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AUTOMATA AND TAME EXPANSIONS OF $(\mathbb{Z}, +)$

BY

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ABSTRACT

The problem of characterizing which automatic sets of integers are stable is here solved. Given a positive integer d and a subset $A \subseteq \mathbb{Z}$ whose set of representations base d is recognized by a finite automaton, a necessary condition is found for $x + y \in A$ to be a stable formula in $\mathrm{Th}(\mathbb{Z}, +, A)$. Combined with a theorem of Moosa and Scanlon this gives a combinatorial characterization of the d-automatic $A \subseteq \mathbb{Z}$ such that $(\mathbb{Z}, +, A)$ is stable. This characterization is in terms of what were called F-sets in [16] and elementary p-nested sets in [10]. Automata-theoretic methods are also used to produce some NIP expansions of $(\mathbb{Z}, +)$, in particular the expansion by the monoid $(d^{\mathbb{N}}, \times)$.

1. Introduction

In [17] Palacín and Sklinos give examples of stable expansions of $\mathrm{Th}(\mathbb{Z}, +)$, and pose the following general question:

Question 1.1: For which $A \subseteq \mathbb{Z}$ is $\mathrm{Th}(\mathbb{Z}, +, A)$ stable?

The project of finding sufficient topological or algebraic conditions on A for stability has been taken up in other recent work; see for example [7] and [13]. The theme also appeared some fifteen years earlier: the results of Moosa and Scanlon in [16] imply that $(\mathbb{Z}, +, A)$ is stable whenever A is what they call an F-set. This includes for example the case $A = d^{\mathbb{N}}$, whose stability was rediscovered in [17].

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In this paper, we consider Question 1.1 when $A \subseteq \mathbb{Z}$ is a d-automatic set for some $d \geq 2$. Automatic sets are reviewed in Section 2, but let us recall here informally that this means there is a finite machine that takes strings of digits from $\{-d+1, \dots, d-1\}$ as input and accepts exactly those strings that are representations base d of an element of A.

Instead of asking when the first-order theory of $(\mathbb{Z}, +, A)$ is stable, we will focus on a local, and hence combinatorial, notion of stability which we now briefly recall. Following [12], an **N-ladder** for a binary relation $R \subseteq X \times X$ on a set X is some $a_0, \dots, a_{N-1}, b_0, \dots, b_{N-1} \in X$ such that $(a_i, b_j) \in R$ if and only if $i \leq j$. The relation R is **N-stable** if there is no N-ladder for R, and is **stable** if it is N-stable for some N. If $(G, +)$ is a group and $A \subseteq G$ we say that A is **stable in** G if $x + y \in A$ is a stable binary relation on G.[1]

Our results will be in terms of the F-sets of [16]; we recall these now in the context of $\mathbb{Z}^m$ where $F = F_d \colon \mathbb{Z}^m \to \mathbb{Z}^m$ is multiplication by d. For $a \in \mathbb{Z}^m$ and r a positive integer we let

$$C(a; r) := \{a + d^r a + \cdots + d^{nr} a : n < \omega\} = a\Big(\frac{(d^r)^{\mathbb{Z}_+} - 1}{d^r - 1}\Big).$$

A **basic groupless F_d-set** in $\mathbb{Z}^m$ is a set of the form

$$\begin{aligned}\alpha + C(a_1; r_1) + \cdots + C(a_n; r_n)& \\ &= \{\alpha + b_1 + \cdots + b_n : b_1 \in C(a_1; r_1), \dots, b_n \in C(a_n; r_n)\}\end{aligned}$$

for some $\alpha, a_1, \dots, a_n \in \mathbb{Z}^m$ and $r_1, \dots, r_n > 0$. A **basic F_d-set** in $\mathbb{Z}^m$ is a set of the form $A + H$ where A is a basic groupless F_d-set and $H \leq \mathbb{Z}^m$. A (**groupless**) **F_d-set** in $\mathbb{Z}^m$ is a finite union of basic (groupless) F_d-sets in $\mathbb{Z}^m$. The **F_d-structure** on $\mathbb{Z}$, denoted $(\mathbb{Z}, \mathcal{F}_d)$, has domain $\mathbb{Z}$ and an atomic relation for every F_d-set in every $\mathbb{Z}^m$. (In fact up to interdefinability $(\mathbb{Z}, \mathcal{F}_d)$ is just $(\mathbb{Z}, +, d^{\mathbb{N}})$ expanded by a predicate for every subgroup of every $\mathbb{Z}^m$.)

F-sets are of Diophantine significance in positive characteristic, appearing in both the isotrivial Mordell–Lang theorem of [16] and the Skolem–Mahler–Lech theorem of [10]; see [4, Section 3] for an account of the connection with the latter. It is shown in [16, Theorem A] that $(\mathbb{Z}, \mathcal{F}_d)$ is stable; so the F_d-sets are

[1] It is worth noting that this terminology conflicts somewhat with the terminology used by Conant in [7], in which he calls $A \subseteq \mathbb{N}$ "stable" if $\mathrm{Th}(\mathbb{Z}, +, A)$ is stable. His is a stronger notion than ours, which is equivalent to saying that $\varphi(x; y)$ given by $x + y \in A$ is a stable formula in $\mathrm{Th}(G, +, A)$.

all stable in $(\mathbb{Z},+)$. Our main result, which appears as Theorem 5.1 below, is a converse to this for d-automatic subsets of $\mathbb{Z}$.

THEOREM A: *Suppose A is d-automatic and stable in $(\mathbb{Z},+)$. Then A is a finite Boolean combination of*

- *cosets of subgroups of $(\mathbb{Z},+)$, and*
- *basic groupless F_d-sets in $\mathbb{Z}$.*

Combined with the work of [16], we get

COROLLARY: *Suppose $A \subseteq \mathbb{Z}$ is d-automatic. Then the following are equivalent:*

(1) $\mathrm{Th}(\mathbb{Z},+,A)$ *is stable.*
(2) *A is stable in $(\mathbb{Z},+)$.*
(3) *A is a finite Boolean combination of cosets of subgroups of $(\mathbb{Z},+)$ and basic groupless F_d-sets in $\mathbb{Z}$.*
(4) *A is definable in $(\mathbb{Z},+,d^{\mathbb{N}})$.*

This appears as Corollary 5.2 below.

Automatic sets separate naturally into sparse and non-sparse sets, with "sparse" meaning that the number of accepted strings grows polynomially with length—see Definition 2.9 for a precise formulation. The first case of the main theorem that we consider is when A is d-sparse. In fact, here we can work more generally in Cartesian powers of $(\mathbb{Z},+)$. It is shown in [4] that groupless F_d-sets are d-sparse, and we stated previously that all F_d-sets, and in particular groupless F_d-sets, are stable in $(\mathbb{Z},+)$. Our result in the sparse case, Theorem 3.1 below, is a partial converse to this:

THEOREM B: *If $A \subseteq \mathbb{Z}^m$ is d-sparse and stable in $(\mathbb{Z}^m,+)$ then it is a finite Boolean combination of basic groupless F_d-sets in $\mathbb{Z}^m$.*

We then turn our attention to d-automatic sets that are not d-sparse. We show that for $A \subseteq \mathbb{Z}$ d-automatic but not d-sparse, if A is stable in $(\mathbb{Z},+)$ then A is generic (i.e., finitely many translates cover $\mathbb{Z}$). This is Theorem 4.2. In particular, every d-automatic subset of $\mathbb{N}$ that is stable in $(\mathbb{Z},+)$ must be d-sparse. Actually, this consequence of our Theorem 4.2 can also be deduced by combining [7, Theorem 8.8] and [3, Theorem 1.1], but the general statement requires significantly more work.

Theorem B and Theorem 4.2, together with some stable group theory, yield Theorem A.

In a somewhat different direction, we conclude the paper by using automata-theoretic methods to produce two NIP expansions of $(\mathbb{Z}, +)$: namely

$$(\mathbb{Z}, +, <, d^{\mathbb{N}}) \quad \text{and} \quad (\mathbb{Z}, +, d^{\mathbb{N}}, \times\restriction d^{\mathbb{N}}),$$

in Theorems 6.1 and 6.9, respectively. The former was recently obtained by Lambotte and Point in [13] using different methods, but the latter is a new example.

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2. Preliminaries on automatic sets

We briefly recall regular languages and finite automata; see [21] for a more detailed presentation.

Definition 2.1: For a finite set Λ viewed as an **alphabet** we denote by Λ^* the set of **words** over Λ, namely finite strings of letters from Λ. The class of **regular languages** over Λ is the smallest subset of $\mathcal{P}(\Lambda^*)$ that contains all finite sets and is such that if A, B are regular then so are $A \cup B$, $AB = \{\sigma\tau : \sigma \in A, \tau \in B\}$, and

$$A^* = \{\sigma_1 \cdots \sigma_n : n < \omega, \sigma_1, \ldots, \sigma_n \in A\}.$$

A **deterministic finite automaton** (DFA) over a finite alphabet Λ is a tuple $\mathcal{A} = (Q, q_0, F, \delta)$ where Q is a finite set of states, $q_0 \in Q$ is the start state, $F \subseteq Q$ is the set of finish states, and $\delta \colon Q \times \Lambda \to Q$ is the transition function: if $\mathcal{A}$ is in state $q \in Q$ and reads the letter $\ell \in \Lambda$ then it moves into state $\delta(q, \ell)$. We identify δ with its natural extension $Q \times \Lambda^* \to Q$ inductively by

$$\delta(q, \ell_1 \cdots \ell_{n+1}) = \delta(\delta(q, \ell_1 \cdots \ell_n), \ell_{n+1}).$$

The set **recognized** by $\mathcal{A}$ is

$$\{\sigma \in \Lambda^* : \delta(q_0, \sigma) \in F\}.$$

A fundamental fact (see [21, Lemma 2.2, Section 3.2, and Section 3.3]) is that the regular languages are precisely the sets recognized by DFAs.

It turns out some behaviour of automata can be captured in Presburger arithmetic:

Proposition 2.2: *Suppose Λ is an alphabet; suppose $L \subseteq \Lambda^*$ is regular and $\sigma_1, \ldots, \sigma_n \in \Lambda^*$. Then*

$$\{(t_1, \ldots, t_n) \in \mathbb{N}^n : \sigma_1^{t_1} \cdots \sigma_n^{t_n} \in L\}$$

is definable in $(\mathbb{N}, +)$.

Proof. Fix an automaton (Q, q_0, F, δ) for L. We apply induction on n to show that $\delta(q_1, \sigma_1^{t_1} \cdots \sigma_n^{t_n}) = q_2$ is definable in $(\mathbb{N}, +)$ for all $q_1, q_2 \in Q$. The case $n = 0$ is vacuous. For the induction step, note that $\delta(q_1, \sigma_1^t)$ is eventually periodic in t, since there are finitely many states; so there are μ, N such that for $t \geq N$ we have that $\delta(q_1, \sigma_1^t) = \delta(q_1, \sigma_1^{t+\mu})$. But then $\delta(q_1, \sigma_1^{t_1} \cdots \sigma_n^{t_n}) = q_2$ is defined by

$$\bigvee_{t<N} ((t_1 = t) \wedge \delta(\delta(q_1, \sigma_1^t), \sigma_2^{t_2} \cdots \sigma_n^{t_n}) = q_2)$$
$$\vee \bigvee_{t<\mu} ((t_1 \in N + t + \mu\mathbb{N}) \wedge \delta(\delta(q_1, \sigma_1^{N+t}), \sigma_2^{t_2} \cdots \sigma_n^{t_n}) = q_2)$$

which is definable in $(\mathbb{N}, +)$ by the induction hypothesis.

But then $\{(t_1, \ldots, t_n) \in \mathbb{N}^n : \sigma_1^{t_1} \cdots \sigma_n^{t_n} \in L\}$ is the union over $q \in F$ of

$$\{(t_1, \ldots, t_n) \in \mathbb{N}^n : \delta(q_0, \sigma_1^{t_1} \cdots \sigma_n^{t_n}) = q\},$$

which is definable in $(\mathbb{N}, +)$. ∎

We are primarily interested in the case where the strings in question are representations of integers. Fix an integer $d \geq 2$. Evaluating a string base d gives a natural map $[\cdot] \colon \mathbb{Z}^* \to \mathbb{Z}$ via

$$[k_0 k_1 \cdots k_{n-1}] = \sum_{i=0}^{n-1} k_i d^i.$$

Note that unlike usual base d representations the most significant digit occurs last, not first.

Definition 2.3: We let $\Sigma = \{0, \ldots, d-1\}$ and $\Sigma_\pm = \{-d+1, \ldots, d-1\}$. We say $A \subseteq \mathbb{Z}$ is d**-automatic** if $\{\sigma \in \Sigma_\pm^* : [\sigma] \in A\}$ is a regular language over $\Sigma_\pm$.

There is a natural extension of this notion to $\mathbb{Z}^m$ for $m \geq 1$. We first extend our map $[\cdot]$ to $(\mathbb{Z}^m)^* \to \mathbb{Z}^m$: we set

$$\left[\begin{pmatrix} k_{10} \\ \vdots \\ k_{m0} \end{pmatrix} \cdots \begin{pmatrix} k_{1(n-1)} \\ \vdots \\ k_{m(n-1)} \end{pmatrix}\right] = \begin{pmatrix} [k_{10} \cdots k_{1(n-1)}] \\ \vdots \\ [k_{m0} \cdots k_{m(n-1)}] \end{pmatrix}.$$

Definition 2.4: We say $A \subseteq \mathbb{Z}^m$ is d**-automatic** if $\{\sigma \in (\Sigma_\pm^m)^* : [\sigma] \in A\}$ is a regular language over $\Sigma_\pm^m$.

