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HAUPT'S THEOREM FOR STRATA OF ABELIAN DIFFERENTIALS

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ABSTRACT

Let S be a closed topological surface. Haupt's theorem provides necessary and sufficient conditions for a complex-valued character of the first integer homology group of S to be realized by integration against a complex-valued 1-form that is holomorphic with respect to some complex structure on S. We prove a refinement of this theorem that takes into account the divisor data of the 1-form.

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1. Introduction

Let S be an oriented connected topological surface without boundary having genus $g \geq 2$. We say that a character $\chi\colon H_1(S;\mathbb{Z}) \to \mathbb{C}$ is **realized** by a complex-valued 1-form ω if and only if for each integral cycle γ we have

$$\int_\gamma \omega = \chi(\gamma).$$

In this case, the image Λ_χ of χ is the set of **periods** of ω.

In 1920, O. Haupt [Hpt20] determined those characters that are realized by some 1-form that is holomorphic with respect to some complex structure on S. More recently, M. Kapovich [Kpv17] rediscovered Haupt's characterization in the following form: A character χ is realized by a holomorphic 1-form ω if and only if

(1) its **area**

$$A(\chi) := \mathrm{Im} \sum \overline{\chi(a_i)}\chi(b_i)$$

is positive where $\{a_i, b_i\}$ is a symplectic basis of $H_1(S;\mathbb{Z})$, and

(2) if Λ_χ is discrete, then Λ_χ is a lattice and the induced homotopy class of maps from S to the torus $\mathbb{C}/\Lambda_\chi$ has degree d_χ strictly greater than 1.

In addition, if Λ_χ is discrete, then the induced map is realized by a branched covering $p\colon S \to \mathbb{C}/\Lambda_\chi$ and the pullback $p^*(dz)$ realizes χ.

In this note we provide a refinement of Haupt's theorem that involves the divisor data of the 1-form. To be precise, let

$$Z(\omega) = \{z_1, z_2, \ldots, z_k\}$$

be the set of zeros of a nontrivial holomorphic 1-form ω, and for each i let α_i denote the multiplicity of the zero z_i. The **divisor data**, $\alpha(\omega)$, is the unordered n-tuple $(\alpha_1, \ldots, \alpha_k)$, whose sum is

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = 2g - 2.$$

Theorem 1.1: *A character $\chi : H_1(S,\mathbb{Z}) \to \mathbb{C}$ is realized by a 1-form ω with divisor data $\alpha(\omega) = (\alpha_1, \ldots, \alpha_k)$ if and only if*

(1) *$A(\chi)$ is positive, and*

(2') *if Λ_χ is discrete, then the induced map $S \to \mathbb{C}/\Lambda_\chi$ has degree*

$$d_\chi > \max\{\alpha_i\}.$$

The proof of the sufficiency is immediate. Indeed, one applies Haupt's theorem and notes that the Riemann–Hurwitz formula shows that the degree of an induced branched covering is at least $1 + \max\{\alpha_i\}$.

To prove the necessity, we will recast the problem in terms of the moduli space theory of 1-forms (see §2). The Hodge bundle $\Omega\mathcal{M}_g$ is the moduli space of complex-valued 1-forms that are holomorphic with respect to some complex structure on S. It is a disjoint union of the strata $\Omega\mathcal{M}_g(\alpha)$, consisting of forms with divisor data α. A connected component of the set of 1-forms that have a prescribed set of periods constitutes a leaf of the 'isoperiodic foliation'. Calsamiglia, Deroin, and Francaviglia [CDF15] classified the closures of the leaves of the isoperiodic foliation. We use this classification to prove the following.

Theorem 1.2: *If L is an isoperiodic leaf whose associated set of periods is not a lattice, then L intersects each connected component of each stratum of the Hodge bundle.*

To prove Theorem 1.1, one combines Theorem 1.2 with the following proposition.

Proposition 1.3: *Let Γ be a lattice in $\mathbb{C}$. For each connected component K of each stratum $\Omega\mathcal{M}_g(\beta)$ of $\Omega\mathcal{M}_g$ and for each integer $d > \max\{\beta_k\}$, there exists a primitive degree d branched covering $p\colon S \to \mathbb{C}/\Gamma$ such that $(S, p^*(dz))$ belongs to K.*

Recall that a branched cover of a torus is primitive if the induced map on homology is surjective.

In §2, we construct the Hodge bundle over Teichmüller space, define the isoperiodic foliation, recall the main result of [CDF15], and prove Theorem 1.2. In §3, we prove Proposition 1.3.

Soon after we posted this paper on the arXiv, Thomas Le Fils shared a preprint [LFs20] containing his independent proof of Theorem 1.1. His proof differs from ours in that it does not pass through Theorem 1.2 and instead uses a study of the mapping class group action on the space of characters in the spirit of [Kpv17]. We note that his paper does not consider connected components of strata.

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2. The Hodge bundle and the isoperiodic foliation

In this section we describe the Hodge bundle and the absolute and relative period mappings. We define the isoperiodic foliation and show that each leaf that passes near a stratum must intersect the stratum. We use this to prove Theorem 1.2. Finally we prove Theorem 1.1 modulo the proof of Proposition 1.3.

We begin by describing the Hodge bundle as a bundle over Teichmüller space. A **marked Riemann surface** is a closed Riemann surface X together with an orientation-preserving homeomorphism $f\colon S \to X$. Two marked surfaces (f_1, X_1) and (f_2, X_2) are considered to be equivalent if $f_2 \circ f_1^{-1}$ is isotopic to a conformal map. The set of equivalence classes of marked genus g surfaces may be given the structure of a complex manifold homeomorphic to $\mathbb{C}^{3g-3}$ called the Teichmüller space $\mathcal{T}_g$.

The **Hodge bundle** $\Omega\mathcal{T}_g \to \mathcal{T}_g$ is the (trivial) vector bundle over $\mathcal{T}_g$ whose fiber above (f, X) consists of (equivalence classes of) holomorphic 1-forms on X. In other words, $\Omega\mathcal{T}_g$ is the space of triples (f, X, ω) up to natural equivalence. The total space of $\Omega\mathcal{T}_g$ is naturally a complex manifold of dimension $4g - 3$. The **absolute period map** $P\colon \Omega\mathcal{T}_g \to H^1(S;\mathbb{C})$ is the holomorphic map that assigns to each triple (f, X, ω) the cohomology class $f^*(\omega)$.

Let $\Omega^*\mathcal{T}_g \subset \Omega\mathcal{T}_g$ denote the set of one-forms that do not vanish identically. The map that assigns divisor data to each 1-form defines a stratification of $\Omega^*\mathcal{T}_g$. In particular, for each partition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of $2g-2$, we define the **stratum** $\Omega\mathcal{T}_g(\alpha)$ to consist of those triples (f, X, ω) such that the divisor data of ω equals α.

One may also define a relative period map in a neighborhood of each nontrivial marked one-form (f_0, X_0, ω_0) in the stratum $\Omega\mathcal{T}_g(\alpha)$. Let $Z \subset S$ be a set of k marked points. Over a contractible neighborhood $U \subset \Omega\mathcal{T}_g(\alpha)$ of (f_0, X_0, ω_0), one may choose representative marking maps to identify Z with the zero sets $Z(\omega)$. Pulling back by these marking maps the class

$$[\omega] \in H^1(X, Z(\omega);\mathbb{C})$$

then defines the **relative period map**

$$P_{\mathrm{rel}}\colon U \to H^1(S, Z;\mathbb{C}).$$

The relative period map is well-known to be a local biholomorphism [Vch90]. Moreover, the relative and absolute period maps are related by $P|_U = r \circ P_{\mathrm{rel}}$ where r is the natural map from $H^1(S, Z;\mathbb{C})$ to $H^1(S;\mathbb{C})$. By considering the long exact sequence in cohomology, one finds that r is surjective, and hence $P|_U$ is a submersion. Since every non-trivial one-form lies in some stratum, we have the following.

Lemma 2.1: *The restriction of the absolute period map P to $\Omega^*\mathcal{T}_g$ is a submersion, as is its restriction to any stratum in $\Omega\mathcal{T}_g$.*

Since P is a submersion, it defines a holomorphic foliation of $\Omega^*\mathcal{T}_g$ called the **isoperiodic** (or **Rel**) **foliation**. Each **isoperiodic leaf** is a connected component of a level set of P.

The mapping class group $\mathrm{Mod}(S)$ naturally acts biholomorphically and properly discontinuously on the Hodge bundle. The quotient of this action is the classical Hodge bundle $\Omega\mathcal{M}_g \to \mathcal{M}_g$ where the base $\mathcal{M}_g$ is the moduli space of Riemann surfaces. In particular, each point in $\Omega\mathcal{M}_g$ may be regarded as (the equivalence class of) a pair (X, ω) where X is a Riemann surface and ω is a holomorphic 1-form on X.

If $\varphi \in \mathrm{Mod}(S)$ then we have

$$P(\varphi^*(\omega)) = \varphi^*(P(\omega)).$$

It follows that the isoperiodic foliation descends to a foliation of $\Omega\mathcal{M}_g$ that we will also refer to as the isoperiodic foliation. Moreover, we have a well-defined map from the set of leaves to the orbit space $H^1(S;\mathbb{C})/\mathrm{Mod}(S)$, and the set of periods

$$\Lambda_L := \left\{ \int_\gamma \omega : \gamma \in H_1(S;\mathbb{Z}) \right\}$$

depends only on the isoperiodic leaf L to which ω belongs.

Each stratum $\Omega\mathcal{T}_g(\alpha)$ is invariant under the action of $\mathrm{Mod}(S)$. Each quotient,

$$\Omega\mathcal{M}_g(\alpha) := \Omega\mathcal{T}_g(\alpha)/\mathrm{Mod}(S),$$

is the **stratum** that consists of pairs (X, ω) with divisor data α.

