ISRAEL JOURNAL OF MATHEMATICS **254** (2023), 243–274
DOI: 10.1007/s11856-022-2396-1

DIFFERENTIAL IDENTITIES AND VARIETIES OF ALMOST POLYNOMIAL GROWTH*

BY

Fabrizio Martino

Dipartimento di Matematica e Informatica, Università degli Studi di Palermo
via Archirafi 34, 90123, Palermo, Italy
e-mail: fabrizio.martino@unipa.it

AND

Carla Rizzo

Departamento de Matemática, CMUC, Universidade de Coimbra
3004-504 Coimbra, Portugal
e-mail: carlarizzo@mat.uc.pt

ABSTRACT

Let $\mathcal{V}$ be an L-variety of associative L-algebras, i.e., algebras where a Lie algebra L acts on them by derivations, and let $c_n^L(\mathcal{V})$, $n \geq 1$, be its L-codimension sequence. If $\mathcal{V}$ is generated by a finite-dimensional L-algebra, then such a sequence is polynomially bounded only if $\mathcal{V}$ does not contain UT_2, the 2×2 upper triangular matrix algebra with trivial L-action, and UT_2^ε where L acts on UT_2 as the 1-dimensional Lie algebra spanned by the inner derivation ε induced by e_{11}. In this paper we completely classify all the L-subvarieties of $\mathrm{var}^L(UT_2)$ and $\mathrm{var}^L(UT_2^\varepsilon)$ by giving a complete list of finite-dimensional L-algebras generating them.

* This work was partially supported by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES.

Received May 14, 2021 and in revised form October 8, 2021

1. Introduction

Let F be a field of characteristic zero, let $F\langle X\rangle$ be the free associative algebra on a countable set X of variables over F and let A be an associative F-algebra. A polynomial of $F\langle X\rangle$ vanishing under every evaluation in A is called a polynomial identity of A and we denote by $\mathrm{Id}(A)$ the T-ideal of polynomial identities satisfied by A. One of the most challenging problems in the theory of algebras with polynomial identities (PI theory) is to find some numerical invariants allowing us to classify such T-ideals of $F\langle X\rangle$. Since there is a one-to-one correspondence between T-ideals and varieties of algebras, often it is convenient to translate a given issue about T-ideals into the language of varieties of algebras.

If P_n is the space of multilinear polynomials in the variables $x_1, \ldots, x_n$, then we set

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)},$$

for all $n \geq 1$, and we call it the codimension sequence of A. If $\mathcal{V}$ is a variety of algebras and $\mathrm{Id}(\mathcal{V})$ is its corresponding T-ideal, then we can similarly define $c_n(\mathcal{V})$. Moreover, if $\mathcal{V} = \mathrm{var}(A)$ is the variety generated by the algebra A, then we refer to the codimension sequence of $\mathcal{V}$ as the one of A. Such a numerical sequence was introduced by Regev in [28] and it measures the rate of growth of the multilinear polynomials lying in the corresponding T-ideal. In the same paper, Regev also showed that if A is an associative algebra satisfying a non-trivial polynomial identity, then $c_n(A)$ is exponentially bounded. Later on, Kemer in [19] and [20] proved several properties about the codimension sequence. On one hand, he showed that $c_n(A)$ is polynomially bounded or grows exponentially, on the other he gave a characterization of the varieties of polynomial growth of the codimension proving that $c_n(A)$ is polynomially bounded if and only if $G, UT_2 \notin \mathrm{var}(A)$, where G is the infinite-dimensional Grassmann algebra and UT_2 is the algebra of 2×2 upper triangular matrices. Hence $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$ are the only varieties of almost polynomial growth, i.e., they grow exponentially but any proper subvariety grows polynomially.

Varieties of poylnomial growth were extensively studied in the past years in various settings. We refer the interested reader to [5], [6], [22] for some results about ordinary algebras; to [8], [23], [24], [32] for superalgebras and more generally group graded algebras; to [3], [7], [10], [16], [17], [25] for algebras with involution, graded involution, superinvolution and pseudoinvolution; to [27] for special Jordan algebras.

In this paper we deal with associative algebras with a Lie algebra action by derivations. If L is such a Lie algebra, then its action can be naturally extended to the action of the universal enveloping algebra $U(L)$ of L and in this case we say that the algebra A is an algebra with derivations or an L-algebra. In this context it is natural to define the differential identities of A, i.e., the polynomials in the variables $x^h = h(x)$, $h \in U(L)$, vanishing on A. In analogy with the ordinary case, one defines the sequence of L-codimensions and studies their asymptotic behavior. In [13] it was proved that, in case of finite-dimensional L-algebras, the sequence of L-codimensions is exponentially bounded or grows polynomially. Moreover, in [9] the authors studied the algebra UT_2^ε of 2×2 upper triangular matrices with the action of the 1-dimensional Lie algebra spanned by the inner derivation ε induced by e_{11}. In that paper, they show that such algebra generates an L-variety of almost polynomial growth. Finally, by using [9] and [20], it can be proved that the L-codimension sequence of an L-variety $\mathcal{V}$ generated by a finite-dimensional L-algebra is polynomially bounded only if $UT_2, UT_2^\varepsilon \notin \mathcal{V}$, where UT_2 stands for the algebra of 2×2 upper triangular matrices and L acts trivially on it.

The main purpose of this paper is to classify, up to PI-equivalence, all the L-subvarieties of $\mathrm{var}^L(UT_2)$ and $\mathrm{var}^L(UT_2^\varepsilon)$ in terms of generators of the corresponding T_L-ideals and to provide a complete list of L-algebras generating such L-subvarieties. Concerning $\mathrm{var}^L(UT_2^\varepsilon)$, the main result is given by Theorem 27 below. We also highlight that if L acts trivially on UT_2, then such a classification coincides with that of the ordinary case given in [22]. We chose to include it here for the sake of completeness.

2. On differential identities

Throughout this paper F will denote a field of characteristic zero and L a Lie algebra over F.

Recall that a derivation of an associative algebra A is a linear map $\delta : A \to A$ such that $(ab)^\delta = a^\delta b + ab^\delta$, for all $a, b \in A$. In particular, an inner derivation induced by $a \in A$ is the derivation ad_a acting on the left on A by

$$b^{\mathrm{ad}_a} = [a, b] = ab - ba,$$

for all $b \in A$. Clearly, the set $\mathrm{Der}(A)$ of all derivations of A is a Lie algebra.

Let $U(L)$ be the universal enveloping algebra of L. By the Poincaré–Birkhoff–Witt Theorem, if L has an ordered basis $\{\delta_i \mid i \in I\}$, then $U(L)$ has a basis $\{\delta_{i_1} \cdots \delta_{i_p} \mid i_1 < \cdots < i_p,\, i_k \in I,\, p \geq 0\}$. Thus if A is an associative F-algebra with an L-action by derivations, then this action can be naturally extended to an $U(L)$-action. In this case we call A an algebra with derivations or L-algebra.

Let $X = \{x_1, x_2, \ldots\}$ be a countable set and $\mathcal{B} = \{d_j \mid j \geq 0\}$ be a basis of $U(L)$. We denote by $F\langle X|L\rangle$ the free associative algebra over F with free formal generators $x_i^{d_j}$, $i > 0$ and $j \geq 0$, where we identify $x_i = x_i^1$, $1 = d_0 \in U(L)$. Notice that $U(L)$ acts on $F\langle X|L\rangle$ by setting

$$\begin{aligned}(x_{i_1}^{d_{j_1}} x_{i_2}^{d_{j_2}} \cdots x_{i_n}^{d_{j_n}})^{\delta} =& x_{i_1}^{\delta d_{j_1}} x_{i_2}^{d_{j_2}} \cdots x_{i_n}^{d_{j_n}} \\ &+ x_{i_1}^{d_{j_1}} x_{i_2}^{\delta d_{j_2}} \cdots x_{i_n}^{d_{j_n}} + \cdots + x_{i_1}^{d_{j_1}} x_{i_2}^{d_{j_2}} \ldots x_{i_n}^{\delta d_{j_n}},\end{aligned}$$

where $\delta \in L$ and $x_{i_1}^{d_{j_1}} x_{i_2}^{d_{j_2}} \ldots x_{i_n}^{d_{j_n}} \in F\langle X|L\rangle$. Thus we call $F\langle X|L\rangle$ the free associative algebra with derivations on X over F and we refer to its elements as differential polynomials or L-polynomials.

Let A be an L-algebra over F. Recall that an L-polynomial

$$f(x_1, \ldots, x_n) \in F\langle X|L\rangle$$

is a differential polynomial identity of A (or simply an L-identity), and we write $f \equiv 0$, if $f(a_1, \ldots, a_n) = 0$ for all $a_i \in A$, $1 \leq i \leq n$. We denote by

$$\mathrm{Id}^L(A) = \{f \in F\langle X|L\rangle \mid f \equiv 0 \text{ on } A\}$$

the T_L-ideal of L-identities of A, i.e., $\mathrm{Id}^L(A)$ is an ideal of $F\langle X|L\rangle$ invariant under all endomorphisms φ of $F\langle X|L\rangle$ such that

$$\varphi(f^h) = \varphi(f)^h,$$

for all $f \in F\langle X|L\rangle$ and $h \in U(L)$ (see for example [14, 21, 29, 30]).

Let H be a Lie subalgebra of L. If A is an L-algebra, then by restricting the action, A can be regarded as an H-algebra. In this case we can identify the T_L-ideal $\mathrm{Id}^L(A)$ and the T_H-ideal $\mathrm{Id}^H(A)$, i.e., in $\mathrm{Id}^L(A)$ we omit the differential identities $x^{\delta} \equiv 0$, for all $\delta \in L \backslash H$. Furthermore, any algebra A can be regarded as an L-algebra by letting L act on A trivially, i.e., L acts on A as the trivial Lie algebra. Hence the theory of differential identities generalizes the ordinary theory of polynomial identities.

As in the ordinary case, in characteristic zero, every L-identity is equivalent to a system of multilinear ones. We denote by

$$P_n^L = \mathrm{span}\{x_{\sigma(1)}^{d_{i_1}} \cdots x_{\sigma(n)}^{d_{i_n}} \mid \sigma \in S_n, d_{i_k} \in \mathcal{B}\}$$

the vector space of multilinear differential polynomials in the variables $x_1, \ldots, x_n$, $n \geq 1$. Since $\mathrm{Id}^L(A)$ is generated, as T_L-ideal, by the multilinear L-polynomials it contains, the study of $\mathrm{Id}^L(A)$ is equivalent to the study of $P_n^L \cap \mathrm{Id}^L(A)$ for all $n \geq 1$. In case $U(L)$ acts on A as a suitable finite-dimensional subalgbera of the endomorphism ring of A, then P_n^L is finite-dimensional and we denote by

$$c_n^L(A) = \dim_F \frac{P_n^L}{P_n^L \cap \mathrm{Id}^L(A)}, \quad n \geq 1,$$

the nth differential codimension of A or the nth L-codimension of A. From now on, we will assume that the action of $U(L)$ is always of this type.

Given a variety $\mathcal{V}$ of L-algebras the growth of $\mathcal{V}$ is defined as the growth of the sequence of differential codimensions of any L-algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^L(A)$. In this case we set

$$c_n^L(\mathcal{V}) = c_n^L(A), \quad n \geq 1.$$

Then we say that $\mathcal{V}$ has polynomial growth if there exist C, t such that

$$c_n^L(\mathcal{V}) \leq Cn^t$$

and that $\mathcal{V}$ has almost polynomial growth if $c_n^L(\mathcal{V})$ is not polynomially bounded but every proper subvariety has polynomial growth.

Next we are going to describe two L-varieties of almost polynomial growth.

Let us denote by UT_2 the L-algebra of 2×2 upper triangular matrices over F where L acts trivially on it. Since $x^\delta \equiv 0$, for all $\delta \in L$, is a differential identity of UT_2, we are dealing with ordinary identities. Thus by [20], it follows that the algebra UT_2 generates an L-variety of almost polynomial growth. Moreover, we have the following result (see [26]).

Theorem 1:

(1) $\mathrm{Id}^L(UT_2) = \langle [x_1, x_2][x_3, x_4] \rangle_{T_L}$;
(2) $c_n^L(UT_2) = 2^{n-1}(n-2) + 2$.

Let us now denote by UT_2^ε the L-algebra of 2×2 upper triangular matrices over F where L acts on it as the 1-dimensional Lie algebra spanned by the inner derivation ε induced by e_{11}, i.e.,

$$(ae_{11} + be_{22} + ce_{12})^\varepsilon = ce_{12},$$

for all $a, b, c \in F$, where e_{ij}'s are the usual matrix units. In [9], the authors proved that UT_2^ε has almost polynomial growth and also they proved the following.