A note on exponential notation: we use Λ^m to denote the alphabet $\Lambda \times \cdots \times \Lambda$. This contrasts its usual meaning in formal languages, namely the set of words over Λ of length m; we use $\Lambda^{(m)}$ to denote this. We will use σ^n to denote the n-fold concatenation of σ with itself; it should be clear from context whether an instance of exponential notation refers to iterated string concatenation or iterated multiplication.

Of course different strings can represent the same integer. It is useful to fix a canonical representation.

Definition 2.5: The **canonical representation** of 0 is the empty word ε. The **canonical representation** of a positive integer a is its usual representation base d in Σ^* (though with the order reversed). The **canonical representation** of a negative integer a is $(-k_0) \cdots (-k_{n-1})$ where $k_0 \cdots k_{n-1}$ is the canonical representation of $-a$. Finally, the **canonical representation** of a tuple $\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$

$$\begin{pmatrix} k_{10} \\ \vdots \\ k_{m0} \end{pmatrix} \cdots \begin{pmatrix} k_{1(n-1)} \\ \vdots \\ k_{m(n-1)} \end{pmatrix},$$

where n is the maximum of the lengths of the canonical representations of the a_i, and $k_{i0} \cdots k_{i(n-1)}$ is the canonical representation of a_i for each i, possibly padded with trailing zeroes to make them of length n.

Note that the canonical representation of an integer base d is a word over $\Sigma_\pm$, and if the integer happens to be non-negative then it is a word over Σ.

Example 2.6: The canonical representation base 10 of $\binom{-23}{432}$ is

$$\begin{pmatrix}-3\\2\end{pmatrix}\begin{pmatrix}-2\\3\end{pmatrix}\begin{pmatrix}0\\4\end{pmatrix}.$$

Remark 2.7: Automaticity is robust under changes in the allowed representations. Indeed, from [14, Proposition 7.1.4] (together with some basic tools of automata theory—see, e.g., [21, Theorems 2.16 and 4.2]) one can show that the following are both equivalent to A being d-automatic:

(1) The set of canonical representations of elements of A is regular. Note that this is essentially the definition given in [1, Section 5.3]. In particular, our definition generalizes the classical notion of d-automatic subsets of $\mathbb{N}$ (see, e.g., [2, Chapter 5]).
(2) For some (equivalently all) finite $\Lambda \subseteq \mathbb{Z}^m$ containing $\Sigma_\pm^m$,

$$\{\sigma \in \Lambda^* : [\sigma] \in A\}$$

is regular over Λ.

We are particularly interested in sparsity among d-automatic sets. If Λ is an alphabet we say $L \subseteq \Lambda^*$ is **sparse** if it is regular and the map

$$k \mapsto |\{\sigma \in L : |\sigma| \leq k\}|$$

grows polynomially in k. Several equivalent formulations of sparsity are known; we will in particular make use of the following characterization:

Fact 2.8 ([4, Proposition 7.1]): *If $L \subseteq \Lambda^*$ then L is sparse if and only if it is a finite union of sets of the form*

$$u_0w_1^*u_1w_2^*\cdots u_{n-1}w_n^*u_n = \{u_0w_1^{r_1}u_1w_2^{r_2}\cdots u_{n-1}w_n^{r_n}u_n : r_1,\dots,r_n \geq 0\}$$

for some $u_0,\dots,u_n,w_1,\dots,w_n \in \Lambda^*$.

Definition 2.9: We say $A \subseteq \mathbb{Z}^m$ is d-**sparse** if the set of canonical representations base d of elements of A is a sparse language over $\Sigma_\pm^m$.

Note by Remark 2.7 that d-sparse sets are d-automatic. In fact, d-sparsity is equivalent to the existence of some finite $\Lambda \supseteq \Sigma_\pm^m$ and some sparse $L \subseteq \Lambda^*$ such that $A = [L]$, but we will not need this.

3. Characterizing stable sparse sets

In this section we give a complete characterization of the d-sparse subsets of $\mathbb{Z}^m$ that are stable in $(\mathbb{Z}^m, +)$. Observe that not every d-sparse set is stable: assuming $d > 2$ the set $A = [0^*10^*2]$ is d-sparse by Fact 2.8, but it is not hard to verify that $d^i + 2d^{j+1} \in A$ if and only if $i \leq j$.

The main result of this section:

Theorem 3.1: *Suppose $A \subseteq \mathbb{Z}^m$ is d-sparse. If A is stable in $(\mathbb{Z}^m, +)$ then A is a finite Boolean combination of basic groupless F_d-sets in $\mathbb{Z}^m$.*

Combined with the results of [16] this theorem yields a complete answer to Question 1.1 for d-sparse subsets of $\mathbb{Z}^m$. See Corollary 3.7 below.

Before proving the theorem let us make some observations that may give the reader a better feel for the automata-theoretic nature of the sets $C(a;r)$.

Remark 3.2:

(1) $d^{\mathbb{N}} = \{1\} \cup (1 + C(d-1;1))$.
(2) If $a = [\sigma]$ where $\sigma \in (\mathbb{Z}^m)^*$ is of length r then $C(a;r) = \{[\sigma^n] : n > 0\}$.
(3) Every groupless F_d-set is d-sparse.
(4) Let $\mathcal{C}$ denote the collection of subsets of $\mathbb{Z}$ of the form $b + C(a;r)$ for some $a, b \in \mathbb{Z}$ and $r > 0$. Let $\mathcal{E}$ be the collection of subsets of $\mathbb{Z}$ of the form $[uv^*w]$ for $u, v, w \in \Sigma^*$ or $u, v, w \in (-\Sigma)^*$. Then up to finite symmetric differences, $\mathcal{C}$ and $\mathcal{E}$ agree.

Proof. Parts (1) and (2) are easily verified by hand. Part (3) is observed for basic groupless F_d-sets in the proof of [4, Theorem 7.4]; the full statement then follows by observing that d-sparse sets are closed under union. We prove (4).

($\supseteq$) We will see in the proof of Lemma 3.4 below that we can write $[uv^*w]$ as a translate of $[\tau^*]$ for some $\tau \in \mathbb{Z}^*$. But then by part (2) this has finite symmetric difference with $C([\tau]; |\tau|)$.

($\subseteq$) Suppose first that we are given $C(a;r)$; by negating if necessary we may assume $a \geq 0$. Note that given any $b \in \mathbb{Z}$ and any length $s > 0$ there is $\sigma \in \mathbb{Z}^*$ of length s such that $[\sigma] = b$; one can, for example, take $\sigma = b0^{s-1}$. Pick a representation $\sigma \in \mathbb{Z}^*$ of a of length r; then by part (2) we are interested in the canonical representations of $[\sigma^* \setminus \{\varepsilon\}]$. For $0 < i < \omega$ write $[\sigma^i] = b_i d^{ir} + c_i$ where $0 \leq c_i < d^{ir}$; so b_i is the "carry" when adding up $a + d^r a + \cdots + d^{(i-1)r} a$ and cutting off after ir

digits. Then

$$[\sigma^{i+1}] = d^r[\sigma^i] + [\sigma] = b_i d^{(i+1)r} + d^r c_i + [\sigma] \geq b_i d^{(i+1)r}$$

so $b_{i+1} \geq b_i$. But $[\sigma^i] = a\frac{d^{ir}-1}{d^r-1} \leq ad^{ir}$; so each $b_i \leq a$. So the b_i are eventually constant, say $b_N = b_{N+1} = \cdots$. Let $p = b_N + [\sigma] \bmod d^r$. Then

$$\begin{aligned}[\sigma^{N+k+1}] &= [\sigma^{N+k}] + d^{(N+k)r}[\sigma] = b_{N+k}d^{(N+k)r} + c_{N+k} + d^{(N+k)r}[\sigma] \\ &= d^{(N+k)r}(b_N + [\sigma]) + c_{N+k}\end{aligned}$$

so

$$c_{N+k+1} = [\sigma^{N+k+1}] \bmod d^{(N+k+1)r} = d^{(N+k)r}p + c_{N+k}$$

(since $c_{N+k} < d^{(N+k)r}$, and thus $d^{(N+k)r}p + c_{N+k} < d^{(N+k+1)r}$). So inductively we get $c_{N+k} = c_N + d^{Nr}p + d^{(N+1)r}p + \cdots + d^{N+k-1}p$. So

$$[\sigma^{N+k}] = b_N d^{(N+k)r} + c_N + d^{Nr}p + \cdots + d^{N+k-1}p.$$

So if $u \in \Sigma^{(Nr)}, v \in \Sigma^{(r)}, w \in \Sigma^*$ represent c_N, p, b_N, respectively, then $[\sigma^{N+k}] = [uv^k w]$. So $C(a;r) = [\sigma^* \setminus \{\varepsilon\}]$ has finite symmetric difference with $[uv^*w]$, as desired.

It remains to show that a translate of a single $C(a;r)$ takes the desired form; by above it suffices to show that a translate of $[uv^*w]$, say by $\gamma \in \mathbb{Z}$, has finite symmetric difference from some $[xy^*z]$. (Again we may assume $[uv^*w] \subseteq \mathbb{N}$.) If $u, v \in (d-1)^*$ then

$$\gamma + [uv^*w] = (\gamma - 1) + [0^{|u|}(0^{|v|})^*\tau]$$

where $[\tau] = [w] + 1$; so we may assume $uv \notin (d-1)^*$. So for some N we get that

$$0 \leq \gamma + [uv^N] < d^{|uv^N|};$$

so if $\sigma \in \Sigma^{(|uv^N|)}$ has $[\sigma] = \gamma + [uv^N]$ then $\gamma + [uv^{N+k}w] = [\sigma v^k w]$. So $\gamma + [uv^*w]$ has finite symmetric difference from $[\sigma v^* w]$. ∎

We begin working towards a proof of Theorem 3.1. Our approach requires that we first understand the stable formulas in $(\mathbb{N}, 0, S, \delta\mathbb{N}, <)$ where S is the successor function and δ is a fixed positive integer. The following proposition, which is of independent interest, is likely known, but as we could find no reference we include a proof here for completeness.

PROPOSITION 3.3: *Fix* $\mathrm{Th}(\mathbb{N}, 0, S, \delta\mathbb{N}, <)$ *as the ambient theory. Let*

$$L_\delta = \{0, S, P_\delta\} \quad \textit{and} \quad L_{\delta,<} = L_\delta \cup \{<\}.$$

Suppose $\varphi(x_1, \ldots, x_n) \in L_{\delta,<}$ *is quantifier-free and stable with respect to any partition of the variables. Then* φ *is equivalent to a quantifier-free* L_δ*-formula.*[2]

Proof. We apply induction on n; the case $n = 0$ is vacuous.

With an eye towards constraining the atomic subformulas of φ, we rewrite φ as follows:

- Replace any occurrence of $S^e x_i < K$ by a disjunction of equalities in the natural way, and of $S^e x_i < S^f x_j$ for $e \geq f$ by $S^{e-f} x_i < x_j$. Using this and the fact that $t_1 \leq t_2 \iff \neg(t_1 > t_2)$, we may assume all atomic inequalities take the form $S^e x_i < x_j$.
- Replace any occurrence of $S^e x_i = K$ by $x_i = K - e$ and of $S^e x_i = S^f x_j$ for $e \geq f$ by $S^{e-f} x_i = x_j$.

So we may assume the atomic subformulas of φ take the following forms:

- $x_i \equiv K \pmod{\delta}$,
- $x_i = K$,
- $S^e x_i < x_j$,
- $S^e x_i = x_j$.

Let M be greater than both the largest K appearing in φ and the largest e with S^e appearing in φ. Note that the truth value of $\varphi(\overline{a})$ is determined by the truth value of the above formulas on $\overline{a}$. Furthermore since we may assume in said formulas that $K < M$ and $e < M$, we get that there are finitely many such formulas; call the set of such formulas Δ. Given $f \colon \Delta \to \{0, 1\}$, let

$$\psi_f = \bigwedge_{\theta \in \Delta} \theta^{f(\theta)}.$$

(Here θ^0 denotes $\neg\theta$ and θ^1 denotes θ.) So φ is equivalent to $\bigvee_{f \in X} \psi_f$ for some $X \subseteq \{0, 1\}^\Delta$. We may assume that ψ_f is consistent for each $f \in X$.

Fix $f \in X$. We will produce an L_δ-formula χ_f such that

$$\models \psi_f \to \chi_f \quad \text{and} \quad \models \chi_f \to \varphi$$

[2] In fact both $\mathrm{Th}(\mathbb{N}, 0, S, \delta\mathbb{N}, <)$ and $\mathrm{Th}(\mathbb{N}, 0, S, \delta\mathbb{N})$ admit quantifier elimination; however we do not make use of this in either the proof or the application of this proposition.