Proposition 2.2: *Let K be a connected component of a stratum. There exists a neighborhood $Z \subset \Omega\mathcal{M}_g$ of K such that if an isoperiodic leaf L intersects Z, then L also intersects K.*

Proof. Let $\widetilde{K}$ be a connected component of the preimage of K in $\Omega^*\mathcal{T}_g$. By Lemma 2.1, the map P is a holomorphic submersion from the $4g-3$ dimensional complex manifold $\Omega^*\mathcal{T}_g$ onto the complex vector space $H^1(S;\mathbb{C})$ which has dimension $2g$. Thus, given (f, X, ω), the inverse function theorem provides an open ball $B^{2g-3} \subset \mathbb{C}^{2g-3}$, an open ball $B^{2g} \subset H^1(S;\mathbb{C})$, and a biholomorphism φ from $B^{2g-3} \times B^{2g}$ onto a neighborhood U of (f, X, ω) so that

$$P \circ \varphi(z, w) = w.$$

Suppose that (f, X, ω) lies in $\widetilde{K}$. Since the restriction of P to $\widetilde{K}$ is a submersion, the image $V := P(U \cap \widetilde{K})$ is open. Note that

$$(P \circ \varphi)^{-1}(V) = B^{2g-3} \times V.$$

If L is a connected component of $P^{-1}(\chi)$ that intersects

$$W := \varphi(B^{2g-3} \times V),$$

then $\chi \in V$ and $L \cap U = \varphi(B^{2g-3} \times \{\chi\})$. In particular, L intersects $\widetilde{K}$.

The neighborhood Z is constructed by taking the image in $\Omega\mathcal{M}_g$ of the union of all such neighborhoods W as (f, X, ω) varies over $\widetilde{K}$. ∎

Next, we describe the result of Casamiglia, Deroin, and Francaviglia [CDF15] that classifies the closures of leaves L in terms of the associated set of periods Λ_L. The closure, $\overline{\Lambda}_L$, is a closed real Lie subgroup of $\mathbb{C} \cong \mathbb{R}^2$. Thus, $\overline{\Lambda}_L$ is either equal to $\mathbb{C}$, is isomorphic to $\mathbb{Z} \oplus \mathbb{R}$, or is discrete.

Let $\Omega_1\mathcal{M}_g \subset \Omega\mathcal{M}_g$ denote the locus of unit-area forms. Since the area functional

$$A(\omega) = \frac{i}{2}\int_S \omega \wedge \overline{\omega}$$

depends only on absolute periods, $\Omega_1\mathcal{M}_g$ is saturated by leaves of the isoperiodic foliation.

Given any closed subgroup $\Gamma \subset \mathbb{C}$, let $\Omega_1^\Gamma\mathcal{M}_g \subset \Omega_1\mathcal{M}_g$ denote the union of the leaves L such that there exists a connected subgroup $\Gamma' \subset \Gamma$ with $\Gamma = \overline{\Lambda}_L + \Gamma'$. If $\Gamma = \mathbb{C}$, then

$$\Omega_1^\Gamma\mathcal{M}_g = \Omega_1\mathcal{M}_g.$$

If Γ is isomorphic to $\mathbb{R} + \sqrt{-1}\cdot\mathbb{Z}$, then $L \subset \Omega_1^\Gamma\mathcal{M}_g$ if either $\overline{\Lambda}_L = \Gamma$ or $\overline{\Lambda}_L$ is a discrete subgroup of Γ with 'primitive imaginary part'. If Γ is discrete, then $\Omega_1^\Gamma\mathcal{M}_g$ is nonempty only if Γ has covolume $1/d$ for some integer $d > 1$, in which case $\Omega_1^\Gamma\mathcal{M}_g$ is a closed isoperiodic leaf which parameterizes primitive degree d branched covers of $\mathbb{C}/\Gamma$.

PROPOSITION 2.3: *If Γ is a lattice, then the space $\Omega_1^\Gamma\mathcal{M}_g$ is connected.*

Proof. By Theorem 9.2 of [GabKaz87], given two primitive, simply branched coverings $p : S \to \mathbb{C}/\Gamma$ and $q : S \to \mathbb{C}/\Gamma$ of the same degree, there exists a homeomorphism $h : S \to S$ and a homeomorphism $k : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma$ isotopic to the identity so that $k \circ p = q \circ h$. Let k_t be the isotopy with $k_0 = k$ and $k_1 = \mathrm{id}$. For each t, the 1-form $(k_t \circ p)^*(dz)$ is holomorphic with respect to the pulled-back complex structure. We have

$$(k_0 \circ p)^*(dz) = h^*(q^*(dz)) \quad \text{and} \quad (k_1 \circ p)^*(dz) = p^*(dz).$$

Hence the path in $\Omega_1^\Gamma\mathcal{M}_g$ associated to $(k_t \circ p)^*(dz)$ joins the point represented by $q^*(dz)$ to the point represented by $p^*(dz)$. Since simply branched coverings are generic, the space $\Omega_1^\Gamma\mathcal{M}_g$ is connected. ∎

Because $\Omega_1^\Gamma\mathcal{M}_g$ is connected, we may simplify the statement of the main theorem of [CDF15].

THEOREM 2.4 ([CDF15]): *Let $L \subset \Omega_1\mathcal{M}_g$ be a leaf of the isoperiodic foliation and let $\Gamma = \overline{\Lambda}_L$. If $g > 2$, then the closure of L is $\Omega_1^\Gamma\mathcal{M}_g$. If $g = 2$, then either the closure of L is $\Omega_1^\Gamma\mathcal{M}_2$ or L lies in the eigenform locus $\mathcal{E} \subset \Omega_1\mathcal{M}_2$.*

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We first suppose that $g > 2$ or $g = 2$ and $L \not\subset \mathcal{E}$. By assumption, L is an isoperiodic leaf such that Λ_L is not a lattice, and so $\overline{\Lambda}_L$ either equals $\mathbb{C}$ or equals $\mathbb{R}\cdot z_1 \oplus \mathbb{Z}\cdot z_2$ where $z_i \in \mathbb{C}$. By Lemma 2.1 the restriction of the absolute period map to a given component K of a given stratum is an open map. It follows that there exists $(X, \omega) \in K$ of area 1 so that the periods of ω lie in $\mathbb{Q}\cdot z_1 \oplus \mathbb{Q}\cdot z_2$. In particular, the set of periods constitute a lattice and there exists $A \in SL_2(\mathbb{R})$ so that the periods of $A\cdot(X, \omega)$ lie in $\overline{\Lambda}_L$. Hence $A\cdot(X, \omega)$ lies in the closure $\overline{L}$ by Theorem 2.4. Thus K intersects $\overline{L}$, and hence K intersects L by Proposition 2.2.

It remains to consider the case where $g = 2$ and $L \subset \mathcal{E}$. In this case, Theorem 1.2 follows from work of McMullen [McM03, McM05]. Indeed, $\Omega\mathcal{M}_2$ consists of two strata, the principal stratum $\Omega\mathcal{M}_2(1,1)$ and the stratum $\Omega\mathcal{M}_2(2)$, and both of these strata are connected. McMullen shows that the eigenform locus $\mathcal{E} \subset \Omega_1\mathcal{M}_2$ is a countable union of orbifolds $\Omega_1 E_D$ where D belongs to a subset of the positive integers. Moreover, each $\Omega_1 E_D$ is saturated by leaves of the isoperiodic foliation. The intersection $\Omega_1 E_D \cap \Omega_1 \mathcal{M}_2(2)$ is his "Weierstrass curve" $\Omega_1 W_D$. The eigenform locus $\Omega_1 E_D$ is a circle bundle over a Hilbert modular surface, which is covered by $\mathbb{H} \times \mathbb{H}$. In this covering, the isoperiodic foliation is simply the "vertical" foliation with leaves $\{c\} \times \mathbb{H}$. Each component of the Weierstrass curve is covered by a graph of a holomorphic function $\mathbb{H} \to \mathbb{H}$ which a fortiori must intersect each vertical leaf, and hence every isoperiodic leaf in $\Omega_1 E_D$ must intersect $\Omega_1 W_D$. Finally, each $\Omega_1 W_D$ is nonempty unless $D = 4$, in which case $\Omega_1 E_4$ parameterizes degree 2 torus-covers, a case that is excluded by hypothesis. ∎

We remark that if Λ_L is a lattice, then the associated space $\Omega_1^{\Lambda_L} \mathcal{M}_2$ need not intersect every stratum $\Omega \mathcal{M}_2(\alpha)$. Indeed, for such an intersection to be nonempty, it is necessary for the covolume of Λ_L to be strictly less than $1/\max \alpha_i$. Proposition 1.3 implies that this condition is also sufficient.

Finally, we prove our variant of Haupt's theorem modulo Proposition 1.3.

Proof of Theorem 1.1. Suppose that $\chi \in \mathrm{Hom}(H^1(S;\mathbb{Z}), \mathbb{C}) \cong H^1(S;\mathbb{C})$ is a character which satisfies the hypotheses of Theorem 1.1. By applying a real rescaling, we may assume moreover that $A(\chi) = 1$. Haupt's theorem then provides a unit-area holomorphic 1-form $(X, \omega) \in \Omega_1 \mathcal{T}_g$ representing χ. Note that each 1-form that lies in the isoperiodic leaf, L, that contains (X, ω) also represents χ. Hence it suffices to show that $\pi(L)$ intersects $\Omega_1 \mathcal{M}(\alpha)$.

If $\Gamma := \overline{\Lambda}_L$ is a lattice, then $\omega = p^*(dz)$ for some degree d primitive branched covering $p : X \to \mathbb{C}/\Gamma$. By Proposition 1.3, there exists a degree d primitive branched covering $q : X' \to \mathbb{C}/\Gamma$ so that

$$q^*(dz) \in \Omega_1 \mathcal{M}(\alpha).$$

In particular, both $\pi(X', q^*(dz))$ and $\pi(X, p^*(dz))$ lie in $\Omega_1^\Gamma \mathcal{M}$. Proposition 2.3 implies that

$$\pi(L) = \Omega_1^\Gamma \mathcal{M},$$

and so $\pi(X', q^*(dz))$ lies in $\pi(L) \cap \Omega_1 \mathcal{M}(\alpha)$.

