3.9.

3: *Braidings between* $M(\mathcal{O}, W)$ *with* $\mathcal{O}$ *not central and other summands.*

We guess that (3.11) holds; this would need an analogue of Theorem 1.11 for finite GK-dim. Clearly this would be related to Conjecture 1.13.

3.7. Finitely generated nilpotent groups whose torsion has order coprime to 6. Let G be a finitely generated nilpotent group with torsion subgroup T, let $\mathcal{O}$ be a finite abelian conjugacy class and $x \in \mathcal{O}$. Assume that $g \in G \backslash G^x$ satisfies (3.7); recall the set $\mathcal{Z}^x_g$ from (3.6).

Lemma 3.13: *If $p = |\mathcal{Z}_g^x|$ is prime, then it divides $|T|$.*

Proof. Let $K = \langle g, G^x\rangle$. As $g^p \in G^x$, $[K : G^x] \in p^{\mathbb{N}}$ by Theorem 2.3, thus p divides $|\mathcal{O}| = [G : G^x] = [G : K][K : G^x]$. Then Lemma 2.10 applies. ∎

Assume now that $|T|$ is coprime to 6. In particular $|T|$ is odd, hence any finite conjugacy class is either abelian or of type C by Proposition 2.9.

Lemma 3.14: *Let $\chi \in \widehat{G^x}$. Then* GK-dim $\mathscr{B}(\mathcal{O}, \chi) < \infty$ *if and only if* (3.4) *holds, in which case $\mathscr{B}(\mathcal{O}, \chi)$ is given by* (3.5).

Proof. If (3.4) does not hold, then GK-dim $\mathscr{B}(\mathcal{O}, \chi) = \infty$ by Lemmas 3.8 and 3.13 (that excludes 2 and 3 by the hypothesis). The converse is clear. ∎

In summary, if G is a finitely generated nilpotent group with torsion T with $|T|$ coprime to 6, and assuming Conjecture 1.13, then the

$$M = M(\mathcal{O}, W) \in \operatorname{Irr} {}^G_G\mathcal{YD}$$

such that GK-dim $\mathscr{B}(M) < \infty$ are either covered by Proposition 3.4, Lemma 3.6 and Lemma 3.14, or else $\mathcal{O}$ is abelian non-central, $\dim W \geq 2$ and $\eta(x) = -1$. For these last, see the discussion in §4.2. Once this is settled, the determination of the semisimple M such that GK-dim $\mathscr{B}(M) < \infty$ can be obtained softly.

3.8. Examples.

3.8.1. *Class 2.* Let G be a finite nilpotent group of odd order. We discuss how to prove the following result from [41] by means of Theorem 3.11: If $[G, G] = Z(G)$, then $\dim \mathscr{B}(M) = \infty$ for any $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\backslash 0$. Equivalently any finite-dimensional pointed Hopf algebra over G is isomorphic to $\Bbbk G$.

The contention $[G, G] \supset Z(G)$ implies that $\dim \mathscr{B}(M) = \infty$ for any $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with central support. By Theorem 3.11, we are reduced to prove:

Claim: *Let $x \in G$ such that $\mathcal{O}_x$ is non-central abelian and let $\chi \in \widehat{G^x}$ with $\chi(x) \neq 1$. Then there exists $g \in G\backslash G^x$ such that $\chi((g^{-1} \triangleright x)(g \triangleright x)) \neq 1$.*

First, let Γ be a group, $x \in \Gamma$ and $g \in N_\Gamma(\Gamma^x)$, such that $[g, [g, x]] = 1$. Then $(g^{-1} \triangleright x)(g \triangleright x) = x^2$. Indeed, clearly $[g, x]^{-1} = [g^{-1}, x]$, hence

$$(g^{-1} \triangleright x)(g \triangleright x) = [g^{-1}, x]x[g, x]x = [g, x]^{-1}[g, x]x^2 = x^2.$$

Now $N_G(G^x) \neq G^x$ since G is nilpotent and any $g \in N_G(G^x)\backslash G^x$ satisfies $[g, [g, x]] = 1$ because $[G, G] \subset Z(G)$.