Case 1: Suppose ψ_f contains a conjunct of the form $S^e x_{j_1} = x_{j_2}$ or $K = x_{j_2}$. Define a term t to be $S^e x_{j_1}$ in the former case and K in the latter case, and consider $\varphi'(\overline{x}') = \varphi'(x_1, \dots, x_{j_2-1}, x_{j_2+1}, \dots, x_n)$ obtained by substituting $x_{j_2} = t$ into φ. This is stable under any partition of the variables because φ is, and because t involves at most one of the x_i. It also contains one fewer variable, so by the induction hypothesis is equivalent to a quantifier-free L_δ-formula $\theta(\overline{x}')$. We can then take χ_f to be $\theta(\overline{x}') \wedge (x_{j_2} = t)$.

Case 2: Suppose ψ_f contains no such conjuncts. Let $\psi_0(\overline{x})$ be the conjunction of the negations of such; so ψ_0 asserts that no $x_i < M$ and no $|x_i - x_j| < M$. Examining Δ we see that since ψ_f is consistent it must take the form

$$\underbrace{\psi_0(\overline{x}) \wedge \left(\bigwedge_{i=1}^{n} x_i \equiv K_i \pmod{\delta}\right)}_{\chi_f} \wedge (x_{\sigma(1)} < \cdots < x_{\sigma(n)})$$

for some $K_i < \delta$ and some $\sigma \in S_n$. (Note that formulas of the form $S^e x_i < x_j$ for $e < M$ are implied by $x_i < x_j$ and $\psi_0(\overline{x})$, so we may safely omit them.) I claim that this choice of χ_f works. It is clear that $\models \psi_f \to \chi_f$; it remains to show that $\models \chi_f \to \varphi$. Consider

$$P := \{\tau \in S_n : \models (\chi_f \wedge (x_{\tau(1)} < \cdots < x_{\tau(n)})) \to \varphi\}.$$

Since $\models \psi_f \to \varphi$ we get that $\sigma \in P$; to show $\models \chi_f \to \varphi$, it suffices to show that $P = S_n$ (since if $\overline{a}$ realizes χ_f then there is $\tau \in S_n$ such that $a_{\tau(1)} < \cdots < a_{\tau(n)}$). Since transpositions of the form $(j-1\ j)$ generate all of S_n, we need only check that if $\tau \in P$ then so is $(j-1\ j)\tau$ for all $1 < j \leq n$.

Suppose then that $\tau \in P$ and $1 < j \leq n$; suppose for contradiction we had $\models \neg\varphi(\overline{a})$ for some realization $\overline{a}$ of

$$\chi_f \wedge (x_{((j-1\ j)\tau)(1)} < \cdots < x_{((j-1\ j)\tau)(n)}).$$

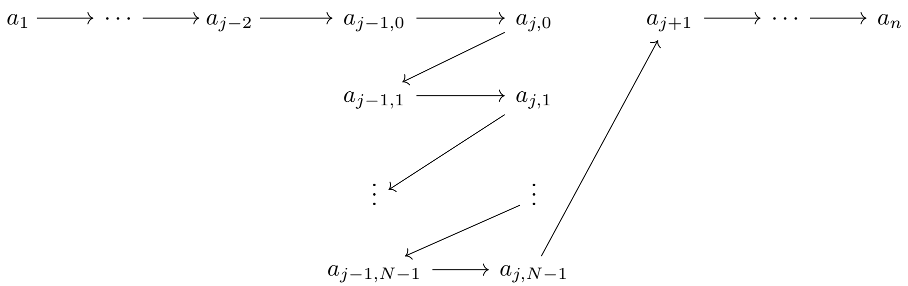
Note that this last formula takes the form $\psi_{f'}$ for some $f' \colon \Delta \to \{0,1\}$. Then since $\models \neg\varphi(\overline{a})$ we get that $f' \notin X$; so $\models \neg\varphi(\overline{a})$ for all realizations $\overline{a}$ of $\psi_{f'}$.

I claim this implies that $\varphi(x_{\tau(1)}, \dots, x_{\tau(j-1)}; x_{\tau(j)}, \dots, x_{\tau(n)})$ has the order property. Indeed, fix $N < \omega$; we construct an N-ladder. For clarity we assume $\tau = \mathrm{id}$; the argument generalizes with little effort. Pick $a_1 \geq M$

such that $a_1 \equiv K_1 \pmod{\delta}$, and inductively pick $a_{i+1} \geq a_i + M$ for $1 < i+1 < j-1$ such that $a_{i+1} \equiv K_{i+1} \pmod{\delta}$. Now pick $a_{j-1,0} \geq a_{j-2} + M$ with $a_{j-1,0} \equiv K_{j-1} \pmod{\delta}$, and inductively choose $a_{j-1,k}$ and $a_{j,k}$ for $k < N$ to satisfy:

- $a_{j,k} \geq a_{j-1,k} + M$,
- $a_{j,k} \equiv K_j \pmod{\delta}$,
- $a_{j-1,k+1} \geq a_{j,k} + M$,
- $a_{j-1,k+1} \equiv K_{j-1} \pmod{\delta}$.

Now pick $a_{j+1} \geq a_{j,N-1} + M$ with $a_{j+1} \equiv K_{j+1} \pmod{\delta}$, and proceed inductively to pick $a_{i+1} \geq a_i + M$ with $a_{i+1} \equiv K_{i+1} \pmod{\delta}$ for $j+1 \leq i < n$. Pictorially:



where an arrow in the diagram indicates that the target is at least the source plus M.

For convenience we let

$$b_k = (a_1, \ldots, a_{j-2}, a_{j-1,k}) \quad \text{and} \quad c_\ell = (a_{j,\ell}, a_{j+1}, \ldots, a_n).$$

Note now that for any $k, \ell < N$ we have $\models \chi_f(b_k, c_\ell)$. Furthermore if $k \leq \ell$ then (b_k, c_ℓ) satisfies $x_1 < \cdots < x_n$; so $\models \psi_f(b_k, c_\ell)$ and thus $\models \varphi(b_k, c_\ell)$. Finally if $k > \ell$ then (b_k, c_ℓ) satisfies

$$x_{(j-1\ j)(1)} < \cdots < x_{(j-1\ j)(n)};$$

so $\models \psi_{f'}(b_k, c_\ell)$, and thus $\models \neg\varphi(b_k, c_\ell)$.

Thus $\models \varphi(b_k, c_\ell)$ if and only if $k \leq \ell$, and we have constructed an N-ladder for φ. So φ has the order property and is thus unstable with respect to this partition of the variables, a contradiction. So no such $\overline{a}$ exists, and P is closed under applying transpositions of adjacent elements. So P is all of S_n, and thus $\models \chi_f \to \varphi$, as desired.

Then φ is equivalent to the L_δ-formula $\bigvee_{f \in X} \chi_f$. ∎

We now describe a simplification of d-sparsity that we will use to connect stable sparse sets to F_d-sets.

Lemma 3.4: *Any d-sparse subset of $\mathbb{Z}^m$ can be written as a finite union of translates of sets of the form*

$$\{[\sigma_1^{e_1}] + \cdots + [\sigma_n^{e_n}] : e_1 \leq \cdots \leq e_n\}$$

where $\sigma_i \in (\mathbb{Z}^m)^*$ all have the same length.

To see how this relates to F_d-sets, recall from Remark 3.2 that

$$[\sigma^*] = C(\sigma; |\sigma|) \cup \{0\};$$

so a set of the above form is (ignoring for the moment the case where some $e_i = 0$) a subset of the groupless F_d-set

$$C(\sigma_1; |\sigma_1|) + \cdots + C(\sigma_n; |\sigma_n|)$$

that is cut out by some kind of order relation.

Proof. By Fact 2.8 we can write A as a finite union of sets of the form

$$[u_0 v_1^* u_1 \cdots v_n^* u_n] \quad \text{for } u_i, v_i \in (\Sigma_{\pm}^m)^*.$$

Note first that we may assume all v_i across the union have the same length N. Indeed, let N be the least common multiple of the lengths of all the v_i across the union. We can then rewrite any v_i^* as

$$\bigcup_{j<\ell_i} v_i^j (v_i^{\ell_i})^*$$

where $\ell_i = \frac{N}{|v_i|}$ (so $|v_i^{\ell_i}| = N$). Using this to replace each v_i^*, we can write

$$[u_0 v_1^* u_1 \cdots v_n^* u_n] = \bigcup_{j_1<\ell_1} \cdots \bigcup_{j_n<\ell_n} [u_0 v_1^{j_1} (v_1^{\ell_1})^* u_1 \cdots v_n^{j_n} (v_n^{\ell_n})^* u_n]$$

and

$$|v_1^{\ell_1}| = \cdots = |v_n^{\ell_n}| = N,$$

as desired.

It then suffices to show that given

$$A = [u_0 v_1^* u_1 \cdots v_n^* u_n]$$

with each $|v_i| = N$ we can write A as a finite union of translates of sets of the form $\{[\sigma_1^{e_1}] + \cdots + [\sigma_n^{e_n}] : e_1 \leq \cdots \leq e_n\}$ with each $|\sigma_i| = N$.

Claim 3.5: *We can write such* $[u_0 v_1^* u_1 \cdots v_n^* u_n]$ *as a finite union of sets of the form* $[a\tau_1^* \cdots \tau_n^*]$ *where* $a \in (\mathbb{Z}^m)^*$ *and each* $\tau_i \in (\mathbb{Z}^m)^*$ *has length* N.

Proof of Claim 3.5. We apply induction on n; the base case $n=0$ is trivial. For the induction step, use the induction hypothesis to write $[u_1 v_2^* u_2 \cdots v_n^* u_n]$ as a finite union of sets of the form $[b\tau_2^* \cdots \tau_n^*]$ with each $|\tau_i| = N$. Then $[u_0 v_1^* u_1 \cdots v_n^* u_n]$ is a finite union of sets of the form $[u_0 v_1^* b \tau_2^* \cdots \tau_n^*]$ with $|v_1| = |\tau_1| = \cdots = |\tau_n| = N$; it suffices to show we can write such $[u_0 v_1^* b \tau_2^* \cdots \tau_n^*]$ in the desired form.

Let $x = [v_1] + d^{|v_1|}[b] - [b]$; then for $k \geq 1$ the following is a telescoping sum:

$$\overbrace{[u_0] + d^{|u_0|}[b] + d^{|u_0|}x}^{y} + d^{|u_0|+|v_1|}x + \cdots + d^{|u_0|+(k-1)|v_1|}x$$
$$= [u_0] + d^{|u_0|}[v_1] + d^{|u_0|+|v_1|}[v_1] + \cdots + d^{|u_0|+(k-1)|v_1|}[v_1] + d^{|u_0|+k|v_1|}[b].$$

(The above equation is taken from a draft of [4]; it was removed from the final paper.) Then if we let

$$a = y \cdot 0^{|u_0|+|v_1|-1},$$
$$\tau_1 = x \cdot 0^{|v_1|-1}$$

(i.e., strings in $(\mathbb{Z}^m)^*$ whose first entries are y and x and whose later entries are the zero tuple), then

$$[u_0 v_1^k b] = \begin{cases} [a\tau_1^{k-1}] & \text{if } k \geq 1, \\ [u_0 b] & \text{else.} \end{cases}$$

Hence if $k \geq 1$ and $w \in (\mathbb{Z}^m)^*$ then

$$[u_0 v_1^k b w] = [u_0 v_1^k b] + d^{|u_0 v_1^k b|}[w] = [a\tau_1^{k-1}] + d^{|a\tau_1^{k-1}|+|b|}[w]$$
$$= [a\tau_1^{k-1}] + d^{|a\tau_1^{k-1}|}[T_{|b|}w]$$
$$= [a\tau_1^{k-1}(T_{|b|}w)]$$

where $T_i\sigma$ is the word obtained by replacing each letter $\ell \in \mathbb{Z}^m$ appearing in σ with $d^i\ell$. Hence

$$[u_0 v_1^* b \tau_2^* \cdots \tau_n^*] = [u_0 b \tau_2^* \cdots \tau_n^*] \cup \{[u_0 v_1^k b w] : k \geq 1, w \in \tau_2^* \cdots \tau_n^*\}$$
$$= [u_0 b \tau_2^* \cdots \tau_n^*] \cup \{[a\tau_1^{k-1}(T_{|b|}w)] : k \geq 1, w \in \tau_2^* \cdots \tau_n^*\}$$
$$= [u_0 b \tau_2^* \cdots \tau_n^*] \cup [a\tau_1^*(T_{|b|}\tau_2)^* \cdots (T_{|b|}\tau_n)^*].$$

And $|\tau_1| = |v_1| = |T_{|b|}\tau_i| = N$ for all i, as desired. ∎

Note that given a set of the form $[a\tau_1^*\cdots\tau_n^*]$ with each $|\tau_i| = N$ we can rewrite it as $[a]+[(T_{|a|}\tau_1)^*\cdots(T_{|a|}\tau_n)^*]$. It then suffices to show that a set of the form $[\tau_1^*\cdots\tau_n^*]$ where each $\tau_i\in(\mathbb{Z}^m)^*$ has length N can be written in the form

$$\{[\sigma_1^{e_1}]+\cdots+[\sigma_n^{e_n}] : e_1\le\cdots\le e_n\}$$

with each $\sigma_i\in(\mathbb{Z}^m)^*$ of length N. For $1\le i\le n$ let $\sigma_i\in(\mathbb{Z}^m)^*$ be any string of length N such that $[\sigma_i]=[\tau_i]-\sum_{j=i+1}^n[\sigma_j]$. (Recall that given any $a\in\mathbb{Z}^m$ there is $\sigma\in(\mathbb{Z}^m)^*$ of length N with $[\sigma]=a$; one can for example let the first character of σ be a and the rest be zeroes.) Then if $e_1\le\cdots\le e_n$ then

$$\begin{aligned}[\sigma_1^{e_1}]+\cdots+[\sigma_n^{e_n}] =&[\sigma_1^{e_1}]\\ &+[\sigma_2^{e_1}]+d^{Ne_1}[\sigma_2^{e_2-e_1}]\\ &\vdots\\ &+[\sigma_n^{e_1}]+d^{Ne_1}[\sigma_n^{e_2-e_1}]+\cdots+d^{Ne_{n-1}}[\sigma_n^{e_n-e_{n-1}}]\\ =&[\tau_1^{e_1}]+d^{Ne_1}[\tau_2^{e_2-e_1}]+\cdots+d^{Ne_{n-1}}[\tau_n^{e_n-e_{n-1}}]\\ =&[\tau_1^{e_1}\tau_2^{e_2-e_1}\cdots\tau_n^{e_n-e_{n-1}}]\end{aligned}$$