THEOREM 2 ([9, Theorem 5]):
(1) $\mathrm{Id}^L(UT_2^\varepsilon) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2^\varepsilon,\ [x_1, x_2]^\varepsilon - [x_1, x_2]\rangle_{T_L}$;
(2) $c_n^L(UT_2^\varepsilon) = 2^{n-1}n - 1$.

Recall that given two L-algebras A and B, A is T_L-equivalent to B and we write $A \sim_{T_L} B$ in case

$$\mathrm{Id}^L(A) = \mathrm{Id}^L(B).$$

As in the ordinary case, a useful tool when studying L-identities of algebras with 1 is provided by the so-called proper polynomials.

Recall that a left normed commutator of length $n \geq 2$ in the variables x_i's is defined inductively by setting

$$[x_1^{h_1}, \ldots, x_{n-1}^{h_{n-1}}, x_n^{h_n}] = -[x_1^{h_1}, \ldots, x_{n-1}^{h_{n-1}}]^{\mathrm{ad}_{x_n^{h_n}}},$$

where $h_1, \ldots, h_n \in U(L)$. An L-polynomial $f(x_1, \ldots, x_n) \in F\langle X|L\rangle$ is a proper L-polynomial if it is a linear combination of elements of the type

$$x_{i_1}^{h_1} \ldots x_{i_k}^{h_k} w_1 \ldots w_m$$

where $h_i \in U(L)$, $h_i \neq 1_{U(L)}$, for all $1 \leq i \leq k$, and $w_1, \ldots, w_m$ are (eventually empty) left normed Lie commutators in x_i's.

In characteristic zero, if A is a unitary L-algebra, then $\mathrm{Id}^L(A)$ is generated, as T_L-ideal, by the multilinear proper L-polynomials (see [1, Lemma 2.1]). Thus if Γ_n^L denotes the subspace of P_n^L of multilinear proper L-polynomials in $n \geq 1$ variables, and $\Gamma_0^L = \mathrm{span}\{1\}$, then we define the sequence of proper L-codimensions of A as

$$\gamma_n^L(A) = \dim_F \frac{\Gamma_n^L}{\Gamma_n^L \cap \mathrm{Id}^L(A)}, \quad n \geq 0.$$

For unitary L-algebra A, the relation between the L-codimensions and the proper L-codimensions is given by the following:

$$c_n^L(A) = \sum_{i=0}^{n} \binom{n}{i} \gamma_i^L(A), \quad n \geq 1.$$

This relation can be proved following closely the proof in the ordinary case in Theorem 4.3.1 of [2].

Next we present a result on proper L-polynomials which will be useful later. Recall that given two sets of L-polynomials $S, S' \subseteq F\langle X|L\rangle$, we say that S' is a consequence of S if $S' \subseteq \langle S\rangle_{T_L}$.

Proposition 3: *Let $i \geq 1$. If k is even, then Γ_{k+i}^L is a consequence of Γ_k^L. Otherwise, Γ_{k+i}^L is a consequence of Γ_k^L plus the polynomial $[x_1, x_2] \cdots [x_k, x_{k+1}]$.*

Proof. We start by proving the statement in case k is even.

Let $w \in \Gamma_{k+i}^L$ be a generator and $i \geq 1$. Suppose first that w is a product of commutators. If w is a product of commutators of length 2, then

$$w = [x_1^{h_1}, x_2^{h_2}] \cdots [x_{k-1}^{h_{k-1}}, x_k^{h_k}] \cdots [x_{k+i-1}^{h_{k+i-1}}, x_{k+i}^{h_{k+i}}],$$

where $h_1, \ldots, h_{k+i} \in U(L)$. Thus w is a consequence of

$$[x_1^{h_1}, x_2^{h_2}] \cdots [x_{k-1}^{h_{k-1}}, x_k^{h_k}] \in \Gamma_k^L$$

and we are done. On the other hand, if w contains a commutator u of length $m > 2$, then u can be viewed as a consequence of a commutator of length $< m$. Thus by the above, we get also that w is a consequence of Γ_k^L. Hence we may assume that

$$w = x_{i_1}^{h_1} \cdots x_{i_t}^{h_t} [\cdots] \cdots [\cdots]$$

with $t > 0$ and $h_1, \ldots, h_t \in U(L)$. If $t \leq i$, then by the previous case, w is a consequence of Γ_k^L. Otherwise, if $t > i$, then w is a consequence of the polynomial $x_{i_r}^{h_r} \cdots x_{i_t}^{h_t} [\cdots] \cdots [\cdots]$, where $r = i+1$, and we are done also in this case.

Now suppose that k is odd.

If we prove that Γ_{k+1}^L is a consequence of Γ_k^L and the polynomial

$$[x_1, x_2] \cdots [x_k, x_{k+1}],$$

by the first part of the proof we reach the desired conclusion. Thus let $w \in \Gamma_{k+1}^L$ be a generator. If either w contains a commutator of length greater than 2 or $w = x_{i_1}^{h_1} \cdots x_{i_t}^{h_t} [\cdots] \cdots [\cdots]$ with $t > 0$, $h_1, \ldots, h_t \in U(L)$, we have, as above, that w is a consequence of Γ_k^L. Thus we may assume that w is a product of commutators of length 2, i.e.,

$$w = [x_1^{h_1}, x_2^{h_2}] \cdots [x_k^{h_k}, x_{k+1}^{h_{k+1}}],$$

where $h_1, \ldots, h_{k+1} \in U(L)$. If $h_i \in \operatorname{span}_F\{1_{U(L)}\}$ for all $1 \leq i \leq k+1$, then $w = \beta[x_1, x_2] \cdots [x_k, x_{k+1}]$, for some $\beta \in F$, and we are done. Hence we may assume that $h_i \notin \operatorname{span}_F\{1_{U(L)}\}$, for some i. We write

$$\begin{aligned} w =&[x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}] x_i^{h_i} x_{i+1}^{h_{i+1}} [x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}] \\ &- [x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}] x_{i+1}^{h_{i+1}} x_i^{h_i} [x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}]. \end{aligned}$$

Since $yz = [y, z] + zy$ and a commutator of length $m > 2$ is a consequence of a commutator of length $< m$, then

$$\begin{aligned} w \equiv& (x_i^{h_i} x_{i+1}^{h_{i+1}} [x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}][x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}] \\ &- x_{i+1}^{h_{i+1}} x_i^{h_i} [x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}][x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}]) \pmod{\langle \Gamma_k \rangle_{T_L}}. \end{aligned}$$

If $h_{i+1} \notin \operatorname{span}_F\{1_{U(L)}\}$, then w is a consequence of

$$y^h [x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}][x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}] \in \Gamma_k^L$$

and we are done. Hence suppose that $h_{i+1} = 1_{U(L)}$, then

$$w \equiv x_i^{h_i} x_{i+1} [x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}][x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}] \pmod{\langle \Gamma_k \rangle_{T_L}}.$$

Without loss of generality, we may assume that $h_i = \delta_1 \cdots \delta_s$, where $\delta_1, \ldots, \delta_s \in L$. Let

$$I = \{i_1, \ldots, i_r\} \quad \text{and} \quad J = \{j_1, \ldots, j_t\}$$

be two disjoint sets such that $I \cup J = \{1, \ldots, s\}$, $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_t$, respectively. We set $c_I = \delta_{i_1} \cdots \delta_{i_r}$ and $c_J = \delta_{j_1} \cdots \delta_{j_t}$, then by definition of derivation, we have the following

$$x_i^{h_i} x_{i+1} = (x_i x_{i+1})^{h_i} - x_i x_{i+1}^{h_i} - \sum_{I,J} x_i^{c_I} x_{i+1}^{c_J}.$$

Since $c_I, c_J \in U(L)$ for all I, J, it follows that w is a consequence of

$$y^h [x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}][x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}] \in \Gamma_k^L$$

and this completes the proof. ∎

As a consequence we have the following.

Corollary 4: *Let A be an L-algebra with 1. If $\gamma_k^L(A) = 0$ for some $k \geq 1$, then $\gamma_m^L(A) = 0$ for all $m \geq k$.*

We remark that these results are general facts not related to the L-identities of UT_2.

One of the main tools in the study of T_L-ideals is the representation theory of the symmetric group S_n. In fact, the natural left S_n-action $\sigma(x_i^h) = x_{\sigma(i)}^h$ turns P_n^L into a S_n-module and therefore the space $P_n^L(A) = P_n^L/(P_n^L \cap \mathrm{Id}^L(A))$ becomes a left S_n-module. Similarly $\Gamma_n^L(A) = \Gamma_n^L/(\Gamma_n^L \cap \mathrm{Id}^L(A))$ is an S_n-module under the induced action. We denote by $\chi_n^L(A)$ and $\psi_n^L(A)$ the S_n-characters of $P_n^L(A)$ and $\Gamma_n^L(A)$, respectively, and we refer to them as the nth L-cocharacter and the nth proper L-cocharacter of A. Since $\operatorname{char} F = 0$, by complete reducibility we can write

$$\chi_n^L(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \quad \psi_n^L(A) = \sum_{\lambda \vdash n} m'_\lambda \chi_\lambda,$$

where λ is a partition of n, χ_λ is the irreducible S_n-character associated to λ, and $m_\lambda, m'_\lambda \geq 0$ are the corresponding multiplicities. It is clear that by knowing the decomposition of the (proper) cocharacter of A, one can get information about the corresponding (proper) codimensions.

3. Constructing L-subvarieties in $\mathrm{var}^L(UT_2^\varepsilon)$

The main goal of this section is to construct some suitable finite-dimensional L-algebras belonging to the L-variety generated by UT_2^ε whose L-codimension sequence grows polynomially.

For all $k \geq 2$ let

$$A_k^\varepsilon = \mathrm{span}_F\{e_{11}, E, E^2, \ldots, E^{k-2}, e_{12}, e_{13}, \ldots, e_{1k}\}$$

be the subalgebra of $UT_k(F)$ where $E = \sum_{i=2}^{k-1} e_{i,i+1}$ and L acts on A_k^ε as the 1-dimensional Lie algebra spanned by the derivation $\varepsilon = \mathrm{ad}_{e_{11}}$. Note that $e_{11}^\varepsilon = (E^j)^\varepsilon = 0$, for all $1 \leq j \leq k-2$, and $e_{1i}^\varepsilon = e_{1i}$, for all $2 \leq i \leq k$.

We also denote by $(A_k^\varepsilon)^*$ the subalgebra of $UT_k(F)$ obtained by flipping A_k^ε along its secondary diagonal. Hence

$$(A_k^\varepsilon)^* = \mathrm{span}_F\{e_{kk}, E, E^2, \ldots, E^{k-2}, e_{1k}, e_{2k}, \ldots, e_{k-1,k}\}.$$

In this case, L acts on $(A_k^\varepsilon)^*$ as the 1-dimensional Lie algebra spanned by the derivation $\varepsilon = \mathrm{ad}_{e_{kk}}$. Notice that one can determine the L-polynomial identities of such an L-algebra via those of A_k^ε. In fact, if $f \in F\langle X|L\rangle$ and f^* is the L-polynomial obtained by reversing the order of the variables in each monomial of f, then one can easily check that $f \in \mathrm{Id}^L(A_k^\varepsilon)$ if and only if $f^* \in \mathrm{Id}^L((A_k^\varepsilon)^*)$. Notice that such a kind of algebras was first studied in the ordinary case in [5]

In what follows, we explicitly describe the L-identities of A_k^ε and $(A_k^\varepsilon)^*$ for any $k \geq 2$.

Lemma 5 ([30], Theorem 3): *Let $k = 2$, then:*

(1) $\mathrm{Id}^L(A_2^\varepsilon) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\, x_1^\varepsilon x_2,\, x_1 x_2^\varepsilon - x_2 x_1^\varepsilon - [x_1, x_2]\rangle_{T_L}$;
(2) $\mathrm{Id}^L((A_2^\varepsilon)^*) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\, x_1 x_2^\varepsilon,\, x_1^\varepsilon x_2 - x_2^\varepsilon x_1 - [x_1, x_2]\rangle_{T_L}$;
(3) $c_n^L(A_2^\varepsilon) = c_n^L((A_2^\varepsilon)^*) = n + 1$.

