ISRAEL JOURNAL OF MATHEMATICS **250** (2022), 85–113
DOI: 10.1007/s11856-022-2332-4

LINES IN AFFINE TORIC VARIETIES

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ABSTRACT

We prove that up to automorphisms of the target the affine line $\mathbb{A}^1$ admits a unique embedding into the regular part of an affine simplicial toric variety of dimension at least 4 which is smooth in codimension 2. This is an analog of the well-known result on the existence of a linearization of any polynomial embedding $\mathbb{A}^1 \hookrightarrow \mathbb{A}^n$ for $n \geq 4$.

1. Introduction

Let $\varphi : C \to C'$ be an isomorphism of two smooth polynomial curves contained in the regular part Y_{reg} of an affine algebraic variety Y over an algebraically closed field $\mathbf{k}$ of characteristic zero. It may happen that φ extends to an automorphism of Y and our first aim is to describe some affine algebraic varieties for which this extension takes place.

This problem was studied in several papers [AMo], [Su], [Cr],[Je], [St], [FS], [Ka20] and [AZ]. It turns out that the answer is positive for some classes of flexible varieties of dimension $n \geq 4$ where Y is flexible if the subgroup $\mathrm{SAut}(Y)$ of the automorphism group $\mathrm{Aut}(Y)$ of Y generated by all one-parameter unipotent subgroups acts transitively on Y_{reg}. Say, this is so if $Y = \mathbb{A}^n$ with $n \geq 4$ [Cr], [Je]. For $n = 3$ the answer is unknown but for $n = 2$ the famous Abhyankar–Moh–Suzuki theorem [AMo], [Su] states that an isomorphism of two smooth

Received February 2021, and in revised form April 25, 2021

plane polynomial curves always extends to an automorphism of the plane $\mathbb{A}^2$. Perhaps, $\mathbb{A}^2$ is the only example of a two-dimensional flexible variety with this property. If Y is an affine simplicial toric variety $\mathbb{A}^2/G$ where G is a finite subgroup of $\mathrm{SL}_2(\mathbf{k})$ acting naturally on $\mathbb{A}^2$, then Arzhantsev and Zaidenberg [AZ] showed that the answer is negative. They actually classified up to automorphisms of Y all smooth polynomial curves in Y_{reg} (there is only a finite number of isomorphism classes of such curves).

In this paper we study the case when Y is an affine simplicial toric variety of dimension $n \geq 4$ (i.e., $Y = \mathbb{A}^n/G$ where G is a finite subgroup of $\mathrm{SL}_n(\mathbf{k})$ acting naturally on $\mathbb{A}^n$). We show that the answer to this extension problem is positive under the assumption of smoothness in codimension 2.

Furthermore, recall that given a subvariety Z of Y with defining ideal I in the algebra $\mathbf{k}[Y]$ of regular functions on Y, its kth infinitesimal neighborhood is the scheme with the defining ideal I^k. In particular, if W is another subvariety of Y with defining ideal J, then an isomorphism $\mathcal{Z} \to \mathcal{W}$ of kth infinitesimal neighborhoods of Z and W is determined by an isomorphism of algebras $\frac{\mathbf{k}[Y]}{I^k} \to \frac{\mathbf{k}[Y]}{J^k}$. There are natural obstacles for extending such isomorphisms to automorphisms of Y. Say, let $Z = W$ be a strict complete intersection given in Y by $u_1 = \cdots = u_m = 0$. Then an automorphism $\psi : \frac{\mathbf{k}[Y]}{I^k} \to \frac{\mathbf{k}[Y]}{I^k}$ over $\frac{\mathbf{k}[Y]}{I}$ is given by polynomials $f_1, \ldots, f_m$ in $u_1, \ldots, u_m$ over $\frac{\mathbf{k}[Y]}{I}$ of degree at most $k-1$. If Y does not admit nonconstant invertible functions and ψ is extendable to an automorphism of Y, then one can see that the Jacobian $\det[\frac{\partial f_i}{\partial u_j}]_{i,j=1}^m$ must be equal to a nonzero constant modulo I^{k-1} in which case we say that ψ has a nonzero constant Jacobian. There is also a notion of a nonzero constant Jacobian of an isomorphism $\mathcal{Z} \to \mathcal{W}$ in the case when both Z and W are smooth polynomial curves in a normal toric variety Y contained in Y_{reg} (see Definition 7.4). The question when such isomorphisms with nonzero constant Jacobians are extendable to automorphisms of Y was considered in [KaUd] and [Ud]. In combinations with the results of [KaUd] and [Ud] we get our first main result (Corollary 7.5).

Theorem 1.1: *Let Y be an affine simplicial toric variety smooth in codimension 2 such that $\dim Y \geq 4$. Let $\varphi : \mathcal{C}_1 \to \mathcal{C}_2$ be an isomorphism of kth infinitesimal neighborhoods of two smooth polynomial curves contained in Y_{reg} such that the Jacobian of φ is a nonzero constant. Then φ extends to an automorphism of Y.*

The second subject of this paper is related to the theorem of Holme [Hol, Theorem 7.4] (later rediscovered in [Ka91] and [Sr]). It states that if Z is an affine algebraic variety with

$$\mathrm{ED}(Z) := \max(2\dim Z + 1, \dim TZ) \leq n,$$

then Z admits a closed embedding into $\mathbb{A}^n$ (the version of this theorem with a smooth Z appeared originally in [Swan, Theorem 2.1]). Recently, Feller and van Santen [FS21] proved that if X is an affine algebraic variety isomorphic to a simple linear algebraic group and Z is smooth, then Z admits a closed embedding into X, provided that $\dim X > \mathrm{ED}(Z) = 2\dim Z + 1$. Since affine spaces, simple linear algebraic groups and normal affine toric varieties are examples of flexible varieties it is natural to look for analogues of Holme's theorem in the flexible case. In this paper we prove the following.

Theorem 1.2 (Theorem 3.7): *Let Z be an affine algebraic variety and X be a smooth quasi-affine flexible variety of dimension at least* $\mathrm{ED}(Z)$*. Then Z admits an injective immersion into X.*

In the case when X is a normal affine toric variety we also find conditions which guarantee that Z admits a closed embedding into X (Theorem 5.3). The formulation of the latter theorem is subtler when X is simplicial and it is a consequence of the following more general fact.

Theorem 1.3 (Corollary 3.3): *Let $\psi : \mathbb{A}^n \to Y$ be a finite morphism where Y is normal. Suppose that Z is an affine algebraic variety such that* $\mathrm{ED}(Z) \leq n$ *and* $\dim Z < \operatorname{codim}_Y Y_{\mathrm{sing}}$*. Then Z admits a closed embedding in Y with the image contained in Y_{reg}.*

The paper is organized as follows. In Section 2 we survey the technique developed in [Ka20] which was later clarified in [KaUd]. In particular, one can find there formal definitions of locally nilpotent vector fields and flexible varieties. Section 2 contains also a modified version of Theorem 4.2 from [Ka20] which is a crucial tool in this paper. Using this result we prove Theorems 1.2 and 1.3 in Section 3. In Section 4 we introduce notations for toric varieties which are used freely throughout the rest of the paper and prove some simple facts about normal affine toric varieties. Section 4 contains an analogue of Holme's theorem for normal affine toric varieties. In Section 5 we study locally nilpotent vector fields on normal affine toric varieties with no torus factors. The properties of locally nilpotent vector fields are crucial for us since compositions of elements of the flows of such vector fields produce automorphisms that extend isomorphisms of smooth polynomial curves. In Section 6 we prove Theorem 1.1.

ACKNOWLEGEMENT. The author is grateful to the referee for very useful comments and corrections.

2. Flexible varieties: preliminaries

In this section we present some technical tools developed in [Ka20] with later clarifications in [KaUd] which we use in this paper. We shall also give a modified version of [Ka20, Theorem 4.2].

Definition 2.1: (1) Given an irreducible algebraic variety $\mathcal{A}$ and a map $\varphi : \mathcal{A} \to \mathrm{Aut}(X)$ (where $\mathrm{Aut}(X)$ is the group of algebraic automorphisms of X) we say that $(\mathcal{A}, \varphi)$ is an **algebraic family** of automorphisms of X if the induced map

$$\mathcal{A} \times X \to X,$$
$$(\alpha, x) \mapsto \varphi(\alpha).x$$

is a morphism (see [Ra]).

(2) If we want to emphasize additionally that $\varphi(\mathcal{A})$ is contained in a subgroup G of $\mathrm{Aut}(X)$, then we say that $\mathcal{A}$ is an **algebraic G-family** of automorphisms of X.

(3) In the case when $\mathcal{A}$ is a connected algebraic group and the induced map $\mathcal{A} \times X \to X$ is not only a morphism but also an action of $\mathcal{A}$ on X, we call this family a **connected algebraic subgroup** of $\mathrm{Aut}(X)$.

(4) Following [AFKKZ, Definition 1.1] we call a subgroup G of $\mathrm{Aut}(X)$ algebraically generated if it is generated as an abstract group by a family $\mathcal{G}$ of connected algebraic subgroups of $\mathrm{Aut}(X)$.

We have the following important fact [AFKKZ, Theorem 1.15] (which is the analogue of the Kleiman transversality theorem [Kl] for algebraically generated groups).

THEOREM 2.2 (Transversality Theorem): *Let a subgroup $G \subseteq \mathrm{Aut}(X)$ be algebraically generated by a system $\mathcal{G}$ of connected algebraic subgroups closed under conjugation in G. Suppose that G acts with an open orbit $O \subseteq X$.*

Then there exist subgroups $H_1, \ldots, H_m \in \mathcal{G}$ such that for any locally closed reduced subschemes Y and Z in O one can find a Zariski dense open subset $U = U(Y,Z) \subseteq H_1 \times \cdots \times H_m$ such that every element $(h_1, \ldots, h_m) \in U$ satisfies the following:

(a) *The translate $(h_1 \cdots h_m).Z_{\mathrm{reg}}$ meets Y_{reg} transversally.*
(b) $\dim(Y \cap (h_1 \cdots h_m).Z) \leq \dim Y + \dim Z - \dim X$.[1] *In particular*

$$Y \cap (h_1 \cdots h_m).Z = \emptyset$$

if $\dim Y + \dim Z < \dim X$.

Definition 2.3: (1) A nonzero derivation δ on the ring A of regular functions on an affine algebraic variety X is called **locally nilpotent** if for every $0 \neq a \in A$ there exists a natural n for which $\delta^n(a) = 0$. This derivation can be viewed as a vector field on X which we also call **locally nilpotent**. The set of all locally nilpotent vector fields on X will be denoted by $\mathrm{LND}(X)$. The flow of $\delta \in \mathrm{LND}(X)$ is an algebraic $\mathbb{G}_a$-action on X, i.e., the action of the group $(\mathbf{k}, +)$ which can be viewed as a one-parameter unipotent group U in the group $\mathrm{Aut}(X)$ of all algebraic automorphisms of X. In fact, every $\mathbb{G}_a$-action is a flow of a locally nilpotent vector field (e.g., see [Fr, Proposition 1.28]).