If Γ is not a lattice and $g > 2$, then Theorem 1.2 implies that the projection $\pi(L)$ is dense in $\Omega_1^\Gamma \mathcal{M}_g$, and hence $\pi(L)$ intersects $\Omega_1 \mathcal{M}(\alpha)$ by Proposition 2.2.

If Γ is not a lattice and $g = 2$, then one can directly construct a 1-form in $\Omega_1 \mathcal{M}(1,1)$ (resp. $\Omega_1 \mathcal{M}(2)$) that represents χ by gluing together two well-chosen slit tori (resp. gluing a cylinder to a slit torus). ∎

3. Primitive torus covers

In this section we complete the proof of Theorem 1.1 by proving Proposition 1.3. That is, for each connected component K of a stratum $\Omega\mathcal{M}(\alpha)$, we construct a primitive branched torus covering $p\colon S \to \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ so that $p^*(dz)$ lies in K. Since the component K is invariant under the $GL_2^+(\mathbb{R})$ action, Proposition 1.3 follows.

To prove Theorem 1.1, we will explicitly construct torus coverings that lie in connected components of strata having one or two zeros, and then we apply a sequence of 'surgeries' to obtain torus coverings with additional zeros. In §3.1 we construct torus coverings for each connected component of each minimal stratum $\Omega\mathcal{M}(2g-2)$. In §3.2 we construct covers for each component of $\Omega\mathcal{M}_g(g-1,g-1)$. In §3.3 we introduce surgeries that add zeros to a torus cover while preserving the degree, and we check the effect of surgery on the spin parity. In §3.4 we construct torus covers such that the 1-form has exactly two zeros and each zero has odd order. We use surgeries to construct torus covers when $\max \alpha_i$ is odd. In §3.5 we describe the algorithm that can be used to construct a torus cover any desired connected component. We also provide some examples.

In what follows we will let T denote the 'unit square' torus $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$.

According to [KoZo03], the connected components of strata are distiguished by hyperellipticity and spin parity. To be precise, we will need to determine whether a torus covering $p\colon S \to T$ admits a **hyperelliptic involution**, a holomorphic involution $\tau\colon S \to S$ such that the quotient $S/\langle\tau\rangle$ is a sphere. Because

$$\tau^*(\omega) = -\omega,$$

a hyperelliptic involution maps each vertical (resp. horizontal) cylinder to a vertical (resp. horizontal) cylinder. Moreover, if τ preserves a vertical or horizontal cylinder C, then τ preserves the central curve of the cylinder and fixes exactly two points on the central curve. The Riemann–Hurwitz formula implies that τ has exactly $2g+2$ fixed points.

We will also need to check the spin parity of a holomomorphic 1-form. Given a Riemann surface X with a holomorphic one-form ω and a loop $\gamma\colon S^1 \to X$ disjoint from the zeros of ω, the Gauss map $G_\gamma\colon S^1 \to S^1$ is defined by

$$G_\gamma(t) = \frac{\omega(\gamma'(t))}{|\omega(\gamma'(t))|}.$$

The **index** of γ is the degree of G_γ. Note that if γ is a geodesic with respect to the natural flat structure on the surface, then G_γ is a constant map and hence $\mathrm{ind}(\gamma) = 0$.

Following Thurston and Johnson [Jns80], Kontsevich and Zorich [KoZo03] gave the following formula for the spin parity of a holomorphic 1-form ω all of whose zeros have even order. Given a symplectic basis $a_1, b_1, \ldots, a_g, b_g$ for $H_1(X;\mathbb{Z})$ consisting of curves that do not pass through a zero, the **spin parity** of ω equals

$$\sum_{i=1}^{g} (\mathrm{ind}(a_i) + 1)(\mathrm{ind}(b_i) + 1) \pmod 2. \tag{1}$$

In particular, this invariant of a holomorphic 1-form with zeros of even order lies in $\mathbb{Z}/2\mathbb{Z}$. We refer to a 1-form as **even** if its spin parity equals 0 mod 2, and as **odd** otherwise.

3.1. Minimal strata. In this subsection, for each $d > 2g-2$, we construct a degree d primitive branched torus covering for each connected component of the 'minimal stratum' $\Omega\mathcal{M}_g(2g-2)$. For $g \geq 4$, the minimal stratum has exactly three connected components [KoZo03]:

- hyperelliptic: The 1-forms in $\Omega\mathcal{M}_g(2g-2)$ that are canonical double covers of meromorphic quadratic differentials on the Riemann sphere with one zero of order $2g-3$ and $2g+1$ simple poles.
- even: The non-hyperelliptic 1-forms with even spin parity.
- odd: The non-hyperelliptic 1-forms with odd spin parity.

Denote these components by

$$\Omega\mathcal{M}_g(2g-2)^{\mathrm{hyp}}, \quad \Omega\mathcal{M}_g(2g-2)^{\mathrm{odd}} \quad \text{and} \quad \Omega\mathcal{M}_g(2g-2)^{\mathrm{even}}.$$

In the case $g = 3$, there is no even component, and in the case $g = 2$, there is only the hyperelliptic component [KoZo03].

For each of the above connected components we will first construct a degree $2g-1$ primitive branched cover p so that $p^*(dz)$ lies in the component. A slight modification of the construction will provide primitive branched coverings of each degree $d > 2g-2$.

For a torus covering to lie in the minimal stratum, it is necessary that it be branched over a single point. To describe such coverings, consider the unbranched covers of the punctured torus $\mathbb{C}/((\mathbb{Z}+i\mathbb{Z}) \setminus \{0\})$. Each such degree d

covering corresponds to a homomorphism ρ from the fundamental group of the once punctured torus to the symmetric group on d letters (the 'monodromy representation'). The fundamental group of the once punctured torus is freely generated by the central curve h of the horizontal cylinder and the central curve v of the vertical cylinder. It follows that each degree d covering that is branched over 0 is determined by $\rho(h)$ and $\rho(v)$. In sum, each branched covering is determined by a pair of permutations that we will denote h and v respectively. This description is unique up to simultaneous conjugation of h and v.

There is a one-to-one correspondence between the zeros of $p^*(dz)$ and the nontrivial cycles of the commutator $[h, v]$. Each cycle of length 1 in $[h, v]$ corresponds to a point in the fiber above $[0]$ that is not ramified. In particular, since in this section, we wish to construct torus coverings with a single ramification point of degree $2g - 1$, we will need to check that $[h, v]$ has one cycle of length $2g - 1$ and $d - (2g - 1)$ cycles of length 1.

Torus coverings branched over one point are often called **square-tiled surfaces**. Indeed, given a pair of permutations h, v of $\{1, \ldots, d\}$, we can construct the covering by gluing together d disjoint unit squares labeled $1, \ldots, d$ as follows: Glue the right side of square i to the left side of square $h(i)$ and the top of square i to the bottom of square $v(i)$. Note that the group generated by h and v must act transitively on $\{1, 2, \ldots, d\}$ for the surface to be connected.

3.1.1. *The hyperelliptic component.* Let $p\colon H_g \to T$ be the degree $d = 2g - 1$ torus covering branched over one point that is defined by the following permutations on $2g - 1$ letters (in cycle notation)

$$h = (1, 2)(3, 4) \cdots (2g - 3, 2g - 2)(2g - 1),$$
$$v = (1)(2, 3)(4, 5) \cdots (2g - 2, 2g - 1).$$

See Figure 1. The commutator $[h, v]$ has order $2g - 1$ and so p has only one ramification point, and thus $p^*(dz)$ has exactly one zero z of order $2g - 2$. Hence each vertical edge (resp. horizontal edge) of each unit square is a 1-cycle in $H_1(H_g; \mathbb{Z})$, and the covering map sends this 1-cycle to the standard vertical (resp. horizontal) generator of $H_1(\mathbb{C}/\mathbb{Z}^2; \mathbb{Z})$. Hence p is primitive.

The 1-form $p^*(dz)$ admits a unique hyperelliptic involution τ. Indeed, the map τ may be constructed by rotating each square in Figure 1 about its center by π radians. The involution τ has $2g + 2$ fixed points consisting of the zero of $p^*(dz)$, the centers of each of the $2g - 1$ squares, the midpoint of the top

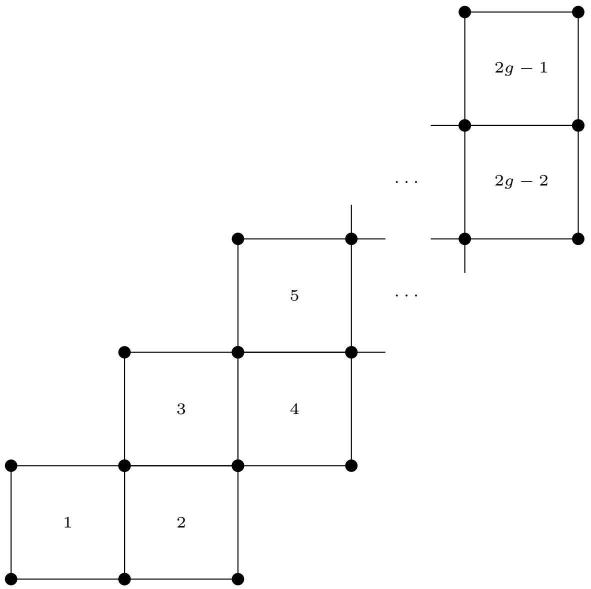

Figure 1. A hyperelliptic surface, H_g, in the minimal stratum that is a degree $2g-1$ primitive branched covering of the torus.