3.8.2. *Heisenberger groups.* Let K be a commutative ring and $n \in \mathbb{N}$. We consider the Heisenberg group $\mathtt{H} = \mathtt{H}_{2n+1}(K)$ that consists of the matrices

$$\begin{pmatrix} 1 & a_1 & a_2 & \dots & a_n & c \\ 0 & 1 & 0 & \dots & 0 & b_1 \\ 0 & 0 & 1 & \dots & 0 & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 1 & b_n \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in GL_{n+2}(K). \tag{3.14}$$

For simplicity, we denote $\mathbf{a} := (a_1, a_2, \dots, a_n)$, $\mathbf{b} := (b_1, b_2, \dots, b_n)$ and the matrix (3.14) by $(\mathbf{a}, \mathbf{b}, c)$. Let $\omega : K^{2n} \times K^{2n} \to K$ be the 'symplectic form'

$$\omega((\mathbf{a}, \mathbf{b}), (\mathbf{r}, \mathbf{s})) = \sum_{1 \le i \le n} (a_i s_i - b_i r_i), \quad \mathbf{a}, \mathbf{b}, \mathbf{r}, \mathbf{s} \in K^n.$$

Then

$$(\mathbf{a}, \mathbf{b}, c) \triangleright (\mathbf{r}, \mathbf{s}, t) = (\mathbf{r}, \mathbf{s}, t + \omega((\mathbf{a}, \mathbf{b}), (\mathbf{r}, \mathbf{s}))).$$

Given $\mathbf{r}, \mathbf{s} \in K^n$, let $(\mathbf{r}, \mathbf{s})^\perp$ be the K-submodule of K^{2n} of those $(\mathbf{a}, \mathbf{b})$ such that $\omega((\mathbf{a}, \mathbf{b}), (\mathbf{r}, \mathbf{s})) = 0$; and let $\langle \mathbf{r}, \mathbf{s} \rangle$ be the ideal of K generated by r_i, s_i, $i \in \mathbb{I}_n$. Clearly $Z(\mathtt{H}) = 0 \times 0 \times K$ and the conjugacy class of $(\mathbf{r}, \mathbf{s}, t)$ is

$$\mathcal{O}_{(\mathbf{r}, \mathbf{s}, t)} := \{(\mathbf{r}, \mathbf{s}, t + \ell) : \ell \in \langle \mathbf{r}, \mathbf{s} \rangle\}.$$

Fix a non-central class $\mathcal{O}$ and $(\mathbf{r}, \mathbf{s}, t) \in \mathcal{O}$, where $(\mathbf{r}, \mathbf{s}) \neq (0, 0)$. Given $\ell \in \langle \mathbf{r}, \mathbf{s} \rangle$ pick $(\mathbf{a}, \mathbf{b})$ such that $\omega((\mathbf{a}, \mathbf{b}), (\mathbf{r}, \mathbf{s})) = \ell$ and set

$$x_\ell = (\mathbf{r}, \mathbf{s}, t + \ell) \in \mathcal{O}, \quad g_\ell = (\mathbf{a}, \mathbf{b}, 0),$$

so that $g_\ell \triangleright x_0 = x_\ell$. Then for $m \in \langle \mathbf{r}, \mathbf{s} \rangle$, the element (1.4) is given by

$$t_{m,\ell} := g_\ell^{-1} x_m g_\ell = \Big(-\mathbf{a}, -\mathbf{b}, \sum_i a_i b_i \Big) (\mathbf{r}, \mathbf{s}, t + m)(\mathbf{a}, \mathbf{b}, 0) = (\mathbf{r}, \mathbf{s}, t + m - \ell).$$