(2) If X is a quasi-affine variety, then an algebraic vector field δ on X is called **locally nilpotent** if δ extends to a locally nilpotent vector field $\tilde{\delta}$ on some affine algebraic variety Y containing X such that $\tilde{\delta}$ vanishes on $Y \setminus X$ where $\mathrm{codim}_Y(Y \setminus X) \geq 2$. Note that under this assumption δ generates a $\mathbb{G}_a$-action on X and we use again the notation $\mathrm{LND}(X)$ for the set of all locally nilpotent vector fields on X.

Definition 2.4: (1) For every locally nilpotent vector field δ and each function $f \in \mathrm{Ker}\,\delta$ from its kernel, the field $f\delta$ is called a **replica** of δ. Recall that such replica is automatically locally nilpotent.

(2) Let $\mathcal{N}$ be a set of locally nilpotent vector fields on X and $G_{\mathcal{N}} \subset \mathrm{Aut}(X)$ denote the group generated by all flows of elements of $\mathcal{N}$. We say that $G_{\mathcal{N}}$ is **generated** by $\mathcal{N}$.

(3) A collection of locally nilpotent vector fields $\mathcal{N}$ is called **saturated** if $\mathcal{N}$ is closed under conjugation by elements in $G_{\mathcal{N}}$ and for every $\delta \in \mathcal{N}$ each replica of δ is also contained in $\mathcal{N}$.

[1] We put the dimension of empty sets equal to $-\infty$.

Definition 2.5: Let X be a normal quasi-affine algebraic variety of dimension at least 2, $\mathcal{N}$ be a saturated set of locally nilpotent vector fields on X and $G = G_{\mathcal{N}}$ be the group generated by $\mathcal{N}$. Then X is called **G-flexible** if for any point x in the smooth part X_{reg} of X the vector space $T_x X$ is generated by the values of locally nilpotent vector fields from $\mathcal{N}$ at x (which is equivalent to the fact that G acts transitively on X_{reg} [FKZ, Theorem 2.12]). In the case of $G = \mathrm{SAut}(X)$ we call X **flexible** without referring to $\mathrm{SAut}(X)$ (recall that $\mathrm{SAut}(X)$ is the subgroup of $\mathrm{Aut}\, X$ generated by all one-parameter unipotent subgroups).

The following is a modified version of [Ka20, Theorem 4.2].

THEOREM 2.6: *Let X be a smooth algebraic variety, Q be a normal algebraic variety, $\varrho : X \to Q$ be a dominant morphism and $G \subset \mathrm{Aut}(X)$ be an algebraically generated group acting 2-transitively on X. Suppose that Q_0 is a smooth open dense subset of Q, $X_0 = \varrho^{-1}(Q_0)$ and Z is a locally closed reduced subvariety of X.*

(i) *Suppose that*

(1) $$\dim X_0 \times_{Q_0} X_0 = 2 \dim X - \dim Q$$

and $\dim Q \geq \dim Z + m$ where $m \geq 1$. Then there exists an algebraic G-family $\mathcal{A}$ of automorphisms of X such that for a general element $\alpha \in \mathcal{A}$ one can find a constructible subset R of $\alpha(Z) \cap X_0$ of dimension $\dim R \leq \dim Z - m$ for which $\varrho(R)$ and $\varrho(\alpha(Z) \setminus R)$ are disjoint and the restriction

$$\varrho|_{(\alpha(Z) \cap X_0) \setminus R} : (\alpha(Z) \cap X_0) \setminus R \to Q_0$$

of ϱ is injective. In particular, if $\dim Q \geq 2 \dim Z + 1$ and $Z'_\alpha = \varrho \circ \alpha(Z)$, then for a general element $\alpha \in \mathcal{A}$ the morphism

$$\varrho|_{\alpha(Z) \cap X_0} : \alpha(Z) \cap X_0 \to Z'_\alpha \cap Q_0$$

is a bijection, while in the case of a pure-dimensional Z and $\dim Q \geq \dim Z + 1$ this morphism is birational.

(ii) *Let G be generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields on X (in particular, X is G-flexible) and*

$$Y = \bigcup_{x \in X_0} \mathrm{Ker}\{\varrho_* : T_x X_0 \to T_{\varrho(x)} Q_0\}$$

Let

(2) $$\dim Y = \dim TX - \dim Q.$$

Suppose that $\dim TZ \le \dim Q$. *Then there exists an algebraic family* $\mathcal{A}$ *of* G*-automorphisms of* X *such that for a general element* $\alpha \in \mathcal{A}$ *and every* $z \in \alpha(Z) \cap X_0$, *the induced map* $\varrho_* : T_z\alpha(Z) \to T_{\varrho(z)}Q$ *of the tangent spaces is injective.*

Proof. For every variety $\mathcal{X}$ denote by $S_{\mathcal{X}}$ the variety $S_{\mathcal{X}} = (\mathcal{X} \times \mathcal{X}) \setminus \Delta_{\mathcal{X}}$ where $\Delta_{\mathcal{X}}$ is the diagonal in $\mathcal{X} \times \mathcal{X}$. Then every automorphism in $\mathrm{Aut}(X)$ can be lifted to an automorphism of S_X. In particular, we have a G-action on S_X and by the assumption this action is transitive on S_X. Consider the subvariety

$$Y = (X_0 \times_{Q_0} X_0) \setminus \Delta_X \subset S_X.$$

By Formula (1), $\dim Y = 2\dim X - \dim Q$ (i.e., the codimension of Y in S_X is $\dim Q$). By Theorem 2.2 (b) we can choose algebraic subgroups $H_1, \ldots, H_m$ of G such that for a general element $(h_1, \ldots, h_m) \in H_1 \times \cdots \times H_m$ one has

$$\begin{aligned}\dim W &\le \dim Y + \dim S_Z - \dim S_X = \dim S_Z - \dim Q \\ &\le 2\dim Z - \dim Q\end{aligned}$$

where $W = Y \cap \alpha(S_Z)$ for $\alpha = h_1 \cdots h_m$. Hence, in case (i) the dimension of W is at most $\dim Z - m$. Let R be the image of W under one of the two natural projections $X \times_Q X \to X$. In particular, R is a constructible set by Chevalley's theorem [Ha, Chap. II, Exercise 3.19], $R \subset \alpha(Z) \cap X_0$ and $\dim R \le \dim Z - m$. Note that for $z \in \alpha(Z) \cap X_0$ one has

$$\varrho^{-1}(\varrho(z)) \cap \alpha(Z) = z \quad \text{iff } z \notin R.$$

Hence, the restriction of ϱ to $(\alpha(Z) \cap X_0) \setminus R$ is injective. Therefore, letting $\mathcal{A} = H_1 \times \cdots \times H_m$, we get (i).

In (ii) for every variety $\mathcal{X}$ and a subvariety $\mathcal{Y}$ of the tangent bundle $T\mathcal{X}$ let $\mathcal{Y}^* = \mathcal{Y} \setminus \mathcal{S}$ where $\mathcal{S}$ is the zero section of the natural morphism $T\mathcal{X} \to \mathcal{X}$. Every automorphism $\alpha \in \mathrm{Aut}(X)$ generates an automorphism of TX. In particular, G acts on $(TX)^*$ and by [AFKKZ, Theorem 4.11 and Remark 4.16] this action is transitive. By Formula (2), $\dim Y^* = \dim TX - \dim Q$. By Theorem 2.2 we can choose one-parameter unipotent algebraic subgroups $\tilde{H}_1, \ldots, \tilde{H}_{\tilde{m}}$ of G such that for a general element $(\tilde{h}_1, \ldots, \tilde{h}_{\tilde{m}}) \in \tilde{H}_1 \times \cdots \times \tilde{H}_{\tilde{m}}$ and $Z'' = (\tilde{h}_1 \cdots \tilde{h}_{\tilde{m}})(Z)$ one has

$$\dim Y^* \cap (TZ'')^* \le \dim Y^* + \dim(TZ)^* - \dim TX^* \le 0.$$

Note that if $Y^* \cap (TZ'')^*$ contains a point, then $\dim Y^* \cap TZ''$ must be at least 1 (since this point is a vector in TZ'' and then $Y^* \cap (T(Z'')^*$ contains all nonzero vectors proportional to that one). That is, $Y^* \cap (T(Z'')^* = \emptyset$. This implies that for every $z \in Z'' \cap X_0$ the restriction of ϱ_* to $T_z Z''$ is injective. Consequently, the restriction of ϱ_* to $T_z Z''$ is injective, i.e., we have (ii). ∎

Let us describe some G-families $\mathcal{A}$ satisfying the conclusions of Theorem 2.6.

Definition 2.7: Let X be a smooth algebraic variety and G be a subgroup of X. Consider $(X \times X) \setminus \Delta$ (where Δ is the diagonal), the complement $(TX)^*$ to the zero section in the tangent bundle of X and the frame bundle $\mathrm{Fr}(X)$ of TX (i.e., the fiber of $\mathrm{Fr}(X)$ over $x \in X$ consists of all bases of $T_x X$). Projectivization of TX replaces $\mathrm{Fr}(X)$ with a bundle $\mathrm{PFr}(X)$ whose fiber over x consists of all ordered n-tuples of points in the projectivization $\mathbb{P}^n$ of $T_x X$ (where $n = \dim X$) that are not contained in the same hyperplane of $\mathbb{P}^n$. Then we have natural G-actions on all these objects. Let Y be either X, or $(X \times_P X) \setminus \Delta$, or $(TX)^*$, or $\mathrm{PFr}(X)$. Suppose that the G-action is transitive on Y. Then we say that an algebraic G-family $\mathcal{A}$ of automorphisms of X is a **regular G-family** for Y if

(i) $\mathcal{A} = H_m \times \cdots \times H_1$ where each H_i belongs to $\mathcal{G}$;
(ii) for a suitable open dense subset $U \subseteq H_m \times \cdots \times H_1$, the map

$$\Psi : H_m \times \cdots \times H_1 \times Y \longrightarrow Y \times Y, \quad (h_m, \ldots, h_1, y) \mapsto ((h_m \cdots h_1).y, y) \tag{3}$$

is smooth on $U \times Y$.

An algebraic G-family $\mathcal{A}$ that is regular for all four varieties X, $(X \times_P X) \setminus \Delta$, $(TX)^*$ and $\mathrm{PFr}(X)$ will be called a **perfect G-family** for Y.