So $[\tau_1^*\cdots\tau_n^*]=\{[\sigma_1^{e_1}]+\cdots+[\sigma_n^{e_n}] : e_1\le\cdots\le e_n\}$, as desired. ∎

The promised connection between stable sparse sets and F_d-sets:

Lemma 3.6: *Suppose $A\subseteq\mathbb{Z}^m$ is d-sparse and stable in $(\mathbb{Z}^m,+)$. Then A is definable in $(\mathbb{Z},\mathcal{F}_d)$.*

Proof. By Lemma 3.4 we can write A as a finite union $\bigcup_{\ell=1}^M B_\ell$ where each $B_\ell\subseteq\mathbb{Z}^m$ takes the form

$$\alpha+\{[\sigma_1^{e_1}]+\cdots+[\sigma_n^{e_n}] : e_1\le\cdots\le e_n\}$$

with $\alpha\in\mathbb{Z}^m$ and each $\sigma_i\in(\mathbb{Z}^m)^*$ has the same length N. We will show that given such a B_ℓ there is $B'_\ell\subseteq\mathbb{Z}^m$ that is definable in $(\mathbb{Z},\mathcal{F}_d)$ such that $B_\ell\subseteq B'_\ell\subseteq A$; it will then follow that $A=\bigcup_{\ell=1}^M B'_\ell$ is definable in $(\mathbb{Z},\mathcal{F}_d)$.

So fix some such $B_\ell=\alpha+\{[\sigma_1^{e_1}]+\cdots+[\sigma_n^{e_n}] : e_1\le\cdots\le e_n\}$; let

$$B'_\ell = A\cap(\alpha+[\sigma_1^*]+\cdots+[\sigma_n^*]).$$

It is clear that $B_\ell\subseteq B'_\ell\subseteq A$; it remains to show that B'_ℓ is definable in $(\mathbb{Z},\mathcal{F}_d)$. To prove this, we will study

$$X := \{(e_1,\ldots,e_n)\in\mathbb{N}^n : [\sigma_1^{e_1}]+\cdots+[\sigma_n^{e_n}]\in A-\alpha\};$$

we will look to apply Proposition 3.3.

Since A is stable in $(\mathbb{Z}^m, +)$ and addition is commutative and associative, we get that $x_0 + x_1 + \cdots + x_n \in A$ is stable under any partition of the variables; thus so too is $x_1 + \cdots + x_n \in A - \alpha$. It follows that X is a stable relation on $\mathbb{N}^n$ under any partition of the variables: large ladders for some partition of X would induce large ladders for the corresponding partition of $x_1 + \cdots + x_n \in A - \alpha$.

To show that Proposition 3.3 applies, it remains to show that X is definable by an $L_{\delta,<}$-formula for some δ. Suppose $f \in S_n$; we will produce an L_δ-formula for X under the assumption that $e_{f(1)} \leq \cdots \leq e_{f(n)}$. Let $\tau_i \in (\mathbb{Z}^m)^*$ be of length N such that

$$[\tau_i] = \sum_{j=i}^{n} [\sigma_{f(j)}].$$

(One can for example take the first character of τ_i to be $\sum_{j=i}^{n}[\sigma_{f(j)}]$ and the rest to be zeroes.) Then as in the proof of Lemma 3.4 we get for $e_{f(1)} \leq \cdots \leq e_{f(n)}$ that

$$(e_1, \ldots, e_n) \in X \iff [\tau_1^{e_{f(1)}} \tau_2^{e_{f(2)} - e_{f(1)}} \cdots \tau_n^{e_{f(n)} - e_{f(n-1)}}] \in A - \alpha.$$

Let $\Lambda \supseteq \Sigma_{\pm}^m$ be any alphabet such that $\tau_1, \ldots, \tau_n \in \Lambda^*$; this is possible since each string $\tau_i \in (\mathbb{Z}^m)^*$ has only finitely many entries. So by Remark 2.7 we get that $\{\mu \in \Lambda^* : [\mu] \in A - \alpha\}$ is regular. So by the proof of Proposition 2.2 we get that $[\tau_1^{t_1} \tau_2^{t_2} \cdots \tau_n^{t_n}] \in A - \alpha$ can be expressed by a Boolean combination of formulas of the form:

- $t_i = K$ for some $K \in \mathbb{N}$, and
- $t_i \equiv K \pmod{\delta}$ for some $K \in \mathbb{N}$ and $\delta \geq 1$.

So for $e_{f(1)} \leq \cdots \leq e_{f(n)}$ we get that $(e_1, \ldots, e_n) \in X$ is characterized by some Boolean combination of formulas of the following forms:

- $e_{f(1)} = K$ for some $K \in \mathbb{N}$.
- $e_{f(1)} \equiv K \pmod{\delta}$ for some $K \in \mathbb{N}$ and $\delta \geq 1$.
- $e_{f(i+1)} - e_{f(i)} = K$ for some $K \in \mathbb{N}$; note that this is equivalent to $e_{f(i+1)} = S^K e_{f(i)}$.
- $e_{f(i+1)} - e_{f(i)} \equiv K \pmod{\delta}$ for some $K \in \mathbb{N}$ and $\delta \geq 1$; note that this is equivalent to

$$\bigvee_{j<\delta} (e_{f(i)} \equiv j \pmod{\delta} \wedge e_{f(i+1)} \equiv K + j \pmod{\delta}).$$

Moreover note that if $\delta \mid \delta'$ then $\delta\mathbb{N}$ is definable in $(\mathbb{N}, 0, S, \delta'\mathbb{N})$. So under the assumption that $e_{f(1)} \leq \cdots \leq e_{f(n)}$ we get that $(e_1, \ldots, e_n) \in X$ is definable by a quantifier-free L_δ-formula for some δ. So

$$(e_1,\dots,e_n)\in X \iff \bigvee_{f\in S_n}(e_{f(1)}\le\cdots\le e_{f(n)}\wedge(e_1,\dots,e_n)\in X)$$

is definable by a quantifier-free $L_{\delta,<}$-formula for some δ. Moreover we saw previously that X is a stable relation on $\mathbb{N}$ under any partition of the variables. So by Proposition 3.3 we get that X can be defined by a quantifier-free L_δ-formula.

Let $\mathbb{1}\in\mathbb{Z}^m$ be the tuple all of whose entries are 1. I claim that the map $\Phi\colon[(\mathbb{1}^N)^*]\to\mathbb{N}$ given by

$$[\mathbb{1}^{Ne}]\mapsto e$$

defines an interpretation of $(\mathbb{N},0,S,\delta\mathbb{N})$ in $(\mathbb{Z},\mathcal{F}_d)$. Indeed, Φ is well-defined, since $e\mapsto[\mathbb{1}^{Ne}]$ is injective; Φ is surjective; and the domain of Φ is

$$[(\mathbb{1}^N)^*]=C([\mathbb{1}^N];N)\cup\{0\},$$

which is definable in $(\mathbb{Z},\mathcal{F}_d)$. Furthermore the sets in $(\mathbb{N},0,S,\delta\mathbb{N})$ defined by $x=y$, $y=Sx$, and $P_\delta(x)$ all have preimages definable in $(\mathbb{Z},\mathcal{F}_d)$: for $e_1,e_2\in\mathbb{N}$ we get that

$$(\mathbb{N},0,S,\delta\mathbb{N})\models e_1=e_2 \iff (\mathbb{Z},\mathcal{F}_d)\models[\mathbb{1}^{Ne_1}]=[\mathbb{1}^{Ne_2}]$$

$$(\mathbb{N},0,S,\delta\mathbb{N})\models e_2=Se_1$$

$$\iff (\mathbb{Z},\mathcal{F}_d)\models\begin{pmatrix}[\mathbb{1}^{Ne_1}]\\ [\mathbb{1}^{Ne_2}]\end{pmatrix}\in\left(C\left(\begin{pmatrix}[\mathbb{1}^N]\\ d^N[\mathbb{1}^N]\end{pmatrix};N\right)\cup\{0\}\right)+\begin{pmatrix}0\\ [\mathbb{1}^N]\end{pmatrix}$$

$$(\mathbb{N},0,S,\delta\mathbb{N})\models P_\delta(e_1) \iff (\mathbb{Z},\mathcal{F}_d)\models[\mathbb{1}^{Ne_1}]\in C([\mathbb{1}^{N\delta}];N\delta)\cup\{0\}$$

Moreover the map $[\mathbb{1}^{Ne}]\mapsto[\sigma_i^e]$ is definable in $(\mathbb{Z},\mathcal{F}_d)$ for each i: its graph is simply

$$C\left(\begin{pmatrix}[\mathbb{1}^N]\\ [\sigma_i]\end{pmatrix};N\right)\cup\{0\}.$$

(Recall that $|\sigma_i|=N$.) Then since X is definable in $(\mathbb{N},0,S,\delta\mathbb{N})$ (and recalling that addition is definable in $(\mathbb{Z},\mathcal{F}_d)$) we get that

$$B'_\ell=A\cap(\alpha+[\sigma_1^*]+\cdots+[\sigma_n^*])=\alpha+\{[\sigma_1^{e_1}]+\cdots+[\sigma_n^{e_n}]:(e_1,\dots,e_n)\in X\}$$

is definable in $(\mathbb{Z},\mathcal{F}_d)$, as desired. ∎

Our theorem now follows:

Proof of Theorem 3.1. By the previous lemma A is definable in $(\mathbb{Z}, \mathcal{F}_d)$. But by [16, Theorem A] $(\mathbb{Z}, \mathcal{F}_d)$ admits quantifier elimination. So A is definable by a Boolean combination of F_d-sets, say in disjunctive normal form; we show the F_d-sets can be taken to be groupless. Take one disjunct

$$\bigcap_{i<k} B_i \setminus \bigcup_{j<\ell} C_j$$

where the B_i, C_j are F_d-sets. By Lemma 3.4, A is contained in a groupless F_d-set $\widehat{A}$. So if $k > 0$ we may replace every B_i and C_j in our disjunct with $B_i \cap \widehat{A}$ and $C_j \cap \widehat{A}$, respectively, and the result of the disjunction will still be A. If $k = 0$ we instead replace our disjunct with $\widehat{A} \setminus \bigcup_{j<\ell}(C_j \cap \widehat{A})$, and again the result of the disjunction is still A. But $B_i \cap \widehat{A}, C_j \cap \widehat{A}$ are intersections of F_d-sets, and hence themselves F_d-sets by [16, Proposition 3.9]. Furthermore $\widehat{A}$ is d-sparse by Remark 3.2; so $B_i \cap \widehat{A}, C_j \cap \widehat{A}$ cannot contain a translate of a subgroup, and hence are groupless F_d-sets. Applying the above replacement to every disjunct, we get that A is a Boolean combination of groupless F_d-sets, and hence a Boolean combination of basic groupless F_d-sets. ∎

Combined with [16], we obtain the following characterization of the stability among d-sparse sets:

Corollary 3.7: *Suppose $A \subseteq \mathbb{Z}^m$ is d-sparse. The following are equivalent:*

(1) $\mathrm{Th}(\mathbb{Z}, +, A)$ *is stable.*
(2) *A is stable in $(\mathbb{Z}^m, +)$.*
(3) *A is a finite Boolean combination of basic groupless F_d-sets in $\mathbb{Z}^m$.*
(4) *A is definable in $(\mathbb{Z}, \mathcal{F}_d)$.*

Proof. (1) $\Longrightarrow$ (2) is clear, (2) $\Longrightarrow$ (3) is Theorem 3.1, (3) $\Longrightarrow$ (4) is clear, and (4) $\Longrightarrow$ (1) is by the fact (Theorem A of [16]) that $\mathrm{Th}(\mathbb{Z}, \mathcal{F}_d)$ is stable. ∎

Note that not all Boolean combinations of basic groupless F_d-sets are d-sparse: consider for example $\mathbb{Z} \setminus C(1;1)$. We can however extend Corollary 3.7 to give a complete description of the stable d-sparse sets.

