

POLYNOMIAL GROWTH AND ASYMPTOTIC DIMENSION

BY

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ABSTRACT

Bonamy et al. [4] showed that graphs of polynomial growth have finite asymptotic dimension. We refine their result showing that a graph of polynomial growth strictly less than n^{k+1} has asymptotic dimension at most k . As a corollary Riemannian manifolds of bounded geometry and polynomial growth strictly less than n^{k+1} have asymptotic dimension at most k .

We show also that there are graphs of growth $< n^{1+\epsilon}$ for any $\epsilon > 0$ and infinite asymptotic Assouad–Nagata dimension.

1. Introduction

Asymptotic dimension is a large scale analog of topological dimension that was introduced by Gromov [12]. It is invariant under quasi-isometries and even stronger under coarse embeddings, so one can think of it as a large scale topological notion (see [2] for an introduction to the subject).

Asymptotic dimension is relevant in several contexts: in geometric group theory, as groups of finite asymptotic dimension satisfy the Novikov conjecture [25], in geometry [17], [7] and in graph theory [21], [11],[4].

The **asymptotic dimension** $\text{asdim } X$ of a metric space X is defined as follows: $\text{asdim } X \leq n$ if and only if for every $m > 0$ there exists $D(m) > 0$ and a covering \mathcal{U} of X by sets of diameter $\leq D(m)$ ($D(m)$ -bounded sets) such

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that any m -ball in X intersects at most $n + 1$ elements of \mathcal{U} . We say that the m -multiplicity of the cover \mathcal{U} is at most $n + 1$. If $D(m)$ in the definition above is a linear function of m , then we say that the **asymptotic Assouad–Nagata dimension** of X is bounded by n ([8], [5]).

Špakula and Tikuisis ask in [23] (see the footnote in page 1021) whether spaces of polynomial growth have finite asymptotic dimension (and apparently as this is a quite natural question, it was considered by other people as well).

A different notion of dimension was considered earlier by Linial, London and Rabinovich [19], and Linial [18], namely one defines the dimension of a graph G to be the smallest n for which there is an embedding $f : G \rightarrow \mathbb{R}^n$ so that $d(f(u), f(v)) \geq 1$ for $u \neq v$ and for some $c > 0$, $d(f(u), f(v)) \leq c$ if u, v are adjacent. Krauthgamer and Lee [16] showed that graphs of polynomial growth $\gamma(r) \leq Cr^k$ embed in this sense in $\mathbb{R}^{O(k \log k)}$.

Using the result of [16] Bonamy et al. prove in [4] that graphs of polynomial growth $\leq Cr^k$ have asymptotic dimension bounded by $O(k \log k)$ answering the question of Špakula–Tikuisis. It is further shown that graphs of superpolynomial growth can have infinite asymptotic dimension.

Benjamini and Georgakopoulos [3] show that planar triangulations of subquadratic growth are quasi-isometric to trees (and so they have $\text{asdim} = 1$).

From the geometric point of view it is interesting to calculate the exact asymptotic dimension of a space. This has been accomplished for several ‘natural’ classes of spaces: It is shown in [7] that the asymptotic dimension of a hyperbolic group G is equal to $\dim(\partial G) + 1$, and in [12] that n -dimensional Hadamard manifolds of pinched negative curvature have asymptotic dimension n (see also [17] for a detailed proof and an extension of this to asymptotic Assouad–Nagata dimension). It is shown in [11], [15],[4] that planar graphs (or more generally planar geodesic metric spaces) have asymptotic dimension at most 2. In this paper we extend this list to the class of spaces with polynomial growth. Note that spaces with polynomial growth appear in several settings. For example, doubling spaces have polynomial growth [14] and manifolds of non-negative Ricci curvature have polynomial volume growth [13]. It is shown in [17] that doubling metric spaces have finite asymptotic dimension (in fact also finite Nagata dimension) and a sharp bound of their Nagata dimension (hence also asymptotic dimension) in terms of Assouad dimension is given in [10]. Tessera in [24] studies geometric properties of general graphs of polynomial growth.

We state now our results. We view a connected graph as a geodesic metric space where each edge has length 1.