Proposition 2.8: *Let X be a smooth algebraic variety and $G \subset \mathrm{Aut}(X)$ be a group algebraically generated by a family $\mathcal{G}$ of algebraic connected subgroups of $\mathrm{Aut}(X)$. Suppose that G acts transitively on X.*

(1) *Then there exists a regular G-family for X (which is of the form $\mathcal{A} = H_1 \times \cdots \times H_m$ where each H_i is an element of $\mathcal{G}$).*
(2) *Every regular G-family for X satisfies the conclusions of Theorem 2.2.*
(3) *If $\mathcal{A}$ is a regular (resp. perfect) G-family for X and H is an element of $\mathcal{G}$, then $H \times \mathcal{A}$ and $\mathcal{A} \times H$ are also regular (resp. perfect) G-families for X.*
(4) *In particular, if X is G-flexible, then there exists a perfect G-family.*

(5) *Let X be G-flexible. Every G-family regular for X (resp. for $(TX)^*$) satisfies the conclusion of Theorem 2.6 (i) (resp. (ii)). In particular, every perfect G-family satisfies the conclusion of Theorem 2.6 (i)–(ii).*

Proof. Statement (1) is proven in [AFKKZ, Proposition 1.15]. Statements (2) and (3) are proven in [Ka20, Proposition 1.10]. The fourth statement follows from the fact that in the flexible case for every $m > 0$ the group G acts m-transitively on X and also transitively on $(TX)^*$ and $\mathrm{PFr}(X)$ [AFKKZ, Theorem 4.11 and Remark 4.16]. Modulo the definitions of regular and perfect families and (2), statement (5) follows from the proof of Theorem 2.6. ∎

Proposition 2.9: *Let X be a smooth algebraic variety, Q be a normal algebraic variety and $\varrho : X \to Q$ be a dominant morphism. Let Q_0 be a smooth dense open subset of Q and $X_0 = \varrho^{-1}(Q_0)$. Suppose that for every $q \in Q_0$ the fiber $\varrho^{-1}(q)$ is smooth and of dimension $\dim X - \dim Q$. Then Formulas (1) and (2) hold. In particular, if X is G-flexible and Z satisfies the assumption of Theorem 2.6(i)–(ii), then for every perfect G-family $\mathcal{A}$ of automorphisms of X and a general $\alpha \in \mathcal{A}$ the morphism $\varrho|_{\alpha(Z)\cap X_0} : \alpha(Z) \cap X_0 \to Q_0$ is an injective immersion.*

Proof. The validity of Formula (1) is straightforward. Consider any irreducible subvariety P of Q_0 and an irreducible component W of $\varrho^{-1}(P)$ whose image is dense in P. Since every fiber of ϱ is smooth so is the generic fiber of $\varrho|_W$ (e.g., see [KR, Lemma 2.1]). Hence, replacing P by its open dense subset we can suppose that W is smooth. Furthermore, by [Ha, Chapter III, Corollary 10.7] we can suppose that $\varrho|_W : W \to P$ is smooth. Therefore,

$$\dim \operatorname{Ker} \varrho_*|_{T_x X} \le \dim X - \dim P$$

for every $x \in W$. This implies that the dimension of

$$Y_W = \bigcup_{x \in W} \operatorname{Ker}\{\varrho_* : T_x X \to T_{\varrho(x)} Q\}$$

is at most $\dim X - \dim P + \dim W$. Since $\varrho|_{X_0}$ is equidimensional, one has

$$\dim W - \dim P = \dim X - \dim Q \quad \text{and} \quad \dim Y_W \le \dim TX - \dim Q.$$

Of course, we can suppose that the latter inequality is true for every irreducible component W in $\varrho^{-1}(P)$ which yields the desired conclusion. ∎

3. Embedding theorems for flexible varieties

Notation 3.1: Let Z be an affine algebraic variety and TZ be its Zariski tangent bundle. Then we let

$$\mathrm{ED}(Z) = \max(2\dim Z + 1, \dim TZ).$$

By [Hol, Theorem 7.4] for every affine algebraic variety Z there exists a closed embedding of Z into $\mathbb{A}^{\mathrm{ED}(Z)}$.

THEOREM 3.2: *Let $\psi : W \to Y$ be a finite morphism where W is a smooth flexible variety and Y is normal. Let Z be a quasi-affine algebraic variety which admits a closed embedding in W. Suppose also that $\dim Z < \operatorname{codim}_Y Y_{\mathrm{sing}}$. Then Z admits a closed embedding in Y with the image contained in Y_{reg}.*

Proof. One can treat Z as a closed subvariety of W. By Theorem 2.2 there exists an algebraic family $\mathcal{A}$ of automorphisms of W such that for a general $\alpha \in \mathcal{A}$ the variety $\alpha(Z)$ does not meet $\psi^{-1}(Y_{\mathrm{sing}})$. By Proposition 2.8(2)–(4), enlarging $\mathcal{A}$, we can suppose that it is a perfect family. Proposition 2.9 implies now that $\psi|_{\alpha(Z)} : \alpha(Z) \to Y_{\mathrm{reg}} \subset Y$ is an injective immersion. Since ψ is finite $\psi|_{\alpha(Z)}$ is also proper. Hence, we are done. ∎

COROLLARY 3.3: *Let $\psi : \mathbb{A}^r \to Y$ be a finite morphism where Y is normal. Suppose that Z is an affine algebraic variety such that $\mathrm{ED}(Z) \le r$ and $\dim Z < \operatorname{codim}_Y Y_{\mathrm{sing}}$. Then Z admits a closed embedding in Y with the image contained in Y_{reg}.*

Remark 3.4: For every $m > 0$ there are examples of affine algebraic varieties of dimension m that cannot be embedded in $\mathbb{A}^{2m}$ [BMS]. In particular, Holme's theorem is sharp and we cannot improve the assumption $\mathrm{ED}(Z) \le r$ in Corollary 3.3. However, the author does not know if the assumption $\dim Z < \operatorname{codim}_Y Y_{\mathrm{sing}}$ is optimal for every Y as in Corollary 3.3 (especially, in the light of Theorem 3.7 below).

PROPOSITION 3.5: *Let X be a G-flexible variety and $H = H_m \times \cdots \times H_1$ be a perfect G-family of automorphisms of X (where $H_1, \ldots, H_m$ are unipotent subgroups of G). Suppose that an open dense supset $U \subset H$ is such that the morphism $\Psi : H \times Y \to Y \times Y$ as in Formula (3) is smooth on $U \times Y$ for every Y equal to one of the varieties X, $(X \times_P X) \setminus \Delta$, $(TX)^*$ and $\mathrm{PFr}(X)$. Then $m, H_1, \ldots, H_m$ and U can be chosen so that the codimension of $H \setminus U$ in H is arbitrarily large.*

Proof. Since one can increase m by Proposition 2.8(3) the desired conclusion follows from [AFKKZ, page 778, footnote]. ∎

Proposition 3.6: *Let H be a smooth flexible variety, X be a normal algebraic variety and $\varphi : H \to X$ be a dominant morphism such that φ is smooth on an open dense subset $U \subset H$ and $\varphi(U) \subset X_{\rm reg}$. Suppose that Z is a closed subvariety of H and $\operatorname{codim}_H(H \setminus U) > \dim Z$. Then for a general element α in a perfect family $\mathcal{A}$ of automorphisms of H the morphism*

$$\varphi|_{\alpha(Z)} : \alpha(Z) \to X$$

is an injective immersion with the image in $X_{\rm reg}$.

Proof. By Theorem 2.2 for a general element α in an algebraic family $\mathcal{A}$ of automorphisms of H the variety $\alpha(Z)$ does not meet $H \setminus U$. By Proposition 2.8(3)–(4) we can suppose that $\mathcal{A}$ is perfect. Theorem 2.6(i)–(ii) and Proposition 2.8(5) imply now that for a general $\alpha \in \mathcal{A}$ the morphism $\varphi|_{\alpha(Z)} : \alpha(Z) \to X$ is an injective immersion which concludes the proof. ∎

Theorem 3.7: *Let Z be an affine algebraic variety and X be a smooth quasi-affine flexible variety of dimension at least $\mathrm{ED}(Z)$. Then Z admits an injective immersion into X.*

Proof. Let U, H, H_i and $\Psi : H \times X \to X \times X$ be as in Proposition 3.5, i.e.,

$$H \simeq \mathbb{A}^t$$

since each H_i is a unipotent group. Restricting Ψ to $H \times x_0$ where x_0 is any point in X we get a morphism $\varphi : \mathbb{A}^t \simeq H \to X$ which is smooth on U. By Holme's theorem we can treat Z as a closed subvariety of $\mathbb{A}^t$. By Proposition 3.5 we can suppose that

$$\operatorname{codim}_H H \setminus U > \dim Z.$$

Since $\mathbb{A}^t$ is a smooth flexible variety we get the desired conclusion by Proposition 3.6. ∎

Remark 3.8: The author does not know if instead of "injective immersion" one can use "closed embedding" in Theorem 3.7. Establishing properness is the bottleneck of the method known to the author. However, in the case of affine toric varieties we managed to cope with this difficulty (see Theorem 5.3 below).

4. Affine toric varieties: preliminaries

We suppose that readers are familiar with toric varieties and all information about toric varieties which is used below can be found in the book of Cox, Little and Schrenck [CLS]). We fix the following notations for the rest of the paper.

- $N \simeq \mathbb{Z}^n$—the standard lattice in $\mathbb{R}^n$;
- $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$—the lattice dual to N;
- $\langle m, u \rangle$—pairing of $m \in M$ or $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ with $u \in N$ or $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$;
- σ—a rational convex polyhedral cone in $N_{\mathbb{R}}$;
- $\sigma^\vee$—the dual cone of σ in $M_{\mathbb{R}}$;
- $\gamma^\perp$—the set $\{m \in M_{\mathbb{R}} | \langle m, \gamma \rangle = 0\}$ where γ is any face of σ;
- X_σ—the toric variety of σ, i.e., X_σ is the spectrum of the group algebra of the semigroup $\sigma^\vee \cap M$;
- $\mathbb{T} = \mathrm{Hom}(M, \mathbb{G}_m)$—the torus acting on X_σ;
- $\{\varrho_1, \ldots, \varrho_r\}$—the set of extremal rays of σ (by abusing notations we also denote by ϱ_i the ray generator, i.e., the primitive lattice vector on the corresponding ray);
- $\sigma(k)$—the set of k-dimensional faces of σ (e.g., $\varrho_i \in \sigma(1)$);
- O_i—the $\mathbb{T}$-orbit in X_σ corresponding to ϱ_i by the orbit-cone correspondence [CLS, Theorem 3.2.6];
- D_i—the irreducible $\mathbb{T}$-invariant Weil divisor in X_σ containing O_i as an open subset (i.e., D_i is the spectrum of the semigroup algebra of $\tau_i = \varrho_i^\perp \cap \sigma^\vee \cap M$);
- H_i–the $\mathbb{G}_m$-subgroup of $\mathbb{T}$ corresponding to ϱ_i, i.e., H_i is a unique $\mathbb{G}_m$-subgroup of $\mathbb{T}$ that acts trivially on D_i and for $t \in H_i$ one has $t.\chi^m = t^{\langle m, \varrho_i \rangle} \chi^m$.