The centralizer of $(\mathbf{r}, \mathbf{s}, t)$ is $\mathtt{H}^{(\mathbf{r}, \mathbf{s}, t)} \simeq (\mathbf{r}, \mathbf{s})^\perp \times K$, which is abelian, so

$$\widehat{\mathtt{H}^{(\mathbf{r}, \mathbf{s}, t)}} \simeq \widehat{(\mathbf{r}, \mathbf{s})^\perp} \times \widehat{K}$$

parametrizes $\operatorname{Irr} \mathtt{H}^{(\mathbf{r}, \mathbf{s}, t)}$. Let $\chi = (\chi_1, \chi_2)$ be such a character and

$$q = \chi_1(\mathbf{r}, \mathbf{s}) \chi_2(t).$$

Then the braiding of $M(\mathcal{O}, \chi)$ is given by

$$c(g_m \otimes g_\ell) = \chi_1(\mathbf{r}, \mathbf{s}) \chi_2(t) \chi_2(m - \ell) g_\ell \otimes g_m, \quad m, \ell \in \langle \mathbf{r}, \mathbf{s} \rangle. \tag{3.15}$$

That is, $M(\mathcal{O},\chi)$ is of diagonal type with matrix twist-equivalent to $\mathbf{q}=(q_{ij})_{i,j\in\mathbb{I}}$, $q_{ij}=q$ for all i,j. Clearly $\dim M(\mathcal{O},\chi)=|\langle\mathbf{r},\mathbf{s}\rangle|$. By Lemma 1.3,

$$\operatorname{GK-dim}\mathscr{B}(\mathcal{O},\chi) = \begin{cases} |\langle\mathbf{r},\mathbf{s}\rangle|, & \mathscr{B}(\mathcal{O},\chi)\simeq_{\text{tw}} S(M(\mathcal{O},\chi)) \quad \text{if } \chi_1(\mathbf{r},\mathbf{s})=\chi_2(t)^{-1};\\ 0, & \mathscr{B}(\mathcal{O},\chi)\simeq_{\text{tw}} \Lambda(M(\mathcal{O},\chi)) \quad \text{if } \chi_1(\mathbf{r},\mathbf{s})=-\chi_2(t)^{-1};\\ 0, & \dim\mathscr{B}(x,\chi)=27 \quad \text{if } \chi_1(\mathbf{r},\mathbf{s})\chi_2(t)\in\mathbb{G}_3' \text{ and } |\langle\mathbf{r},\mathbf{s}\rangle|=2;\\ \infty & \text{in any other case.}\end{cases}$$

The first two lines correspond to the situation in Lemma 3.7 while the third one is covered by Lemma 3.8 (i); this last case occurs, e.g., with $K=\mathbb{Z}/6$, $n=1$, $\mathbf{r}=(3,0)=\mathbf{s}$, $t=\overline{2}$, χ_1 trivial and χ_2 non-trivial.

Another family of examples arises taking an ideal I of K and the quotient

$$\widetilde{\mathtt{H}} := \mathtt{H}_{2n+1}(K)/0\times 0\times I \simeq K^{2n}\times K/I.$$

Clearly $Z(\widetilde{\mathtt{H}})=I^{2n}\times K/I$. Let $\pi: K\to K/I$, $t\mapsto\overline{t}$ be the natural projection and let $\overline{\omega}: K^{2n}\times K^{2n}\to K/I$,

$$\overline{\omega}=\pi\omega.$$

If $(\mathbf{r},\mathbf{s})\in K^{2n}$, and $t\in K$, then

$$\widetilde{\mathtt{H}}^{(\mathbf{r},\mathbf{s},\overline{t})}=(\mathbf{r},\mathbf{s})^{\perp,I}\times K/I;\quad \mathcal{O}_{(\mathbf{r},\mathbf{s},\overline{t})}=(\mathbf{r},\mathbf{s},\overline{t+\langle\mathbf{r},\mathbf{s}\rangle}),$$

where

$$(\mathbf{r},\mathbf{s})^{\perp,I}=\{(\mathbf{a},\mathbf{b})\in K^{2n}:\omega((\mathbf{a},\mathbf{b}),(\mathbf{r},\mathbf{s}))\in I\}.$$

Let $\chi=(\chi_1,\chi_2)\in\widehat{(\mathbf{r},\mathbf{s})^{\perp,I}}\times\widehat{K/I}$ and $q=\chi_1(\mathbf{r},\mathbf{s})\chi_2(t)$. Then $M(\mathcal{O},\chi)$ is of diagonal type with matrix twist-equivalent to $\mathbf{q}=(q_{ij})_{i,j\in\mathbb{I}}$, $q_{ij}=q$ for all i,j and $\mathscr{B}(\mathcal{O},\chi)$ is determined by a similar analysis.