(and bottom) edge of square 1, and the midpoint of the left (and right) edge of square $2g-1$. The quotient $H_g/\langle\tau\rangle$ is a sphere and it follows that $p^*(dz)$ is hyperelliptic.

To construct primitive branched covers of degree $d > 2g-1$, we lengthen one of the vertical cylinders by placing $d-(2g-1)$ additional squares on top of the square $2g-1$ in Figure 1. To be precise, let $p\colon H_g^d \to \mathbb{C}/\mathbb{Z}$ be the covering determined by the permutations

$$h = (1,2)(3,4)\cdots(2g-3,2g-2)(2g-1)(2g-2)\cdots(d-1)(d),$$
$$v = (1)(2,3)(4,5)\cdots(2g-2,2g-1,\ldots,d-1,d).$$

The commutator $[h,v]$ has one cycle of length $2g-1$ and $d-(2g-1)$ cycles of length 1. In other words, $p^*(z)$ has a single zero of order $2g-2$. The covering p is primitive for the same reason that the covering $H_g \to T$ is primitive. The surface H_g^d admits a hyperelliptic involution τ which rotates by π each of the squares labeled 1 through $2g-2$ about their respective centers. The involution τ preserves the horizontal Euclidean cylinder C consisting of the squares $2g-1,\dots,d$, and its restriction to C has two fixed points. The only remaining fixed point of τ is the unique zero of $p^*(dz)$.

3.1.2. *The odd component.* Let $p\colon O_g \to T$ be the degree $d = 2g-1$ torus covering branched over one point that is defined by the following permutations on $2g-1$ letters (in cycle notation):

$$h = (1,3,5,\dots,2g-1)\cdot(2)\cdot(4)\cdots(2g-2),$$
$$v = (1,2)(3,4)\cdots(2g-3,2g-2).$$

See Figure 2.

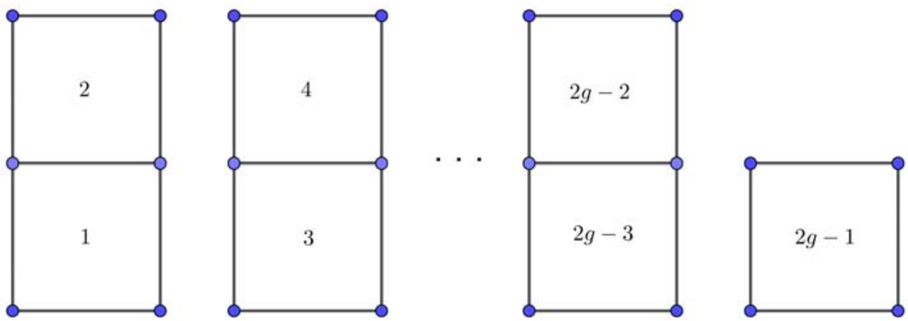

Figure 2. The odd spin parity torus cover O_g in the minimal stratum. The odd numbered squares together form a horizontal cylinder of length g and each even numbered square corresponds to a horizontal cylinder of length 1.

To see that the surface is not hyperelliptic, we suppose that O_g admits a hyperelliptic involution τ and then derive a contradiction. The horizontal cylinder C that consists of the odd numbered squares is the only horizontal cylinder of length greater than 1, and hence $\tau(C) = C$.[1] In particular, the map τ preserves the union V of the vertical saddle connections that are contained in C. The complement of V consists of the vertical cylinder corresponding to the square labeled $2g-1$ and the $g-1$ slit tori S_i corresponding to the cycles $(i, i+1)$ for i odd. If $\tau(S_i) = S_j$ for some $i \neq j$, then the sphere $O_g/\langle\tau\rangle$ would contain a once holed torus. This is not possible, and so $\tau(S_i) = S_i$ for each i. In

[1] In fact, for each 1-form in the minimal stratum, each cylinder is preserved by the hyperelliptic involution. See, for example, the proof of Lemma 8 in [KoZo03].

particular, τ preserves each odd numbered square, and the center of each odd numbered square is a fixed point. That is, τ has at least $g-1>2$ fixed points. But each (non-null homologous) cylinder C has exactly 2 fixed points, and this is the desired contradiction.

To see that the spin parity of O_g is odd, we exhibit O_g as the slit tori decomposition mentioned in the previous paragraph. See Figure 3. Here we have chosen a symplectic basis $\{a_i, b_i\}$ for the first homology of O_g. The curve a_g 'turns' once as it traverses each slit torus and hence has index equal to $g-1$. All other curves in this symplectic basis are geodesics and hence have index equal to zero. Thus, it follows from formula (1) that the spin parity equals $2g-1$ mod 2.

The map p is primitive because, for example, the classes $p_*(a_1)$ and $p_*(b_g)$ generate the first homology of $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$.

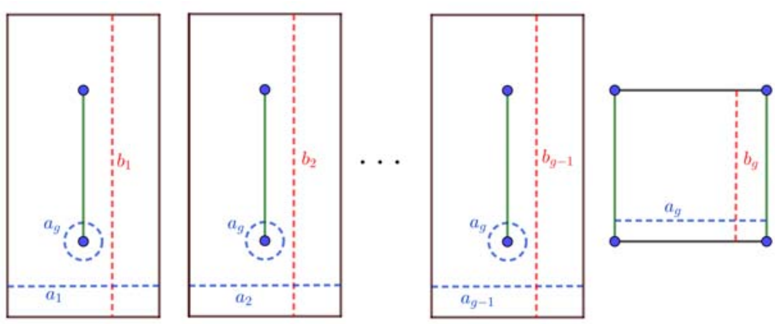

Figure 3. The simple closed curves $\{a_i, b_i\}$ form a symplectic basis for the first homology of O_g. Note that the curve a_g intersects each slit torus, and each intersection contributes 1 to the index of a_g.

To construct a primitive branched cover $p : O_g^d \to \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ of degree $d>2g-1$, replace the cycle $(2g-1)$ in the horizontal permutaion h with $(2g-1, 2g, \ldots, d-1, d)$. This is equivalent to replacing the vertical cylinder that corresponds to the square labeled $2g-1$ with a vertical cylinder of width $d-(2g-2)$.

3.1.3. *The even component.* For $g \geq 4$, let $p\colon E_g \to T$ be the degree $d = 2g-1$ branched covering of $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ defined by the permutations

$$h = (1, 3, 5, \dots, 2g-1, 4),$$
$$v = (1, 2)(3, 4)\cdots(2g-3, 2g-2)(2g-1).$$

See Figure 4. The surface E_g differs from O_g in the way that the squares labeled 3 and 4 are attached. Arguments similar to the ones given in §3.1.2 show that p is not hyperelliptic, is of even spin parity, and is primitive. For example, the horizontal cylinder C consisting of the square labeled 4 and the odd numbered squares would be preserved by a hyperelliptic involution, and one can use this to argue that E_g is not hyperelliptic. All of the elements in the symplectic basis in Figure 4 have index zero except for a_2 and a_g which have indices 1 and $g-1$ respectively. In particular, the spin parity is $2g \mod 2$.

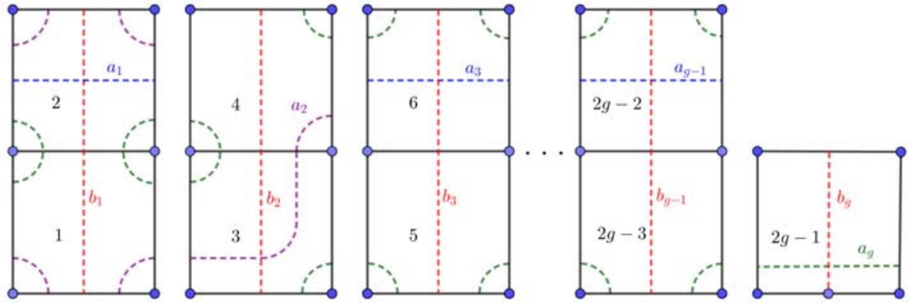

Figure 4. The even parity torus cover E_g in the minimal stratum. The simple closed curves $\{a_i, b_i\}$ form a symplectic basis for the first homology of O_g. The intersection of a_g with each vertical cylinder of height two contributes 1 to the index of a_g. All other basis elements have index 0 except for a_2 which has index 1.

To obtain a degree d cover $p : E_g^d \to \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$, replace the vertical cylinder of width 1 corresponding to the square labeled $2g-1$ with a vertical cylinder of width $d-(2g-2)$. In other words, replace the cycle $(2g-1)$ that appears in v with the cycle $(2g-1, 2g, \dots, d-1, d)$.

3.2. The strata with two zeros of equal order. According to [KoZo03], if $g \geq 5$ is odd, then the stratum $\Omega\mathcal{M}(g-1, g-1)$ has three connected components: hyperelliptic; even spin parity and non-hyperelliptic; and odd parity and non-hyperelliptic. When $g = 2, 3$ or $g \geq 4$ and even, the stratum has exactly two components: hyperelliptic and non-hyperelliptic. In §3.2.1 we exhibit a surface in each hyperelliptic component, regardless of the parity of g, and then in §3.2.2 we construct examples in the remaining non-hyperelliptic component(s). Our constructions will be based on gluing together surfaces with slits.

3.2.1. $\Omega\mathcal{M}(g-1, g-1)^{\mathrm{hyp}}$. If $g = 2m$ is even, we construct a degree g hyperelliptic torus cover as follows. First, create a genus two surface by gluing together two copies of $\mathbb{C}/\mathbb{Z}^2$ that each have a horizontal slit. Take m distinct copies, $S_1, \ldots, S_m$, of this genus two surface. Each genus two surface S_i has exactly four horizontal saddle connections, two that correspond to the slits and two that do not. See Figure 5.