We would like to remind that since

$$\sigma^\vee \cap M \quad \text{and} \quad \tau_i = \varrho_i^\perp \cap \sigma^\vee \cap M$$

are saturated affine semigroups, the varieties X_σ and D_i are always normal (e.g., see [CLS, Theorem 1.3.5]). Furthermore, we consider only the case when X_σ has no torus factors (or, equivalently, every invertible function on X_σ is constant).

Let X_Σ be the toric variety of a fan Σ and r be the cardinality of one-dimensional cones in Σ (in particular, if $\Sigma = \sigma$ then r is the number of the ray generators ϱ_i of σ). If torus factors are absent, then by [CLS, Theorem 5.1.10] there exists a subgroup G of $\mathbb{G}_m^r$ which is a quasitorus (i.e., the direct product of a finite subgroup and a subtorus of $\mathbb{G}_m^r$) and a closed subvariety $Z(\Sigma)$ of $\mathbb{A}^r$ such that $Z(\Sigma)$ is invariant under the natural action of G on $\mathbb{A}^r$ and X_Σ is isomorphic to $(\mathbb{A}^r \setminus Z(\Sigma))/\!\!/G$ (while $\mathbb{G}_m^r/G$ is isomorphic to the torus $\mathbb{T}$ acting on X_Σ). We are dealing with the situation when $\Sigma = \sigma$ and in this case $Z(\sigma)$ is empty by construction (see the definition of $Z(\sigma)$ on [CLS, page 206]). Thus, we have the quotient morphism

$$\pi : \mathbb{A}^r \to \mathbb{A}^r /\!\!/ G \simeq X_\sigma \tag{4}$$

which is $\mathbb{G}_m^r$-equivariant. In connection with this formula we fix the following notations.

- $x_1, \ldots, x_r$—a fixed coordinate system on $\mathbb{A}^r$;
- $\tilde{T} = \mathbb{G}_m^r$—the standard torus (with respect to the coordinate system) acting on $\mathbb{A}^r$;
- $\tilde{D}_i$—the hyperplane in $\mathbb{A}^r$ given by $x_i = 0$;
- $\tilde{H}_i$—the $\mathbb{G}_m$-subgroup of $\tilde{T}$ acting trivially on $\tilde{D}_i$, i.e., this action is the flow of the semisimple vector field $x_i \frac{\partial}{\partial x_i}$;
- U—the subset of X_σ consisting of all points $u \in X_\sigma$ for which $\pi^{-1}(u)$ is a G-orbit (in particular, this orbit is closed in $\mathbb{A}^r$);
- U_0—the subset of X_σ consisting of all points $u \in U$ for which the orbit $\pi^{-1}(u)$ has a trivial stabilizer.

Let us list some properties of the morphism $\pi : \mathbb{A}^r \to X_\sigma$ and the objects introduced before.

Lemma 4.1: (i) *The morphism π is an almost geometric quotient, i.e., U is an open dense subset of X_σ and, consequently, general orbits of G in $\mathbb{A}^r$ are closed and isomorphic to G.*[2] *Furthermore, for every u in U the fiber $\pi^{-1}(u)$ is isomorphic as a homogeneous G-space to G/F where F is a finite subgroup of G.*

(ii) *The group $G \cap \tilde{H}_i$ is trivial.*

(iii) *Let $\theta \in \sigma(k)$ be regular, i.e., the set $\{\varrho_{i_1}, \ldots, \varrho_{i_k}\}$ of the generators of the extremal rays of θ can be extended to a basis of N. Then G meets the group $\tilde{F}$ generated by $\tilde{H}_{i_1}, \ldots, \tilde{H}_{i_k}$ at identity only.*

[2] Note that this implies that U_0 is also open dense subset of X_σ by [PV, Theorem 6.3].

(iv) *The homomorphism* $\pi^* : \mathbf{k}[X_\sigma] \to \mathbf{k}[x_1, \ldots, x_r]$ *induced by* π *is determined by the formula*

$$\pi^*(\chi^m) = \prod_{l=1}^{r} x_l^{\langle m, \varrho_l \rangle}. \tag{5}$$

(v) *The image of* $\tilde{H}_i$ *in* $\mathbb{T} = \frac{\tilde{T}}{G}$ *coincides with* H_i *and* $\pi(\tilde{D}_i) = D_i$.

Proof. For the first statement of (i) see [CLS, Theorem 5.1.10]. This implies that

$$\dim G = \dim A^r - \dim X_\sigma = r - n.$$

For every $u \in U$ one has $\dim \pi^{-1}(u) = r - n$ since it cannot be less by Chevalley's theorem [Ha, Chapter II, Exercise 3.22] and it cannot be larger since $\pi^{-1}(u)$ is a G-orbit. Being a homogeneous space $\pi^{-1}(u)$ is of the form G/F where $F \subset G$ is the stabilizer of the orbit $\pi^{-1}(u)$ and the condition on the dimension implies that F is finite. This concludes (i).

By [CLS, Lemma 5.1.1] we have

$$G = \left\{ \bar{t} = (t_1, \ldots, t_r) \in \mathbb{G}_m^r | \prod_{i=1}^{r} t_i^{\langle \varrho_i, m \rangle} = 1 \text{ for all } m \in M \right\}. \tag{6}$$

Suppose that $G \cap \tilde{H}_i$ contains a finite subgroup of d-roots of unity. Assume that $d > 1$ and let ε be a primitive d-root of unity. Then Formula (6) implies that $\varepsilon^{\langle m, \varrho_i \rangle} = 1$ for every $m \in M$, i.e., $\langle m, \varrho_i \rangle$ is divisible by d. Hence, $\frac{\varrho_i}{d} \in N$ contrary to the fact that ϱ_i is a primitive vector in the lattice N. This yields (ii).

In (iii) let $G \cap \tilde{F}$ contain a subgroup isomorphic to the group of d-roots of unity. Then a similar argument implies that for a collection $\{l_1, \ldots, l_s\}$ of integers with greatest common divisor 1 the sum $\sum_{s=1}^{k} l_s \varrho_{i_s}$ is divisible by d. However, if $d > 1$, then this is contrary to the fact that the set $\{\varrho_{i_1}, \ldots, \varrho_{i_k}\}$ is extendable to a basis of N. Thus, we have (iii).

For (iv) see [CLS, page 209]. Statement (v) follows from the explicit construction of π in [CLS, Proposition 5.1.9] which implies, in particular, (iv). Vice versa, (v) can be also illustrated by Formula (5). Indeed, this formula implies that for every χ^m, $m \in \tau_i = \varrho^\perp \cap \sigma^\vee \cap M$ the function $\pi^*(\chi^m)$ is independent of x_i, i.e., it is fixed under the $\tilde{H}_i$-action. In particular, χ^m is fixed under the action of the image H_i' of $\tilde{H}_i$ in $\mathbb{T} = \frac{\tilde{T}}{G}$ (which is isomorphic to $\tilde{H}_i$ by (ii)). Hence, the H_i'-action on D_i is trivial. Since H_i is a unique $\mathbb{G}_m$-subgroup of $\mathbb{T}$ with this property we see that $H_i' = H_i$. In particular, the divisor $\pi^{-1}(D_i)$ must be fixed under the $\tilde{H}_i$-action which implies that $\pi^{-1}(D_i) = \tilde{D}_i$ and we are done. ∎

Consider an affine algebraic group H acting on an affine variety Z, an affine variety Y with a trivial H-action and an H-equivariant morphism $\varrho : Z \to Y$. Recall that under these assumptions Z together with the morphism $\varrho : Z \to Y$ is called an H-torsor (or a principal H-bundle) if for every $y \in Y$ there exists an étale morphism $\varphi_y : W_y \to Y$ such that $y \in \operatorname{Im} \varphi_y$ and $W_y \times_Y Z$ becomes a trivial principal H-bundle under the natural H-action. If each φ_y is injective, then Z is called a locally trivial principal H-bundle.

Proposition 4.2: (1) *The morphism $\pi|_{\pi^{-1}(U_0)} : \pi^{-1}(U_0) \to U_0$ is a principal G-bundle (in particular, π is smooth over U_0). Furthermore, if G is irreducible, then this principal G-bundle is locally trivial.*

(2) *Let E be the subset of $\mathbb{A}^r$ consisting of all $\bar{x} = (x_1, \ldots, x_r) \in \mathbb{A}^r$ such that at most one coordinate x_i is equal to zero. Then*

$$\pi(E) \subset U_0.$$

(3) *U_0 is the regular part of X_σ.*

Proof. If $u \in U_0$ and $w \in \pi^{-1}(u_0)$, then by the Luna étale slice theorem [Lu] (see also [PV, Theorem 6.4]) there exists a smooth subvariety V of X_σ transversal to $\pi^{-1}(u)$ at w such that $\pi|_V : V \to U_0$ is étale. This implies that U_0 is contained in the regular part of X_σ and that π is smooth over U_0. Furthermore, the natural G-action makes $V \times_{U_0} \mathbb{A}^r$ a trivial G-bundle. Hence, $\pi^{-1}(U_0) \to U_0$ is a principal G-bundle. Recall also that if G is connected, then G is a special group in the sense of Serre (see [Gro58, Def. 2 and page 16]) and for any special group K every K-principal bundle is locally trivial [Gro58, Theorem 3]. Thus, we have (1).

For (2) and (3) we need to recall that by orbit-cone correspondence [CLS, Theorem 3.2.6] every $\theta \in \sigma(k)$ corresponds to a $\mathbb{T}$-orbit $O(\theta) \subset X_\sigma$ of dimension $n-k$ where $O(\theta)$ is the orbit of a so-called distinguished point. The description of this point [CLS, page 116] implies that $O(\theta)$ consists of all points $u \in X_\sigma$ such that $\chi^m(u) \neq 0$ if and only if $m \in \theta^\perp \cap M$. In particular, the ring $\mathbf{k}[R]$ of regular functions on the closure R of $O(\theta)$ in X_σ can be viewed as the semigroup algebra of $\theta^\perp \cap M$. Let $\varrho_{i_1}, \ldots, \varrho_{i_k}$ be the extremal rays generating θ (i.e., $\theta^\perp = \varrho_{i_1}^\perp \cap \cdots \cap \varrho_{i_k}^\perp$) and F be the subgroup of $\mathbb{T}$ generated by $H_{i_1}, \ldots, H_{i_k}$. Note that the natural inclusion $\mathbf{k}[R] \hookrightarrow \mathbf{k}[X_\sigma]$ makes $\mathbf{k}[R]$ the subring of F-invariants and R is given in X_σ by the ideal generated by

$$\{\chi^m | m \in (\sigma^\perp \setminus \theta^\perp) \cap M\}.$$

Hence, R is the fixed point set of the F-action since for every $v \in X_\sigma \setminus R$ one can find $m \in (\sigma^\perp \setminus \theta^\perp) \cap M$ with $\chi^m(v) \neq 0$. The difference between the points of $O(\theta)$ and $R \setminus O(\theta)$ is that for $w \in R \setminus O(\theta)$ there exists $j \in \{1, \dots, r\} \setminus \{i_1, \dots, i_k\}$ such that w is also fixed under the H_j-action (because w is contained in a $\mathbb{T}$-orbit of a smaller dimension corresponding to a cone in σ containing θ and some ϱ_j), whereas for any point in $O(\theta)$ such j does not exist.