Definition 1.1: We define the **growth function** of a connected graph $G = (V, E)$ to be

$$\gamma(r) = \sup\{|B_v(r)| : v \in V\},$$

where we denote above by $|X|$ the number of vertices of a subset X of G . Due to the independence from a base vertex, some authors call $\gamma(r)$ the uniform growth function.

We prove the following:

THEOREM 2.4: *Let $G = (V, E)$ be a connected graph with growth function $\gamma(r)$ satisfying*

$$\lim_{r \rightarrow \infty} \frac{\gamma(r)}{r^{k+1}} = 0 \quad \text{for some } k \in \mathbb{N}.$$

Then $\text{asdim } G \leq k$.

As a corollary we have:

COROLLARY 3.3: *If M^n is a Riemannian manifold of bounded geometry and volume growth function $\text{Vol}(r)$ satisfying*

$$\lim_{r \rightarrow \infty} \frac{\text{Vol}(r)}{r^{k+1}} = 0 \quad \text{for some } k \in \mathbb{N},$$

then $\text{asdim } M^n \leq k$.

We define the volume growth function for M^n as for graphs

$$\text{Vol}(r) = \sup\{\text{Vol}(B_x(r)) : x \in M^n\}.$$

We say that M^n is of bounded geometry if there are $a > 0, b > 1$ such that for any open ball of radius a in M^n there is a b -bilipschitz map to a Euclidean open ball of radius 1. Recall that a map $f : X \rightarrow Y$ is b -bilipschitz if it is onto and

$$1/b d(x, y) \leq d(f(x), f(y)) \leq b d(x, y) \quad \text{for all } x, y \in X.$$

We remark that other common definitions of bounded geometry for non-compact manifolds imply ours.

It turns out that Theorem 2.4 applies more generally to metric spaces for an appropriate definition of volume (see section 3). However, one does not have a similar bound for the asymptotic Assouad–Nagata dimension. We have:

THEOREM 3.5: *There is a metric space X with growth function satisfying*

$$\lim_{r \rightarrow \infty} \frac{\gamma(r)}{r} = 0$$

and infinite asymptotic Assouad–Nagata dimension.

There is a graph (G, E) with growth function $\gamma(r)$ satisfying

$$\lim_{r \rightarrow \infty} \frac{\gamma(r)}{r^{1+\epsilon}} = 0$$

for any $\epsilon > 0$ and infinite asymptotic Assouad–Nagata dimension.

We note it follows from this theorem that the bound on the asymptotic dimension in terms of Assouad dimension implied by [10] is far from optimal. Since the Assouad dimension bounds the Nagata dimension, for the graphs of the theorem the Assouad dimension is infinite while the asymptotic dimension is equal to 1.

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2. Graphs of polynomial growth

Definition 2.1: Let G be a metric space and $r > 0$. We say that $X \subset G$ is **r -scale connected** if for any $x, y \in X$ there is a sequence $x_1 = x, x_2, \dots, x_n = y$ in X such that $d(x_i, x_{i+1}) \leq r$ for all $i = 1, \dots, n - 1$. If $A \subset X$ we say that A is an **r -connected component** of X if A is a maximal r -scale connected subset of X .

So, for example, if $G = (V, E)$ is a connected graph, V has a single 1-connected component (equal to itself) and each singleton $\{v\}$ is a $1/2$ -connected component of V .

Definition 2.2: Let G be a metric space and $r > 0$. We say that $X \subset G$ has r -dim $X \leq n$ if there is a $D > 0$ and a cover \mathcal{U} of X by sets of diameter $\leq D$ such that any r -ball intersects at most $n + 1$ elements of \mathcal{U} .

Clearly $\text{asdim } G \leq n$ if r -dim $G \leq n$ for all r .

Definition 2.3: Let $G = (V, E)$ be a connected graph and let $X \subset G$. We define the **growth function** of X to be

$$\gamma_X(t) = \sup\{|B_v(t) \cap X| : v \in V\}.$$

LEMMA 2.1: *Let $G = (V, E)$ be a connected graph and let $X \subset G$. If there is some t so that $\gamma_X(t) < \frac{1}{2r}t$ then $r\text{-dim } X = 0$.*

Proof. It suffices to show that the r -connected components of X are bounded. By hypothesis there is a t such that $\gamma_X(t) < t/2r$. It follows that any r -connected component of X has diameter bounded by t . ■

Definition 2.4: Let $G = (V, E)$ be a connected graph and let $Y \subset V$. We say that Y is **(D, r) -separating** if all r -connected components of $V \setminus Y$ have diameter bounded by D .