Lemma 6: *Let $k \geq 3$, then:*

(1) $\mathrm{Id}^L(A_k^\varepsilon) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\, x_1^\varepsilon x_2^\varepsilon,\, [x_1, x_2]^\varepsilon - [x_1, x_2],\; x_1^\varepsilon x_2 \cdots x_k\rangle_{T_L}$;
(2) $c_n^L(A_k^\varepsilon) = 2 + n + \sum_{l=0}^{k-2} \binom{n}{l}(n - l + 1) + \sum_{l=1}^{k-2} \sum_{j=2}^{n-l+1} \binom{n-j}{l-1}(j - 1)$
$\approx qn^{k-1}$,
for some $q > 0$.

Hence

$$\mathrm{Id}^L((A_k^\varepsilon)^*) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\, x_1^\varepsilon x_2^\varepsilon,\, [x_1, x_2]^\varepsilon - [x_1, x_2],\; x_1 \cdots x_{k-1} x_k^\varepsilon\rangle_{T_L}$$

and

$$c_n^L((A_k^\varepsilon)^*) = c_n^L(A_k^\varepsilon) \approx qn^{k-1}.$$

Proof. Write $I = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\, x_1^\varepsilon x_2^\varepsilon,\, [x_1, x_2]^\varepsilon - [x_1, x_2],\; x_1^\varepsilon x_2 \cdots x_k\rangle_{T_L}$. It is clear that $I \subseteq \mathrm{Id}^L(A_k^\varepsilon)$. In order to prove the opposite inclusion, first we find a set of generators of P_n^L modulo $P_n^L \cap I$, for all $n \geq 1$.

Let $f \in P_n^L$ be a multilinear L-polynomial of degree n. Because of the L-identities $x_1^{\varepsilon^2} - x_1^\varepsilon \equiv 0$ and $x_1^\varepsilon x_2^\varepsilon \equiv 0$, in each monomial of f there can occur at most one differential variable x_j^ε. Moreover,

$$[x_1, x_2]x_3^\varepsilon \equiv 0 \quad \text{and} \quad x_3^\varepsilon[x_1, x_2] \equiv 0$$

are consequences of $x_1^\varepsilon x_2^\varepsilon \equiv 0$ and $[x_1, x_2]^\varepsilon - [x_1, x_2] \equiv 0$. Furthermore, from $[x_1, x_2]^\varepsilon - [x_1, x_2] \equiv 0$ and $x_1^\varepsilon x_2 \cdots x_k \equiv 0$, it follows also that

$$[x_1, x_2]x_3 \cdots x_{k+1} \equiv 0.$$

Finally, since

$$[x_1, x_2][x_3, x_4] \equiv 0$$

is a consequence of $x_1^\varepsilon x_2^\varepsilon \equiv 0$ and $[x_1, x_2]^\varepsilon - [x_1, x_2] \equiv 0$, every left normed commutator $[x_{j_1}, \dots, x_{j_t}]$ can be written as a linear combination of $[x_{i_1}, \dots, x_{i_t}]$ where $i_1 > i_2 < \cdots < i_t$ (see for instance [2, Theorem 5.2.1]).

By taking into account the previous remarks plus the Poincaré–Birkhoff–Witt theorem, modulo I, f is a linear combination of L-polynomials of the type

$$\begin{aligned} &x_1 \cdots x_n, \qquad x_{i_1} \cdots x_{i_t} [x_i, x_j] x_{j_1} \cdots x_{j_l}, \\ &x_2 \cdots x_n x_1^\varepsilon, \quad x_{p_1} \cdots x_{p_r} x_m^\varepsilon x_{q_1} \cdots x_{q_s}, \end{aligned} \tag{1}$$

where $t + l = n - 2$, $r + s = n - 1$, $l < k - 1$, $s < k - 1$, $i > j < i_1 < \cdots < i_t$, $j_1 < \cdots < j_l$, $m < p_1 < \cdots < p_r$ and $q_1 < \cdots < q_s$. It follows that the space P_n^L is generated modulo $P_n^L \cap I$ by the above polynomials.

We next show that they are linearly independent modulo $\mathrm{Id}^L(A_k^\varepsilon)$. To that end, let $f \in \mathrm{Id}^L(A_k^\varepsilon)$ be a linear combination of such polynomials and write

$$\begin{aligned} f = &\alpha x_1 \cdots x_n + \beta x_2 \cdots x_n x_1^\varepsilon + \sum_{l<k-1} \sum_{I,J} \alpha_{I,J} x_{i_1} \cdots x_{i_t} [x_i, x_j] x_{j_1} \cdots x_{j_l} \\ &+ \sum_{s<k-1} \sum_{P,Q} \beta_{P,Q} x_{p_1} \cdots x_{p_r} x_m^\varepsilon x_{q_1} \cdots x_{q_s}, \end{aligned}$$

where

$$I = \{i, j, i_1, \dots, i_t\}, \quad J = \{j_1, \dots, j_l\}, \quad P = \{m, p_1, \dots, p_r\} \\ \text{and } Q = \{q_1, \dots, q_s\}$$

are disjoint sets of indices subjected to the above conditions.

First suppose that $\alpha \neq 0$. Then by making the evaluation $x_1 = \cdots = x_n = e_{11}$ one gets $\alpha = 0$, a contradiction.

Suppose that there exists $\alpha_{I,J} \neq 0$ for some $l < k-1$, I and J. Then by making the evaluation $x_i = e_{12}$, $x_j = x_{i_1} = \cdots = x_{i_t} = e_{11}$ and $x_{j_1} = \cdots = x_{j_l} = E$, we get $\alpha_{I,J} = 0$, a contradiction.

Now suppose that $\beta \neq 0$. Then if one considers the evaluation $x_1 = e_{12}$ and $x_2 = \cdots = x_n = e_{11}$, we get $\beta = 0$, a contradiction.

Finally, if $\beta_{P,Q} \neq 0$ for some $s < k - 1$, P and Q, then let $x_m = e_{12}$, $x_{p_1} = \cdots = x_{p_r} = e_{11}$ and $x_{q_1} = \cdots = x_{q_s} = E$, obtaining $\beta_{P,Q} = 0$, a contradiction.

Therefore the elements in (1) are linearly independent modulo $P_n^L \cap \mathrm{Id}^L(A_k^\varepsilon)$ and, since $P_n^L \cap \mathrm{Id}^L(A_k^\varepsilon) \supseteq P_n^L \cap I$, they form a basis of P_n^L modulo $P_n^L \cap \mathrm{Id}^L(A_k^\varepsilon)$ and $\mathrm{Id}^L(A_k^\varepsilon) = I$.

Thus, by counting we get

$$c_n^L(A_k^\varepsilon) = 2 + n + \sum_{l=0}^{k-2} \binom{n}{l}(n-l+1) + \sum_{l=1}^{k-2}\sum_{j=2}^{n-l+1} \binom{n-j}{l-1}(j-1) \approx qn^{k-1},$$

for some $q > 0$ and we are done.

Notice that, from the previous results, it follows also that

$$\mathrm{Id}^L((A_k^\varepsilon)^*) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2^\varepsilon,\ [x_1, x_2]^\varepsilon - [x_1, x_2],\ x_1 \cdots x_{k-1} x_k^\varepsilon \rangle_{T_L}$$

and $c_n^L((A_k^\varepsilon)^*) = c_n^L(A_k^\varepsilon) \approx qn^{k-1}$. ∎

We now introduce, for any fixed $k \geq 2$, a unitary L-algebra in $\mathrm{var}^L(UT_2^\varepsilon)$ the codimension sequence of which grows as n^{k-1}.

To this end, for all $k \geq 2$, let

$$N_k^\varepsilon = \mathrm{span}_F\{I, E, E^2, \ldots, E^{k-2}, e_{12}, e_{13}, \ldots, e_{1k}\}$$

where I is the identity $k \times k$ matrix and L acts on N_k^ε as the 1-dimensional Lie algebra spanned by the derivation $\varepsilon = \mathrm{ad}_{e_{11}}$. In this case $I^\varepsilon = (E^j)^\varepsilon = 0$, for all $1 \leq j \leq k-1$, and $e_{1i}^\varepsilon = e_{1i}$, for all $2 \leq i \leq k$.

Lemma 7: *Let $k \geq 2$, then:*

1. $\mathrm{Id}^L(N_k^\varepsilon) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2^\varepsilon,\ [x_1, x_2]^\varepsilon - [x_1, x_2],\ [x_1, \ldots, x_k] \rangle_{T_L}$;
2. $c_n^L(N_k^\varepsilon) = 1 + \sum_{j=1}^{k-1} \binom{n}{j} j \approx qn^{k-1}$, *for some $q > 0$.*

Proof. Let $Q = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2^\varepsilon,\ [x_1, x_2]^\varepsilon - [x_1, x_2],\ [x_1, \ldots, x_k] \rangle_{T_L}$. It is easily proved that $Q \subseteq \mathrm{Id}^L(N_k^\varepsilon)$.

Let now f be an L-identity of N_k^ε. We may assume that f is multilinear, and since N_k^ε is a unitary algebra, we may take f proper.

After reducing f modulo Q, we get that f is the zero polynomial if $\deg f \geq k$ and it is a linear combination of commutators

$$[x_1^\varepsilon, x_2, \ldots, x_n] \quad [x_i, x_1, \ldots, \widehat{x}_i, \ldots, x_n]$$

if $\deg f < k$, where $2 \leq i \leq n$ and the symbol $\widehat{x}_i$ means that the variable x_i is omitted.

Hence, modulo Q,

$$f = \alpha [x_1^\varepsilon, x_2, \ldots, x_n] + \sum_{i=2}^{n} \beta_i [x_i, x_1, \ldots, \widehat{x}_i, \ldots, x_n],$$

where $n \leq k-1$. We claim that such commutators are linearly independent modulo $\mathrm{Id}^L(N_k^\varepsilon)$, i.e., f is the zero polynomial modulo $\mathrm{Id}^L(N_k^\varepsilon)$ and this will imply that $Q = \mathrm{Id}^L(N_k^\varepsilon)$, as required.

Suppose that $\beta_i \neq 0$ for some i. Then we consider the evaluation $x_i = e_{12}$, $x_j = E$ for all $j \neq i$ and we get $\beta_i = 0$, a contradiction. Now, if $\alpha \neq 0$, then we make the evaluation $x_1 = \cdots = x_n = E$ and we get $\alpha = 0$, a contradiction. This says that $f \in Q$ and so $Q = \mathrm{Id}^L(N_k^\varepsilon)$ as claimed.

The arguments above also prove that

$$\gamma_j^L(N_k^\varepsilon) = \begin{cases} j, & \text{if } j \leq k-1, \\ 0, & \text{if } j \geq k. \end{cases}$$

Hence we also get that

$$c_n^L(N_k^\varepsilon) = 1 + \sum_{j=1}^{k-1} \binom{n}{j} \gamma_j^L(N_k^\varepsilon) = 1 + \sum_{j=1}^{k-1} \binom{n}{j} j \approx qn^{k-1},$$

for some $q > 0$. ∎

We want to highlight that the case $k=2$ was already studied in [30, Theorem 1]. Moreover, it is clear that if $k=1$, then $N_1^\varepsilon = F$, $\mathrm{Id}^L(N_1^\varepsilon) = \langle [x_1, x_2], x_1^\varepsilon \rangle_{T_L}$ and $c_n^L(N_1^\varepsilon) = 1$ for all $n \geq 1$.

4. On the structure of algebras generating L-subvarieties of $\mathrm{var}^L(UT_2^\varepsilon)$

In this section we shall study the structure of L-algebras belonging to the L-variety generated by UT_2^ε.

Notice that in what follows we may assume, without loss of generality, that L is a 1-dimensional Lie algebra spanned by ε.

We start by proving that any L-algebra inside $\mathrm{var}^L(UT_2^\varepsilon)$ satisfies the same L-identities of a finite-dimensional L-algebra.

Theorem 8: *If $A \in \mathrm{var}^L(UT_2^\varepsilon)$ is a finitely generated L-algebra over an algebraically closed field F of characteristic zero, then A is T_L-equivalent to a finite-dimensional L-algebra over F.*

Proof. If $A \in \mathrm{var}^L(UT_2^\varepsilon)$, then by Theorem 2, $x^{\varepsilon^2} - x^\varepsilon \in \mathrm{Id}^L(A)$. Hence $U(L)$ acts on A as the 2-dimensional semisimple Hopf algebra H with basis $\{1_H, \bar{\varepsilon}\}$ where $\bar{\varepsilon}^2 = \bar{\varepsilon}$. Thus A can be regarded as an algebra with H-action and we may restrict the T_L-ideal $\mathrm{Id}^L(A)$ to the T_H-ideal $\mathrm{Id}^H(A)$. Thus the claim follows from [18, Theorem 1.1]. ∎

We refer the reader to [13, 18] for an account on algebras with a Hopf algebra action and the related theory of polynomial identities.