Let

$$\bar{x} \in \tilde{T} = E \setminus \bigcup_{i=1}^{r} \tilde{D}_i.$$

Since π is $\tilde{T}$-equivariant $G.\bar{x}$ is a general orbit, i.e., $\pi(\bar{x}) \in U_0$ by Lemma 4.1(i). Since $E \cap \tilde{D}_{i_1}$ is a $\tilde{T}$-orbit dense in $\tilde{D}_{i_1}$ and $\pi(\tilde{D}_{i_1}) = D_{i_1}$ by Lemma 4.1 (v) we see that $\pi(E \cap \tilde{D}_{i_1}) = O_{i_1}$. For every $\bar{x} \in E \cap \tilde{D}_{i_1}$ its $\tilde{T}$-orbit Q is naturally isomorphic to G by Lemma 4.1 (ii) and $\pi(x) = u \in O_{i_1}$. Note that if Q is not closed, then its closure contains a point with some coordinates $x_j = 0$ where $j \neq i_1$. However, this implies that u is a fixed point under both H_{i_1}-action and H_j-action contrary to the argument before. Hence, Q is closed. Furthermore, Q is a unique closed G-orbit in $\pi^{-1}(u)$ by [CLS, Theorem 5.0.7]. If there exists another orbit Q' in $\pi^{-1}(u)$, then the closure of Q' contains Q (e.g., see [PV, Theorem 4.7 and Corollary]). However, this is impossible since $\dim Q' \leq \dim Q = \dim G$. Hence, $\pi^{-1}(u) = Q$ and $u \in U_0$ which is (2).

Consider u in the smooth part of X_σ. Then u is contained in some $O(\theta)$ as before where θ must be regular by [CLS, Theorem 1.3.12 and Example 1.2.20]. Let $\tilde{O}(\theta) \subset \mathbb{A}^r$ be the $\tilde{T}$-orbit consisting of all points $\bar{x}$ whose zero coordinates are exactly $x_{i_1}, \dots, x_{i_k}$. Let θ' be a cone in σ properly contained in θ, i.e., $O(\theta)$ is contained in the closure of $O(\theta')$. Let us, say, that θ' is generated by extremal rays $\varrho_{i_2}, \dots, \varrho_{i_k}$. Then we can suppose by induction that such θ' is regular and that $\pi(\tilde{O}(\theta')) = O(\theta') \subset U_0$. In particular, $\pi^{-1}(u)$ belongs to the closure of $\tilde{O}(\theta')$ and $\pi^{-1}(u) \cap \tilde{O}(\theta') = \emptyset$. This implies that every $\bar{x} \in \pi^{-1}(u)$ cannot have a nonzero coordinate x_{i_1}. Hence, $\bar{x}$ must be contained in $\tilde{O}(\theta)$ (indeed, if $\bar{x}$ has a zero coordinate x_j with $j \notin \{i_1, \dots, i_k\}$, then u is fixed under the H_j-action contrary to the argument before). This implies that the $\tilde{T}$-orbit Q of $\bar{x}$ is closed since otherwise its closure contains a point with an undesirable zero coordinate. By Lemma 4.1(iii), Q is naturally isomorphic to G and arguing as before we see that $\pi^{-1}(u) = Q$. Hence, $u \in U_0$ which yields (3) and concludes the proof. ∎

COROLLARY 4.3: *Let Y be an open subset of X_σ such that* $\operatorname{codim}_{X_\sigma}(X_\sigma \setminus Y) \geq 2$. *Then $\pi^{-1}(X_\sigma \setminus Y)$ has codimension at least 2 in $\mathbb{A}^r$.*

Proof. Note that $\pi^{-1}(Y) \subset \pi^{-1}(U \cap Y) \cup (\mathbb{A}^r \setminus E)$. The definition of U implies that $\pi^{-1}(U \setminus Y)$ has codimension at least 2 in $\mathbb{A}^r$ and the same is true for $A^r \setminus E$. Hence, we have the desired conclusion. ∎

COROLLARY 4.4: *Let C be a closed curve in X_σ contained in U_0. Suppose that either*

(1) *C is isomorphic to the affine line $\mathbb{A}^1$, or*
(2) *G is connected and C is a smooth rational curve.*

Then there exists a closed curve $\tilde{C} \subset \mathbb{A}^r$ such that $\pi|_{\tilde{C}} : \tilde{C} \to C$ is an isomorphism.

Proof. By Proposition 4.2, $\pi^{-1}(C)$ is a locally trivial principal G-bundle. Statement (1) now follows from [FS, Theorem A.1] which states that for each affine algebraic group F every principal F-bundle over the affine line admits a section.

In (2) by Proposition 4.2(1) we can find an open cover $\{V_i\}$ of C for which $\pi^{-1}(V_i)$ is naturally isomorphic to $V_i \times G$. In particular, one has sections $s_i : V_i \to \pi^{-1}(V_i)$ and $s_j|_{V_i \cap V_j} = g_{ij} s_i$ where $g_{ij} : V_i \cap V_j \to G$ is a morphism. Since $G \simeq \mathbb{G}_m^{r-n}$ we see that g_{ij} can be presented as a collection of $r-n$ sections of $\mathcal{O}_C^*$ over $V_i \cap V_j$. Hence, $H^1(C, G)$ is the direct sum of $r-n$ samples of $H^1(C, \mathcal{O}_C^*)$. Since C is a smooth rational curve we have $H^1(C, \mathcal{O}_C^*) = \operatorname{Pic} C = 0$ and, hence,

$$H^1(C, G) = 0.$$

Thus, we can suppose that every pair of sections s_i and s_j agree on $V_i \cap V_j$. Consequently, we have a global section of $\pi|_{\pi^{-1}(C)} : \pi^{-1}(C) \to C$ which yields the desired conclusion. ∎

5. Embedding theorems for affine toric varieties

Notation 5.1: In this section X_σ is an affine toric variety without torus factors. In particular, $X_\sigma \simeq \mathbb{A}^r /\!\!/ G$ where $G \subset \tilde{T} = \mathbb{G}_m^r$ is a quasitorus acting naturally on $\mathbb{A}^r$. We also denote by $\pi : \mathbb{A}^r \to X_\sigma$ the quotient morphism as in Formula (4) with U (resp. U_0) being the dense open subset of X_σ consisting of all points $u \in X_\sigma$ for which $\pi^{-1}(u)$ is a G-orbit (resp. a G-orbit with a trivial stabilizer). That is, U_0 is the regular part of X_σ by Proposition 4.2.

LEMMA 5.2: *Let $\bar{0}$ be the origin in $\mathbb{A}^r$, $o = \pi(\bar{0})$ and $A = \mathbf{k}[\mathbb{A}^r]^G$. Then one can choose a collection of monomials as generators of A and the set V of common zeros of this collection is contained in $\pi^{-1}(o)$. In particular, $V \subset \pi^{-1}(X_\sigma \setminus U)$ unless $X_\sigma = U$.*[3]

Proof. Since the natural $\tilde{T}$-action respects monomials the same is true for the G-action. Thus, any G-invariant polynomial is the sum of G-invariant monomials which yields the first claim. Since $V \subset \mathbb{A}^r$ is closed and G-invariant $Z = \pi(V)$ is closed in X_σ and for every $z \in Z$ the only closed orbit in $\pi^{-1}(z)$ is contained also in V. Assume that Z contains two distinct points z_1 and z_2 and L_i is the closed orbit of $\pi^{-1}(z_i)$. Note that the restriction of every polynomial from A to V is constant. Hence, elements of A do not separate L_1 and L_2 contrary to the fact that the regular functions on X_σ separate z_1 and z_2. Thus, Z is at most a singleton. Since $\bar{0} \in V$ and $o \in Z$ we see that $V \subset \pi^{-1}(o)$. Let $\dim \pi^{-1}(z) > r - n$ for some $z \in X_\sigma \setminus o$. Since π is $\tilde{T}$-equivarinat the same is true for all points in the $\mathbb{T}$-orbit P of z in X_σ. The closure of P contains a $\mathbb{T}$-orbit Q of a smaller dimension and for every $w \in Q$ one has $\dim \pi^{-1}(w) > r - n$ by Chevalley's theorem. Reducing the dimension of such $\mathbb{T}$-orbits further we see that

$$\dim \pi^{-1}(o) > r - n,$$

i.e., $o \notin U$. This concludes the proof. ∎

THEOREM 5.3: *Let X_σ be a normal affine toric variety without torus factors and $l = \operatorname{codim}_{\mathbb{A}^r} \pi^{-1}(X_\sigma \setminus U_0)$. Suppose that Z is an affine algebraic variety such that $\mathrm{ED}(Z) \leq \dim X_\sigma$ and $\dim Z < l$. Then there exists a closed embedding $\iota : Z \hookrightarrow X_\sigma$ such that $\iota(Z)$ is contained in the regular part U_0 of X_σ. Furthermore, $l \geq 2$ and, in particular, for every affine curve C with $\mathrm{ED}(C) \leq \dim X_\sigma$ there exists a closed embedding of C in X_σ with the image in U_0.*

Proof. By Proposition 4.2(1) the morphism $\pi|_{\pi^{-1}(U_0)} : \pi^{-1}(U_0) \to U_0$ is smooth and $l \geq 2$ by Corollary 4.3. By Holme's theorem Z can be treated as a closed subvariety of $\mathbb{A}^r$. Proposition 3.6 implies now that for a general element α in a perfect family $\mathcal{A}$ of automorphisms of $\mathbb{A}^r$ the morphism $\pi|_{\alpha(Z)} : \alpha(Z) \to U_0$ is an injective immersion.

[3] Recall that if $X_\sigma = U$, then σ is simplicial by [CLS, Theorem 5.1.10].

Furthermore, consider the natural embedding $\mathbb{A}^r \hookrightarrow \mathbb{P}^r$, $D = \mathbb{P}^r \setminus \mathbb{A}^r \simeq \mathbb{P}^{r-1}$ and $H = \mathrm{GL}_r(\mathbf{k})$. Then we have the natural H-action on $\mathbb{P}^r$ such that D is invariant under it. By Proposition 2.8(3) we can replace $\mathcal{A}$ by the family $H \times \mathcal{A}$. That is, for a general h in H and a general α in $\mathcal{A}$ the morphism

$$\pi|_{h\circ\alpha(Z)} : h \circ \alpha(Z) \to U_0$$

is still an injective immersion.