More generally, if $X \subset V$ and $Y \subset X$ we say that that Y is a **(D, r) -separating** subset of X if all r -connected components of $X \setminus Y$ have diameter bounded by D .

Before proceeding to give the formal proof of Theorem 2.4 we give here a rough sketch. We argue by induction where the case $n = 0$ is done in Lemma 2.1. Our aim is to show that a set of vertices $X \subset V$ that has growth $\lesssim t^k$, has $r\text{-dim } X \leq k$. In order to do this it suffices to find a (D, r) -separating subset $Y \subset X$ of growth $\lesssim t^{k-1}$ (for an appropriately chosen $D \gg r$) and then apply induction. Indeed by induction we have $r\text{-dim } Y \leq k - 1$, and using this and the r -connected components of $X \setminus Y$ one produces a cover showing that $r\text{-dim } X \leq k$. So the problem is reduced to showing that there is such a Y with growth $\lesssim t^{k-1}$.

It is natural to use a (D, r) -separating subset of ‘minimal growth’. Suppose say that we have a (D, r) -separating subset Y with the property that

$$(1) \qquad |B_e(n) \cap Y| \leq |B_e(n) \cap Z|$$

for any other (D, r) -separating subset Z , where the vertex e is fixed and the inequality holds for any $n \in \mathbb{N}$. We show that such a subset has growth $\gamma_Y(t) \lesssim t^{k-1}$. We note that since for the growth of X we have $\gamma_X(t) \lesssim t^k$, we can find around any point v for any sufficiently large s an r -thick annulus A_s of X of radii $\sim s$ and volume $\lesssim s^{k-1}$. If now around a point v of Y the growth is greater than $\sim s^{k-1}$, we replace the ball of radius s of Y around v

by A_s , i.e., we replace Y by the set

$$Y_1 = (Y \setminus B_v(s)) \cup A_s.$$

This produces a (D, r) -separating subset of X violating inequality (1) (for $D \sim s$), which shows that indeed the growth of Y is $\lesssim t^{k-1}$.

One technical complication arises: it is not possible to show that there is a (D, r) -separating subset Y that satisfies (1) for every n . However, it is easy to see that such a set exists for any fixed n . So we define minimal sets instead as ‘limits’ of subsets Y_n that are minimizing for fixed n ’s. It turns out that this weaker definition of minimality is sufficient for our purposes.

We present now the formal proof starting by giving this somewhat technical definition of minimality.

Definition 2.5: We fix a vertex $e \in G$. We say that a (D, r) -separating subset Y of V is **minimal** if there is a sequence of (D, r) -separating subsets Y_n of V such that the following hold:

- (1) For any (D, r) -separating subset Z of V and for all $n \in \mathbb{N}$

$$|B_e(n) \cap Z| \geq |B_e(n) \cap Y_n|.$$

- (2) For any $k > 0$ there is some $n_k \geq k$ such that

$$B_e(k) \cap Y = B_e(k) \cap Y_{n_k+t} \quad \text{for all } t \in \mathbb{N}.$$

More generally, if $Y \subset X \subset V$ and Y is a (D, r) -separating subset of X , we say that Y is minimal if the same two conditions are satisfied for (D, r) -separating subsets of X .

LEMMA 2.2: Let $G=(V, E)$ be a locally finite connected graph. For any $D, r > 0$, minimal (D, r) -separating subsets exist. The same is true for minimal (D, r) -separating subsets of a subset X .

Proof. Clearly (D, r) -separating sets exist, e.g., take the complement of a $D/2$ -ball. Let Y_n be a (D, r) -separating subset of V for which $|B_e(n) \cap Y_n|$ attains the minimal value among all (D, r) -separating subsets. Since G is locally finite we can pass to a subsequence Y_{n_k} such that

$$Y_{n_k} \cap B_e(k) = Y_{n_k+t} \cap B_e(k)$$

for all $t \in \mathbb{N}$. Then set

$$Y = \bigcup_{k=1}^{\infty} (Y_{n_k} \cap B_e(k)).$$

Clearly Y is a minimal (D, r) -separating subset of V .