Suppose that $K=\mathbb{Z}$ and $I=\mathbb{Z}/N$ where $N\geq 2$; then $\mathtt{H}$ is an FC-group with torsion $\simeq\mathbb{Z}/N$. Assume further that $N=2d$ is even, $n=1$, $\mathbf{r}=(d,0)\neq\mathbf{s}=(0,0)$, $t=\overline{1}$; thus

$$(\mathbf{r},\mathbf{s})^{\perp,I}\simeq\frac{N}{(d,N)}\mathbb{Z}\times\mathbb{Z}^3\simeq\mathbb{Z}^4.$$

Then $|\overline{\langle\mathbf{r},\mathbf{s}\rangle}|=2$ and $q=\chi_1(\mathbf{r},\mathbf{s})\chi_2(t)$ might be in $\mathbb{G}_3'$ even if N is coprime to 3.

3.8.3. *Unitriangular groups.* We show examples of classes of type C. Let K be a commutative ring and $\mathtt{G} = \mathtt{UT}_4(K)$, consisting of the matrices

$$\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL_4(K). \tag{3.16}$$

For simplicity, we denote matrices (3.16) by

$$\mathbf{a} := \begin{pmatrix} a_{12} & & a_{23} & & a_{34} \\ & a_{13} & & a_{24} & \\ & & a_{14} & & \end{pmatrix},$$

etc. Then

$$\mathbf{a} \triangleright \mathbf{b} = \begin{pmatrix} b_{12} & b_{23} & b_{34} \\ b_{13} + a_{12}b_{23} - b_{12}a_{23} & & b_{24} + a_{23}b_{34} - b_{23}a_{34} \\ b_{14} + b_{12}(a_{23}a_{34} - a_{24}) - b_{13}a_{34} - b_{23}a_{12}a_{34} + b_{24}a_{12} + b_{34}a_{13} & & \end{pmatrix}.$$

Let

$$\mathbf{r} := \begin{pmatrix} r_{12} & & r_{23} & & r_{34} \\ & 0 & & 0 & \\ & & 0 & & \end{pmatrix}$$

where $r_{12}, r_{23}, r_{34} \in \mathcal{U}(K)$ (the group of units of K). Then

$$\mathcal{O}_{\mathbf{r}} = \left\{ \begin{pmatrix} r_{12} & & r_{23} & & r_{34} \\ & c_{13} & & c_{24} & \\ & & c_{14} & & \end{pmatrix} : c_{13}, c_{24}, c_{14} \in K \right\}.$$

Then $\mathcal{O}_{\mathbf{r}}$ is not an abelian, e.g.,

$$\mathbf{s} := \begin{pmatrix} r_{12} & & r_{23} & & r_{34} \\ & 1 & & 0 & \\ & & 0 & & \end{pmatrix}$$

satisfies

$$[\mathbf{r}, \mathbf{s}] = \begin{pmatrix} 0 & & 0 & & 0 \\ & 0 & & 0 & \\ & & -r_{34} & & \end{pmatrix}.$$

Hence $H := \langle \mathbf{r}, \mathbf{s} \rangle \simeq \mathtt{H}_3(K)$ via $(1, 0, 0) \mapsto \mathbf{r}$, $(0, 1, 0) \mapsto \mathbf{s}$, and $\mathcal{O}_{\mathbf{r}}^H \cap \mathcal{O}_{\mathbf{s}}^H = \emptyset$. Thus, if K is finite and $|K|$ is odd, then $\mathcal{O}_{\mathbf{r}}$ is of type C.