Now we recall the following result characterizing the nth L-cocharacter of UT_2^ε.

Theorem 9 ([9], Theorem 12): *If $\chi_n^L(UT_2^\varepsilon) = \sum_{\lambda\vdash n} m_\lambda\chi_\lambda$ is the nth cocharacter of UT_2^ε, then*

$$m_\lambda = \begin{cases} n+1, & \lambda = (n), \\ 2(q+1), & \lambda = (p+q,p), \\ q+1, & \lambda = (p+q,p,1), \\ 0, & \text{otherwise}, \end{cases}$$

where $p, q \geq 0$.

In order to characterize the L-subvariety of $\mathrm{var}^L(UT_2^\varepsilon)$ we are going to prove the following.

Theorem 10: *If $A \in \mathrm{var}^L(UT_2^\varepsilon)$, then A is T_L-equivalent to a finitely generated L-algebra.*

Proof. Let B be the relatively free algebra of $\mathrm{var}^L(A)$ with 3 generators. We claim that $\mathrm{Id}^L(A) = \mathrm{Id}^L(B)$. Clearly $\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(B)$, thus we shall prove the opposite inclusion.

Let $f \in \mathrm{Id}^L(B)$ be a multilinear polynomial of degree n and let M be the S_n-module generated by f. Without loss of generality, we may assume that M is irreducible. In fact, if $M = M_1 \oplus \cdots \oplus M_k$ is the decomposition into irreducible components, where M_i is generated by f_i as S_n-module, $1 \leq i \leq k$, then $f_i \in \mathrm{Id}^L(A)$ for all i implies that also $f \in \mathrm{Id}^L(A)$.

Let χ_λ be the irreducible character of M, where $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, and let

$$e_{T_\lambda} = \sum_{\substack{\tau\in R_{T_\lambda} \\ \sigma\in C_{T_\lambda}}} (\operatorname{sgn}\sigma)\tau\sigma$$

be the corresponding essential idempotent (see for instance [12, Chapter 2]). Here recall that R_{T_λ} and C_{T_λ} stand for the row-stabilizer and the column-stabilizer of the Young tableau T_λ, respectively.

If $\lambda_4 \neq 0$ or $\lambda_3 > 1$, then by Theorem 9, it follows that $f \in \mathrm{Id}^L(A)$. Thus we may assume that $\lambda_4 = 0$ and $\lambda_3 \leq 1$.

Now consider $g = (\sum_{\tau\in R_{T_\lambda}} \tau)f$ and notice that g is symmetric in at most two disjoint subsets X_1, X_2 of differential variables. If we identify all the variables of X_1 with x_1 and all the variables of X_2 with x_2 in g, we obtain the homogeneous polynomial $p = p(x_1, x_2, x_3)$ which is still an L-identity of B. But from the definition of relatively free algebra, it follows that $p \in \mathrm{Id}^L(A)$. By multilinearizing the polynomial p, we get the polynomial $\lambda_1!\lambda_2!g(x_1, \dots x_n)$. Hence $g \in \mathrm{Id}^L(A)$ and, since M is irreducible and $g \neq 0$, it follows that also $f \in \mathrm{Id}^L(A)$. This completes the proof. ∎

As a consequence of Theorems 8 and 10 we have the following.

Corollary 11: *If $A \in \mathrm{var}^L(UT_2^\varepsilon)$ is an L-algebra over an algebraically closed field F of characteristic zero, then $\mathrm{var}^L(A) = \mathrm{var}^L(B)$ for some finite-dimensional L-algebra B over F.*

According to Corollary 11, from now on we will always assume, without loss of generality, that if $\mathcal{V} \subseteq \mathrm{var}^L(UT_2^\varepsilon)$, then $\mathcal{V} = \mathrm{var}^L(A)$ where A is a finite-dimensional L-algebra.

Now we are going to describe the structure of such finite-dimensional L-algebras belonging to $\mathrm{var}^L(UT_2^\varepsilon)$.

First we recall some definitions. A subalgebra (ideal) B of A is an L-subalgebra (ideal) if it is a subalgebra (ideal) such that $B^L \subseteq B$, where B^L denotes the set of all $h(b)$, for all $b \in B$ and $h \in U(L)$.

Let A be a finite-dimensional L-algebra over an algebraically closed field. By the Wedderburn–Malcev Theorem for associative algebras, we can write

$$A = B + J \tag{2}$$

where B is a maximal semisimple unitary subalgebra of A and $J = J(A)$ is its Jacobson radical. Notice that although J is an L-invariant ideal of A (see [15]), it may not exist as an L-invariant Wedderburn–Malcev decomposition, i.e., it may happen that all semisimple subalgebras B of A that satisfy (2) are not L-subalgebras of A. For example, the algebra UT_2^δ of 2×2 upper triangular matrices where L acts as the 1-dimensional Lie algebra spanned by the inner derivation δ induced by e_{12} has no L-invariant Wedderburn–Malcev decomposition (see [31, Example 3]). Things are different inside $\mathrm{var}^L(UT_2^\varepsilon)$, in fact at the end of the section, we will prove that, up to T_L-equivalence, we can always assume that a subvariety of $\mathrm{var}^L(UT_2^\varepsilon)$ is generated by an L-algebra with an L-invariant Wedderburn–Malcev decomposition.

To this end, first recall that J can be decomposed into a direct sum of B-bimodules

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$$

where, for $i,k \in \{0,1\}$, J_{ik} is a left faithful module or a 0-left module according as $i=1$ or $i=0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k=1$ or $k=0$, respectively. Moreover, for $i,k,l,m \in \{0,1\}$, $J_{ik}J_{lm} \subseteq \delta_{k,l}J_{im}$ where $\delta_{k,l}$ is the Kronecker delta and $J_{11} = BN$ for some nilpotent subalgebra N of A commuting with B. For a proof of this result see [11, Lemma 2].

Proposition 12: *Let $A = B + J$ be a finite-dimensional L-algebra. If $\delta \in \mathrm{Der}(A)$, then $1_B^\delta \in J_{01} + J_{10}$. Moreover, $J_{00}^\delta \subseteq J_{00} + J_{01} + J_{10}$, $J_{01}^\delta \subseteq J_{00} + J_{01} + J_{11}$, $J_{10}^\delta \subseteq J_{00} + J_{10} + J_{11}$ and $J_{11}^\delta \subseteq J_{01} + J_{10} + J_{11}$.*

Proof. Since $\delta \in \mathrm{Der}(A)$, by [15, Theorem 4.3] $\delta = \mathrm{ad}_b + \mathrm{ad}_j + \delta'$, where $b \in B$, $j \in J$ and $\delta' \in \mathrm{Der}(A)$ is such that $B^{\delta'} = 0$. Thus since

$$[J_{11}, 1_B] = J_{00}1_B = 1_B J_{00} = 0,$$

we get $1_B^\delta = 1_B^{\mathrm{ad}_j} \in J_{01} + J_{10}$.

Let $j_{00} \in J_{00}$. Since $1_B J_{00} = 0$, we get $0 = 1_B^\delta j_{00} + 1_B j_{00}^\delta$ and so it follows that $1_B j_{00}^\delta \in J_{10}$. On the other hand, since $J_{00}1_B = 0$, we have $0 = j_{00}^\delta 1_B + j_{00}1_B^\delta$. Then

$$j_{00}^\delta 1_B \in J_{01}.$$

Thus it follows that $J_{00}^\delta \subseteq J_{00} + J_{01} + J_{10}$.

Let now $j_{11} \in J_{11}$. Then

$$j_{11}^\delta = (j_{11}1_B)^\delta = j_{11}^\delta 1_B + j_{11}1_B^\delta \in J_{01} + J_{10} + J_{11}.$$

Thus we get $J_{11}^\delta \subseteq J_{01} + J_{10} + J_{11}$. Similarly it can be proved for J_{01} and J_{10}. ∎

In case of algebras belonging to $\mathrm{var}^L(UT_2^\varepsilon)$, the action of L on J and its components can be assumed to be much more simpler.

Lemma 13: *If $A = B + J \in \mathrm{var}^L(UT_2^\varepsilon)$ with $J = J_{00} + J_{10} + J_{01} + J_{11}$, then $j^\varepsilon = j$ for all $j \in J_{01} \cup J_{10}$.*

Proof. If $j \in J_{01}$ (resp. $j \in J_{10}$), then $j = [j, 1_B]$ (resp. $j = [1_B, j]$). Thus the claim follows since $[x_1,x_2]^\varepsilon - [x_1,x_2] \in \mathrm{Id}^L(A)$. ∎

Lemma 14: *Let $A = B + J \in \mathrm{var}^L(UT_2^\varepsilon)$. Then*

$$\begin{aligned} J_{00}^\varepsilon J_{01} &= J_{10} J_{00}^\varepsilon = J_{11}^\varepsilon J_{10} \\ &= J_{01} J_{11}^\varepsilon = J_{01} J_{10} \\ &= J_{10} J_{01} = J_{01}[J_{11}, J_{11}] \\ &= [J_{11}, J_{11}] J_{10} = [J_{00}, J_{00}] J_{01} \\ &= J_{10}[J_{00}, J_{00}] = 0. \end{aligned}$$

Proof. Since $[x_1, x_2]^\varepsilon - [x_1, x_2] \equiv 0$ and $x_1^\varepsilon x_2^\varepsilon \equiv 0$ on $\mathrm{var}^L(UT_2^\varepsilon)$, the result immediately follows applying Lemma 13. ∎

Theorem 15: *If $A = B + J \in \mathrm{var}^L(UT_2^\varepsilon)$ then, up to T_L-equivalence, $B^L \subseteq B$.*

Proof. If $J = 0$ there is nothing to prove, so let $J \neq 0$ and since $A \in \mathrm{var}^L(UT_2^\varepsilon)$ then $B = F \oplus \cdots \oplus F$. By Proposition 12 it readily follows that if either $1_B^\varepsilon = 0$ or $J_{01} = J_{10} = 0$, then $B^\varepsilon \subseteq B$ and we are done also in this case.

So, suppose $1_B^\varepsilon = j \neq 0$, where $j \in J_{01} + J_{10}$, and consider

$$\varepsilon' = \varepsilon - \mathrm{ad}_j \in \mathrm{Der}(A).$$

Note that $\varepsilon' \neq 0$, in fact if $\varepsilon' = 0$, then $\varepsilon = \mathrm{ad}_j$ and $1_B^{\varepsilon^2} = j^\varepsilon = 0$, since $J_{01}^2 = J_{10}^2 = 0$ and, by Lemma 14, $J_{01} J_{10} = J_{10} J_{01} = 0$. This is a contradiction since $x^{\varepsilon^2} - x^\varepsilon \in \mathrm{Id}^L(A)$.

Let now $A_{\varepsilon'}$ be the L-algebra A where L acts on it as the 1-dimensional Lie algebra spanned by ε'. Clearly $1_B^{\varepsilon'} = 0$, $B^{\varepsilon'} \subseteq B$ and a straightforward computation can also prove that $A_{\varepsilon'} \in \mathrm{var}^L(UT_2^\varepsilon)$. We claim that $\mathrm{Id}^L(A) = \mathrm{Id}^L(A_{\varepsilon'})$ and this will complete the proof.

Let $f \in \mathrm{Id}^L(A)$ be a multilinear polynomial of degree n. According to [9, Theorem 5] we can write f as

$$\begin{aligned} f = &\alpha x_1 \cdots x_n + \sum_{k=1}^{n} \beta_k x_{i_1} \cdots x_{i_{n-1}} x_k^\varepsilon \\ &+ \sum_{P,t} \gamma_{P,t} x_{p_1} \cdots x_{p_m} [x_t^\varepsilon, x_{j_1}, \ldots, x_{j_{n-m-1}}] + g, \end{aligned}$$

where

$$g \in \mathrm{Id}^L(UT_2^\varepsilon) \subseteq \mathrm{Id}^L(A),$$

and $i_1 < \cdots < i_{n-1}$, $p_1 < \cdots < p_m$, $j_1 < \cdots < j_{n-m-1}$. Notice that if we make the evaluation $x_1 = \cdots = x_n = 1_B$, we get $\alpha = 0$. Thus

$$\begin{aligned} f = & \sum_{k=1}^{n} \beta_k x_{i_1} \cdots x_{i_{n-1}} x_k^{\varepsilon} \\ & + \sum_{P,t} \gamma_{P,t} x_{p_1} \cdots x_{p_m} [x_t^{\varepsilon}, x_{j_1}, \ldots, x_{j_{n-m-1}}] + g \in \mathrm{Id}^L(A). \end{aligned} \tag{3}$$

In order to prove that $f \in \mathrm{Id}^L(A_{\varepsilon'})$, we have to show that

$$f = \sum_{k=1}^{n} \beta_k x_{i_1} \cdots x_{i_{n-1}} x_k^{\varepsilon'} + \sum_{P,t} \gamma_{P,t} x_{p_1} \cdots x_{p_m} [x_t^{\varepsilon'}, x_{j_1}, \ldots, x_{j_{n-m-1}}] + \tilde{g}$$

vanishes under every evaluation of elements of A. Here $\tilde{g}$ stands for the polynomial g in which we substituted every differential variable x^{ε} with $x^{\varepsilon'}$.