Corollary 3.8: *Suppose $A \subseteq \mathbb{Z}^m$. Then the following are equivalent:*

(1) *A is d-sparse and* $\mathrm{Th}(\mathbb{Z}, +, A)$ *is stable.*
(2) *A is a finite union of sets of the form $(B_1 \cap \cdots \cap B_k) \setminus (C_1 \cup \cdots \cup C_\ell)$ where $k > 0$, $\ell \geq 0$, and each B_i, C_j is a basic groupless F_d-set.*

Proof. By Corollary 3.7, it suffices to show the following: given a finite Boolean combination A of basic groupless F_d-sets, say with disjunctive normal form

$$\bigcup_{i=1}^{n}\Big(\bigcap_{j=1}^{k_i} B_{ij}\cap\bigcap_{j=1}^{\ell_i} C_{ij}^c\Big)$$

with B_{ij}, C_{ij} basic groupless F_d-sets, we have that A is d-sparse if and only if each $k_i \neq 0$.

If some $k_i = 0$ then

$$A \supseteq \Big(\bigcup_{j=1}^{\ell_i} C_{ij}\Big)^c$$

and by Remark 3.2 the latter is the complement of a d-sparse set; hence A is not d-sparse. Suppose conversely that each $k_i \neq 0$. Then A is d-automatic, as a Boolean combination of d-automatic sets: this can be shown using the fact that regular languages are closed under Boolean combinations (see, e.g., [21, Theorem 4.1]). Moreover

$$A \subseteq \bigcup_{i=1}^{n} B_{i1}$$

and the latter is d-sparse by Remark 3.2. So A is d-sparse. ∎

In fact these sets coincide with the **generalized groupless F_d-sets** of [16]: these are finite unions of sets of the form $B\setminus C$ where B, C are groupless F_d-sets. This can be seen using [16, Remark 3.11].

4. Beyond sparsity: the non-generic case

In the previous section we characterized the d-sparse sets that are stable in $(\mathbb{Z}^m,+)$. So the question of which automatic sets are stable in $(\mathbb{Z}^m,+)$ reduces to the non-sparse case. We begin to study this problem in this section, restricting our attention to subsets of $\mathbb{Z}$.

As an example of a non-sparse automatic set that is stable in $(\mathbb{Z},+)$, consider a coset of a subgroup, say $A = r + s\mathbb{Z}$ where $s > 0$. Then A is stable in $(\mathbb{Z},+)$ since it is definable in $(\mathbb{Z},+)$. It is not d-sparse: the number of $a \in A$ with $d^{-k} < a < d^k$ grows exponentially with k, so the set of canonical representations of A is not sparse. It is d-automatic: see [2, Theorem 5.4.2] (though recall as mentioned in Remark 2.7 that they use a different convention for representing integers, so the automaton will be slightly different).

One can also take Boolean combinations of cosets and the stable sparse sets of the previous section to get further examples, as long as the result is not d-sparse. But all examples produced in this way will be "generic":

Definition 4.1: We say $A \subseteq \mathbb{Z}$ is **generic** if some finite union of additive translates of A covers $\mathbb{Z}$.

We show that in the non-sparse setting all stable automatic sets are generic.

THEOREM 4.2: *Suppose $A \subseteq \mathbb{Z}$ is d-automatic and not d-sparse. If A is stable in $(\mathbb{Z}, +)$ then A is generic.*

It will be easier to work first in $\mathbb{N}$, and in particular to use $\Sigma = \{0, \ldots, d-1\}$ for our representations rather than $\Sigma_{\pm} = \{-d+1, \ldots, d-1\}$; the main advantage to doing so is that whenever $\sigma, \tau \in \Sigma^*$ have the same length we have

$$\sigma = \tau \iff [\sigma] = [\tau].$$

(Note that the same does not hold in $\Sigma_{\pm}$: for example, $[(d-1)0] = [(-1)1]$.) Recall from Remark 2.7 that $A \subseteq \mathbb{N}$ is a d-automatic subset of $\mathbb{Z}$ if and only if it is a d-automatic subset of $\mathbb{N}$ in the classical sense; i.e., $\{\sigma \in \Sigma^* : [\sigma] \in A\}$ is regular. Note also that if $A \subseteq \mathbb{N}$ then the canonical representations of the elements of A all lie in Σ^*, and up to trailing zeroes these are the only representations over Σ of elements of A. So $A \subseteq \mathbb{N}$ is d-sparse as a subset of $\mathbb{Z}$ if and only if

$$\{\sigma \in \Sigma^* : [\sigma] \in A, \sigma \text{ has no trailing zeroes}\}$$

is sparse. On the other hand stability and genericity when relativized to $\mathbb{N}$ give something new:

Definition 4.3: We say $A \subseteq \mathbb{N}$ is **stable in** $\mathbb{N}$ if $x + y \in A$ is a stable relation on $\mathbb{N}$. We say A is **generic in** $\mathbb{N}$ if some finite union of (possibly negative) translates of A covers $\mathbb{N}$.

We will first focus on proving:

PROPOSITION 4.4: *Suppose $A \subseteq \mathbb{N}$ is d-automatic and not d-sparse. If A is stable in $\mathbb{N}$ then A is generic in $\mathbb{N}$.*

We begin by recalling the pumping lemma for regular languages; see for example [21, Lemma 4.1]. This will prove useful both in characterizing genericity (Lemma 4.6) and in constructing the ladders in Lemma 4.7.

Fact 4.5 (Pumping lemma): *If $R \subseteq \Sigma^*$ is regular then there is a* pumping length $p > 0$ *such that if $\mu \in R$ has length $\geq p$ then we can write $\mu = uvw$ for some $u, v, w \in \Sigma^*$ such that*

- $v \neq \varepsilon$,
- $|uv| \leq p$,
- $uv^*w \subseteq R$.

Informally, there is an infix v of μ that can be "pumped" without leaving R.

We now give a characterization of the generic d-automatic sets.

Lemma 4.6: *Suppose $A \subseteq \mathbb{N}$ is d-automatic; let $L \subseteq \Sigma^*$ be the set of representations of elements of A. Then the following are equivalent:*

(1) *A is generic in $\mathbb{N}$.*
(2) *For any $r, s \in \mathbb{N}$, every $\tau \in \Sigma^*$ occurs as a suffix of a word in L of length $r + sk$ for some $k \geq 0$.*

In other words, A is not generic in $\mathbb{N}$ if and only if there are $r, s \in \mathbb{N}$ such that $L \cap \Sigma^{(r+s\mathbb{N})}$ has a forbidden suffix.

Proof. Note first that A is not generic in $\mathbb{N}$ if and only if there are arbitrarily large gaps in A (i.e., runs of naturals not in A).

(1) $\Longrightarrow$ (2): Suppose we are given τ, r, s such that τ is a forbidden suffix for $L \cap \Sigma^{(r+s\mathbb{N})}$. Then if $r + sk > |\tau|$ then A is disjoint from

$$[\Sigma^{(r+sk-|\tau|)}\tau] = \{b \in \mathbb{N} : d^{r+sk-|\tau|}[\tau] \leq b < d^{r+sk-|\tau|}([\tau]+1)\}.$$

So A has a gap of size $d^{r+sk-|\tau|}$. So as $k \to \infty$ we get arbitrarily large gaps in A; so A is not generic in $\mathbb{N}$.

(2) $\Longrightarrow$ (1): Suppose A is not generic in $\mathbb{N}$. Let \$ be a letter not in Σ; we will use \$ as a separator. Consider the set $S \subseteq (\Sigma \cup \{\$\})^*$ of $0^m\$\tau$ for $m < \omega$ and $\tau \in \Sigma^*$ with the property that

$$\Sigma^{(m)}\tau \cap L = \emptyset;$$

in other words, if we replace each zero with any letter and delete the separator, the result is never in L. So $0^m\$\tau \in S$ if and only if τ is a forbidden suffix for $L \cap \Sigma^{(m+|\tau|)}$. Then S is regular: it is not too hard to construct a nondeterministic finite automaton (NFA) for the complement, which suffices (see, e.g., [21, Section 2.2]). Since there are arbitrarily large gaps in A we get that there are elements $0^m\$\tau \in S$ with m arbitrarily large. Indeed, suppose we are given m. Find a gap of size $2d^m$; then this gap will contain two multiples of d^m, say $a, a+d^m$. Then if $\tau \in \Sigma^*$ is such that $[\tau] = \frac{a}{d^m}$ then τ is a forbidden suffix for $L \cap \Sigma^{(m+|\tau|)}$; so $0^m\$\tau \in S$.

Pick $0^m\$\tau \in S$ with m bigger than the pumping length of S (as defined in Fact 4.5). Then by the pumping lemma we can write $m = r + s$ so that $0^r(0^s)^*\$\tau \subseteq S$; so τ is a forbidden suffix for $L \cap \Sigma^{(r+|\tau|+s\mathbb{N})}$. ∎

The following technical lemma is the source of instability in Proposition 4.4. For $K < \omega$ we define a partial binary operation $+_K$ on Σ^* by setting $\sigma +_K \tau$ to be the unique representation of $[\sigma] + [\tau]$ of length K, if one exists.

Lemma 4.7: *Suppose $L \subseteq \Sigma^*$ is regular but not sparse, and satisfies $L = L^*$ and*

(†) *there are $r, s \in \mathbb{N}$ such that $L \cap \Sigma^{(r+s\mathbb{N})}$ is infinite and has a forbidden suffix σ.*

Then for all $N < \omega$ there is $K < \omega$ such that the binary relation $x +_K y \in L$ on Σ^* has an N-ladder.

Proof. Pick σ, r, s as in (†). Since $L \cap \Sigma^{(r+s\mathbb{N})}$ is infinite, there is $a \in \Sigma^{(|\sigma|)}$ that occurs as a suffix of some element of $L \cap \Sigma^{(r+s\mathbb{N})}$. Suppose $[a] \leq [\sigma]$; we will see at the end how to modify the argument in the case $[a] > [\sigma]$. Note since σ is a forbidden suffix of $L \cap \Sigma^{(r+s\mathbb{N})}$ that $a \neq \sigma$; so, since $|a| = |\sigma|$ and we are working over $\Sigma = \{0, \ldots, d-1\}$, we get that $[a] \neq [\sigma]$. Our assumption that $[a] \leq [\sigma]$ then yields that $[a] < [\sigma]$.

Let $\leq_{\mathbb{N}}$ be the preorder on Σ^* induced by the ordering on $\mathbb{N}$ via $[\cdot]$: that is, $\tau_1 \leq_{\mathbb{N}} \tau_2$ if $[\tau_1] \leq [\tau_2]$. We may assume a is maximal under $\leq_{\mathbb{N}}$ among the $b \in \Sigma^{(|\sigma|)}$ that occur as a suffix of some element of $L \cap \Sigma^{(r+s\mathbb{N})}$ and satisfy $[b] \leq [\sigma]$. Fix $a' \in \Sigma^{(|\sigma|)}$ such that $[a'] = [a] + 1$; note such a' exists since $[a] < [\sigma] < d^{|\sigma|}$, and hence $[a] + 1 < d^{|\sigma|}$ can be represented by a string of length $|\sigma|$. Then by maximality of a we get that a' does not occur as a suffix of some element of $L \cap \Sigma^{(r+s\mathbb{N})}$.

Consider the set S of $\tau \in L \cap \Sigma^{(r+s\mathbb{N})}$ with a as a suffix such that τ is $\leq_{\mathbb{N}}$-maximal among the elements of L ending in a that are of the same length as τ. Then S is infinite: since $L = L^*$ and a occurs as a suffix of some $\mu \in L \cap \Sigma^{(r+s\mathbb{N})}$, we get for $k < \omega$ that $\mu^{1+sk} \in L \cap \Sigma^{(r+s\mathbb{N})}$ also has a as a suffix, and hence that S contains a word of length $(1 + sk)|\mu|$. Furthermore S is regular: using the fact that

$$\left\{ \binom{\ell_1}{\ell'_1} \cdots \binom{\ell_n}{\ell'_n} \in (\Sigma^2)^* : [\ell_1 \cdots \ell_n] < [\ell'_1 \cdots \ell'_n] \right\}$$

and $\Sigma^* a$ are regular, one can construct an NFA for the complement of S. So by the pumping lemma (Fact 4.5) S contains a set of the form uv^*w with $v \neq \varepsilon$. By prepending a power of v to w we may assume $|w| \geq |a|$; in particular, since $uw \in S$ must have a as a suffix, we get that w has a as a suffix (and is non-empty).

Since $L = L^*$ and $uv^*w \subseteq S \subseteq L$ we get that $L \supseteq (uv^*w)^* \supseteq u\{wu, v\}^*w$. This, together with the maximality of elements of S, the fact that a' is a forbidden suffix for $L \cap \Sigma^{(r+s\mathbb{N})}$, and the fact that $|uv^*w| \in r + s\mathbb{N}$, will be enough to construct our ladder.

Pick n, m such that $n|wu| = m|v|$; then $u(wu)^n w \in L$ and ends in a, so since $uv^m w \in S$ and $|uv^m w| = |u(wu)^n w|$ we get that $[u(wu)^n w] \leq [uv^m w]$, and hence that $[(wu)^n] \leq [v^m]$.