By Lemma 5.2 we can find generators $g_1, \ldots, g_s$ of $\mathbf{k}[X_\sigma]$ such that the polynomials $f_i = g_i \circ \pi$ are monomials and the codimension (in $\mathbb{A}^r$) of the variety given by $f_1 = \cdots = f_s = 0$ is at least l. Note also that $f_1, \ldots, f_s$ can be viewed as coordinate functions of $\pi : \mathbb{A}^r \to X_\sigma \subset \mathbb{A}^s$ and they can be extended to rational functions on $\mathbb{P}^r$. Since each f_i is homogeneous with respect to the standard degree function the intersection R of the indeterminacy sets of these extensions is given by the common zeros of $f_1, \ldots, f_s$ in D. In particular, R has codimension at least l in D. Let P be the intersection of D with the closure of $h \circ \alpha(Z)$ in $\mathbb{P}^r$, i.e., $\dim P \leq \dim Z - 1 < l - 1$. Since the restriction of the H-action to D is transitive P does not meet R for general $h \in H$ and $\alpha \in \mathcal{A}$ by Theorem 2.2. Hence, $\pi|_{h\circ\alpha(Z)} : h \circ \alpha(Z) \to X_\sigma$ is a proper morphism by [Ka20, Corollary 5.4]. Consequently, it is a closed embedding which concludes the proof. ∎

In particular, we have the following fact which is also a trivial consequence of Corollary 3.3.

Corollary 5.4: *Let X_σ be an affine simplicial toric variety. Let Z be an affine algebraic variety such that* $\mathrm{ED}(Z) \leq \dim X_\sigma$ *and* $\dim Z$ *is less than the codimension of the singularities of X_σ in X_σ. Then there is a closed embedding of Z into X_σ with the image in U_0.*

6. Locally nilpotent vector fields on affine toric varieties

We use a combinatorial description of locally nilpotent vector fields on X_σ given by Liendo in his paper [Li] in which he rediscovered Demazure roots [De, Section 3.1] (see also [AKuZ19, Definition 4.2]). Recall that a Demazure root associated with some ϱ_i is any element e of M such that $\langle e, \varrho_i \rangle = -1$ and $\langle e, \varrho_j \rangle$ is nonnegative for every $j \neq i$. The vector field on X_σ defined by

$$\partial_{\varrho_i,e}(\chi^m) = \langle m, \varrho_i \rangle \chi^{m+e} \tag{7}$$

is locally nilpotent and up to a constant factor every homogeneous locally nilpotent vector field is of this form. For a Demazure root $e \in M$ associated with ϱ_i one has $\tilde{e} := \pi^*(\chi^e) = (\tilde{e}_1, \ldots, \tilde{e}_r)$ where by Formula (5) the i-th coordinate $\tilde{e}_i$ is equal to -1. Let $\tilde{e}' = (\tilde{e}'_1, \ldots, \tilde{e}'_r)$ where the i-th coordinate $\tilde{e}'_i$ is equal to zero, whereas $\tilde{e}'_l = \tilde{e}_l$ for $l \neq i$. Formulas (5) and (7) imply now the following fact which was first discovered in [AKuZ19]).

LEMMA 6.1: *The polynomial $\pi^*(\partial_{\varrho_i,e}(\chi^m))$ coincides with $\tilde{\partial}_{\varrho_i,e}(\pi^*(\chi^m))$ where the locally nilpotent vector field $\tilde{\partial}_{\varrho_i,e}$ on $\mathbb{A}^r$ is given by $\bar{x}^{\tilde{e}'} \frac{\partial}{\partial x_i}$ with*

$$\bar{x}^{\tilde{e}'} = \prod_{l=1}^{r} x_l^{\tilde{e}'_l},$$

i.e., the flow of $\tilde{\partial}_{\varrho_i,e}$ is given by

$$\bar{x} = (x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_{i-1}, x_i + t\bar{x}^{\tilde{e}'}, x_{i+1}, \ldots, x_r) \tag{8}$$

where t is the time parameter.

The algebra $\mathbf{k}[D_i]$ of regular functions on D_i can be viewed as the semigroup algebra of $\tau_i = \varrho_i^\perp \cap \sigma^\vee \cap M$. Note that $\mathbf{k}[D_i]$ is the kernel of $\partial_{\varrho_i,e}$ viewed as a derivation on $\mathbf{k}[X_\sigma]$. The natural embedding $\mathbf{k}[D_i] \hookrightarrow \mathbf{k}[X_\sigma]$ yields a dominant $\mathbb{T}$-equivariant morphism $\kappa_i : X_\sigma \to D_i$ that is the categorical quotient of the $\mathbb{G}_a$-action associated with $\partial_{\varrho_i,e}$. Note that it is also the categorical quotient of the natural H_i-action on X_σ since $\mathbf{k}[D_i]$ is the subring of H_i-invariants of $\mathbf{k}[X_\sigma]$. Furthermore, as we mentioned before in the proof of Proposition 4.2, D_i is the fixed point set of the H_i-action on X_σ.

Notation 6.2: Similarly, consider a cone $\theta \in \sigma(2)$ containing two extremal rays ϱ_i and ϱ_j and the subgroup H_{ij} of $\mathbb{T}$ generated by H_i and H_j. The dual cone of θ meets M along $\tau_i \cap \tau_j$. The semigroup algebra of $\tau_i \cap \tau_j$ can be viewed as the algebra of regular functions on

$$D_{ij} = D_i \cap D_j.$$

As before, one can see that $D_{ij} = X_\sigma /\!\!/ H_{ij}$ and D_{ij} is the fixed point set of the H_{ij}-action on X_σ. Since $\mathbf{k}[D_{ij}]$ has no zero divisors and its transcendence degree is $n-2$, one can see that D_{ij} is an irreducible $\mathbb{T}$-invariant Weil divisor in D_i. In particular, D_{ij} contains a dense $\mathbb{T}$-orbit $O(\theta)$ (which is associated with θ via the orbit-cone correspondence).

LEMMA 6.3: *Let Notation 6.2 hold and θ be regular. Then κ_i is smooth over $O(\theta)$ and $\kappa_i^{-1}(u)$ is isomorphic to $\mathbb{A}^1$ for every $u \in O(\theta)$.*

Proof. Since θ is regular $O(\theta)$ is contained in the regular part U_0 of X_σ by [CLS, Theorem 1.3.12 and Example 1.2.20]. Let $u \in O(\theta)$. Then $T_u X_\sigma$ is equipped with the induced linear H_{ij}-action. By the Luna slice étale theorem for smooth points (e.g., see [PV, Theorem 6.4]) there exists an H_{ij}-equivariant étale morphism $\varphi : Y \to T_u X_\sigma$ from a dense open H_{ij}-invariant subset Y of X_σ containing u. Hence, since the map $T_u X_\sigma \to T_u X_\sigma /\!\!/ H_i$ is smooth so is $\kappa_i|_Y : Y \to Y /\!\!/ H_i$. For every point $w \in \kappa_i^{-1}(u)$ the closure of its H_i-orbit contains the fixed point u (e.g., [PV, Theorem 4.7 and Corollary]), i.e., this orbit is contained in Y. Hence, $\kappa_i^{-1}(u) \subset Y$ (and, consequently, $\kappa_i^{-1}(u)$ is isomorphic via φ to an affine line through the origin in $T_u X_\sigma$). This yields the desired conclusion. ∎

LEMMA 6.4: *Suppose that for some ϱ_i every $\theta \in \sigma(2)$ containing ϱ_i is regular. Then there exists an open subset V_i of $D_i \cap U_0$ such that $\operatorname{codim}_{D_i} D_i \setminus V_i \geq 2$ and for every $v \in V_i$ one can find a locally nilpotent vector field δ of the form $g\partial_{\varrho_i,e}$ where $g \in \mathbf{k}(D_i) \subset \mathbf{k}(X_\sigma)$ which does not vanish on $\kappa_i^{-1}(v)$.*

Proof. Let $\varrho_{j_1}, \ldots, \varrho_{j_k}$ be the collection of all extremal rays distinct from ϱ_i such that for every $s = 1, \ldots, k$ there exists $\theta \in \sigma(2)$ containing ϱ_{j_s} and ϱ_i. Formula (7) implies that $\partial_{\varrho_i,e}$ does not vanish over

$$O_i = D_i \setminus \bigcup_{s=1}^{k} D_{j_s}.$$

Thus, we have to consider the case when v is a general point of some D_{j_s}. Choose a rational function f_s on D_i with poles on $D_i \cap D_{j_s}$ only such that these poles are simple at general points of $D_i \cap D_{j_s}$. Let l_s be the zero multiplicity of $\partial_{\varrho_i,e}$ at general points of D_{j_s}. Then the vector field

$$\delta = f_s^{l_s} \partial_{\varrho_i,e}$$

is regular, locally nilpotent, tangent to the fibers of κ_i and it does not vanish at general points of D_{j_s}. By Lemma 6.3, $\kappa_i^{-1}(v)$ is isomorphic to the affine line and since $v \in D_i \cap D_{j_s}$ one has $\kappa_i^{-1}(v) \subset D_{j_s}$. Thus, $\delta|_{\kappa_i^{-1}(v)}$ does not vanish since it is tangent to the line $\kappa_i^{-1}(v)$ and nonzero at a general point of $\kappa_i^{-1}(v)$. This yields the desired conclusion. ∎

Proposition 6.5: *Let every $\theta \in \sigma(2)$ containing ϱ_i be regular and let $V_i \subset D_i$ be as in Lemma 6.4. Let Z be a closed subvariety of $D_i \subset X_\sigma$ which is contained in V_i. Let $s : Z \to \mathbb{A}^r$ be a section of $\pi : \mathbb{A}^r \to X_\sigma$ over Z for which $\tilde{Z} = s(Z)$ is closed in $\mathbb{A}^r$. Then one can find a locally nilpotent vector field δ equivalent to $\partial_{\varrho_i,e}$ and such that δ does not vanish on $\kappa_i^{-1}(Z)$.*[4]