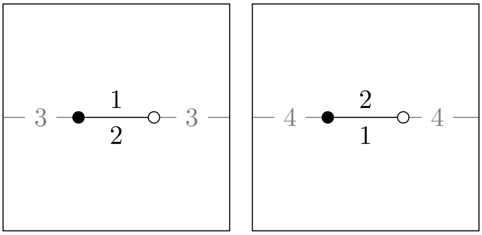

Figure 5. A genus two surface constructed from two slit tori has four horizontal saddle connections.

From both S_1 and S_m remove one of the ones that do not correspond to a slit and from each of the remaining genus two surfaces, $S_2, \ldots, S_{m-1}$, remove both of the horizontal saddle connections that do not correspond to a slit. Glue the top (resp. bottom) of the new slit on S_1 to the bottom (resp. top) of one of the (new) slits on S_2. Then, inductively, glue the top (resp. bottom) of the remaining slit on S_i to the bottom (resp. top) of one of the slits on S_{i+1}. Let X_g denote the resulting degree g cover of $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ when g is even.

If $g = 2m + 1$ is odd, then remove the horizontal saddle connection of X_{2m} that lies in S_m and then glue in an additional horizontally slit torus to obtain the torus cover X_{2m+1}. The surfaces X_6 and X_7 are described in Figure 6.

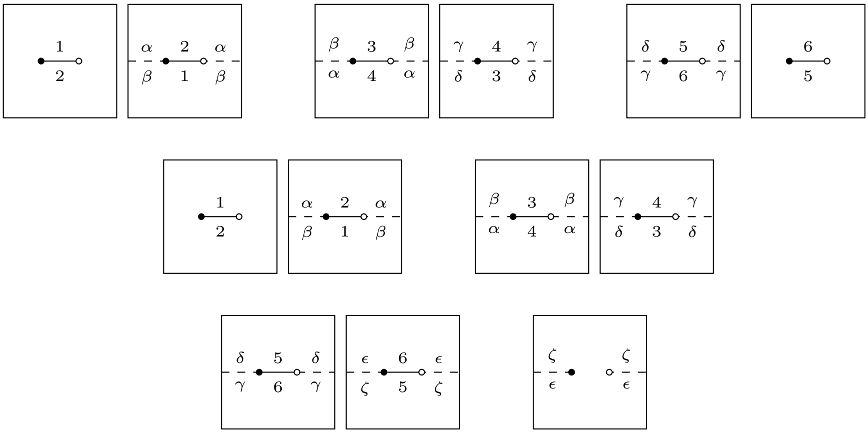

Figure 6. Primitive degree g torus covers in $\Omega\mathcal{M}_g(g-1,g-1)^{\text{hyp}}$ in the cases $g=6$ and $g=7$. Each square corresponds to a slit torus.

A torus cover X_g^d of degree $d = k+g-1$ can be constructed in the same way if one replaces a slit copy of $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ in the construction of the genus two surface S_1 with a slit copy of $\mathbb{C}/(k\mathbb{Z}+i\mathbb{Z})$. The hyperelliptic involution on X_g^d corresponds to the elliptic involution of each slit torus that fixes the center of each slit. A vertical curve in S_1 (resp. horizontal curve in S_2) is mapped to the standard vertical (resp. horizontal) generator of $H_1(\mathbb{C}/(\mathbb{Z}+i\mathbb{Z}),\mathbb{Z})$. Hence the covering is primitive.

Remark 3.1: A degree g, primitive, hyperelliptic torus covering can also be defined in terms of the classical Chebyshev polynomial P_g, the unique polynomial satisfying

$$P_g(\cos\theta) = \cos(g\cdot\theta)$$

for each $\theta \in \mathbb{R}$. Given $a \in (0,1)$ such that $P_g(a) \neq \pm 1$, let q be the unique quadratic differential on the Riemann sphere $\widehat{\mathbb{C}}$ with simple poles at $\{\pm 1, \pm a\}$. The set $P_g^{-1}\{+1,-1\}$ consists of all $g-1$ critical points of degree two together with the two additional points at which P_g is not branched. The map P_g is not branched at any of the $2g$ points in $P_g^{-1}\{+a,-a\}$. It follows that $P_g^*(q)$ has $2+2g$ simple poles and one zero of degree $2g-2$ at $\infty \in \widehat{\mathbb{C}}$. Let (X,ω) and $(\mathbb{C}/\Lambda, dz)$ be the respective canonical double covers of $(\widehat{\mathbb{C}}, P_g^*(q))$ and $(\widehat{\mathbb{C}}, q)$. The map $P_g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ lifts to a primitive degree g branched cover $\widetilde{P}_g\colon X \to \mathbb{C}/\Lambda$

so that $\widetilde{P}_g^*(dz) = \omega$. It follows that (X, ω) lies in the hyperelliptic component of $\Omega\mathcal{M}(g-1, g-1)$.

3.2.2. *Non-hyperelliptic components of* $\Omega\mathcal{M}_g(g-1, g-1)$. Recall that if $g = 3$ or $g \geq 4$ and g is even, then there is exactly one non-hyperelliptic component. If $g \geq 5$ and g is odd, then there are exactly two non-hyperelliptic components, one consisting of odd spin parity 1-forms and one consisting of even spin parity 1-forms. We first construct a torus covering that is non-hyperelliptic in each genus and then observe that if g is odd, then its spin parity is odd. Then we separately construct an even spin torus covering for g odd.

For each $g \geq 2$, define a degree g torus cover X_g by cyclically gluing together distinct horizontally slit tori $S_1, \ldots, S_g$. To be more precise, glue the top of the slit on S_i to the bottom of the slit on S_{i+1}. The case of $g = 5$ is illustrated in Figure 7.

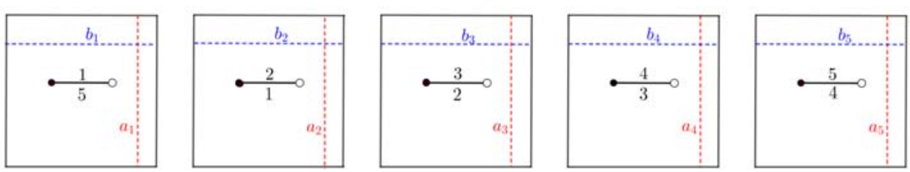

Figure 7. A cyclically glued g-slit torus cover X_g when $g = 5$.

To prove that the surface X_g is not hyperelliptic, let us assume to the contrary that a hyperelliptic involution τ exists and derive a contradiction. Let C be the vertical cylinder that contains each of the slits $s_i \subset S_i$. The cylinder C is the only vertical cylinder that has length greater than one, and hence it would be preserved by a hyperelliptic involution τ. Thus, τ would preserve the union of horizontal saddle connections that belong to C, and hence would preserve the complement A, that is the disjoint union of the slit tori S_i. If τ were to map one slit torus S_i onto a distinct slit torus S_j, then the quotient $X_g/\langle\tau\rangle$ would contain the embedded one-holed torus $S_i \cup S_j/\langle\tau\rangle$, and hence the quotient would not be a sphere. Thus the hyperelliptic involution τ would have to preserve each S_i, and hence would act as an elliptic involution on each S_i. It follows that the involution $\tau|_{S_i}$ has a fixed point $x_i \in C$. Hence C contains g fixed points, and since $g \geq 3$, this is the desired contradiction.

When g is odd, then the spin parity of X_g is well-defined, and a straightforward argument shows that the spin parity of X_g is odd. Indeed, choose a homology basis for each slit torus S_i consisting of a vertical and a horizontal curve. The index of each of these curves is zero. Thus, the spin parity of X_g is $\sum_{i=1}^{g} 1 \equiv g \mod 2$.

To obtain non-hyperelliptic covers $p\colon X_g^d \to T$ of degree $d = g - 1 + k$, one may modify the construction by replacing, for example, S_1 with the slit torus obtained by removing a horizontal slit s from the torus $\mathbb{C}/(k\mathbb{Z} + i\mathbb{Z})$. Similar arguments show that X_g^d is not hyperelliptic and has spin parity equal to g mod 2.

It remains to construct, for each odd $g \geq 5$ and each $d \geq g$, a non-hyperelliptic, even spin parity, torus cover in $\Omega\mathcal{M}(g-1, g-1)$ of degree d. To construct it for degree $d = g$, we will perform a surgery to the surfaces X_2 and X_{g-2}. More precisely, we begin with the disjoint union of slit tori $S_1, \ldots, S_g$ as above. We construct the surface X_2 by cyclically gluing together S_1 and S_2 as above. We construct the surface X_{g-2} by cyclically gluing the slit tori $S_3, \ldots, S_{g-1}$.

Let $\delta_2 \subset S_2 \subset X_2$ denote the unique horizontal saddle connection that is parallel but disjoint from the horizontal saddle connections associated to the slit σ_2. Let $\delta_3 \subset S_3 \subset X_{g-3}$ denote the unique horizontal saddle connection that is parallel but disjoint from the horizontal saddle connections associated to the slit σ_3. Remove δ_2 from X_2 and remove δ_3 from X_{g-2}. Glue the top (resp. bottom) of δ_2 to the bottom (resp. top) of δ_3. See Figure 8 for the case of $g = 5$. The resulting surface Y_g covers T, and using a homology basis like the one illustrated in Figure 8, one finds that the spin parity is $g - 2 + 2 + 3 \equiv g + 3$ mod 2.

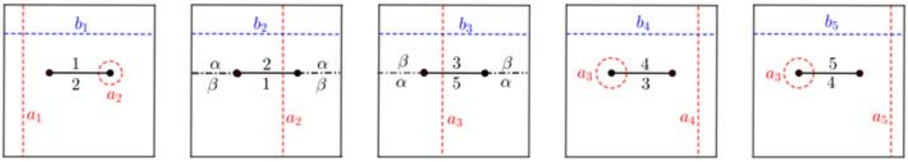

Figure 8. A genus 5 torus covering Y_g in the even spin parity component of $\Omega\mathcal{M}(4, 4)$. The top of the slit δ_2 and the bottom of the slit δ_3 are labeled with α, and the bottom of δ_2 and the top of δ_3 are labeled by β.