Since $\varepsilon' = \varepsilon - \mathrm{ad}_j$, it is enough to prove that

$$\sum_{k=1}^{n} \beta_k x_{i_1} \cdots x_{i_{n-1}} x_k^{\mathrm{ad}_j} + \sum_{P,t} \gamma_{P,t} x_{p_1} \cdots x_{p_m} [x_t^{\mathrm{ad}_j}, x_{j_1}, \ldots, x_{j_{n-m-1}}] \in \mathrm{Id}^L(A_{\varepsilon'}).$$

But by definition of inner derivation, the claim follows since

$$[j, x]^{\varepsilon} - [j, x] \in \mathrm{Id}^L(A)$$

and (3) holds. Hence $f \in \mathrm{Id}^L(A_{\varepsilon'})$.

Similarly it can be proved that $\mathrm{Id}^L(A_{\varepsilon'}) \subseteq \mathrm{Id}^L(A)$. Thus $A \sim_{T_L} A_{\varepsilon'}$ and the claim is proved. ∎

As a consequence of Proposition 12, Lemma 13 and Theorem 15 we get the following.

Corollary 16: *If $A = B + J \in \mathrm{var}^L(UT_2^{\varepsilon})$ with $J = J_{00} + J_{10} + J_{01} + J_{11}$, then $J_{ik}^L \subseteq J_{ik}$, for all $i, k \in \{0, 1\}$.*

According to the previous results, from now on we will assume that the Wedderburn–Malcev decomposition and the Jacobson radical decomposition into bimodules of every considered L-algebra are L-invariant.

5. On minimal L-subvarieties in $\mathrm{var}^L(UT_2^\varepsilon)$

In this section we shall prove that the L-algebras A_k^ε, $(A_k^\varepsilon)^*$ and N_k^ε introduced in Section 3 generate minimal L-varieties of polynomial growth. We start with the definition of minimal L-variety.

Definition 1: An L-variety $\mathcal{V}$ is said to be minimal of polynomial growth if $c_n^L(\mathcal{V}) \approx qn^k$, for some $q > 0$, and for any proper L-subvariety $\mathcal{U} \subsetneq \mathcal{V}$, we have that $c_n^L(\mathcal{U}) \approx q'n^t$ with $t < k$.

Algebras generating minimal varieties will play an important role in the main result, since we shall prove that any L-algebra inside $\mathrm{var}^L(UT_2^\varepsilon)$ has the same differential identities of a direct sum of such kind of algebra plus a nilpotent algebra, eventually.

Remark 17 ([25], Remark 2): Let $A = F + J$ be an L-algebra with

$$J = J_{00} + J_{10} + J_{01} + J_{11}.$$

If A satisfies the identity $[x_1, \dots, x_t] \equiv 0$ for some $t \geq 2$, then $J_{01} = J_{10} = 0$.

Proof. The proof immediately follows from the fact that $J_{10} = [J_{10}, \underbrace{F, \dots, F}_{t-1}]$ and $J_{01} = [J_{01}, \underbrace{F, \dots, F}_{t-1}]$. ∎

THEOREM 18: *For all $k \geq 2$, N_k^ε generates a minimal L-variety of polynomial growth.*

Proof. Suppose that $A \in \mathrm{var}^L(N_k^\varepsilon)$ and $c_n^L(A) \approx qn^{k-1}$ for some $q > 0$. We shall prove that $A \sim_{T_L} N_k^\varepsilon$ and this will complete the proof.

Since $c_n^L(A)$ is polynomially bounded, by [31, Theorem 10] we may assume that

$$A = B_1 \oplus \cdots \oplus B_m$$

where $B_1, \dots, B_m$ are finite-dimensional L-algebras such that

$$\dim_F \frac{B_i}{J(B_i)} \leq 1,$$

for all $1 \leq i \leq m$. This implies that either $B_i \cong F + J(B_i)$ or $B_i \cong J(B_i)$ is a nilpotent algebra. Moreover, since

$$c_n^L(A) \leq c_n^L(B_1) + \cdots + c_n^L(B_m),$$

then there exists B_i such that $c_n^L(B_i) \approx bn^{k-1}$, for some $b > 0$. Thus

$$\mathrm{var}^L(N_k^\varepsilon) \supseteq \mathrm{var}^L(A) \supseteq \mathrm{var}^L(F + J(B_i)) \supseteq \mathrm{var}^L(F + J_{11}(B_i))$$

and $c_n^L(F+J(B_i)) \approx bn^{k-1.}$ Here we remark that $F+J_{11}(B_i)$ is an L-subalgebra of $F+J(B_i)$ since, according to Theorem 15, in $\mathrm{var}^L(UT_2^\varepsilon)$ we may assume $F^\varepsilon=0$ and $J_{ij}^\varepsilon \subseteq J_{ij}$ for all $i, j \in \{0, 1\}$.

Moreover, by Remark 17, we get that $J_{10}(B_i) = J_{01}(B_i) = 0$ and so

$$F + J(B_i) = \Big(F + J_{11}(B_i)\Big) \oplus J_{00}(B_i),$$

as L-algebras, and $c_n^L(F + J(B_i)) = c_n^L(F + J_{11}(B_i))$ for n large enough.

It turns out that, in order to prove $A \sim_{T_L} N_k^\varepsilon$, it suffices to show that $F + J_{11}(B_i) \sim_{T_L} N_k^\varepsilon$. Hence we assume, as we may, that A is a unitary L-algebra and we shall look at its proper codimension and cocharacter sequences.

Since $c_n^L(A) \approx qn^{k-1}$, then

$$c_n^L(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^L(A)$$

and, by Corollary 4, $\gamma_i^L(A) \neq 0$ for all $0 \leq i \leq k-1$ and $\gamma_i^L(A) = 0$ for all $i \geq k$.

Moreover, recall that since $\mathrm{Id}^L(N_k^\varepsilon) \subseteq \mathrm{Id}^L(A)$, then $\frac{\Gamma_i^L}{\Gamma_i^L \cap \mathrm{Id}^L(A)}$ is isomorphic to a quotient module of

$$\frac{\Gamma_i^L}{\Gamma_i^L \cap \mathrm{Id}^L(N_k^\varepsilon)}.$$

Thus if

$$\psi_i^L(A) = \sum_{\lambda \vdash i} m_\lambda \chi_\lambda \quad \text{and} \quad \psi_i^L(N_k^\varepsilon) = \sum_{\lambda \vdash i} m'_\lambda \chi_\lambda$$

are the i-th proper L-cocharacters of A and N_k^ε, respectively, then $m_\lambda \leq m'_\lambda$ for all $\lambda \vdash i$.

From now on, suppose $k \geq 3$. For all $3 \leq i \leq k-1$, let

$$f_1 = [x_1, \underbrace{x_2, \dots, x_2}_{i-1}] \quad \text{and} \quad f_2 = [x_1^\varepsilon, \underbrace{x_1, \dots, x_1}_{i-1}]$$

be the highest weight vectors corresponding to the partitions $\lambda_1 = (i-1, 1)$ and $\lambda_2 = (i)$, respectively. It is clear that f_1 and f_2 are not differential identities of N_k^ε, thus $\chi_{(i-1,1)}$ and $\chi_{(i)}$ participate in the i-th proper L-cocharacter of N_k^ε with non-zero multiplicities.

Moreover, since $\gamma_i^L(N_k^\varepsilon) = i = \deg \chi_{(i-1,1)} + \deg \chi_{(i)}$, for all $2 \le i \le k-1$, we get that

$$\psi_i^L(N_k^\varepsilon) = \chi_{(i-1,1)} + \chi_{(i)}.$$

Now, since $\gamma_{k-1}^L(A) \neq 0$ then either $\psi_{k-1}^L(A) = \chi_{(k-1)}$ or $\psi_{k-1}^L(A) = \chi_{(k-2,1)}$ or $\psi_{k-1}^L(A) = \chi_{(k-1)} + \chi_{(k-2,1)}$.

First suppose that $\psi_{k-1}^L(A) = \chi_{(k-1)}$. Then $[x_1, \underbrace{x_2, \dots, x_2}_{k-2}] \equiv 0$ on A and this trivially implies $[x_1^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}] \equiv 0$ on A. Thus $\psi_{k-1}^L(A) = 0$ and $\gamma_{k-1}^L(A) = 0$, a contradiction.

Now suppose $\psi_{k-1}^L(A) = \chi_{(k-2,1)}$, then $[x_1^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}] \equiv 0$ on A. Let us substitute the variable x_1 with $x_1 + x_2$ and consider the multihomogeneous component with degree $k-2$ in x_1 and 1 in x_2. As a consequence, we get the following identity modulo $\mathrm{Id}^L(UT_2^\varepsilon)$:

$$[x_2^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}] + (k-2)[x_1^\varepsilon, x_2, \underbrace{x_1, \dots, x_1}_{k-3}] \equiv 0. \tag{4}$$

Since $[x_1, x_2] - [x_1^\varepsilon, x_2] - [x_1, x_2^\varepsilon] \in \mathrm{Id}^L(UT_2^\varepsilon) \subseteq \mathrm{Id}^L(A)$, we get

$$[x_2, x_1^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-3}] \equiv [x_2, \underbrace{x_1, \dots, x_1}_{k-2}] - [x_2^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}].$$

By putting together the latter relation with (4) we get the identity

$$(k-2)[x_2, \underbrace{x_1, \dots, x_1}_{k-2}] \equiv (k-1)[x_2^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}]. \tag{5}$$

Moreover, by substituting the variable x_2 with x_2^ε in (4) and recalling that $x_2^{\varepsilon^2} \equiv x_2^\varepsilon$, we also obtain $[x_2^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}] \equiv 0$. From this plus identity (5), we finally get the identity $[x_2, \underbrace{x_1, \dots, x_1}_{k-2}] \equiv 0$, thus $\psi_{k-1}^L(A) = 0$ and $\gamma_{k-1}^L(A) = 0$, a contradiction. Hence we must have $\psi_{k-1}^L(A) = \chi_{(k-1)} + \chi_{(k-2,1)}$.

Now, for all $2 \le i \le k-2$, as before either $\psi_i^L(A) = \chi_{(i)}$ or $\psi_i^L(A) = \chi_{(i-1,1)}$ or $\psi_i^L(A) = \chi_{(i)} + \chi_{(i-1,1)}$.

If $\psi_i^L(A) = \chi_{(i)}$ then $[x_1, \underbrace{x_2, \dots, x_2}_{i-1}] \equiv 0$ on A. Thus also $[x_1, \underbrace{x_2, \dots, x_2}_{k-2}] \equiv 0$, which is absurd, for the first part of the proof. Analogously, if $\psi_i^L(A) = \chi_{(i-1,1)}$ then $[x_1^\varepsilon, \underbrace{x_1, \dots, x_1}_{i-1}] \equiv 0$ on A and so $[x_1^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}] \equiv 0$, a contradiction.

Thus $\psi_i^L(A) = \chi_{(i)} + \chi_{(i-1,1)} = \psi_i^L(N_k^\varepsilon)$, for all $1 \le i \le k-1$ and

$$c_n^L(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^L(A) = 1 + \sum_{i=1}^{k-1} \binom{n}{i} i = c_n^L(N_k^\varepsilon).$$

Hence A and N_k^ε have the same codimension sequence and, since

$$\mathrm{Id}^L(N_k^\varepsilon) \subseteq \mathrm{Id}^L(A),$$

we get the equality $\mathrm{Id}^L(N_k^\varepsilon) = \mathrm{Id}^L(A)$, as required.

Notice that if $k=2$, then $\psi_2(N_2^\varepsilon) = \chi_{(1,1)} + \chi_{(2)}$ and with similar arguments as in the first part of the proof we get $\psi_2(A) = \psi_2(N_2^\varepsilon)$, $c_n^L(A) = c_n^L(N_2^\varepsilon)$ and so $A \sim_{T_L} N_2^\varepsilon$. ∎

We now recall a result about the Jacobson radical of an algebra belonging to $\mathrm{var}^L(A_k^\varepsilon)$ that will be very useful hereafter.