The same proof applies for a (D, r) -separating subset of subsets of V . ■

Definition 2.6: Let $G = (V, E)$ be a graph. An **annulus** with center $v \in G$ and radii $m < n$ is the set

$$A(v, m, n) = \{x \in V : m \leq d(v, x) \leq n\}.$$

We say that $n - m$ is the **thickness** of the annulus.

LEMMA 2.3: Let $k, r \in \mathbb{N}$ and let $G = (V, E)$ be a connected locally finite graph. If $X \subset G$ is such that for some m the growth at $t_0 = 4^m r$ satisfies

$$\gamma_X(t_0) \leq \left(\frac{1}{4r}\right)^k t_0^k,$$

then $r\text{-dim}(X) \leq k - 1$.

Proof. We will prove this by induction on k . By Lemma 2.1 the assertion holds for $k = 1$. Assume inductively that the assertion holds for $k - 1$. Let X be a subset satisfying

$$\gamma_X(t_0) \leq \left(\frac{1}{4r}\right)^k t_0^k.$$

Let $D = 2t_0$ and let Y be a minimal (D, r) -separating subset of X . If $t_1 = t_0/4$ we claim that

$$(*) \quad \gamma_Y(t_1) \leq \left(\frac{1}{4r}\right)^{k-1} t_1^{k-1}.$$

Indeed assume that this is not the case, so there is some v such that

$$|B_v(t_1) \cap Y| > \left(\frac{1}{4r}\right)^{k-1} t_1^{k-1}.$$

Consider the annulus of X , $A(v, t_1, t_0)$. Since $t_0 = 4t_1$, $A(v, t_1, t_0)$ contains $t_0/2r$ disjoint annuli A_i of thickness r . Clearly

$$\sum |A_i| \leq \left(\frac{1}{4r}\right)^k t_0^k,$$

so some annulus, say A_j , satisfies

$$|A_j| \leq \frac{2r}{t_0} \left(\frac{1}{4r}\right)^k t_0^k < \left(\frac{1}{4r}\right)^{k-1} t_1^{k-1}.$$

Let $N = d(e, v) + 4t_0$ (where e is as in Definition 2.5).

Since Y is minimal there is a sequence Y_n of (D, r) -separating subsets of X satisfying the properties of Definition 2.5. If we set $Y' = Y_{n_N}$, then by the property (1) of the definition it follows that

$$B_e(n_N) \cap Y = B_e(n_N) \cap Y'.$$

By the property (2) of the definition we have that for any (D, r) -separating subset Z of X

$$|B_e(N) \cap Z| \geq |B_e(N) \cap Y'|.$$

Recall that $n_N \geq N$, so we have

$$B_e(N) \cap Y = B_e(N) \cap Y'.$$

In particular

$$B_v(4t_0) \cap Y = B_v(4t_0) \cap Y'.$$

Consider now the set $Y_1 = (Y' \setminus B_v(t_1)) \cup A_j$. We show that Y_1 is also a (D, r) -separating subset of X . Note that the thickness of A_j is r and $t_1 = D/8$, so the r -connected component of $X \setminus Y_1$ containing v has diameter $< D$. Assume that v_1, v_2 lie in distinct r -connected components of $X \setminus Y'$ and neither of them lies in the r -connected component of $X \setminus Y_1$ containing v . We will show that v_1, v_2 lie in distinct r -connected components of $X \setminus Y_1$. Indeed for any sequence of points $x_1 = v_1, \dots, x_m = v_2$ that joins them with $d(x_i, x_{i+1}) \leq r$ for $i \leq m - 1$ we have that some x_j lies on Y' . Since $Y_1 \supset Y' \setminus B_v(t_1)$ either x_j lies on Y_1 or x_j lies in $B_v(t_1)$. But since A_j is r -thick this implies that one of v_1, v_2 is in the r -connected component of $X \setminus Y_1$ containing v . Hence $x_j \in Y_1$ so v_1, v_2 lie in distinct connected components of $X \setminus Y_1$. This completes the proof that Y_1 is a (D, r) -separating subset of X .