Case 1: Suppose $[(wu)^n] < [v^m]$; then since $[v^m] - [(wu)^n] \leq [v^m]$ there is $\alpha \in \Sigma^{(m|v|)}$ such that $[\alpha] = [v^m] - [(wu)^n] > 0$. Fix $N < \omega$; to avoid notational clutter, we will produce K for which there exists an $(N+1)$-ladder, not just an N-ladder. We let

$$K = |uv^{mN}w| \in r + s\mathbb{N},$$
$$d_i = u(wu)^{n(N-i)}v^{mi}w,$$
$$e_i = 0^{|u|}\alpha^{N-i},$$

for $i \leq N$. Then $d_i +_K e_j$ is defined for all i, j; i.e., $[d_i]+[e_j]$ has a representation of length K. Indeed, $|e_j| \leq K - |w| \leq K - |a|$; so $[e_j] < d^{K-|a|}$. So if we write $d_i = \tau a$ for some τ (possible since d_i has w, and hence a, as a suffix) then

$$[d_i] + [e_j] < [\tau a] + d^{K-|a|} = [\tau a] + d^{|\tau|} = [\tau a'] < d^K$$

since $|\tau a'| = |d_i| = K$. So $[d_i]+[e_j]$ has a representation of length K, and $d_i +_K e_j$ is defined. In fact the above proof shows that $d_i +_K e_j$ has either a or a' as a suffix: by the above we get that $[d_i] + [e_j]$ satisfies

$$[0^{|\tau|}a] \leq [\tau a] = [d_i] \leq [d_i] + [e_j] < [\tau a'] \leq [(d-1)^{|\tau|}a']$$

and $|\tau a| = |\tau a'| = K$; so, since $[a'] = [a] + 1$, we get that $d_i +_K e_j$ has a' as a suffix if $[d_i] + [e_j] \geq [0^{|\tau|}a']$, and otherwise has a as a suffix.

An important property of the d_i, e_j is that if $i \le j$ then

$$d_i +_K e_j = uv^{m(N-j)}(wu)^{n(j-i)}v^{mi}w.$$

Indeed, recall that

$$[(wu)^n] + [\alpha] = [v^m];$$

then writing out the sum $[d_i]+[e_j] = [u(wu)^{n(N-i)}v^{mi}w]+[0^{|u|}\alpha^{N-j}]$ and lining up substrings of equal lengths, we get:

$$\begin{array}{cccccccccc} & & \multicolumn{3}{c}{\overbrace{\hphantom{(wu)^n \quad \cdots \quad (wu)^n}}^{N-j\text{ copies}}} & \multicolumn{3}{c}{\overbrace{\hphantom{(wu)^n \quad \cdots \quad (wu)^n}}^{j-i\text{ copies}}} & \\ & u & (wu)^n & \cdots & (wu)^n & (wu)^n & \cdots & (wu)^n & v^{mi}w \\ + & 0^{|u|} & \alpha & \cdots & \alpha & & & & \\ \hline & u & v^m & \cdots & v^m & (wu)^n & \cdots & (wu)^n & v^{mi}w \end{array}$$

So $[d_i] + [e_j] = [uv^{m(N-j)}(wu)^{n(j-i)}v^{mi}w]$; thus since

$$|uv^{m(N-j)}(wu)^{n(j-i)}v^{mi}w| = |d_i| = K,$$

we indeed get that $d_i +_K e_j = uv^{m(N-j)}(wu)^{n(j-i)}v^{mi}w$.

We now show that the d_i, e_j form an $(N+1)$-ladder. Since $[\alpha] > 0$ it is clear that the e_i are strictly decreasing. Suppose $i > j$; then

$$[d_i +_K e_j] = [d_i] + [e_j] > [d_i] + [e_i] = [d_i +_K e_i] = [uv^{mN}w].$$

So if $d_i +_K e_j$ has a as a suffix then, since $uv^{mN}w \in S$ and $d_i +_K e_j$ has the same length, has a as a suffix, and represents a strictly larger number, we get that $d_i +_K e_j \notin L$. Otherwise as noted above we get that $d_i +_K e_j$ has a' as a suffix, in which case $d_i +_K e_j \notin L$ since a' is a forbidden suffix for $L \cap \Sigma^{(r+s\mathbb{N})}$ and $|d_i +_K e_j| = K \in r + s\mathbb{N}$. Conversely suppose $i \le j$; then

$$d_i +_K e_j = uv^{m(N-j)}(wu)^{n(j-i)}v^{mi}w \in u\{wu, v\}^*w \subseteq L.$$

So the d_i, e_i form an $(N+1)$-ladder for $x +_K y \in L$.

Case 2: Suppose $[(wu)^n] = [v^m]$; so $(wu)^n = v^m$. Then

$$uv^*w \supseteq u((wu)^n)^*w = ((uw)^n)^*uw;$$

so if we let $u' = \varepsilon$, $v' = (uw)^n$, and $w' = uw$ then $u'(v')^*w' \subseteq uv^*w \subseteq S$. Furthermore $v' \neq \varepsilon$ since $|v'| = n|uw| = m|v| > 0$, and $v' \in L$ since $L = L^*$ and $uw \in uv^*w \subseteq L$. So we may replace u, v, w with u', v', w' respectively, and we may thus assume that $u = \varepsilon$ and $v, w \in L$. (Recall that the only requirements we had of u, v, w were that $uv^*w \subseteq S$, that $v \neq \varepsilon$, and that w have a as a suffix.)

By [4, Proposition 7.1] since L is not sparse there are $x, y_1, y_2, z \in \Sigma^*$ with y_1, y_2 distinct, non-trivial, and of the same length such that $x\{y_1, y_2\}^* z \subseteq L$. Let $b = xy_1z$ and $c = xy_2z$; so $|b| = |c|$ with $b, c \in L$ and $b \neq c$. By replacing b, c, v with powers thereof we may assume $|b| = |c| = |v|$. Then since $b \neq c$ we get that one of b, c, without loss of generality say b, has $b \neq v$, and thus $[b] \neq [v]$. Note since $L = L^*$ that $L \supseteq \{b, v\}^* w$.

Since $vw \in S$ and since bw has the same length as vw, has a as a suffix, and lies in L, we get that $[bw] \leq [vw]$. So $[b] \leq [v]$, and since $b \neq v$ we get $[b] < [v]$. Then since $[v] - [b] < [v]$ there is $\alpha \in \Sigma^{(|v|)}$ such that $[\alpha] = [v] - [b]$. Fix $N < \omega$; we again show there exists K for which we can produce an $(N+1)$-ladder. We then let

$$K = |v^N w| \in r + s\mathbb{N},$$
$$d_i = b^{N-i} v^i w,$$
$$e_i = \alpha^{N-i},$$

for $i \leq N$. Then by an argument identical to the previous case the d_i, e_i form an $(N+1)$-ladder for $x +_K y \in L$. The case $[a] > [\sigma]$ is similar; we outline it here. We take minimal such a under $\leq_{\mathbb{N}}$, and define S to be the set of $\tau \in L \cap \Sigma^{(r+s\mathbb{N})}$ ending in a that are $\leq_{\mathbb{N}}$-minimal among the elements of L ending in a that are of the same length as τ. Then S is again infinite and regular, and thus contains a set of the form uv^*w; we again assume w has a as a suffix. If $n|wu| = m|v|$ then dually to before we get $[(wu)^m] \geq [v^n]$. Fix $N < \omega$; again our goal will be to find K for which we can produce an $(N+1)$-ladder. If $[(wu)^n] > [v^m]$, say with $\alpha \in \Sigma^{(m|v|)}$ with $[\alpha] = [(wu)^n] - [v^m] > 0$, then we'd like to let

$$K = |uv^{mN} w|,$$
$$d_i = u(wu)^{n(N-i)} v^{mi} w,$$
$$e_i = 0^{|u|}(-\alpha)^{N-i},$$

and claim this as our ladder. Unfortunately we're working over Σ, not $\Sigma_\pm$, so we cannot allow the e_i to use negative digits. This is easily fixed, however: note for all i, j that $[d_i] \geq d^{|u|+Nm|v|} \geq -[e_j]$ (since $[a] \neq 0$ and w, and hence d_i, has a as a suffix). So we can take $d'_i, e'_i \in \Sigma^*$ such that $[d'_i] = [d_i] - d^{|u|+Nm|v|}$ and $[e'_i] = e_i + d^{|u|+Nm|v|}$. Then $[d'_i] + [e'_j] = [d_i] + [e_j]$, and now as before one can show that $d'_i +_K e'_j$ is always defined and is in L if and only if $i \leq j$.

If $[(wu)^n] = [v^m]$ we do a similar trick. As before we may assume $u = \varepsilon$ and $v, w \in L$, and we get some $b \in L$ with $|b| = |v|$ and $[b] \neq [v]$; dually to before we get $[b] > [v]$, say with $\alpha \in \Sigma^{(|v|)}$ such that $[\alpha] = [b] - [v]$. Our initial attempt at a ladder will now be

$$K = |v^N w|,$$
$$d_i = b^{N-i} v^i w,$$
$$e_i = (-\alpha)^{N-i}.$$

Now we have $[d_i] \geq d^{N|v|} \geq -[e_j]$; so we can pull the same trick to turn the d_i, e_i into a ladder. ∎

Suppose $M = (Q, q_0, F, \delta)$ is a DFA over Σ. For $q \in Q$ we let

$$L_q = \{\sigma \in \Sigma^* : \delta(q, \sigma) = q\};$$

that is, L_q is the set of words which take state q back to state q in M. Note that L_q is regular: it is recognized by the automaton $(Q, q, \{q\}, \delta)$.

We will primarily be interested in L_q for q a **non-dead** state:

Definition 4.8: Suppose $M = (q, q_0, F, \delta)$ is a DFA over Σ. We say $q \in Q$ is a **dead state** if there is no $\sigma \in \Sigma^*$ such that $\delta(q, \sigma) \in F$.

Lemma 4.9: *Suppose $A \subseteq \mathbb{N}$ is d-automatic but not d-sparse; suppose A is not generic in $\mathbb{N}$. Fix an automaton $M = (Q, q_0, F, \delta)$ that recognizes $\{\sigma \in \Sigma^* : [\sigma] \in A\}$. Then there is a non-dead $q \in Q$ such that L_q satisfies the hypotheses of Lemma 4.7: namely L_q is regular but not sparse, $L_q = L_q^*$, and L_q satisfies (†).*

Proof. Note we always have that L_q is regular and $L_q = L_q^*$; so we only need non-sparsity and (†). We first note some facts about how non-sparsity and (†) interact with L_q. Recall that (†) applied to L_q states that there are $r, s \in \mathbb{N}$ such that $L_q \cap \Sigma^{(r+s\mathbb{N})}$ is infinite and has a forbidden suffix.

Claim 4.10:

(1) *If q is a finish state of M and L_q is infinite then L_q satisfies (†).*
(2) *There is a non-dead q such that L_q is not sparse.*
(3) *If q, q' are states in M with a path from q to q' and vice-versa then L_q is sparse if and only if $L_{q'}$ is.*

Proof of Claim 4.10. (1) We may assume that for every $q' \in Q$ there is $\mu \in \Sigma^*$ such that $\delta(q_0,\mu) = q'$; that is, every state is reachable from the start state. Indeed, if X is the set of q' for which there is no such μ, then

$$(Q \setminus X, q_0, F \setminus X, \delta \restriction (Q \setminus X))$$

is a DFA recognizing the same language that has the desired property.

Fix $\mu \in \Sigma^*$ such that $\delta(q_0,\mu) = q$; let $L = \{\sigma \in \Sigma^* : [\sigma] \in A\}$. By non-genericity of A in $\mathbb{N}$ and Lemma 4.6 there are $r,s \in \mathbb{N}$ for which there is a forbidden suffix for $L \cap \Sigma^{(r+s\mathbb{N})}$. Note that if τ is a forbidden suffix for $L \cap \Sigma^{(r+s\mathbb{N})}$ then $\tau 0^t$ is a forbidden suffix for $L \cap \Sigma^{(r+t+s\mathbb{N})}$ (since L is closed under removing trailing zeroes). So there is a forbidden suffix for $L \cap \Sigma^{(r'+s\mathbb{N})}$ for any $r' \geq r$. Since L_q is infinite we can find $r' \geq \max\{r, |\mu|\}$ such that $L_q \cap \Sigma^{(r'-|\mu|+s\mathbb{N})}$ is infinite. Then since q is a finish state, the forbidden suffix for $L \cap \Sigma^{(r'+s\mathbb{N})}$ is also a forbidden suffix for $L_q \cap \Sigma^{(r'-|\mu|+s\mathbb{N})}$. So L_q satisfies (†).

(2) By [4, Proposition 7.1] there is a non-dead state q and distinct non-empty $u,v \in \Sigma^*$ such that $\delta(q,u) = \delta(q,v) = q$ and $\delta(q,x) \neq q$ for x any proper non-empty prefix of u or v. Then taking b,c to be powers of u,v respectively such that $|b| = |c|$, we get that $b \neq c$ (otherwise u or v would be a prefix of the other); also $\delta(q,a) = \delta(q,b) = q$, so $b,c \in L_q$. So $L_q \supseteq \{b,c\}^*$, and a quick computation shows that L_q is not sparse.

(3) Let $\delta(q,\mu) = q'$ and $\delta(q',\nu) = q$. Suppose L_q is not sparse. Then

$$L_{q'} \supseteq \nu L_q \mu$$

is also not sparse. ∎

By Claim 4.10 (2) there is q such that L_q is not sparse and q has a path to a finish state q'. If $L_{q'}$ is not sparse then by Claim 4.10 (1) q' satisfies the desired properties, and we're done; suppose then that it is sparse. We show in this case that there is a forbidden infix for L_q (i.e., some τ that does not appear as a substring of any element of L_q), and hence in particular that L_q satisfies (†) with $r = 0$ and $s = 1$.