Lemma 19: *Let $A = F + J \in \mathrm{var}^L(A_k^\varepsilon)$ (resp. $A = F + J \in \mathrm{var}^L((A_k^\varepsilon)^*)$). Then*

$$J_{11}^\varepsilon = 0 \quad \textit{and} \quad J_{01} = [J_{11}, J_{11}] = 0$$

(resp. $J_{10} = [J_{11}, J_{11}] = 0$).

Proof. We will prove the statement in case $A \in \mathrm{var}^L(A_k^\varepsilon)$. The other one will follow analogously.

Recall that according to Corollary 16, $J_{11}^\varepsilon \subseteq J_{11}$. Moreover, since

$$x_1^\varepsilon x_2 \cdots x_k \in \mathrm{Id}^L(A_k^\varepsilon) \subseteq \mathrm{Id}^L(A),$$

for all $j \in J_{11}$ we get $j^\varepsilon \underbrace{1_F \cdots 1_F}_{k-1} = 0$. Thus, if we let $j^\varepsilon = \tilde{j} \in J_{11}$ then

$$0 = \tilde{j} \underbrace{1_F \cdots 1_F}_{k-1} = \tilde{j},$$

since 1_F acts as a unit element on J_{11}. Now, due to the identity

$$[x_1, x_2]^\varepsilon - [x_1, x_2] \equiv 0,$$

we get also $[J_{11}, J_{11}] = 0$.

Finally, by Lemma 13, for all $a \in J_{01}$, $a^\varepsilon = a$. Thus by using the same argument as before, we get that $J_{01} = J_{01}^\varepsilon = 0$. ∎

LEMMA 20: *Let $A = F + J \in \mathrm{var}^L(A_k^\varepsilon)$ (resp. $A = F + J \in \mathrm{var}^L((A_k^\varepsilon)^*)$). If $c_n^L(A) \approx qn^{k-1}$, for some $q > 0$, then $A \sim_{T_L} A_k^\varepsilon$ (resp. $A \sim_{T_L} (A_k^\varepsilon)^*$).*

Proof. We prove the statement for

$$A = F + J \in \mathrm{var}^L(A_k^\varepsilon).$$

The case $A = F + J \in \mathrm{var}^L((A_k^\varepsilon)^*)$ will follow with similar arguments.

By the previous Lemma,

$$J_{01} = [J_{11}, J_{11}] = 0,$$

so we may assume $A = F + J_{00} + J_{10} + J_{11}$ and J_{11} commutative. Moreover, $J_{11}^\varepsilon = 0$.

First suppose that $J_{10}J_{00}^{k-2} = 0$.

If $J^m = 0$, then for all $n \geq m$ we shall prove that

$$g = x_k \cdots x_n x_1^\varepsilon x_2 \cdots x_{k-1} \in \mathrm{Id}^L(A).$$

Since such a monomial is multilinear, we can evaluate each variable in a basis of A consisting of a union of a basis of J_{00}, J_{10}, J_{11} and 1_F. Since $n \geq m$ and $J^m = 0$, if we evaluate all the variables in J then we get zero, thus at least one variable must be evaluated in 1_F.

Let us focus our attention on the variable x_1. It is clear that if x_1 is evaluated in 1_F or on J_{11}, then g vanishes since $F^\varepsilon = J_{11}^\varepsilon = 0$. If we evaluate x_1 in an element $j_{10} \in J_{10}$, then $j_{10}^\varepsilon = j_{10}$ and we are forced to evaluate $x_2, \dots, x_{k-1}$ on elements of J_{00}. Since $J_{10}J_{00}^{k-2} = 0$, we get zero. Finally, let us evaluate x_1 on an element $j_{00} \in J_{00}$. Then $j_{00}^\varepsilon \in J_{00}$ and since there exists t such that x_t is evaluated in 1_F, also in this case we get zero.

Therefore we have proved that $x_k \cdots x_n x_1^\varepsilon x_2 \cdots x_{k-1} \in \mathrm{Id}^L(A)$. From this identity and from $[x_1, x_2]^\varepsilon - [x_1, x_2] \equiv 0$ it follows also that

$$x_{k+1} \cdots x_n [x_1, x_2] x_3 \cdots x_k \in \mathrm{Id}^L(A).$$

Since $A \in \mathrm{var}^L(A_k^\varepsilon)$, if $f \in P_n^L$ with $\deg f = n \geq m$, then after reducing f modulo the T_L-ideal generated by the differential identities of A_k^ε and by g, by using also Lemma 6, we have that f is a linear combination of the L-polynomials

$$x_1 \cdots x_n, \quad x_{i_1} \cdots x_{i_t}[x_i, x_j]x_{j_1} \cdots x_{j_l},$$
$$x_2 \cdots x_n x_1^\varepsilon, \quad x_{p_1} \cdots x_{p_r} x_m^\varepsilon x_{q_1} \cdots x_{q_s},$$

where $t+l=n-2$, $r+s=n-1$, $l<k-2$, $s<k-2$, $i>j<i_1<\cdots<i_t$, $j_1<\cdots<j_l$, $m<p_1<\cdots<p_r$ and $q_1<\cdots<q_s$. Note that $l,s<k-2$ since $g\equiv 0$ on A.

Therefore

$$c_n^L(A)\leq 2+n+\sum_{l=0}^{k-3}\binom{n}{l}(n-l+1)+\sum_{l=1}^{k-3}\sum_{j=2}^{n-l+1}\binom{n-j}{l-1}(j-1)\approx q'n^{k-2},$$

for some $q'>0$. This is a contradiction, since we are assuming that $c_n^L(A)\approx qn^{k-1}$.

Thus $J_{10}J_{00}^{k-2}\neq 0$ and there exist $a\in J_{10}$ and $b_1,\dots,b_{k-2}\in J_{00}$ such that $ab_1\cdots b_{k-2}\neq 0$. Let $f\in \mathrm{Id}^L(A)$ be a multilinear L-polynomial of degree n. By Lemma 6, f modulo $\mathrm{Id}^L(A_k^\varepsilon)$ can be written as

$$\begin{aligned} f=&\alpha x_1\cdots x_n+\beta x_2\cdots x_n x_1^\varepsilon+\sum_{l<k-1}\sum_{I,J}\alpha_{I,J}x_{i_1}\cdots x_{i_t}[x_i,x_j]x_{j_1}\cdots x_{j_l}\\ &+\sum_{s<k-1}\sum_{P,Q}\beta_{P,Q}x_{p_1}\cdots x_{p_r}x_m^\varepsilon x_{q_1}\cdots x_{q_s}+f', \end{aligned}$$

where $f'\in\mathrm{Id}^L(A_k^\varepsilon)$, $I=\{i,j,i_1,\dots,i_t\}$, $J=\{j_1,\dots,j_l\}$, $P=\{m,p_1,\dots,p_r\}$ and $Q=\{q_1,\dots,q_s\}$ with $t+l=n-2$, $r+s=n-1$, $l<k-1$, $s<k-1$, $i>j<i_1<\cdots<i_t$, $j_1<\cdots<j_l$, $m<p_1<\cdots<p_r$ and $q_1<\cdots<q_s$.

By choosing $x_1=\cdots=x_n=1_F$ we get $\alpha=0$. Moreover, by induction on l, for fixed I and J, the evaluation $x_i=a$, $x_j=x_{i_1}=\cdots=x_{i_t}=1_F$ and $x_{j_h}=b_h$, for all $1\leq h\leq l$, gives $\alpha_{I,J}=0$. If we let $x_1=a$ and $x_2=\cdots=x_n=1_F$, then we get $\beta=0$. Finally, by induction on s, for fixed P and Q, the evaluation $x_m=a$, $x_{p_1}=\cdots=x_{p_r}=1_F$ and $x_{q_h}=b_h$, for all $1\leq h\leq s$, gives $\beta_{P,Q}=0$.

Thus $f=f'\in\mathrm{Id}^L(A_k^\varepsilon)$ and $\mathrm{Id}^L(A_k^\varepsilon)=\mathrm{Id}^L(A)$, as claimed. ∎

We are now in a position to prove that A_k^ε and $(A_k^\varepsilon)^*$ generate minimal L-varieties.

Theorem 21: *For all $k\geq 2$, A_k^ε and $(A_k^\varepsilon)^*$ generate minimal L-varieties of polynomial growth.*

Proof. Let $A\in\mathrm{var}^L(A_k^\varepsilon)$ such that $c_n^L(A)\approx qn^{k-1}$, for some $q>0$. By [31, Theorem 10] we assume

$$A=B_1\oplus\cdots\oplus B_m$$

where $B_1,\dots,B_m$ are finite-dimensional L-algebras such that $\dim_F\frac{B_i}{J(B_i)}\leq 1$. This says that either $B_i\cong F+J(B_i)$ or $B_i\cong J(B_i)$ is a nilpotent L-algebra, for all $1 \leq i \leq m$. Since

$$c_n^L(A) \leq c_n^L(B_1) + \cdots + c_n^L(B_m),$$

there exists B_i such that $c_n^L(B_i) \approx bn^{k-1}$, for some $b > 0$. Thus $B_i = F + J(B_i)$ and, by the previous Lemma, $B_i \sim_{T_L} A_k^\varepsilon$. Hence

$$\mathrm{var}^L(A_k^\varepsilon) = \mathrm{var}^L(B_i) \subseteq \mathrm{var}^L(A) \subseteq \mathrm{var}^L(A_k^\varepsilon)$$

and so $\mathrm{var}^L(A) = \mathrm{var}^L(A_k^\varepsilon)$.

Similarly one can prove the statement for $(A_k^\varepsilon)^*$. ∎

6. Classifying subvarieties of var$^L(UT_2^\varepsilon)$

In this section we present the main result about the L-variety generated by UT_2^ε, i.e., we will classify up to T_L-equivalence all the L-algebras generating L-subvarieties of $\mathrm{var}^L(UT_2^\varepsilon)$.

To this end, we start with the following lemma concerning algebras with slow codimension growth.

Lemma 22: *Let $A = F + J_{10} + J_{11} \in \mathrm{var}^L(UT_2^\varepsilon)$ with $J_{10} \neq 0$ (resp. $A = F + J_{01} + J_{11} \in \mathrm{var}^L(UT_2^\varepsilon)$ with $J_{01} \neq 0$). If $J_{11}^\varepsilon = 0$, then $A \sim_{T_L} A_2^\varepsilon$ (resp. $A \sim_{T_L} (A_2^\varepsilon)^*$).*

Proof. Since $F^\varepsilon = J_{11}^\varepsilon = 0$ and $J_{10}^2 = 0$, it is clear that

$$x_1x_2^\varepsilon - x_2x_1^\varepsilon - [x_1, x_2] \in \mathrm{Id}^L(A),$$

thus $\mathrm{Id}^L(A_2^\varepsilon) \subseteq \mathrm{Id}^L(A)$.

In order to prove the opposite inclusion, let $f \in \mathrm{Id}^L(A)$ be a multilinear L-polynomial of degree n. By [30, Theorem 3], f can be written as

$$f = \sum_{j=1}^{n} \alpha_j x_{i_1} \cdots x_{i_{n-1}} x_j + \beta x_2 \cdots x_n x_1^\varepsilon + g,$$

where $g \in \mathrm{Id}^L(A_2^\varepsilon)$ and $i_1 < \cdots < i_{n-1}$.

Suppose that there exists $j \neq 1$ such that $\alpha_j \neq 0$. Then by making the evaluation $x_j = b \in J_{10}$, for some $b \neq 0$, and $x_{i_1} = \cdots = x_{i_{n-1}} = 1_F$, we get $\alpha_j = 0$, a contradiction. Now, if $\alpha_1 \neq 0$, then by making the evaluation $x_1 = \cdots = x_n = 1_F$ we get $\alpha_1 = 0$, a contradiction. Finally, if $\beta \neq 0$, then we let $x_1 = b$ and $x_2 = \cdots = x_n = 1_F$ getting $\beta = 0$, a contradiction.

Hence $f = g \in \mathrm{Id}^L(A_2^\varepsilon)$ and so $A \sim_{T_L} A_2^\varepsilon$.