Proof. Lemma 6.4 implies that for every $u \in Z$ one can find a locally nilpotent vector field δ_z of the form $g_z \delta_{\varrho_i,e}$, $g_z \in \mathbf{k}(D_i) \subset \mathbf{k}(X_\sigma)$ which does not vanish on $\kappa_i^{-1}(u)$. Recall that g_z as an element of $\mathbf{k}(X_\sigma)$ is invariant under the H_i-action. Hence, by Lemma 6.1, $\delta_z = \pi_*(\tilde{\delta}_z)$ where $\tilde{\delta}_z$ is of the form $\tilde{f}_z \frac{\partial}{\partial x_i}$ with $\tilde{f}_z$ being a polynomial independent of x_i since x_i is not invariant under the $\tilde{H}_i$-action. By assumption $\tilde{\delta}_z$ and, therefore also $\tilde{f}_z$, does not vanish at $\pi^{-1}(u) \cap \tilde{Z}$. Hence, by the Nullstellensatz one can find polynomials $\tilde{h}_z$ such that only a finite number of them are nonzero and $\sum_z \tilde{h}_z \tilde{f}_z|_{\tilde{Z}} = 1$. By the assumption every regular function on $\tilde{Z}$ is a lift of a regular function on Z which extends to an element

$$\mathbf{k}[D_i] = \operatorname{Ker} \partial_{\varrho_i,e} \subset \mathbf{k}[X_\sigma].$$

In particular, one can suppose that $\tilde{f}_z = \pi^* f_z$ and $\tilde{h}_z = \pi^* h_z$ where $f_z, h_z \in \mathbf{k}[D_i]$. Hence, $\delta = \sum_z h_z \delta_z$ (resp. $\tilde{\delta} = \sum_z \tilde{h}_z \tilde{\delta}_z$) is a locally nilpotent non-vanishing vector field on $\kappa_i^{-1}(Z)$ (resp. $\pi^{-1}(\kappa_i^{-1}(Z))$) which yields the desired conclusion. ∎

Corollary 6.6: *Let the assumptions of Proposition 6.5 hold and Z (and, therefore, $\tilde{Z}$) be isomorphic to the affine line $\mathbb{A}^1$ equipped with a coordinate t. Let $\tilde{\delta}$ be the locally nilpotent vector field on $\mathbb{A}^r$ as in the proof of Proposition 6.5 (i.e., $\pi_*(\tilde{\delta}) = \delta$). Then for every polynomial $h(t)$ there exists a function $g \in \mathbf{k}[D_i] \subset \mathbf{k}[X_\sigma]$ such for the flow $\tilde{\beta}_h^i$ of the locally nilpotent vector field $\pi^*(g)\tilde{\delta}$ at time 1*[5] *one has $x_i \circ \tilde{\beta}_h^i(t) = h(t)$, $t \in \tilde{Z}$.*

Proof. Since δ is equivalent to $\partial_{\varrho_i,e}$ and does not vanish on Z one can suppose (by Lemma 6.1) that the restriction of $\tilde{\delta}$ to $\tilde{Z}$ coincides with $\frac{\partial}{\partial x_i}$. Let

$$\breve{g}(t) = h(t) - x_i(t).$$

Note that $\breve{g}(t)$ (as a function on Z) admits an extension to a function $g \in \mathbf{k}[D_i] = \operatorname{Ker} \delta$. This extension yields the desired function. ∎

[4] Two locally nilpotent derivations are equivalent if they have the same kernels.

[5] That is, $\tilde{\beta}_h^i : \mathbb{A}^r \to \mathbb{A}^r$ is defined by $(\tilde{\beta}_h^i)_*(\lambda) = \exp(\pi^*(g)\tilde{\delta})(\lambda)\ \forall \lambda \in \mathbb{A}^{[r]}$.

7. Affine simplicial toric varieties

Recall that an affine toric variety X_σ is simplicial if every face of σ is a simplex, i.e., $n = r$ and G is a finite group. This implies, in particular, that for every $j \neq l$ the extremal rays ϱ_l and ϱ_j are contained in some $\theta \in \sigma(2)$ and $D_{lj} = D_l \cap D_j$ is always a Weil divisor in D_l.

Lemma 7.1: *Let X_σ be a simplicial toric variety of dimension at least 4 which is smooth in codimension 2 and C be a smooth polynomial curve in the regular part of X_σ. Let V_i be as in Lemma 6.4,*

$$V = \bigcap_{i=1}^{r} \kappa_i^{-1}(V_i), \quad W_l = \kappa_l^{-1}(V_l) \setminus \bigcap_{j \neq l} \kappa_j^{-1}(V_j) \quad \text{and} \quad W_l' = \kappa_l(W_l).$$

For every $\theta \in \sigma(2)$ containing extremal rays ϱ_l and ϱ_j let $\psi_\theta : X_\sigma \to D_{lj}$ be the morphism induced by the homomorphism of the semigroup algebras associated with the natural embedding $\tau_l \cap \tau_j \hookrightarrow \sigma$. Then replacing C with its image under an automorphism of X_σ one can suppose that

(i) *C is contained in V;*
(ii) *$C_l = \kappa_l(C)$ meets W_l' at a finite set for every $l = 1, \ldots, r$;*
(iii) *$\kappa_l|_C : C \to D_l$ is a closed embedding for every $l = 1, \ldots, r$;*
(iv) *$\psi_\theta : C \to \psi_\theta(C)$ is a birational morphism for every $\theta \in \sigma(2)$ containing ϱ_l and ϱ_j.*

Proof. Since D_{lj} is a divisor in D_l and $D_l \setminus V_l$ has codimesion at least 2 in D_l we see that V_l contains an open subset of D_{lj} and $\kappa_l^{-1}(V_l)$ contains an open part of D_j. Hence, $X_\sigma \setminus \kappa_l^{-1}(V_l)$ does not contain Weil divisors in X_σ, i.e., it is of codimension at least 2. Consequently,

$$\operatorname{codim}_{X_\sigma} X_\sigma \setminus V \geq 2.$$

Recall that X_σ is flexible by [AKuZ] and, therefore, U_0 and V are flexible by [FKZ, Theorem 2.6]. By Theorem 2.2 for a general α in any perfect family $\mathcal{A}$ of automorphisms of U_0 (which are extendable to automorphisms of X_σ by the Hartogs' theorem) $\alpha(C)$ is contained in V and $\alpha(C)$ meets every $\kappa_l^{-1}(W_l')$ at a finite set which yields (i) and (ii). Lemma 6.3 implies that every κ_l is smooth over $\kappa_l(V)$. Thus, by Theorem 2.6 and Proposition 2.8(5) for a general $\alpha \in \mathcal{A}$ each morphism $\kappa_l : \alpha(C) \to D_l$ is a closed embedding and each morphism $\psi_\theta : \alpha(C) \to \psi_\theta(\alpha(C))$ is birational which yields (iii)–(iv) and the desired conclusion. ∎

Lemma 7.2: *Let the assumptions of Lemma 7.1 hold and C satisfy conditions (i)–(iv). Suppose that i, δ, h and g are as in Corollary 6.6 and β^i_h is flow of the locally nilpotent vector field $g\delta$ a time 1. Suppose further that $h(t) = ct+d$ where c and d are general constants. Then the curve $\beta^i_h(C)$ also satisfies conditions (i)–(iv).*

Proof. Let $\tilde{V} = \pi^{-1}(V)$. By Corollary 4.4 there exists a curve $\tilde{C} \subset \tilde{V}$ such that $\pi|_{\tilde{C}} : \tilde{C} \to C$ is an isomorphism. Note that $\pi \circ \tilde{\beta}^i_h = \beta^i_h \circ \pi$ where $\tilde{\beta}^i_h$ is as in Corollary 6.6. Hence, besides conditions (i) and (ii) for $\beta^i_h(C)$ it suffices to prove that

(iii′) for $\tilde{\kappa}_l = \kappa_l \circ \pi$ the morphism

$$\tilde{\kappa}_l|_{\tilde{\beta}^i_h(\tilde{C})} : \tilde{\beta}^i_h(\tilde{C}) \to D_l$$

is a closed embedding for every $l = 1, \ldots, r$;

(iv′) for $\tilde{\psi}_\theta = \psi_\theta \circ \pi$ the morphism

$$\tilde{\psi}_\theta|_{\tilde{\beta}^i_h(\tilde{C})} : \tilde{\beta}^i_h(\tilde{C}) \to \tilde{\psi}_\theta(\tilde{\beta}^i_h(\tilde{C}))$$

is birational for every $\theta \in \sigma(2)$ containing ϱ_l and ϱ_j.

Let us start with (iv′). One can choose coordinate functions of $\tilde{\psi}_\theta$ in the form $\pi^*(\chi^m)$ where $m \in \tau_l \cap \tau_j$. By Formula (5), $\pi^*(\chi^m)$ is of the form $x_i^{k_m} y_m$ where y_m is a monomial independent of x_i. Condition (iv) implies that there exist $m', m'' \in \tau_l \cap \tau_j$ such that for $t \in \mathbb{A}^1 \simeq \tilde{C}$ the functions $x_i^{k_{m'}}(t) y_{m'}(t)$ and $x_i^{k_{m''}}(t) y_{m''}(t)$ are not proportional and, in particular, $\frac{y_{m'}(t)}{y_{m''}(t)}$ is a nonzero rational function. Hence, for general c and d the functions $(ct+d)^{k_{m'}} y_{m'}(t)$ and $(ct+d)^{k_{m''}}(t) y_{m''}(t)$ are not proportional and Corollary 6.6 implies that the morphism $\tilde{\psi}_\theta|_{\tilde{\beta}^i_h(\tilde{C})} : \tilde{\beta}^i_h(\tilde{C}) \to \tilde{\psi}_\theta(\tilde{\beta}^i_h(\tilde{C}))$ is birational which is (iv′).

Let S_{lj} be a finite subset of $\tilde{C}$ for which $\tilde{\psi}_\theta|_{\tilde{\beta}^i_h(\tilde{C} \setminus S_{lj})} : \tilde{\beta}^i_h(\tilde{C} \setminus S_{lj}) \to D_{lj}$ is an embedding. Then condition (iii) implies that for every $t_0 \in S_{lj}$ there exists $m \in \tau_l$ such that $\frac{\mathrm{d}}{\mathrm{d}t} x_i^{k_m} y_m|_{t=t_0}$ is nonzero. This implies that either $y_m(t_0) \neq 0$ or $\frac{\mathrm{d}}{\mathrm{d}t} y_m|_{t=t_0} \neq 0$. Consequently, $\frac{\mathrm{d}}{\mathrm{d}t}(ct+d)^{k_m} y_m|_{t=t_0}$ is nonzero for general c and d. Hence, we can suppose that $\tilde{\kappa}_l|_{\tilde{\beta}^i_h(\tilde{C})} : \tilde{\beta}^i_h(\tilde{C}) \to D_l$ is an immersion. Furthermore, for every $t_0 \neq t_1 \in S_{lj}$ there exists $m \in \tau_l$ such that $x_i^{k_m}(t_0) y_m(t_0) \neq x_i^{k_m}(t_1) y_m(t_1)$. Again for general c and d this implies that

$$(ct_0+d)^{k_m} y_m(t_0) \neq (ct_1+d)^{k_m} y_m(t_1).$$

Hence, $\tilde{\kappa}_l|_{\tilde{\beta}^i_h(\tilde{C})} : \tilde{\beta}^i_h(\tilde{C}) \to D_l$ is a closed embedding which is (iii′).