However, this contradicts the second property of Definition 2.5 for Y' since clearly

$$|B_e(N) \cap Y_1| < |B_e(N) \cap Y'|.$$

This proves the inequality (*) of our claim. Note now that Y satisfies the inductive hypothesis so r -dim $Y \leq k - 2$. Take now a uniformly bounded cover \mathcal{V} of Y of r -multiplicity $\leq k - 1$ and add to it the r -connected components of $X \setminus Y$, that have diameter at most D by hypothesis. We obtain a uniformly bounded cover \mathcal{U} of X of r -multiplicity $\leq k$. We note that the diameter of the sets in the cover is bounded by $2t_0$. In particular, the diameter of the sets in the cover that we constructed depends only on the function γ_X (and r) and not on X . ■

THEOREM 2.4: *Let $G = (V, E)$ be a connected graph with growth function $\gamma(t)$ satisfying*

$$\lim_{t \rightarrow \infty} \frac{\gamma(t)}{t^{k+1}} = 0 \quad \text{for some } k \in \mathbb{N}.$$

Then $\text{asdim } G \leq k$.

Proof. By assumption, for any $r > 0$ there is an m such that for $t_0 = 4^m r$

$$\gamma(t_0) \leq \left(\frac{1}{4r}\right)^{k+1} t_0^{k+1}.$$

So by lemma 2.3 $r\text{-dim } G \leq k$. Since this holds for every r , $\text{asdim } G \leq k$. ■

Example 2.5: The Cayley graph of \mathbb{Z}^k has growth $\gamma(r) \sim r^k$. The above theorem implies that there is no graph of growth between r^k and r^{k+1} that has asymptotic dimension $k + 1$; i.e., there is nothing ‘smaller’ than the Cayley graph of \mathbb{Z}^{k+1} with $\text{asdim} = k + 1$. It is important of course for this that we define growth ‘uniformly’ independently of a base vertex. If one defines growth with respect to a base point, paraboloids of dimension k have strictly smaller growth than \mathbb{R}^k but asymptotic dimension = k .

Example 2.6: It is clear that there is no lower bound on asdim in terms of volume. For example, 3-regular trees have exponential volume growth and $\text{asdim} = 1$. It is easy to create similar examples of trees with polynomial growth as well by taking a sparse set of branch points.

For example, take an infinite complete binary tree T , with root e (so every vertex has two ‘children’), and subdivide edges so that an edge at distance n from e in T has length $2^{\lfloor n/k \rfloor}$ after the subdivision, for some fixed k . Then

$$\gamma(r) \sim r^{k+1}.$$

3. Assouad–Nagata dimension and metric spaces

There are several ways to assign volume functions to general metric spaces. A quite naive definition is appropriate for this paper:

Definition 3.1: Let X be a metric space and $\epsilon, \delta > 0$. Then N is an (ϵ, δ) -net of X if for any $x \in X$ there is $n \in N$ with $d(x, n) \leq \epsilon$ and for any $n_1, n_2 \in N$, $d(n_1, n_2) \geq \delta$.

Using Zorn’s lemma it is easy to see that any metric space X contains an (ϵ, ϵ) -net for any $\epsilon > 0$.

Definition 3.2: Let X be a metric space and let $t \in \mathbb{R}$. We define the t -**growth function** of a metric space to be

$$\gamma^t(r) = \sup\{|B_v(r) \cap N| : v \in X, N (t, t)\text{-net}\}.$$

The supremum is over all (t, t) -nets (and $v \in X$).

We say that X has **polynomial growth** if there are $t > 0, k, C > 0$ such that

$$\gamma^t(r) < Cr^k$$

for all r .

Observe that $\gamma^t(r) \leq \gamma^s(r)$ if $t > s$. Note also that if $\gamma^t(r) \neq \infty$ for any r , then for any $s > t$ there is some $C_s > 0$ such that $\gamma^t(r) \leq C_s \gamma^s(r)$. Indeed this is because there is a $C > 0$ such that if N is a (t, t) -net, any ball of radius $2s$ contains at most C points of N .

It follows that if $\gamma^t(r) \neq \infty$ for any r for some t , then the specific ‘scale’ $s > t$ that we pick to define the growth function does not affect much the asymptotic behavior of the growth function. For these reasons we will omit below the reference to scale and we will denote the growth function for a specific scale simply by $\gamma(r)$.