Similarly, if $A = F + J_{01} + J_{11}$, we get $A \sim_{T_L} (A_2^\varepsilon)^*$. ∎

Lemma 23: *Let $A = F + J_{11} \in \mathrm{var}^L(UT_2^\varepsilon)$. Then $A \sim_{T_L} N_k^\varepsilon$, for some $k \geq 1$.*

Proof. Since $A \in \mathrm{var}^L(UT_2^\varepsilon)$, then $c_n^L(A) \approx qn^{k-1}$ for some $q > 0$ and $k \geq 1$.

If $J_{11}^\varepsilon = 0$, then $x^\varepsilon \equiv 0$ on A and so $[x_1, x_2] \in \mathrm{Id}^L(A)$. This trivially implies that A is a commutative algebra with trivial derivation, i.e., $A \sim_{T_L} N_1^\varepsilon = F$.

Let now $J_{11}^\varepsilon \neq 0$. Since A is a unitary algebra, we can consider the proper L-codimension sequence and write

$$c_n^L(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^L(A),$$

with $\gamma_i^L(A) = 0$ for all $i \geq k$. In particular,

$$\gamma_k^L(A) = 0$$

and so $[x_1, \ldots, x_k] \in \mathrm{Id}^L(A)$. Hence $\mathrm{Id}^L(N_k^\varepsilon) \subseteq \mathrm{Id}^L(A)$ and by Theorem 18, since $c_n^L(A) \approx qn^{k-1}$, it follows that $A \sim_{T_L} N_k^\varepsilon$. ∎

We now prove some auxiliary lemmas very useful in the proof of the main theorem. We start with the following that allows us to reduce our problem to the study of a variety generated by an L-algebra with either $J_{01} = 0$ or $J_{10} = 0$.

Lemma 24: *Let $A = F + J \in \mathrm{var}^L(UT_2^\varepsilon)$. Then*

$$A \sim_{T_L} (F + J_{00} + J_{10} + J_{11}) \oplus (F + J_{00} + J_{01} + J_{11}).$$

Proof. Let $B_1 = F + J_{00} + J_{10} + J_{11}$ and $B_2 = F + J_{00} + J_{01} + J_{11}$. Since $F^\varepsilon = 0$ and $J_{ij}^\varepsilon \subseteq J_{ij}$ for all $i, j \in \{0, 1\}$, it is clear that B_1 and B_2 are L-subalgebras of A. Then $\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(B_1 \oplus B_2) = \mathrm{Id}^L(B_1) \cap \mathrm{Id}^L(B_2)$.

Moreover, since $J_{01}J_{10} = J_{10}J_{01} = 0$, it turns out that also

$$\mathrm{Id}^L(B_1 \oplus B_2) \subseteq \mathrm{Id}^L(A)$$

holds. Thus $A \sim_{T_L} B_1 \oplus B_2$ as claimed. ∎

Lemma 25: *Let $A = F + J_{00} + J_{10} + J_{11} \in \mathrm{var}^L(UT_2^\varepsilon)$ with $J_{10} \neq 0$ (resp. $A = F + J_{00} + J_{01} + J_{11} \in \mathrm{var}^L(UT_2^\varepsilon)$ with $J_{01} \neq 0$).*

1. *If $J_{11}^\varepsilon = 0$, then $A \sim_{T_L} A_k^\varepsilon \oplus N$ (resp. $A \sim_{T_L} (A_k^\varepsilon)^* \oplus N$), for some $k \geq 2$ where N is a nilpotent L-algebra.*
2. *If $J_{11}^\varepsilon \neq 0$, then $A \sim_{T_L} A_k^\varepsilon \oplus N_u^\varepsilon \oplus N$ (resp. $A \sim_{T_L} (A_k^\varepsilon)^* \oplus N_u^\varepsilon \oplus N$), for some $u \geq 2$ and $k \geq 2$ where N is a nilpotent L-algebra.*

Proof. Let $A = F + J_{00} + J_{10} + J_{11} \in \mathrm{var}^L(UT_2^\varepsilon)$ with $J_{10} \neq 0$. The other case will follow with similar arguments.

Suppose first $J_{11}^\varepsilon = 0$ and let $t \geq 0$ be the greatest integer such that $J_{10}J_{00}^t \neq 0$. Notice that if $t = 0$ then $J_{10}J_{00} = 0$ and $A = (F + J_{10} + J_{11}) \oplus J_{00}$ as L-algebras. By Lemma 22 we get $F + J_{10} + J_{11} \sim_{T_L} A_2^\varepsilon$, hence $A \sim_{T_L} A_2^\varepsilon \oplus J_{00}$, where J_{00} is a nilpotent L-algebra.

So let us assume $t > 0$, i.e., $J_{10}J_{00}^t \neq 0$ (that is, in particular, $J_{10}J_{00} \neq 0$) and $J_{10}J_{00}^{t+1} = 0$.

Suppose that $J_{00}^\varepsilon J_{00}^{t+1} = 0$. Then it is easy to check that

$$x_1^\varepsilon x_2 \cdots x_{t+2} \in \mathrm{Id}^L(A),$$

thus $\mathrm{Id}^L(A_{t+2}^\varepsilon) \subseteq \mathrm{Id}^L(A)$. Furthermore, since $J_{10}J_{00}^t \neq 0$, there exist $a \in J_{10}$ and $b_1, \ldots, b_t \in J_{00}$ such that $ab_1 \cdots b_t \neq 0$. Therefore, as in the proof of Lemma 20, one can prove that $A \sim_{T_L} A_{t+2}^\varepsilon$.

Suppose now $J_{00}^\varepsilon J_{00}^{t+1} \neq 0$. Note that, since $J_{00}^\varepsilon \subseteq J_{00}$, $\varepsilon^2 = \varepsilon$ and Lemma 14 holds, J_{00}^ε is an L-ideal of A, thus we can consider $\bar{A} = A/J_{00}^\varepsilon$. As before, since $J_{10}J_{00}^{t+1} = 0$, it follows that $x_1^\varepsilon x_2 \cdots x_{t+2} \in \mathrm{Id}^L(\bar{A})$, $\mathrm{Id}^L(A_{t+2}^\varepsilon) \subseteq \mathrm{Id}^L(\bar{A})$ and so $\bar{A} \sim_{T_L} A_{t+2}^\varepsilon$.

Notice that $\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(\bar{A}) = \mathrm{Id}^L(A_{t+2}^\varepsilon)$ and, since J_{00} is an L-subalgebra of A, $\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(J_{00})$. Therefore $\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(A_{t+2}^\varepsilon \oplus J_{00})$.

Conversely, let $f \in \mathrm{Id}^L(A_{t+2}^\varepsilon \oplus J_{00})$ be a multilinear L-polynomial of degree n. We can write f as

$$\begin{aligned} f =& \alpha x_1 \cdots x_n + \beta x_2 \cdots x_n x_1^\varepsilon + \sum_{l<t+1} \sum_{I,J} \alpha_{I,J} x_{i_1} \cdots x_{i_k} [x_i, x_j] x_{j_1} \cdots x_{j_l} \\ &+ \sum_{s<t+1} \sum_{P,Q} \beta_{P,Q} x_{p_1} \cdots x_{p_r} x_m^\varepsilon x_{q_1} \cdots x_{q_s} \\ &+ \sum_{l'>t} \sum_{I',J'} \alpha_{I',J'} x_{i'_1} \cdots x_{i'_k} [x_{i'}, x_{j'}] x_{j'_1} \cdots x_{j'_l} \\ &+ \sum_{s'>t} \sum_{P',Q'} \beta_{P',Q'} x_{p'_1} \cdots x_{p'_r} x_{m'}^\varepsilon x_{q'_1} \cdots x_{q'_s} + g, \end{aligned} \tag{6}$$

where $g \in \mathrm{Id}^L(UT_2^\varepsilon) \subseteq \mathrm{Id}^L(A)$ and the indices of the variables are subjected to the conditions as in Lemma 6.

We remark that g and the last two summands of f are L-identities of A_{t+2}^ε. Moreover, in Lemma 6 it was also proved that the first four summands of f are linearly independent modulo $\mathrm{Id}^L(A_{t+2}^\varepsilon)$, hence $\alpha = \beta = \alpha_{I,J} = \beta_{P,Q} = 0$ for all I, J, P and Q, and

$$f = \sum_{l'>t} \sum_{I',J'} \alpha_{I',J'} x_{i'_1} \cdots x_{i'_k} [x_{i'}, x_{j'}] x_{j'_1} \cdots x_{j'_l} + \sum_{s'>t} \sum_{P',Q'} \beta_{P',Q'} x_{p'_1} \cdots x_{p'_r} x^{\varepsilon}_{m'} x_{q'_1} \cdots x_{q'_s} + g. \tag{7}$$

Since $f \in \mathrm{Id}^L(J_{00})$, if we evaluate all the variables on J_{00}, we get zero. Now, since $J_{10}J_{00}^{t+1} = J_{11}^{\varepsilon} = [J_{11}, J_{11}] = 0$, every evaluation of f into elements of A gives the zero value, hence $f \in \mathrm{Id}^L(A)$. So $\mathrm{Id}^L(A^{\varepsilon}_{t+2} \oplus J_{00}) \subseteq \mathrm{Id}^L(A)$ and $A \sim_{T_L} A^{\varepsilon}_{t+2} \oplus J_{00}$ follows.

Suppose now $J^{\varepsilon}_{11} \neq 0$.

Let $B = F + J_{00} + J_{10}$ and $D = F + J_{11}$. It is clear that B and D are L-subalgebras of A. Moreover, for the first part of the proof, $B \sim_{T_L} A^{\varepsilon}_{t+2} \oplus N$, for some $t \geq 0$, and by Lemma 23, $D \sim_{T_L} N^{\varepsilon}_u$ for some $u \geq 2$. Thus

$$\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(B \oplus D) = \mathrm{Id}^L(A^{\varepsilon}_{t+2} \oplus N^{\varepsilon}_u \oplus N).$$

Conversely, let $f \in \mathrm{Id}^L(A^{\varepsilon}_{t+2} \oplus N^{\varepsilon}_u \oplus N)$ be a multilinear polynomial of degree n and write f as in (6). As in the previous case, since $f \in \mathrm{Id}^L(A^{\varepsilon}_{t+2})$, we can reduce f as in (7). Notice that $f \in \mathrm{Id}^L(B) \cap \mathrm{Id}^L(D)$, thus any evaluation of f in B or in D gives zero. Furthermore, since $J_{10}J_{00}^{t+1} = J^{\varepsilon}_{11}J_{10} = 0$, we get that f vanishes under any evaluation on elements of A.

Thus $f \in \mathrm{Id}^L(A)$ and $\mathrm{Id}^L(A) = \mathrm{Id}^L(B \oplus D) = \mathrm{Id}^L(A^{\varepsilon}_{t+2} \oplus N^{\varepsilon}_u \oplus N)$. This immediately implies $A \sim_{T_L} A^{\varepsilon}_{t+2} \oplus N^{\varepsilon}_u \oplus N$ and we are done. ∎

By putting together the previous results, we get the following.

Lemma 26: *Let $A = F + J \in \mathrm{var}^L(UT_2^{\varepsilon})$ with $J_{10} \neq 0$ and $J_{01} \neq 0$. Then either $A \sim_{T_L} A^{\varepsilon}_k \oplus (A^{\varepsilon}_r)^* \oplus N$ or $A \sim_{T_L} A^{\varepsilon}_k \oplus (A^{\varepsilon}_r)^* \oplus N^{\varepsilon}_u \oplus N$, for some $k, r, u \geq 2$, where N is a nilpotent L-algebra.*

Proof. By Lemma 24, $A \sim_{T_L} B_1 \oplus B_2$ where $B_1 = F + J_{00} + J_{10} + J_{11}$ and $B_2 = F + J_{00} + J_{01} + J_{11}$. Moreover, by the previous Lemma, $B_1 \sim_{T_L} A^{\varepsilon}_k \oplus N$ or $B_1 \sim_{T_L} A^{\varepsilon}_k \oplus N^{\varepsilon}_u \oplus N$ and $B_2 \sim_{T_L} (A^{\varepsilon}_r)^* \oplus N$ or $B_2 \sim_{T_L} (A^{\varepsilon}_r)^* \oplus N^{\varepsilon}_u \oplus N$, for some $k, r, u \geq 2$ and N a nilpotent L-algebra. It readily follows that

$$A \sim_{T_L} A^{\varepsilon}_k \oplus (A^{\varepsilon}_r)^* \oplus N \quad \text{or}$$
$$A \sim_{T_L} A^{\varepsilon}_k \oplus (A^{\varepsilon}_r)^* \oplus N^{\varepsilon}_u \oplus N,$$

as claimed. ∎

We are now in a position to prove the main theorem of the paper.