By (ii), for every l the curve C_l meets W_l' at a finite set Q_l' or, equivalently, for a general point $z \in C$ one has $\kappa_l(z) \notin W_l'$. The same remains true for a general point $\beta_h^i(z)$ of the curve $\beta_h^i(C)$. Indeed, it suffices to show that it is true for some point of $\beta_h^i(C)$. Let $t \in \tilde{C} \simeq \mathbb{A}^1$ be the preimage of z in $\tilde{C}$. Note that $\beta_h^i(C)$ meets C at the points where $x_i(t) = ct + d$. Since c and d are general the solutions of the latter equation yield general points of C and, hence, condition (ii) for the curve $\beta_h^i(C)$.

Recall that β_h^i is the flow at time 1 of a locally nilpotent vector field $g\delta$ as in Corollary 6.6 which is equivalent to a vector field $\partial_{\varrho_i,e}$ and, therefore, tangent to the fibers of κ_i. In particular, $\beta_h^i(C)$ is contained in $\kappa_i^{-1}(C_i)$ where $C_i = \kappa_i(C)$. By construction $\kappa_i^{-1}(C_i \setminus Q_i') \subset V$. By Lemma 6.3 every fiber L of $\kappa_l|_{\kappa_l^{-1}(Q_i')} : \kappa^{-1}(Q_i') \to Q_i'$ is isomorphic to the affine line and such L meets V since C does at some point $z_0 \in C$ (where this z_0 is unique since $\kappa_l : C \to D_l$ is a closed embedding). Let t be a coordinate on $\tilde{C} \simeq C \simeq \mathbb{A}^1$. Recall that by construction in Corollary 6.6, $g \subset \mathbf{k}[D_i]$ is an extension of the function $ct + d - x_i(t)$. Hence, $g\delta = \delta_1 + d\delta$ where the locally nilpotent vector field δ_1 commutes with δ. In particular, the flow of $g\delta$ at time 1 is the composition of the flows of δ_1 at time 1 and δ at time d. Since by Proposition 6.5, δ does not vanish on L and d is general, we see that $\beta_h^i(z_0)$ is a general point of L and, therefore, it belongs to V. This yields condition (i) for $\beta_h^i(C)$ and the desired conclusion. ∎

THEOREM 7.3: *Let X_σ be an affine simplicial toric variety of dimension at least 4 such that X_σ is smooth in codimension 2. Suppose that $\mathbb{T}_0$ is an algebraic torus and $Y = \mathbb{T}_0 \times X_\sigma$. Let $\varphi : C \to C'$ be an isomorphism of two smooth polynomial curves contained in the regular part of Y. Then φ extends to an automorphism of Y.*

Proof. Let $\nu : Y \to \mathbb{T}_0$ be the natural projection. Since C and C' are polynomial curves their images $z = \nu(C)$ and $z' = \nu(C')$ are singletons. Consider an automorphism α_0 of $\mathbb{T}_0$ sending z' to z and its natural lift to an automorphism α of Y. Replacing C' by $\alpha(C')$ and φ by $\varphi \circ \alpha^{-1}$ we can suppose that $z = z'$. Hence, $C, C' \subset \nu^{-1}(z) \simeq X_\sigma$ and it suffices to consider the case of $Y = X_\sigma$ only.

Suppose that $\pi : \mathbb{A}^r \to X_\sigma$ is as in Formula (4). By Corollary 4.4 one can find a curve $\tilde{C}$ (resp. $\tilde{C}'$) in $\mathbb{A}^r$ such that $\pi|_{\tilde{C}} : \tilde{C} \to C$ (resp. $\pi|_{\tilde{C}'} : \tilde{C}' \to C'$) is an isomorphism. Let t' be a coordinate on $C' \simeq \tilde{C}'$ and $t = \varphi^*(t')$ be the coordinate on $C \simeq \tilde{C}$. Applying consequently automorphisms β_h^i as in Lemma 7.2 with i running over $\{1, \ldots, r\}$ one can suppose that $\tilde{C}$ is a curve such that $x_i(t) = c_i t + d_i$ for every i where $(c_1, d_1, \ldots, c_r, d_r)$ is a general point in $\mathbb{A}^{2r}$. Similarly, one can suppose that $\tilde{C}'$ is a curve such that $x_i(t') = c_i' t' + d_i'$ for every i where $(c_1', d_1', \ldots, c_r', d_r')$ is a general point in $\mathbb{A}^{2r}$. Choosing these two general points equal we get the desired conclusion. ∎

We need to recall the following [KaUd, Definition 8.2].

Definition 7.4: Let C_1 and C_2 be smooth curves in a smooth quasi-affine variety Y with defining ideals I_1 and I_2 in $\mathbf{k}[Y]$. We suppose also that C_1 and C_2 are closed in an affine variety containing Y. Let Y possess a volume form ω (i.e., ω is a nonvanishing section of the canonical bundle on Y) and let each conormal bundle $\frac{I_j}{I_j^2}$ of C_j in Y be trivial. By [KaUd, Lemma 6.3] there is a neighborhood W_j of C_j in Y in which C_j is a strict complete intersection given by $u_{1,j} = \cdots = u_{n-1,j} = 0$ where $u_{1,j}, \ldots, u_{n-1,j} \in I_j$ and $n = \dim Y$. That is, for $A_j = \frac{\mathbf{k}[Y]}{I_j}$ we have the graded algebra

$$\frac{\mathbf{k}[Y]}{I_j^k} \simeq A_j \oplus \bigoplus_{l=1}^{k-1} \frac{I_j^l}{I_j^{l+1}}$$

which can be viewed as the algebra of polynomials in $u_{1,j}, \ldots, u_{n-1,j}$ over A_j of degree at most $k-1$. Consider an isomorphism $\varphi : \frac{\mathbf{k}[Y]}{I_1^k} \to \frac{\mathbf{k}[Y]}{I_2^k}$ of these algebras for a natural k. Up to the induced isomorphism $A_1 \simeq A_2$ this isomorphism φ is determined by its values $\varphi(u_{i,1})$, $i = 1, \ldots, n-1$. These values can be viewed as polynomials in $u_{1,2}, \ldots, u_{n-1,2}$ over A_2, i.e., one has the matrix

$$\Big[\frac{\partial \varphi(u_{l,1})}{\partial u_{s,2}}\Big]_{l,s=1}^{n-1}.$$

Since the normal bundle $N_Y C_j$ is trivial, the existence of ω implies the existence of a volume form on C_j. Fix volume forms ω_j on C_j such that $\tilde{\varphi}^* \omega_1 = \omega_2$ where the isomorphism $\tilde{\varphi} : C_2 \to C_1$ is induced by φ. Choose a section $\mathrm{pr}_j : TY|_{C_j} \to TC_j$ of the canonical inclusion $TC_j \hookrightarrow TY|_{C_j}$ and consider the section $\tilde{\omega}_j = \omega_j \circ \mathrm{pr}_j$ of the dual bundle $(TY|_{C_j})^\vee$ of $TY|_{C_j}$. Then one can require that $\omega|_{C_j}$ coincides with $\tilde{\omega}_j \wedge \mathrm{d}u_{1,j} \wedge \cdots \wedge \mathrm{d}u_{n-1,j}$. Under this requirement the determinant of $[\frac{\partial \varphi(u_{l,1})}{\partial u_{s,2}}]_{l,s=1}^{n-1}$ is well-defined modulo I_2^{k-1} (i.e., it is independent of the choice of coordinates $u_{1,j}, \ldots, u_{n-1,j}$). Hence, we say that φ has Jacobian $a \in \mathbf{k} \setminus \{0\}$ if the determinant of $[\frac{\partial \varphi(u_{l,1})}{\partial u_{s,2}}]_{l,s=1}^{n-1}$ is equal to a modulo I_2^{k-1}.

Note that Definition 7.4 is applicable in the case when C_1 and C_2 are smooth polynomial curves in a simplicial toric variety X_σ contained in its regular part. Indeed, U_0 as the regular part of X_σ is flexible [AKuZ]. Recall that $\mathbb{A}^r$ admits a volume form invariant under the natural $\mathrm{SL}_r(\mathbf{k})$-action. Hence, we can push this volume form down to a volume form ω on U_0 since $\pi|_{\pi^{-1}(U_0)} : \pi^{-1}(U_0) \to U_0$ is an unramified covering by Proposition 4.2. Furthermore, since the normal bundles of smooth polynomial curves are always trivial, we are under the assumptions of Definition 7.4.

Corollary 7.5: *Let $\varphi : \mathcal{C} \to \mathcal{C}'$ be an isomorphism of kth infinitesimal neighborhoods of two smooth polynomial curve C and C' contained in the regular part U_0 of an affine simplicial toric variety X_σ of dimension at least 4 which is smooth in codimension 2. Suppose that the Jacobian of φ is a nonzero constant a. Then φ extends to an automorphism Φ of X_σ.*

Proof. Recall that by [KaUd, Lemma 6.2] every automorphism α of U_0 has a constant Jacobian where the Jacobian is computed as $\frac{\alpha^*(\omega)}{\omega}$ and, if α is a composition of elements of flows of locally nilpotent vector fields, then its Jacobian is 1. Let $\psi : C' \to C$ be an isomorphism. By Theorem 7.3, ψ extends to an automorphism Ψ of X_σ which in turn induces an automorphism $\mathcal{C}' \to \mathcal{C}$ also denoted by ψ. By construction Ψ is a composition of elements of flows of locally nilpotent vector fields. Hence, its Jacobian is 1. Taking a composition of φ with the action of an appropriate element of $\mathbb{T}$ and replacing C' with its image under this action we can suppose that the Jacobian of φ is also 1 modulo $(I')^{k-1}$ in the sense of Definition 7.4 where I' (resp. I) is the defining ideal of C' (resp. C) in $\mathbf{k}[X_\sigma]$. Then the automorphism $\lambda := \psi \circ \varphi : \mathcal{C} \to \mathcal{C}$ has Jacobian 1 modulo I^{k-1}. By [KaUd, Theorem 6.6] λ extends to an automorphism Λ of X_σ. It remains to note that $\Psi^{-1} \circ \Lambda$ is the desired extension of φ and we are done. ∎

Remark 7.6: Let $\mathcal{N}$ be the smallest saturated set of locally nilpotent vector fields on X_σ that contains all vector fields of the form $\partial_{\varrho_i,e}$ as in Formula (7). Consider the subgroup $G \subset \mathrm{SAut}(X_\sigma)$ generated by $\mathcal{N}$. Assume that in Theorem 7.3 $Y = X_\sigma$. Then it follows from the proof that an automorphism extending φ can be chosen in G. Furthermore, one can check that V as in Lemma 7.1 is G-flexible (recall that we can suppose that such V contains C and C'). Hence, if the Jacobian of φ from Corollary 7.5 is 1, then [KaUd, Theorem 6.6] implies that Φ can be also chosen in G.

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