To obtain torus covers of higher degree one need only, as above, replace one of the slit tori with the slit torus coming from $\mathbb{C}/(k\mathbb{Z}+i\mathbb{Z})$. To see that the surface Y_g is not hyperelliptic, apply the argument used for X_g to the unique vertical cylinder C in X_{g-2} that has circumference greater than 2.

3.3. Surgeries that add zeros and preserve degree. Thus far, we have produced torus coverings in each connected component of both the minimal stratum $\Omega\mathcal{M}_g(2g-2)$ and the stratum $\Omega\mathcal{M}_g(g-1,g-1)$. To obtain torus covers in all connected components of all other strata, we will perform certain 'surgeries' on the torus covers E_g^d and O_g^d in the minimal strata as well as a variant, Z_g^d, of these that will be described in §3.4. Each surgery described here modifies the torus covering by adding zeros, increasing genus, and preserving degree. Each surgery can be performed on a torus cover branched over one point that has at least one vertical cylinder of circumference one and that has sufficiently many vertical cylinders of circumference at least two.

To be precise, let $p : X \to T$ be a torus covering of degree d such that there exists a vertical (open) cylinder $C \subset T$ that does not contain a branch point of p.[2] We will say that the torus covering p is **surgery admissible with respect to C and k** if the components of $p^{-1}(C)$ consist of

- at least k cylinders each having circumferences at least two, and
- at least one nonseparating cylinder whose circumference equals one.

In particular, to be surgery admissible p must have degree $d \geq 2k+1$.

In §3.3.1, we show how to add a zero of order $2k$ to a surgery admissible covering, and in §3.3.2 we show how to add a pair of odd order zeros. Each of these surgeries produces a surgery admissible torus covering. Therefore, we may apply any finite sequence of these surgeries. In §3.3.3, we show how to calculate the change in the spin parity so as to be sure that we can obtain a torus covering in each connected component of a stratum.

According to Theorem 1 in [KoZo03], components consisting of hyperelliptic surfaces only occur in the strata $\Omega\mathcal{M}(2g-2)$ and $\Omega\mathcal{M}(g-1,g-1)$. Thus, we will not need to consider the effect of surgeries on hyperellipticity.

[2] The boundary of C may contain a branch point.

3.3.1. *Adding a zero of order* $2k$. Let $p : X \to T$ be a surgery admissible torus cover. In this subsection, we describe a 'surgery' on this torus covering that yields a surgery admissible torus covering $\overline{p} : \overline{X} \to T$ with the same degree d and an additional zero of order $2k$.

Let $C \subset T$ be the vertical (open) cylinder that does not contain a branch point of p. Let C_0 be a component of $p^{-1}(C)$ that has circumference 1, and let $C_1, \ldots, C_k$ be components of circumference at least two. Choose a vertical closed geodesic $\sigma \subset C$ and choose $P \in \sigma$. The inverse image $p^{-1}(\sigma)$ consists of disjoint closed geodesics. The set $p^{-1}(\sigma \setminus \{P\})$ consists of d disjoint vertical segments. Choose exactly one segment σ_i from each cylinder C_i. See Figure 9. Cut along each σ_i and glue the left side of σ_i to the right side of σ_{i+1}. Let $\overline{X}$ be the resulting surface.

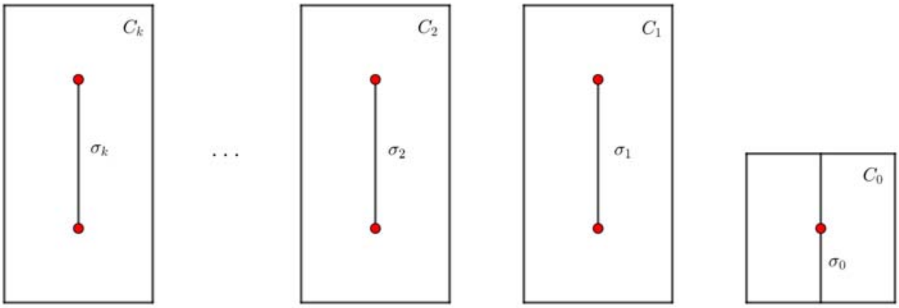

Figure 9. Adding a zero of order $2k$. Cut along each σ_i and identify the left side of σ_i to the right side of σ_j. The red endpoints are thus all identified with one another and they represent a ramification point of local index $2k + 1$ over P.

The covering p determines a surgery admissible torus covering $\overline{p} : \overline{X} \to T$ of degree d that is branched over 0 and P. Moreover, the 1-form $\overline{p}^*(dz)$ has an additional zero of order $2k$, and the genus of $\overline{X}$ is k greater than the genus of X.

3.3.2. *Adding a pair of zeros of odd order.* In this subsection we describe a surgery on $p : X \to T$ that adds a zero of order $2k-1$ and a zero of order $2k'-1$ where $k' \leq k$.[3] We will first assume that $k' = k$ and then show how to modify this surgery when $k' < k$.

[3] The orders of zeros of a holomorphic 1-form ω on a genus g Riemann surface must sum to $2g - 2$, and so ω has even number of zeros of odd order. Thus, any surgery that increases

To add a pair of zeros that have the same order $2k-1$, choose a horizontal segment τ that lies in C and has length strictly less than the width of C. Let τ_0 be the unique component of $p^{-1}(\tau)$ that lies in C_0. For each $i \in \{1, \ldots, k-1\}$, choose two connected components, τ_{2i-1} and τ_{2i}, of $p^{-1}(\tau)$ that lie in C_i, and choose one component, τ_{2k-1}, of $p^{-1}(\tau)$ that lies in C_k. See Figure 10. Cut the surface X along each τ_i. Then identify the top of τ_i with the bottom of τ_{i+1}. The resulting surface is a degree d torus cover with two new zeros of order $2k-1$.

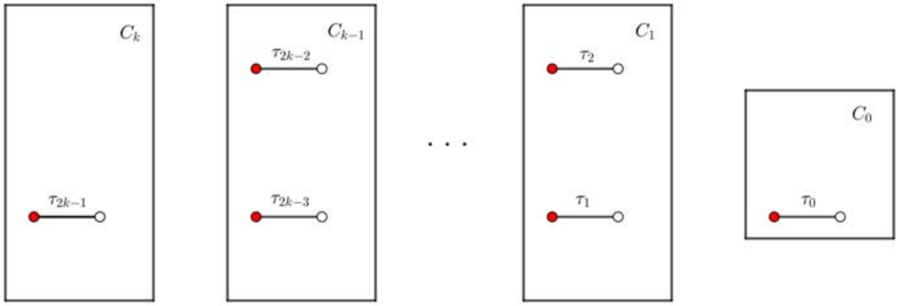

Figure 10. The white points are ramification points over one endpoint of τ, and the red points are ramifications over the other endpoint of τ.

We next modify the construction to show how to add one zero of order $2k-1$ and one zero of order $2k'-1$ where $k' < k$. Roughly speaking, the surgery is a combination of the surgery that adds two zeros of order $2k'-1$ and the surgery that adds a zero of order $2(k-k')$. To be precise, let σ be a vertical closed geodesic that lies in C and let τ be a horizontal segment in C that has one endpoint P on σ. Let τ_0 be the component of $p^{-1}(\tau)$ that lies in C_0 and let σ_0 be the lift of σ to C_0. For $i \in \{1, \ldots, k'-1\}$ choose two components, τ_{2i-1} and τ_{2i}, of $p^{-1}(\tau)$ that lie in C_i, choose one component, $\tau_{2k'-1}$, of $p^{-1}(\tau)$ that lies in $C_{k'}$. For $i \in \{k'+1, \ldots, k\}$ choose one component, σ_i, of $p^{-1}(\sigma \setminus \{P\})$ that lies in C_i. Cut along each σ_i and each τ_i, cyclically reglue the σ_i, and cyclically reglue the τ_i. The new zero that corresponds to the point P has order $2k-1$ and the new zero that corresponds to the other endpoint, Q, of τ has order $2k'-1$. See Figure 11 for an example of this construction.

the number of odd order zeros will necessarily increase the number of odd order zeros by an even integer.

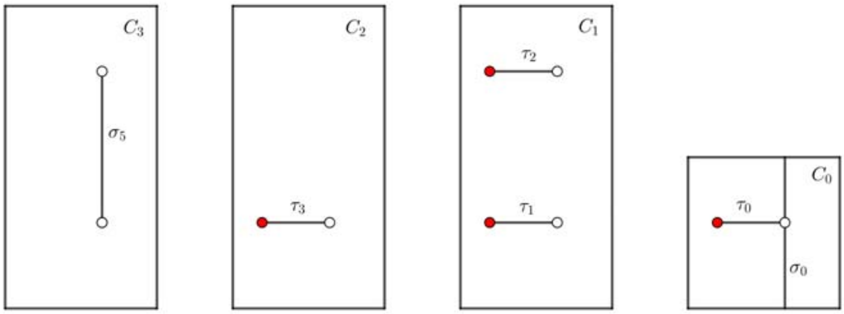

Figure 11. Adding zeros of different odd orders to X. The new zero that corresponds to the white points has order 5, and the zero corresponds to the red points has order 3.

3.3.3. *Change of parity computations.* In this subsection we consider how the spin parity changes when the surgery described in §3.3.1 is applied. In particular, we find that adding a zero of order $2k$ preserves the spin parity if k is even and it changes the spin parity if k is odd. Recall that the spin parity is not defined for 1-forms with zeros of odd order, and hence we will not consider the surgery of §3.3.2.

Let $p : X \to T$ be a surgery admissible torus covering, and let $\overline{p} : \overline{X} \to T$ be the result of applying the surgery of §3.3.1.