Theorem 27: *If $A \in \mathrm{var}^L(UT_2^\varepsilon)$, then A is T_L-equivalent to one of the following L-algebras: UT_2^ε, N, $N_t^\varepsilon \oplus N$, $A_k^\varepsilon \oplus N$, $(A_r^\varepsilon)^* \oplus N$, $A_k^\varepsilon \oplus N_u^\varepsilon \oplus N$, $(A_r^\varepsilon)^* \oplus N_u^\varepsilon \oplus N$, $A_k^\varepsilon \oplus (A_r^\varepsilon)^* \oplus N$, $A_k^\varepsilon \oplus (A_r^\varepsilon)^* \oplus N_u^\varepsilon \oplus N$, where N is a nilpotent algebra and $k, r, u \geq 2$, $t \geq 1$.*

Proof. If $A \sim_{T_L} UT_2^\varepsilon$ there is nothing to prove, so suppose that A generates a proper L-subvariety of $\mathrm{var}^L(UT_2^\varepsilon)$. Thus, by [9, Theorem 15], $c_n^L(A)$ is polynomially bounded and by [31, Theorem 10] we may assume that

$$A = B_1 \oplus \cdots \oplus B_m,$$

where $B_1, \ldots, B_m$ are finite-dimensional L-subalgebras of A such that

$$\dim_F \frac{B_i}{J(B_i)} \leq 1,$$

for all $1 \leq i \leq m$.

If for all i, $\dim_F \frac{B_i}{J(B_i)} = 0$, then $B_i = J(B_i)$ is a nilpotent L-algebra and $A \sim_{T_L} N$ where $N = B_1 \oplus \cdots \oplus B_m$.

Thus suppose that there exists i such that $\dim_F \frac{B_i}{J(B_i)} = 1$, that is

$$B_i = F + J(B_i).$$

Write

$$J(B_i) = J_{00} \oplus J_{10} \oplus J_{01} \oplus J_{11}.$$

If $J_{10} = J_{01} = 0$, then by Lemma 23, $A \sim_{T_L} N_{u_i}^\varepsilon \oplus N$ for some $u_i \geq 1$, where N is a nilpotent L-algebra. If either $J_{10} \neq 0$ or $J_{01} \neq 0$, then by Lemmas 25 and 26, B_i is T_L-equivalent to one of the following L-algebras: $A_{k_i}^\varepsilon \oplus N$, $(A_{r_i}^\varepsilon)^* \oplus N$, $A_{k_i}^\varepsilon \oplus N_{u_i}^\varepsilon \oplus N$, $(A_{r_i}^\varepsilon)^* \oplus N_{u_i}^\varepsilon \oplus N$, $A_{k_i}^\varepsilon \oplus (A_{r_i}^\varepsilon)^* \oplus N$ or $A_{k_i}^\varepsilon \oplus (A_{r_i}^\varepsilon)^* \oplus N_{u_i}^\varepsilon \oplus N$, for some $k_i, r_i, u_i \geq 2$.

Since $A = B_1 \oplus \cdots \oplus B_m$, by taking into account the previous possibilities, we get the desired conclusion. ∎

As a direct consequence of the previous Theorem and Lemmas 18 and 21, we get the following corollary that classifies, up to T_L-equivalence, all L-algebras generating minimal varieties of polynomial growth inside $\mathrm{var}^L(UT_2^\varepsilon)$.

Corollary 28: *Let $A \in \mathrm{var}^L(UT_2^\varepsilon)$. Then A generates a minimal L-variety if and only if either $A \sim_{T_L} N_u^\varepsilon$ or $A \sim_{T_L} A_k^\varepsilon$ or $A \sim_{T_L} (A_k^\varepsilon)^*$, for some $u \geq 1$, $k \geq 2$.*

7. Classifying subvarieties of $\mathrm{var}^L(UT_2)$

In this section we classify, up to T_L-equivalence, all the L-subvarieties of $\mathrm{var}^L(UT_2)$. As we remarked before, since L acts trivially on UT_2, this is equivalent to the classification of the algebras inside the variety generated by UT_2 in the ordinary case given in [22]. In what follows we present such results in the language of L-algebras for the convenience of the reader.

For $k \geq 2$, let A_k, A_k^* and N_k be the algebras A_k^ε, $(A_k^\varepsilon)^*$ and N_k^ε constructed in Section 3, respectively, where L acts trivially on them.

Since $x^\delta \equiv 0$ for all $\delta \in L$, in this case we are dealing with ordinary identities. Thus we have the following results characterizing the L-identities and the growth of the L-codimensions of the above algebras.

Theorem 29 ([4, Lemma 3]):

(1) $\mathrm{Id}^L(A_2) = \langle [x_1, x_2]x_3 \rangle_{T_L}$ *and* $\mathrm{Id}^L(A_2^*) = \langle x_1[x_2, x_3] \rangle_{T_L}$.
(2) $c_n^L(A_2) = c_n^L(A_2^*) = n$.

Theorem 30 ([22], Lemma 3.1): *Let* $k \geq 3$, *then:*

(1) $\mathrm{Id}^L(A_k) = \langle [x_1, x_2][x_3, x_4],\ [x_1, x_2]x_3 \cdots x_{k+1} \rangle_{T_L}$;
(2) $c_n^L(A_k) = \sum_{l=0}^{k-2} \binom{n}{l}(n-l-1) + 1 \approx qn^{k-1}$, *for some* $q > 0$.

Hence $\mathrm{Id}^L(A_k^*) = \langle [x_1, x_2][x_3, x_4],\ x_1 \cdots x_{k-2}[x_{k-1}, x_k] \rangle_{T_L}$ *and*

$$c_n^L(A_k^*) = c_n^L(A_k) \approx qn^{k-1}.$$

Theorem 31 ([5, Theorem 3.4]): *Let* $k \geq 3$, *then:*

(1) $\mathrm{Id}^L(N_k) = \langle [x_1, x_2][x_3, x_4],\ [x_1, \ldots, x_k] \rangle_{T_L}$;
(2) $c_n^L(N_k) = 1 + \sum_{j=2}^{k-1} \binom{n}{j}(j-1) \approx qn^{k-1}$, *for some* $q > 0$.

Moreover, $N_2 \sim_{T_L} F$.

The following result classifies the subvarieties of $\mathrm{var}^L(UT_2)$.

Theorem 32 ([22, Theorem 5.4]): *If* $A \in \mathrm{var}^L(UT_2)$, *then* A *is* T_L*-equivalent to one of the following* L*-algebras:* UT_2, N, $N_u \oplus N$, $A_k \oplus N$, $A_r^* \oplus N$, $A_k \oplus N_u \oplus N$, $A_r^* \oplus N_u \oplus N$, $A_k \oplus A_r^* \oplus N$, $A_k \oplus A_r^* \oplus N_u \oplus N$, *where* N *is a nilpotent algebra and* $k, r, u \geq 2$.

As a consequence of the previous theorems, we can also get the classification of all L-algebras generating minimal varieties.

Corollary 33: *An L-algebra $A \in \mathrm{var}^L(UT_2)$ generates a minimal variety of polynomial growth if and only if either $A \sim_{T_L} N_u$ or $A \sim_{T_L} A_k$ or $A \sim_{T_L} A_k^*$, for some $u \geq 2$, $k \geq 2$.*

References

[1] O. M. Di Vincenzo and V. Nardozza, *Differential polynomial identities of upper triangular matrices under the action of the two-dimensional metabelian Lie algebra*, Algebras and Representation Theory **25** (2022), 187–209.

[2] V. Drensky, *Free Algebras and PI-algebras*, Springer, Singapore, 2000.

[3] A. Giambruno, A. Ioppolo and D. La Mattina, *Varieties of algebras with superinvolution of almost polynomial growth*, Algebras and Representation Theory **19** (2016), 599–611.

[4] A. Giambruno and D. La Mattina, *PI-algebras with slow codimension growth*, Journal of Algebra **284** (2005), 371–391.

[5] A. Giambruno, D. La Mattina and V. Petrogradsky, *Matrix algebras of polynomial codimension growth*, Israel Journal of Mathematics **158** (2007), 367–378.

[6] A. Giambruno, D. La Mattina and M. Zaicev, *Classifying the minimal varities of polynomial growth*, Canadian Journal of Mathematics **66** (2014), 625–640.

[7] A. Giambruno and S. Mishchenko, *On star-varieties with almost polynomial growth*, Algebra Colloquium **8** (2001), 33–42.

[8] A. Giambruno, S. Mishchenko and M. Zaicev, *Polynomial identities on superalgebras and almost polynomial growth*, Communications in Algebra **29** (2001), 3787–3800.

[9] A. Giambruno and C. Rizzo, *Differential identities, 2×2 upper triangular matrices and varieties of almost polynomial growth*, Journal of Pure and Applied Algebra **223** (2019), 1710–1727.

[10] A. Giambruno, R. B. dos Santos and A. C. Vieira, *Identities of $*$-superalgebras and almost polynomial growth*, Linear and Multilinear Algebra **64** (2016), 484–501.

[11] A. Giambruno and M. Zaicev, *Asymptotics for the standard and the Capelli identities*, Israel Journal of Mathematics **135** (2003), 125–145.

[12] A. Giambruno and M. Zaicev, *Polynomial Identities and Asymptotic Methods*, Mathematical Surveys and Monographs, Vol. 122, American Mathematical Society, Providence, RI, 2005.

[13] A. S. Gordienko, *Asymptotics of H-identities for associative algebras with an H-invariant radical*, Journal of Algebra **393** (2013), 92–101.

[14] A. S. Gordienko and M. V. Kochetov, *Derivations, gradings, actions of algebraic groups, and codimension growth of polynomial identities*, Algebras and Representation Theory **17** (2014), 539–563.

[15] G. Hochschild. *Semi-simple algebras and generalized derivations*, American Journal of Mathematics **64** (1942), 677–694.

[16] A. Ioppolo and D. La Mattina, *Polynomial codimension growth of algebras with involutions and superinvolutions*, Journal of Algebra **472** (2017), 519–545.

[17] A. Ioppolo and F. Martino, *Varieties of algebras with pseudoinvolution and polynomial growth*, Linear and Multilinear Algebra **66** (2018), 2286–2304.

[18] Y. Karasik, *Kemer's theory for H-module algebras with application to the PI exponent*, Journal of Algebra **457** (2016), 194–227.

[19] A. R. Kemer, *T-ideals with power growth of the codimensions are Specht*, ibirskiĭ Matematičeskiĭ Žurnal **19** (1978), 54–69; English translation: Siberian Mathematical Journal **19** (1978), 37–48.

[20] A. R. Kemer, *Varieties of finite rank*, in *Abstracts of the 15-th All-Union Algebraic Conference, Krasnoyarsk. Part 2*, 1979, p. 73.

[21] V. K. Kharchenko, *Differential identities of semiprime rings*, Algebra i Logika **18** (1979) 86-–119.

[22] D. La Mattina, *Varieties of almost polynomial growth: classifying their subvarieties*, Manuscripta Mathematica **123** (2007), 185–203.

[23] D. La Mattina, *Polynomial codimension growth of graded algebras*, in *Groups, Rings and Group Rings*, Contemporary Mathematics, Vol. 499, American Mathematical Society, Providence, RI, 2009, pp. 189–197.

[24] D. La Mattina, *Varieties of superalgebras of almost polynomial growth*, Journal of Algebra **336** (2011), 209–226.

[25] D. La Mattina and F. Martino, *Polynomial growth and star-varieties*, Journal of Pure and Applied Algebra **220** (2016), 246–262.

[26] J. N. Malcev, *A basis for the identities of the algebra of upper triangular matrices*, Algebra i Logika **10** (1971), 393–400.

[27] F. Martino, *Varieties of special Jordan algebras of almost polynomial growth*, Journal of Algebra **531** (2019), 184–196.

[28] A. Regev, *Existence of identities in $A \otimes B$*, Israel Journal of Mathematics **11** (1972), 131–152.

[29] C. Rizzo, *The Grassmann algebra and its differential identities*, Algebras and Representation Theory **23** (2020), 125–134.

[30] C. Rizzo, *Growth of differential identities*, in *Polynomial Identities in Algebras*, Springer INdAM Series, Vol. 44, Springer, Cham, 2021, pp. 383–400.

[31] C. Rizzo, R. B. dos Santos and A. C. Vieira, *Differential identities and polynomial growth of the codimensions*, Algebras and Representation Theory, `https://doi.org/10.1007/s10468-022-10163-0`.

[32] A. Valenti, *Group graded algebras and almost polynomial growth*, Journal of Algebra **334** (2011), 247–254.