4. Conclusions

We discuss the problems that remain open as well as some applications.

4.1. Representations. In the previous sections we have argued assuming some information about the representations and conjugacy classes of finitely generated nilpotent groups. But the representation theory of such groups is not completely known to our knowledge. Let $H \leq G$ and $W \in \operatorname{Rep} H$. Then $\operatorname{Ind}_H^G W = \Bbbk G \otimes_{\Bbbk H} W$ is called the induced representation of W. We need two definitions. Let $\pi : G \to GL(V)$ be a representation.

- π is **monomial** if there are $H \leq G$ and $\chi \in \widehat{H}$ such that $\pi \simeq \operatorname{Ind}_H^G \chi$.
- π has **finite weight** if there are $K \leq G$ and $\eta \in \widehat{K}$ such that

$$V_{K,\eta} := \{v \in V : \pi(k)(v) = \eta(k)v \ \forall k \in K\}$$

is non-zero and finite-dimensional. Generalizing classical results of Dixmier, Kirillov and Brown, Parshin [35] conjectured the following result.

Theorem 4.1 ([19]): *Let G be a finitely generated nilpotent group. An irreducible representation of G is monomial iff it has a finite weight.*

Thus finite-dimensional irreducible representations of G are monomial, but this was already known [20, Lemma 1].

4.2. Finite nilpotent groups of even order. In order to deal with Nichols algebras over a finite nilpotent group of even order with finite dimension or finite GK-dim extending Theorem 3.11 (see also Section 3.6), the following points need to be addressed:

- Conjugacy classes: It suffices to consider 2-groups. We need to keep track of the conjugacy classes that are neither abelian nor of type C; the information from [31, 32] would be crucial.
- Irreducible Yetter–Drinfeld modules, $\dim W \geq 2$: In the setting of Lemma 3.6 we still have to consider the case $\eta(x) = -1$. We shall need:

Lemma 4.2: *Let Γ be an abelian group and $V = V_g \oplus V_h \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ such that $\operatorname{GK-dim} \mathscr{B}(V) < \infty$, where $g \neq h$. Then the action of h on V_g is locally finite.* ∎

Therefore we may consider eigenvectors and blocks; see the proof of Lemma 3.6. Since the classification of the Nichols algebras of blocks & pale blocks & points with finite GK-dim is not yet finished, we could not carry out a complete analysis that eventually would be straightforward.

- ◦ Irreducible Yetter–Drinfeld modules, $\dim W = 1$: We need to explore further the possibilities $|\mathcal{Z}_g^x| \in \{2, 3\}$ in Lemma 3.8, together with other restrictions.
- ◦ Braiding between simple modules: The condition (3.11) has to be adapted, cf. Theorem 1.11; in the even case it does not follow from finite dimension.

4.3. Finitely generated nilpotent groups with even torsion. The strategy would be parallel to the one in the previous Subsection.

4.4. Finitely generated nilpotent-by-finite groups. Assume that G is nilpotent-by-finite and fix $N \lhd G$ a normal nilpotent subgroup of finite index. Then the image $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ in G/N is a conjugacy class. As said, the knowledge of the Nichols algebras over finite groups is still incomplete, but if $\widetilde{\mathcal{O}}$ is of type C, D or F, then so would be $\mathcal{O}$ and then any Nichols algebra over G with support $\mathcal{O}$ would have infinite dimension, and infinite GK-dim, if Conjecture 1.13 and its analogues for types D and F are true. See [8] and references therein.

4.5. Applications. Once all the Nichols algebras over G with finite GK-dim or finite dimension are known, one still needs to (i) compute all post-Nichols algebras with finite GK-dim and (ii) compute all liftings. For (i), when the braided vector spaces come from the abelian setting, we know: a finite-dimensional Nichols algebra does not have post-Nichols algebras with finite dimension (except itself) [13]. For finite GK-dim see [10, 15]. As for (ii), see [16]. Finally observe that when G is torsion free, our results contribute to the classification of pointed Hopf algebras with finite GK-dim that are domains.

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