Similarly one sees that the growth does not depend much on the net we pick. If $\gamma^t(r) \neq \infty$ for all r and N_1, N_2 are (t, t) -nets, then there is $C > 0$ such that

$$|B_v(r) \cap N_1| \leq C|B_v(r) \cap N_2|.$$

Note however that if $\gamma^t(r) = \infty$ for some r , it is possible that $|B_v(r) \cap N| < \infty$ for some specific net N :

Example 3.1: Consider the linear graph with vertices \mathbb{N} where we attach n extra edges to the vertex n for every n . Then $\gamma^1(1) = \infty$, but if we take \mathbb{N} as a $(1, 1)$ -net, $|B_v(1) \cap \mathbb{N}| \leq 3$ for all v .

We note that for a graph, $\gamma(r)$ is finite for all $r > 0$ if and only if it has uniformly bounded degree if and only if $\gamma(r)$ is finite for some $r > 0$.

LEMMA 3.2: *Let X be a metric space with 1-growth function $\gamma(r)$ satisfying*

$$\lim_{r \rightarrow \infty} \frac{\gamma(r)}{r^{k+1}} = 0 \quad \text{for some } k \in \mathbb{N}.$$

Then $\text{asdim } X \leq k$.

Proof. Let N be a $(1, 1)$ -net of X . Given $m > 0$ we define a graph $G = (N, E)$ where $(x, y) \in E$ if and only if $d(x, y) \leq 2m$. Now G could have several connected components, however for each connected component Γ we have that its growth function satisfies

$$\gamma_\Gamma(r) \leq \gamma(2mr),$$

so

$$\lim_{r \rightarrow \infty} \frac{\gamma_\Gamma(r)}{r^{k+1}} = 0.$$

By Theorem 2.4 the asymptotic dimension of Γ is at most k , so the m -dim of Γ is bounded by k . In fact as we showed in Lemma 2.3, there is a cover \mathcal{U} of Γ with m -multiplicity $\leq k + 1$ such that the diameter of every set in \mathcal{U} is bounded by a constant that depends only on γ (see the last line of the proof of Lemma 2.3).

Since this is true for every connected component of G we have that m -dim of X is bounded by k . As this is true for every m , $\text{asdim } X \leq k$. ■

COROLLARY 3.3: *If M^n is a Riemannian manifold of bounded geometry and volume growth function $\text{Vol}(r)$ satisfying*

$$\lim_{r \rightarrow \infty} \frac{\text{Vol}(r)}{r^{k+1}} = 0 \quad \text{for some } k \in \mathbb{N},$$

then $\text{asdim } M^n \leq k$.

Proof. Let us say that any a -ball $B_x(a)$ of M^n is b -bilipschitz with the 1-ball of \mathbb{R}^n . Then

$$c_1 \leq \text{Vol } B_x(a) \leq c_2$$

for some $c_1, c_2 > 0$ and for any $x \in M^n$.

If N is an (a, a) -net of M^n and $B_v(r + a)$ is a ball of radius $r + a$, we consider the set $C = B_v(r) \cap N$. The open balls $B_c(a)$ where $c \in C$ are disjoint, each has volume $\geq c_1$ and they are all contained in $B_v(r + a)$. It follows that

$$c_1 \cdot |C| \leq \text{Vol}(B_v(r + a)).$$

Hence if we see M^n as a metric space, its a -growth function satisfies

$$\gamma^a(r) \leq \frac{\text{Vol}(r + a)}{c_1},$$

so by Lemma 3.2, $\text{asdim } M^n \leq k$. ■

Example 3.4: The bounded geometry hypothesis is necessary: Consider any graph G of bounded degree with infinite asymptotic dimension. ‘Thicken’ the graph to a 2-manifold S (so edges become thin cylinders). By replacing the edges by thinner and thinner ‘tubes’ as we go to infinity, we obtain a 2-manifold of finite area and infinite asymptotic dimension.

THEOREM 3.5: *There is a metric space X with 1-growth function $\gamma(r)$ satisfying*

$$\lim_{r \rightarrow \infty} \frac{\gamma(r)}{r} = 0$$

and infinite asymptotic Assouad–Nagata dimension.

There is a connected graph $G = (V, E)$ with growth function $\gamma(r)$ satisfying

$$\lim_{r \rightarrow \infty} \frac{\gamma(r)}{r^{1+\epsilon}} = 0$$

for any $\epsilon > 0$ and infinite asymptotic Assouad–Nagata dimension.