Lemma 3.2: *If k is even, then the spin parity of $p^*(dz)$ equals the spin parity of $\overline{p}^*(dz)$. If k is odd, then the spin parity of $p^*(dz)$ does not equal the spin parity of $\overline{p}^*(dz)$.*

Proof. Let $C, C_0, C_1, \ldots, C_k$, $\sigma, \sigma_0, \sigma_1, \ldots, \sigma_k$, and P be as in §3.3.1.

We first prove the statement in the special case of $k = 1$. Let b be a vertical closed geodesic in C that is disjoint from σ and let b_0 be the component of $p^{-1}(b)$ that lies in C_0. Since X is surgery admissible, the simple closed curve b_0 is not null-homologous. Let a_0 be a simple closed curve on X so that the geometric intersection number $i(a_0, b_0) = 1$, so that a_0 does not intersect a ramification point of p, and so that $a_0 \cap C_0$ is a horizontal segment. We further suppose that a_0 intersects σ_1 orthogonally at a point in $p^{-1}(a_0 \cap \sigma)$. Thus, after cutting along σ_0 and σ_1 and regluing as described in §3.3.1, the closed curve a_0 becomes two simple closed curves, a_0^+ and a_0^-. Let a_0^+ be the resulting simple closed curve that intersects $b_0^+ := b_0$ and let a_0^- be the other curve. Let b_0^- be a vertical geodesic in C_0 that intersects a_0^-. See Figure 12.

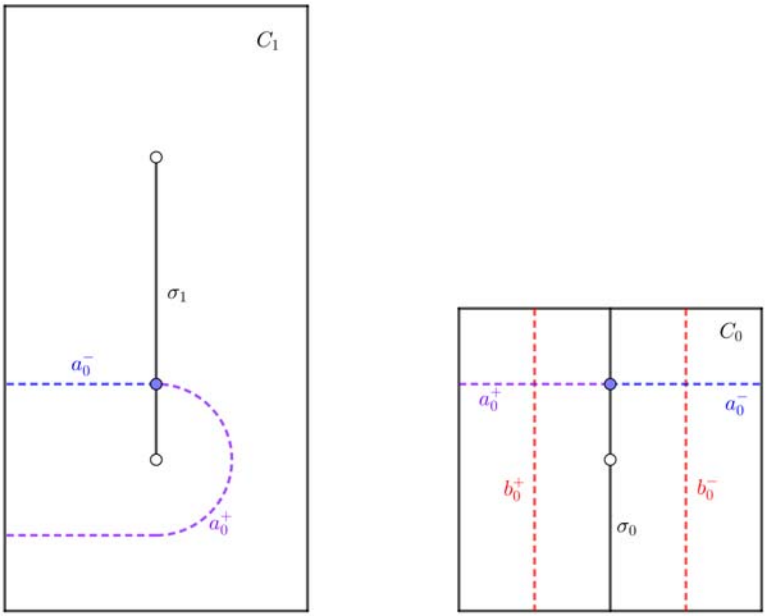

Figure 12. The first four elements of symplectic basis for the surface that results from adding one zero of order two.

Complete $\{a_0, b_0\}$ to a symplectic basis $\{a_0, b_0, \dots, a_{g-1}, b_{g-1}\}$ for $H_1(X;\mathbb{Z})$ so that no a_i nor b_i intersects a ramification point or σ_1 if $i > 0$. Then the collection

$$\{a_0^+, b_0^+, a_0^-, b_0^-, a_1, b_1, \dots, a_{g-1}, b_{g-1}\}$$

is a symplectic basis for the surface that results from the surgery. The curves a_i and b_i do not change if $i > 0$ and hence their indices do not change. The curves $b_0 = b_0^+$ and b_0^- are geodesics and hence their indices equal zero. The index of a_0 equals the sum of the indices of a_0^+ and a_0^-. It follows that the spin parity 'increases' by 1. Hence the claim is proven in the case $k = 1$.

To prove the claim for $k > 1$, we will consider, for $i \leq k$, the result $\overline{p}_i : \overline{X}_i \to T$ of adding a zero of order $2i$ using $C_0, C_1, \ldots, C_i$ and curves $\sigma_0, \ldots, \sigma_i$, and we will consider the result $\overline{q}_i : \overline{Y}_i \to T$ of adding a zero of order 2 to $\overline{X}_{i-1}$. It suffices to show that for each $i \leq k$, the spin parity of $\overline{p}_i^*(dz)$ equals the spin parity of $\overline{q}_i^*(dz)$. Indeed, an inductive argument using the case $k = 1$ would then imply the claim.

To prove that the spin parity $\overline{p}_i^*(dz)$ equals the spin parity of $\overline{q}_i^*(dz)$, we realize $\overline{Y}_i$ as an arbitrarily small perturbation of $\overline{X}_i$. In particular, we choose $\delta > 0$ and add a zero of order two to $\overline{X}_{i-1}$ as follows: Let $\sigma' \subset C$ be the vertical geodesic to the 'right' of σ such that the distance between σ and σ' equals δ. Let $\alpha \subset T$ be the horizontal geodesic that intersects σ at P. Let P' denote the intersection point of α and σ'. Let σ_0' be the component of $p^{-1}(\sigma') \cap C_0$ that has distance δ from σ_0. Let R_i be the connected component of $p^{-1}(C \setminus \alpha) \cap C_i$ that contains σ_i, and let σ_i' be the component of $p^{-1}(\sigma' \setminus P)$ contained in R_i that has distance δ from σ_i. Cut along σ_0' and σ_1' and glue the left (resp. right) side of σ_0' to the right (resp. left) side of σ_1'. The resulting torus covering is

$$\overline{q}_i : \overline{Y}_i \to T.$$

We next construct a piecewise differentiable homeomorphism $f : \overline{Y}_i \to \overline{X}_i$ as follows. Define f to be the identity on the complement of $C_0 \cup C_i$. The right side of the segment σ_i corresponds to a simple closed curve $\gamma \subset \overline{X}_i$. Let A be the annular neighborhood of γ consisting of points of distance at most $\delta/2$ from γ. Let A^+ be the connected component of $A \setminus \gamma$ that lies in $C_0 \cup C_i$. Define f so that it maps the annulus $A' \subset C_0$ bounded by σ_0 and the right by σ_0' onto the annulus A_+. Define f so that it maps the cylinder in C_0 that lies to the right of σ_0' to the part of the cylinder in C_0 that lies to the right of σ_0 that is exterior to A^+. Define f to map the thrice holed sphere $C_i \setminus \sigma_i'$ to the thrice holed sphere $C_i \setminus A^+$. See Figure 13.

The construction of f can be made to depend continuously on the parameter δ. By pulling back the 1-form $\overline{q}^*(dz)$ using the inverse f^{-1} we obtain a continuous family of 1-forms ω_δ on $\overline{X}_i$. Each zero of ω_δ defines a simple arc on $\overline{X}_i$ that is parametrized by δ. These arcs are disjoint, and hence we may choose a symplectic basis for $H_1(\overline{X}_i; \mathbb{Z})$ that avoids these arcs. It follows that the spin parity of ω_δ is constant in δ. Thus, for each δ, the spin parity of $\overline{q}^*(dz)$ equals the spin parity of $\overline{p}^*(\delta)$. ∎

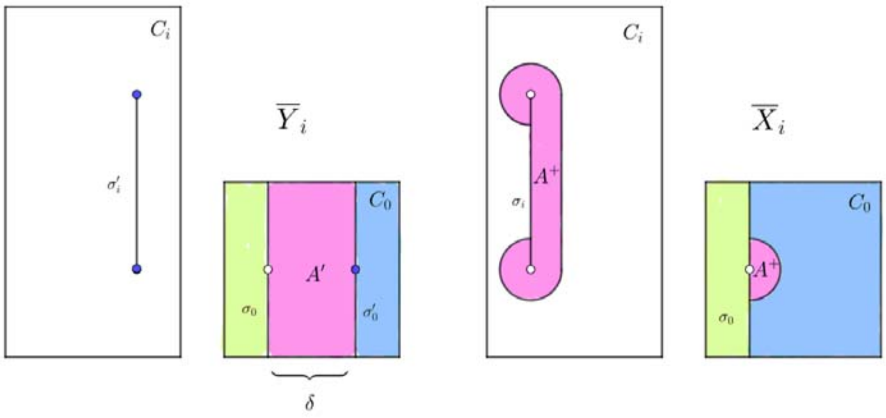

Figure 13. The homeomorphism f that maps $\overline{Y}_i$ to $\overline{X}_i$. The map f is the identity on the complement of the cylinders C_0 and C_i. Each colored region in $C_0 \cup C_i$ is mapped to the region of the same color in $C_0 \cup C_i$.

3.4. Surgery admissible torus covers with highest order odd. In the next subsection, we will describe an algorithm which produces a degree d torus cover in a prescribed connected component of a stratum that satisfies the hypotheses of Proposition 1.3. The algorithm will be based on the surgeries described above. If the highest order of a zero in the prescribed stratum is even, then we will choose the initial surface to be either the torus covering E_g^d or the torus covering O_g^d (see §3.1) depending on the desired spin parity. In this section we construct a surgery admissible degree d torus covering $q\colon Z_{m,n}^d \to T$ which will be the starting point of the algorithm when the zero of highest order in the prescribed stratum is odd. The surface $Z_{m,n}^d$ will have genus $m+n$ and the associated 1-form $q^*(dz)$ will have exactly two zeros, one with order $2m-1$ and the other with order $2n-1$. We will assume that $m \geq n$.

We will describe the construction of $Z_{m,n}^d$ in the case where $d = 2m$. See Figure 14 for an example with $m = 3$. The surfaces for $d > 2m$ are obtained by adding additional squares as before. Let $p\colon O_m \to T$ be the degree $2m-1$ torus covering described in §3.1.2. Let S denote the disjoint union of O_m and T. Define the degree $2m$ torus covering $\widetilde{p} : S \to T$ by letting $\widetilde{p}(z) = p(z)$ for each $z \in O_m$, and by letting $\widetilde{p}(z) = z$ for each $z \in T$.