Note that there is no path from q' to q, else by Claim 4.10 (3) $L_{q'}$ would not be sparse. Enumerate the states of M with a path to q (and hence to q') as $(q_i : i < n)$. Inductively pick $\sigma_i \in \Sigma^*$ as follows: if $\delta(q_i, \sigma_0 \cdots \sigma_{i-1})$ has no path to q we let $\sigma_i = \varepsilon$, and otherwise we pick σ_i such that $\delta(q_i, \sigma_0 \cdots \sigma_i) = q'$. Note then that $\delta(q_i, \sigma_0 \cdots \sigma_i)$ has no path to q; hence neither does $\delta(q_i, \sigma_0 \cdots \sigma_{n-1})$.

Let $\tau = \sigma_0 \cdots \sigma_{n-1}$. We have shown that if r is a state with a path to q (so one of the q_i) then $\delta(r, \tau)$ has no path to q. Clearly if r has no path to q then neither does $\delta(r, \tau)$. Hence for all $r \in Q$ we get that $\delta(r, \tau)$ has no path to q; that is, τ is a forbidden infix for L_q. ∎

With the above lemmas, we are almost ready to prove Proposition 4.4. We will need the notion of a minimal automaton:

Definition 4.11: The **minimal automaton** of a regular language $L \subseteq \Sigma^*$ is an automaton (Q, q_0, F, δ) recognizing L satisfying the following:

- all states are reachable from the start state, and
- given distinct $q, q' \in Q$ there is $\nu \in \Sigma^*$ such that $\delta(q, \nu) \in F$ if and only if $\delta(q', \nu) \notin F$.

Such automata exist and are unique: see the proof of the right-to-left direction of [21, Theorem 4.7].

Proof of Proposition 4.4. Suppose $A \subseteq \mathbb{N}$ is d-automatic and neither d-sparse nor generic in $\mathbb{N}$; we wish to show that A is not stable in $\mathbb{N}$. Fix a minimal automaton $M = (Q, q_0, F, \delta)$ for the set of representations over Σ of elements of A. By Lemma 4.9 there is a non-dead q such that L_q satisfies the hypotheses of Lemma 4.7. Using minimality, for each $q' \neq q$ let $\sigma_{q'} \in \Sigma^*$ and $\varepsilon_{q'} \in \{0, 1\}$ be such that

$$(\delta(q, \sigma_{q'}) \in F)^{\varepsilon_{q'}} \wedge (\delta(q', \sigma_{q'}) \in F)^{1-\varepsilon_{q'}}$$

holds (where, as before, φ^0 denotes $\neg\varphi$ and φ^1 denotes φ). If $\theta \in Q$ then $\theta = q$ if and only if

$$\bigwedge_{q' \neq q} (\delta(\theta, \sigma_{q'}) \in F)^{\varepsilon_{q'}}$$

holds. Consider then the following formula in the variables $\overline{x} = (x_{q'} : q' \neq q)$ and y:

$$\varphi(\overline{x}; y) = \bigwedge_{q' \neq q} (x_{q'} + y \in A)^{\varepsilon_{q'}}.$$

We show that φ is unstable in $\mathbb{N}$, and hence since φ is a Boolean combination of instances of $x + y \in A$ that A is unstable in $\mathbb{N}$.

Fix $N < \omega$; we show there exists an N-ladder for the relation $x + y \in A$. Recall that L_q satisfies the hypotheses of Lemma 4.7; so for some $K < \omega$ there is an N-ladder $(d_i, e_i : i < N)$ for $x +_K y \in L_q$. We may assume each $|d_i| = K$. Take any $\mu \in \Sigma^*$ such that $\delta(q_0, \mu) = q$, and let

$$b_{i,q'} = [\mu d_i \sigma_{q'}],$$
$$c_i = [0^{|\mu|} e_i].$$

These

$$\overline{b_i} := (b_{i,q'} : q' \neq q), c_i$$

will be our N-ladder for φ. Note that $b_{i,q'} + c_j = [\mu(d_i +_K e_j)\sigma_{q'}]$. Then

$$\begin{aligned} \varphi(\overline{b_i}; c_j) &\iff \bigwedge_{q' \neq q} (b_{i,q'} + c_j \in A)^{\varepsilon_{q'}} \\ &\iff \bigwedge_{q' \neq q} (\delta(q_0, \mu(d_i +_K e_j)\sigma_{q'}) \in F)^{\varepsilon_{q'}} \\ &\iff \bigwedge_{q' \neq q} (\delta(\delta(q, d_i +_K e_j), \sigma_{q'}) \in F)^{\varepsilon_{q'}} \\ &\iff \delta(q, d_i +_K e_j) = q \\ &\iff d_i +_K e_j \in L_q \\ &\iff i \leq j. \end{aligned}$$

So φ is unstable in $\mathbb{N}$, and thus A is unstable in $\mathbb{N}$. ∎

We can now do the case $A \subseteq \mathbb{Z}$:

Proof of Theorem 4.2. Suppose $A \subseteq \mathbb{Z}$ is d-automatic but neither d-sparse nor generic in $\mathbb{Z}$; we wish to show that A is not stable in $(\mathbb{Z}, +)$.

Case 1: Suppose one of $A \cap \mathbb{N}$ and $-A \cap \mathbb{N}$ is generic in $\mathbb{N}$ and the other is d-sparse. Then taking finitely many translates and unioning we get a set B where (say) $B \cap \mathbb{N}$ is d-sparse and $B \supseteq -\mathbb{N}$. (Note that d-sparsity is closed under translation and finite union.) In fact this is enough to deduce that $B \cap \mathbb{N}$ is disjoint from some coset:

Claim 4.12: *If $C \subseteq \mathbb{N}$ is d-sparse then there are $r, s \in \mathbb{N}$ with $r < s$ such that*

$$C \cap (r + s\mathbb{N}) = \emptyset.$$

Proof. Let $L \subseteq \Sigma^*$ be the set of canonical representations of elements of C; so L is sparse. Recall by [21, Theorem 3.8] that the set of prefixes of a sparse set is also sparse, and in particular is not all of Σ^*. So there is some $\sigma \in \Sigma^*$ that does not occur as a prefix of an element of L; i.e., σ is a forbidden prefix for L. Note that if σ is a forbidden prefix for L then so is $\sigma 1$; so, by possibly appending a 1, we may assume that σ has no trailing zeroes. Let $r = [\sigma]$ and $s = d^{|\sigma|}$; so $r < s$. Then if $a \in r + s\mathbb{N}$ then the canonical representation of a is $\sigma\tau$, where τ is the canonical representation of

$$\frac{a-r}{s} = \frac{a-[\sigma]}{d^{|\sigma|}}.$$

In particular, the canonical representation of a begins with σ, and is thus not in L; so $a \notin C$. So $(r + s\mathbb{N}) \cap C = \emptyset$, as desired. ∎

Pick such $r, s \in \mathbb{N}$ for $B \cap \mathbb{N}$; so $(r + s\mathbb{Z}) \cap B = r + s\mathbb{Z}_{<0}$. It follows that $(r + s\mathbb{Z}) \cap B$ is unstable in $(\mathbb{Z}, +)$: if $x, y \in s\mathbb{Z}$ then

$$x \leq y \iff r + x - y - s \in r + s\mathbb{Z}_{<0}.$$

So $(x + y \in r + s\mathbb{Z}) \wedge (x + y \in B)$ is unstable in $(\mathbb{Z}, +)$. But $x + y \in r + s\mathbb{Z}$ is stable in $(\mathbb{Z}, +)$, since it is definable in $(\mathbb{Z}, +)$; so B is unstable in $(\mathbb{Z}, +)$. So since B is a finite union of translates of A we get that A is unstable in $(\mathbb{Z}, +)$.

Case 2: Suppose otherwise. I claim that one of $A \cap \mathbb{N}$ and $-A \cap \mathbb{N}$ is neither generic in $\mathbb{N}$ nor d-sparse. Indeed, suppose otherwise; so $A \cap \mathbb{N}$ is either generic in $\mathbb{N}$ or d-sparse, and likewise with $-A \cap \mathbb{N}$. Since A is not generic in $\mathbb{Z}$, at most one of $A \cap \mathbb{N}$ or $-A \cap \mathbb{N}$ is generic in $\mathbb{N}$; likewise with d-sparse. So one of $A \cap \mathbb{N}, -A \cap \mathbb{N}$ is generic in $\mathbb{N}$, and the other is d-sparse; so we are in the previous case, a contradiction.

So one of $A \cap \mathbb{N}$ or $-A \cap \mathbb{N}$ is neither generic in $\mathbb{N}$ nor d-sparse. Note that A is stable in $(\mathbb{Z}, +)$ if and only if $-A$ is. Hence replacing A by $-A$ if necessary we may assume $A \cap \mathbb{N}$ is neither generic in $\mathbb{N}$ nor d-sparse. Then by Proposition 4.4 there are arbitrarily large ladders in $\mathbb{N}$ for $x + y \in A \cap \mathbb{N}$; since $\mathbb{N}$ is closed under addition, we get that these are also ladders in $\mathbb{Z}$ for $x + y \in A$. Hence A is unstable in $(\mathbb{Z}, +)$. ∎

As an illustration of our theorem we note that the following automatic sets are not stable in $(\mathbb{Z}, +)$. Indeed, it is easily checked that they are all neither sparse nor generic.

Corollary 4.13: *The following automatic sets are unstable in* $(\mathbb{Z},+)$*:*

- *The set of* $a \in \mathbb{Z}$ *such that the canonical base-*d *representation of* a *ends in* ± 1 *(assuming* $d > 2$*).*
- *The set of* $a \in \mathbb{Z}$ *such that the canonical base-*d *representation of* a *does not contain a* 0 *(assuming* $d > 2$*).*
- *The set of* $a \in \mathbb{Z}$ *such that the canonical base-*d *representation of* a *is of even length.*
- *The set of* $a \in \mathbb{Z}$ *such that the canonical binary representation of* a *takes the form* $0^{k_0}10^{k_1}1\cdots 10^{k_m}1$ *or* $0^{k_0}(-1)0^{k_1}(-1)\cdots(-1)0^{k_m}(-1)$ *for some even* $k_0,\ldots,k_m$ *(possibly zero); i.e., does not contain a block of zeroes of odd length. These are precisely the* $a \in \mathbb{Z}$ *such that the Baum–Sweet sequence has a 1 in the* $|a|^{\text{th}}$ *position. See* [2, Section 5.1] *for more details on the Baum–Sweet sequence.*

The converse of Theorem 4.2 is certainly false. For example, let $A \subseteq \mathbb{Z}$ be as in the example at the beginning of Section 3; so A is d-sparse and unstable in $(\mathbb{Z},+)$. Then the complement of A remains unstable, and is generic since A does not contain a pair of adjacent integers.

5. The general case

Gabriel Conant pointed out to me in private communications that Theorems 3.1 and 4.2, together with [9, Theorem 2.3 (iv)], allow us to deal with arbitrary d-automatic stable subsets of $\mathbb{Z}$.

Theorem 5.1: *Suppose* $A \subseteq \mathbb{Z}$ *is* d*-automatic and stable in* $(\mathbb{Z},+)$*. Then* A *is a finite Boolean combination of*

- *cosets of subgroups of* $(\mathbb{Z},+)$*, and*
- *basic groupless* F_d*-sets in* $\mathbb{Z}$*.*

Proof. It is known that stable subsets of a group are close to being a finite union of cosets, in the sense that they have non-generic symmetric difference with such; see [9, Theorem 2.3 (iv)] (taking $\delta(x,y)$ to be $x+y \in A$ and $\varphi(x) \in \mathrm{Def}_\delta(G)$ to be $x \in A$). So there is a subgroup $H \leq \mathbb{Z}$ and a union Y of cosets of H such that $Z := A \triangle Y$ is non-generic in $\mathbb{Z}$. Since Y is a union of cosets it is also stable in $(\mathbb{Z},+)$ and d-automatic. Hence Z is both d-automatic and stable in $(\mathbb{Z},+)$. Theorem 4.2 yields that Z is d-sparse, and then Theorem 3.1 yields that Z is a finite Boolean combination of basic groupless F_d-sets. Hence $A = Z \triangle Y$ is a finite Boolean combination of sets of the desired form. ∎

Corollary 5.2: *Suppose $A \subseteq \mathbb{Z}$ is d-automatic. The following are equivalent:*

(1) $\mathrm{Th}(\mathbb{Z},+,A)$ *is stable.*
(2) A *is stable in* $(\mathbb{Z},+)$.
(3) A *is a finite Boolean combination of*
  - *cosets of subgroups of* $(\mathbb{Z},+)$, *and*
  - *basic groupless* F_d*-sets in* $\mathbb{Z}$.
(4) A *is definable in* $(\mathbb{Z},+,d^{\mathbb{N}})$.