Proof. We give first a sketch of this proof. Both parts are similar so we explain the idea for the case of graphs, which is more involved. To ensure that G has infinite asymptotic Assouad–Nagata dimension it suffices that for every n , G will contain a subgraph isomorphic to an n -ball of the Cayley graph of \mathbb{Z}^n . Of course such balls would lead to big growth, however this can be corrected by rescaling the length of the edges (or equivalently subdividing each edge many times so that most vertices have degree 2 after this subdivision). There remains the problem that the degree of the original vertices (before subdivisions) is n (so unbounded), but this can be mended by replacing the vertices with trees with sublinear growth and n -end points.

We proceed now with the details. Let Γ_n be the Cayley graph of \mathbb{Z}^n with respect to the standard generating set. Let $C_e(n)$ be the n -‘cube’ of side length $2n$. Let G_n be the metric space obtained by $C_e(n)$ by changing the length of edges from 1 to 2^{n^2} (so we rescale the metric by a factor of 2^{n^2}). We consider a metric on

$$\bigcup_{n \in \mathbb{N}} G_n$$

so that the G_n ’s are far apart. For example, we may consider the linear graph with vertex set \mathbb{N} and identify an arbitrary vertex of G_n with 10^{n^2} . In this way we get a graph Γ , which we see as a metric space containing $\bigcup_{n \in \mathbb{N}} G_n$, and so we obtain an induced metric on $\bigcup_{n \in \mathbb{N}} G_n$. Let X be the vertex set of $\bigcup_{n \in \mathbb{N}} G_n$.

We note that for the metric we just defined on X , the 1-growth function $\gamma(r)$ of X satisfies

$$\lim_{r \rightarrow \infty} \frac{\gamma(r)}{r} = 0.$$

As X contains bigger and bigger copies of the vertex sets of G_n 's it is easy to see that the asymptotic Assouad–Nagata dimension of X is infinite. Indeed suppose that the Assouad–Nagata dimension of X is equal to k . Then there is a $C > 0$ such that for any sufficiently big r there is a cover of X by sets of diameter $\leq Cr$ such that any r -ball intersects at most $k + 1$ of these sets. Let us denote by $V(n)$ the vertex set of the $[n]^{k+1}$ grid in Γ_{k+1} and let d_n be the induced metric on $V(n)$ by the inclusion in Γ_{k+1} ; then it is clear that X contains copies of $(V(n), 2^{n^2}d_n)$ for any $n > 0$. So for any sufficiently big r there is a cover \mathcal{U}_n of $(V(n), 2^{n^2}d_n)$ by sets of diameter $\leq Cr2^{n^2}$ such that any $2^{n^2}r$ -ball intersects at most $k + 1$ of these sets. By rescaling the metric we have that $(V(n), d_n)$ has a cover \mathcal{U}'_n by sets of diameter $\leq Cr$ such that any r -ball intersects at most $k + 1$ of these sets. However, as this is true for any n it implies that the asymptotic dimension of Γ_{k+1} is at most k , a contradiction. This proves the first part of the theorem.

To prove the second part we make a similar construction. Let $C_\epsilon(n, k)$ be the n -cube of side length $2k$ in Γ_n (the Cayley graph of \mathbb{Z}^n). Let $G(n, k)$ be the metric space obtained by $C_\epsilon(n, k)$ by changing the length of edges from 1 to 2^k (so we rescale the metric by a factor of 2^k). Clearly we can turn this into a graph by subdividing the original edges into 2^k edges. We consider the linear graph with vertex set \mathbb{N} and identify a vertex of $G(n, k)$ with 10^k . We obtain in this way a graph Y_n with asymptotic Assouad–Nagata dimension at least n and growth $\gamma(r)$ that satisfies

$$\lim_{r \rightarrow \infty} \frac{\gamma(r)}{r^{1+\epsilon}} = 0$$

for any $\epsilon > 0$. We note that the maximum degree of a vertex in Y_n is $2n + 2$, so it is not possible to obtain a graph of bounded degree taking a ‘union’ of Y_n 's. For this reason the growth function of such a union would be infinite for any $r \geq 1$.