Choose a horizontal segment τ in T of length $\epsilon < 1$ that has one endpoint at the origin in T. The inverse image $\widetilde{p}^{-1}(\tau)$ has $2m$ connected components. Let τ_{2n} denote the unique connected component of $\widetilde{p}^{-1}(\tau)$ that lies in T, and choose components $\tau_1, \ldots, \tau_{2n-1}$, from among the remaining $2m-1$ components that lie in O_m.

We will cut along each τ_i and reglue, but the choice of gluing must be made with some care to ensure that the resulting 1-form has no more than two zeros. To describe precisely the gluing, we will suppose that for $i < 2n$, the τ_i are labeled in the order that they appear as one winds clockwise around the zero of degree $2m-2$ on O_m. More precisely, suppose that $\gamma : \mathbb{R}/\mathbb{Z} \to T$ gives the standard clockwise parameterization of the circle of radius $\epsilon/2$ centered at the origin $0 \in T$. Then γ lifts via p to a map

$$\widetilde{\gamma} : \mathbb{R}/(2m-1)\mathbb{Z} \to O_m,$$

and for each $i \in \{1, \ldots, 2n-1\}$, there exists a unique $t_i \in \mathbb{R}/(2m-1)\mathbb{Z}$ such that

$$\widetilde{\gamma}(t_i) \in \tau_i.$$

By relabeling if necessary, we may assume that $i < j$ if and only if $t_i < t_j$.

Cut along each τ_i and reglue the bottom of τ_i to the top of τ_{i+1}. Let

$$q : Z^d_{m,n} \to \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$$

denote the resulting (connected) torus covering of degree d. Because $2n$ is even, the point $q^{-1}(0)$ is a zero of order $2m-1$, and if Q denotes the other endpoint of τ, then $q^{-1}(Q)$ is a single zero of order $2n-1$. The genus of $Z^d_{m,n}$ is $m+n$.

3.5. An algorithm and examples. In this section we describe an algorithm for constructing a primitive torus cover of degree d in any desired connected component K of a stratum $\Omega\mathcal{M}_g(\alpha)$. We then illustrate the algorithm with some examples.

Otherwise, we suppose that the desired divisor data $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfies

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n,$$

and if each α_i is even, we define $\theta \in \mathbb{Z}/2\mathbb{Z}$ by

$$\theta := \text{spin} + \frac{\alpha_1}{2} + \frac{\alpha_2}{2} + \cdots + \frac{\alpha_{n-1}}{2} \mod 2$$

where 'spin' denotes the desired spin parity.

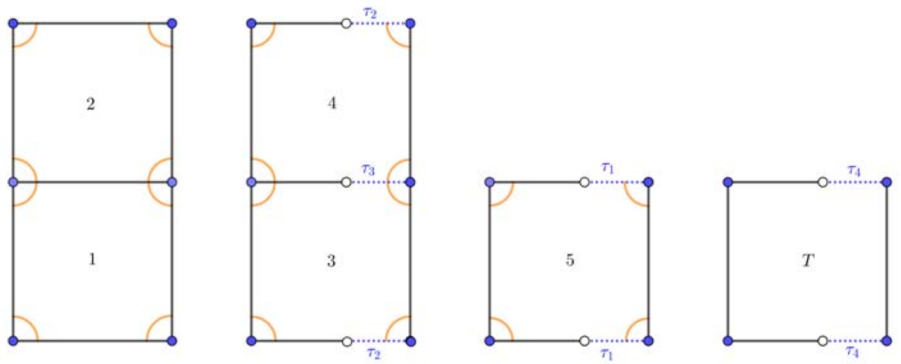

Figure 14. The degree 6 torus cover $Z_{3,2} \in \Omega\mathcal{M}(5,3)$ obtained from the surface $O_3 \in \Omega\mathcal{M}(4)^{\text{odd}}$. The purple points correspond to the zero of order 5, and the white points correspond to the zero of order 3. The labeling of the τ_i for $i < 2n$ is induced by the simple closed curve $\widetilde{\gamma}$ (in orange) on O_3 that winds clockwise around the pre-image of 0.

- If each α_i is an even integer, and
  - $n = 1$, then apply one of the constructions in §3.1,
  - $n = 2$ and $\alpha_1 = \alpha_2$, then apply one of the constructions in §3.2,
  - otherwise
    - if $\theta = 0 \bmod 2$, then apply the surgery of §3.3.1 to the torus cover E_g^d to add zeros of order $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$,
    - if $\theta = 1 \bmod 2$, then apply the surgery of §3.3.1 to the torus cover O_g^d to add zeros of order $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$;
- otherwise (when some α_i is odd)
  - if α_n is even, then apply the surgeries of §3.3.1 and §3.3.2 to the torus cover O_g^d to add zeros of order $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$,
  - if α_n is odd, then some other zero, say α_j, is odd. Begin with the torus cover $Z_{m,n}^d$ where $m = (\alpha_n + 1)/2$ and $n = (\alpha_j + 1)/2$, and apply the surgeries of §3.3.1 and §3.3.2 to add zeros of order α_i for $i \neq j$ or n.

3.5.1. *Torus covers in* $\Omega\mathcal{M}_4(1,2,3)$. Suppose that we wish to construct a degree 4 torus cover p so that $p^*(dz)$ has a zero of order 3, a zero of order 2, and a zero of order 1. That is, we have $\alpha_3 = 3$ which is odd, and $\alpha_1 = 1$ is odd as well. Hence we begin with the torus cover $Z^4_{2,1}$ whose construction is described in §3.4. Then we perform the surgery described in §3.3.1 to add a zero of order two. See Figure 15. In more detail, the surface $Z^4_{2,1}$ is obtained by slitting the L-shaped surface O_2 that lies in $\Omega\mathcal{M}_2(2)$ along the segment τ_2, slitting a square torus along a segment τ_2, and then gluing the top (resp. bottom) of τ_1 to the bottom (resp. top) of τ_2. The resulting surface is cut along the segments σ_0 and σ_1 and the the left (resp. right) of σ_0 to the right (resp. left) of σ_1.

Note that by adjoining $d-4$ squares to the right—that is, by replacing the unit square torus with the rectangular torus $\mathbb{C}/((d-3)\mathbb{Z}+i\mathbb{Z})$—one obtains a degree $d > 4$ torus covering in $\Omega\mathcal{M}_4(1,2,3)$.

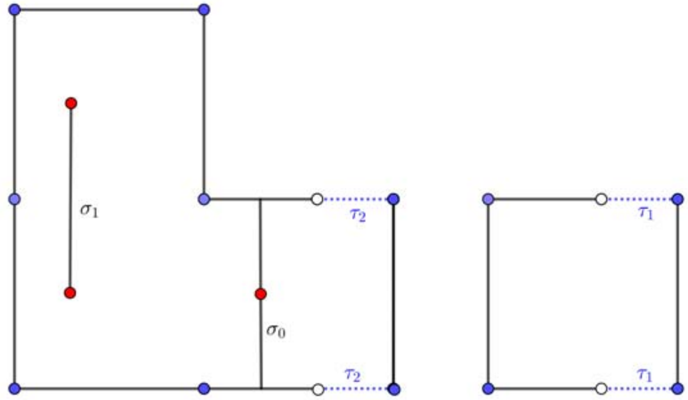

Figure 15. A degree 4 torus cover in the stratum $\Omega\mathcal{M}_4(1,2,3)$. The blue points correspond to a zero of order 3, the white points correspond to a zero of order 2, and the red points correspond to a zero of order 1.

3.5.2. *Torus covers in* $\Omega\mathcal{M}_7^{\text{odd}}(2,4,6)$. We describe the construction of a torus cover of degree 7 that has odd spin parity and has one zero of order 6, one zero of order 4, and one zero of order 2. Since $\theta = 1+2+1 = 0 \bmod 2$, we begin with the degree 7 torus cover $p : E^7_4 \to T$ such that $p^*(dz) \in \Omega\mathcal{M}_4(6)$, and we then we use the surgery of §3.3.1 to add zeros of order 2 and 4 while preserving the degree. See Figure 16. In detail, 'opposing sides' of the polygon in Figure 16 are identified with the exception of the sides labeled γ_i in which case we identify γ_1 with γ_3 and identify γ_2 with γ_4. To add the additional zero of order 4 (resp. 2) we cut along σ_0 and σ_1 (resp. σ_0' and σ_1') and reglue.

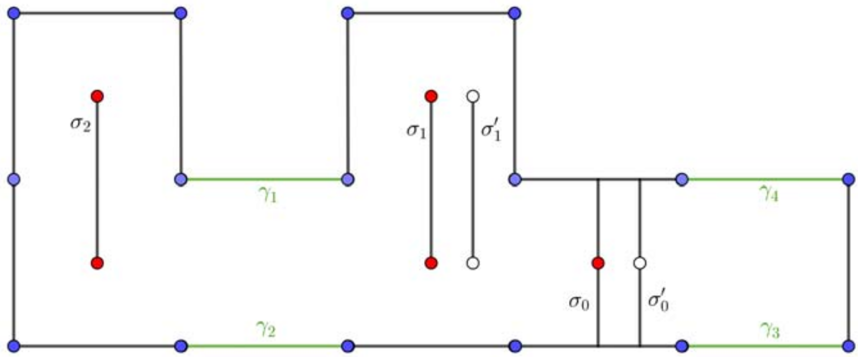

Figure 16. A degree 7 torus cover in the odd component of $\Omega\mathcal{M}_7(2,4,6)$. The blue points correspond to the zero of order 6, the red points correspond to the zero of order 4, the white points correspond to the zero of order 2.

By adjoining $d-7$ additional squares to the right, one obtains a degree $d > 7$ torus covering in $\Omega\mathcal{M}_7^{\rm odd}(2,4,6)$.

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