To correct this we replace each vertex v of degree $2n > 2$ of $C_\epsilon(n, k)$ in Y_n by a finite binary tree T_n with diameter 2^n and $2n$ end vertices. Then we identify each end-vertex of T_n with a vertex of an edge adjacent to v . We call the graph obtained by replacing vertices in this way X_n . More specifically, T_n is a finite rooted binary tree with $\sim \log n$ branch points and edges

of length $\sim 2^n / \log(\log n)$. As usual we subdivide the edges of T_n into edges of length 1 to get a simplicial graph.

The growth function γ_n of X_n satisfies

$$\lim_{r \rightarrow \infty} \frac{\gamma_n(r)}{r^{1+\epsilon}} = 0$$

for any $\epsilon > 0$. We claim that X_n has Assouad–Nagata dimension at least equal to n . Indeed we have a map $f_n : X_n \rightarrow Y_n$ obtained by collapsing the T_n ’s to points. Clearly f_n is a quasi-isometry as the fibers of this map have uniformly bounded diameter, so X_n and Y_n have the same asymptotic Assouad–Nagata dimension.

Finally we may take X to be the ‘union’ of X_n ’s. More precisely take L to be the linear graph with vertex set n and identify a vertex of X_n with the vertex 2^{2^n} of L . Each vertex of the graph X obtained in this way has bounded degree. We note that if we consider the union of T_n ’s, its growth is bounded by $10r^{1+\epsilon}$ for any $\epsilon > 0$ and any $r \geq 1$. It is easy to see that the growth function $\gamma(r)$ of X satisfies

$$\lim_{r \rightarrow \infty} \frac{\gamma(r)}{r^{1+\epsilon}} = 0$$

and X has infinite asymptotic Assouad–Nagata dimension since it contains a copy of X_n for all n . ■

4. Discussion and questions

We say that a proper metric space M_n is universal for a class \mathcal{C} of proper metric spaces of asymptotic dimension n if any metric space in \mathcal{C} admits a coarse embedding in M_n . This is in analogy to the topological dimension theory where compact spaces of dimension n embed in the Menger compactum μ^n . It is shown in [8] that there is no such space for $n = 1$ if we take \mathcal{C} to be the class of all proper metric spaces of asymptotic dimension n .

On the other hand, for hyperbolic spaces Buyalo, Dranishnikov and Schroeder [6] show that if X is visual hyperbolic metric space such that its boundary ∂X is a doubling metric space and $\text{asdim } X = n$, then X quasi-isometrically embeds in a product of $n + 1$ -binary metric trees.

As the asymptotic dimension is a ‘coarse topology’ notion, it makes sense to consider coarse embeddings of such spaces rather than quasi-isometries.

It is not clear how much bigger dimension one needs in order to achieve a coarse embedding of a graph of growth $\sim r^k$ in \mathbb{R}^n instead of an embedding considered by Linial, London and Rabinovich [19] and Krauthgamer–Lee. Note that the embeddings of [16] are weaker than coarse embeddings, as one only requires that distinct vertices map at distance ≥ 1 from each other. For example, there is an onto embedding in their sense from the linear graph with vertices \mathbb{N} to the standard Cayley graph of \mathbb{Z}^2 —so these embeddings may raise asdim .

Question 4.1: Let $G = (V, E)$ be a graph such that its growth function satisfies $\gamma(n) \leq Cn^k$ for some $C > 0$. Is there a coarse embedding $f : G \rightarrow \mathbb{R}^{O(k \log k)}$?

One can ask also whether there is a ‘universal’ space for spaces with polynomial growth:

Question 4.2: Is there a proper metric space P_k such that if X is a metric space of polynomial growth $< Cn^k$, then X coarsely embeds in P_k ? If so, can one take P_k so that $\text{asdim } P_k = k$ and P_k is of polynomial growth $\sim n^k$? (it might be necessary here to fix C).

Manifolds with nonnegative Ricci curvature have been extensively studied (see, e.g., [1], [22], [20]). Our result implies that if M^n is a complete Riemannian manifold of nonnegative Ricci curvature and bounded geometry, then $\text{asdim } M^n \leq n$. One wonders whether the bounded geometry assumption is necessary:

Question 4.3: Let M^n be a complete Riemannian manifold of nonnegative Ricci curvature. Is it true that $\text{asdim } M^n \leq n$?

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