Proof. That (1) $\implies$ (2) is clear; that (2) $\implies$ (3) is Theorem 5.1; and that (4) $\implies$ (1) is [16, Theorem A]. For (3) $\implies$ (4), it suffices to show that each

$$C(a;r) = a\Big(\frac{(d^r)^{\mathbb{N}} - 1}{d^r - 1}\Big) \setminus \{0\}$$

is definable in $(\mathbb{Z},+,d^{\mathbb{N}})$; for this it suffices to show that each $(d^r)^{\mathbb{N}}$ is definable in $(\mathbb{Z},+,d^{\mathbb{N}})$. Fix $r > 1$, and let $\varphi(x)$ be

$$(x \in d^{\mathbb{N}}) \wedge ((d^r - 1) \mid x - 1).$$

(Note that $(d^r - 1) \mid x - 1$ is definable in $(\mathbb{Z},+,d^{\mathbb{N}})$ since r is fixed.) It is clear that $(d^r)^{\mathbb{N}} \subseteq \varphi(\mathbb{Z})$. Conversely, suppose we are given some element d^n of $\varphi(\mathbb{Z})$; write $n = qr + s$ for some $0 \le s < r$. Then $1 \equiv d^n \equiv d^s \pmod{d^r - 1}$; so, since $1 \le d^s < d^r - 1$, we get that $d^s = 1$, and $s = 0$. So $d^n = d^{qr} \in (d^r)^{\mathbb{N}}$, and $(d^r)^{\mathbb{N}}$ is definable in $(\mathbb{Z},+,d^{\mathbb{N}})$. ∎

6. Two NIP expansions of $(\mathbb{Z},+)$

In this final section we show how to apply automata-theoretic methods to produce some NIP expansions of $(\mathbb{Z},+)$; see [20] for background on NIP.

6.1. $(\mathbb{Z},+,<,d^{\mathbb{N}})$ is NIP. Fix $d > 0$. That $\mathrm{Th}(\mathbb{Z},+,<,d^{\mathbb{N}})$ is NIP was shown recently by Lambotte and Point (it is an instance of [13, Corollary 2.33]), but our proof is novel and short. It will be convenient to work in $(\mathbb{N},+)$ rather than $(\mathbb{Z},+,<)$. Since $(\mathbb{Z},+,<,d^{\mathbb{N}})$ is interpretable in $(\mathbb{N},+,d^{\mathbb{N}})$, it will suffice to prove:

Theorem 6.1: $\mathrm{Th}(\mathbb{N},+,d^{\mathbb{N}})$ *is NIP.*

Before proving the theorem, let us observe that since all d-sparse subsets of $\mathbb{N}$ are definable in $(\mathbb{N},+,d^{\mathbb{N}})$—see [19, Theorem 5]—and as $A \subseteq \mathbb{Z}$ is d-sparse if and only if both $A \cap \mathbb{N}$ and $-A \cap \mathbb{N}$ are, we get:

COROLLARY 6.2: *The expansion of $(\mathbb{Z},+)$ by all d-sparse subsets is NIP.*

Our proof of Theorem 6.1 will make use of a result of Chernikov and Simon on NIP pairs of structures; we briefly recall their setup and result. We let $L = \{+\}$ and $\mathcal{N} = (\mathbb{N},+)$; we fix $\mathrm{Th}(\mathcal{N})$ as our ambient theory.

Definition 6.3: Let L_P be L expanded by a unary predicate P. A **bounded L_P-formula** is one of the form $(Q_1 x_1 \in P)\cdots(Q_n x_n \in P)\varphi$ for some quantifiers Q_i and some $\varphi \in L$. If M is an L-structure and $A \subseteq M$ we say A is **bounded in** M if every L_P-formula is $\mathrm{Th}(M,A)$-equivalent to a bounded one.

Definition 6.4: Suppose M is a structure and $A \subseteq M$. The **induced structure** A_M of M on A has domain A and atomic relations $D \cap A^n$ for each $\emptyset$-definable $D \subseteq M^n$.

FACT 6.5 ([6, Corollary 2.5]): *Suppose M is a structure and $A \subseteq M$ is bounded in M. If $\mathrm{Th}(M)$ and $\mathrm{Th}(A_M)$ are NIP then so is $\mathrm{Th}(M,A)$.*

We wish to apply this to $(M,A) = (\mathcal{N}, d^{\mathbb{N}})$. Boundedness follows from earlier work of Point:

PROPOSITION 6.6: *$d^{\mathbb{N}}$ is bounded in $\mathcal{N}$.*

Proof. Propositions 9 and 11 of [18] state that

$$\mathrm{Th}\left(\mathbb{N},+,\dot{-},<,0,1,\frac{\cdot}{n},\lambda_d,S,S^{-1}\right)_{n\geq 1}$$

admits quantifier elimination, where

- $a \dot{-} b = \max\{a-b,0\}$,
- $S(d^n) = d^{n+1}$ and $S(a) = a$ for other a,
- $S^{-1}(d^{n+1}) = d^n$ and $S^{-1}(a) = a$ for other a,
- $\lambda_d(x) = d^{\lfloor \log_d(x) \rfloor}$ for $x > 0$ and $\lambda_d(0) = 0$.

It then remains to show that any quantifier-free formula in this signature is equivalent to a bounded L_P-formula. But a quantifier-free formula

$$\varphi(\ldots,\lambda_d(t),\ldots)$$

involving λ_d is equivalent to

$$(\exists x \in d^{\mathbb{N}})\ ((x \leq t) \wedge (\forall y \in d^{\mathbb{N}})\neg(x < y \leq t) \wedge \varphi(\ldots,x,\ldots))$$
$$\vee\ ((t = 0) \wedge \varphi(\ldots,0,\ldots))$$

So at the cost of quantifying over $d^{\mathbb{N}}$ we can eliminate occurrences of λ_d; we can similarly dispense with occurrences of S and S^{-1}. Repeatedly applying this yields that any quantifier-free formula φ in $\{+, \dot{-}, <, 0, 1, \frac{\cdot}{n}, \lambda_d, S, S^{-1}\}_{n\geq 1}$ is equivalent to one of the form $(Q_1 x_1 \in d^{\mathbb{N}}) \cdots (Q_n x_n \in d^{\mathbb{N}})\psi$ where ψ is a formula in $\{+, \dot{-}, <, 0, 1, \frac{\cdot}{n}\}_{n\geq 1}$. But since $(\mathbb{N}, +, \dot{-}, <, 0, 1, \frac{\cdot}{n})_{n\geq 1}$ is a definitional expansion of $(\mathbb{N}, +)$, we get that φ is equivalent to a bounded L_P-formula. ∎

It is well-known that $\mathcal{N}$ is NIP; it is definable in $(\mathbb{Z}, +, <)$, which is NIP as all ordered abelian groups are (see [11]). It remains to show that the induced structure $(d^{\mathbb{N}})_{\mathcal{N}}$ is NIP.

The following is well-known; see, e.g., [5, Theorem 6.1], of which it is a weakening.

Fact 6.7: *All definable subsets of $\mathcal{N}$ are d-automatic.*

We therefore wish for a description of how d-automatic sets can intersect $d^{\mathbb{N}}$.

Proposition 6.8: *If $X \subseteq \mathbb{N}^n$ is d-automatic then the relation*

$$\{(k_1, \ldots, k_n) \in \mathbb{N}^n : (d^{k_1}, \ldots, d^{k_n}) \in X\}$$

is definable in $(\mathbb{N}, +)$.

Proof. By symmetry and disjunction it suffices to check the case $k_1 \leq \cdots \leq k_n$.

It will be convenient to work with $(X \cap \mathbb{N}^n_{>0}) - \mathbb{1}$, where $\mathbb{1} \in \mathbb{N}^n$ is the tuple all of whose entries are 1; note that this is also d-automatic. Then taking

$$\sigma_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d-1 \\ \vdots \\ d-1 \end{pmatrix}$$

with $i-1$ zeroes, we get for $k_1 \leq \cdots \leq k_n$ that

$$\begin{aligned}(d^{k_1}, \ldots, d^{k_n}) \in X &\iff (d^{k_1} - 1, \ldots, d^{k_n} - 1) \in (X \cap \mathbb{N}^n_{>0}) - \mathbb{1} \\ &\iff [\sigma_1^{k_1} \sigma_2^{k_2 - k_1} \cdots \sigma_n^{k_n - k_{n-1}}] \in (X \cap \mathbb{N}^n_{>0}) - \mathbb{1}\end{aligned}$$

(since the base-d representation of $d^{k_i} - 1$ consists of $d-1$ repeated k_i times). But by Proposition 2.2 the last condition is definable in $(\mathbb{N}, +)$, as desired. ∎

Our theorem now follows easily:

Proof of Theorem 6.1. Proposition 6.8 and Fact 6.7 imply that the map $k \mapsto d^k$ induces an interpretation of $(d^{\mathbb{N}})_{\mathcal{N}}$ in $(\mathbb{N},+)$. But $\mathrm{Th}(\mathbb{N},+)$ is NIP; so $\mathrm{Th}(d^{\mathbb{N}})_{\mathcal{N}}$ is NIP. But $d^{\mathbb{N}}$ is bounded in $\mathcal{N}$ by Proposition 6.6, and $\mathcal{N}$ is NIP. So $\mathrm{Th}(\mathcal{N}, d^{\mathbb{N}})$ is NIP by Fact 6.5. ∎

6.2. $(\mathbb{Z},+,d^{\mathbb{N}},\times\restriction d^{\mathbb{N}})$ IS NIP. Next we consider the expansion of $(\mathbb{Z},+)$ by the monoid $(d^{\mathbb{N}},\times)$; that is, we consider the structure $(\mathbb{Z},+,d^{\mathbb{N}},\times\restriction d^{\mathbb{N}})$, viewing $\times\restriction d^{\mathbb{N}}$ as a ternary relation on $\mathbb{Z}$. Note that as the ordering on $d^{\mathbb{N}}$ is definable here, $(\mathbb{Z},+,d^{\mathbb{N}},\times\restriction d^{\mathbb{N}})$ is not stable. However:

THEOREM 6.9: *$\mathrm{Th}(\mathbb{Z},+,d^{\mathbb{N}},\times\restriction d^{\mathbb{N}})$ is NIP.*

We first note that this is not a consequence of Theorem 6.1; that is, $(\mathbb{Z},+,d^{\mathbb{N}},\times\restriction d^{\mathbb{N}})$ is not a reduct of $(\mathbb{Z},+,<,d^{\mathbb{N}})$. Indeed, we remarked above that [5, Theorem 6.1] implies that every definable subset of $\mathcal{N}$ is d-automatic; in fact, it can be used to show that every definable subset of $(\mathbb{Z},+,<,d^{\mathbb{N}})$ is d-automatic. But $\times\restriction d^{\mathbb{N}}$ itself is not d-automatic: since

$$\left[\begin{pmatrix} 0^i1 \\ 0^i1 \\ 0^{i+1} \end{pmatrix} \cdot \begin{pmatrix} 0^{j+1} \\ 0^{j+1} \\ 0^j1 \end{pmatrix}\right] \in \times\restriction d^{\mathbb{N}} \iff i = j+1$$

it follows from the Myhill-Nerode theorem (see, e.g., [21, Theorem 4.7]) that the set of canonical representations of elements of $\times\restriction d^{\mathbb{N}}$ is not regular. So $\times\restriction d^{\mathbb{N}}$ is not definable in $(\mathbb{Z},+,<,d^{\mathbb{N}})$.

It is perhaps surprising that our methods are useful even though we are expanding by a set that is not automatic. The reason automatic methods still apply is Fact 6.7, together with the following generalization of Proposition 6.8, which tells us that the interaction between iterated concatenation and membership in automatic sets can be described using Presburger arithmetic.

LEMMA 6.10: *Suppose $X \subseteq \mathbb{Z}^m$ is d-automatic and*

$$(\ell_{11},\ldots,\ell_{1n_1}),\ldots,(\ell_{m1},\ldots,\ell_{mn_m})$$

are tuples from $(\Sigma_\pm)^{<\omega}$. Then the relation

$$\left\{(k_{ij}) : \begin{pmatrix} [\ell_{11}^{k_{11}} \cdots \ell_{1n_1}^{k_{1n_1}}] \\ \vdots \\ [\ell_{m1}^{k_{m1}} \cdots \ell_{mn_m}^{k_{mn_m}}] \end{pmatrix} \in X\right\} \subseteq \mathbb{N}^{n_1}\times\cdots\times\mathbb{N}^{n_m}$$

is definable in $(\mathbb{N},+)$.

Proof. We define a map $P\colon (\Sigma_\pm^*)^m \to (\Sigma_\pm^m)^*$ as follows. Take as input

$$\sigma_1, \dots, \sigma_m \in \Sigma_\pm^*,$$

and let $N = \max\{|\sigma_1|, \dots, |\sigma_m|\}$. Then

$$\begin{pmatrix} \sigma_1 0^{N-|\sigma_1|} \\ \vdots \\ \sigma_m 0^{N-|\sigma_m|} \end{pmatrix}$$

is a tuple of strings all of which have length N. We can view such a tuple as a string over $\Sigma_\pm^m$: if we write

$$\begin{pmatrix} \sigma_1 0^{N-|\sigma_1|} \\ \vdots \\ \sigma_m 0^{N-|\sigma_m|} \end{pmatrix} = \begin{pmatrix} \ell'_{11} \cdots \ell'_{1N} \\ \vdots \\ \ell'_{m1} \cdots \ell'_{mN} \end{pmatrix}$$

for $\ell'_{ij} \in \Sigma_\pm$ then

$$\begin{pmatrix} \ell'_{11} \\ \vdots \\ \ell'_{m1} \end{pmatrix} \cdots \begin{pmatrix} \ell'_{1N} \\ \vdots \\ \ell'_{mN} \end{pmatrix} \in (\Sigma_\pm^m)^*.$$

We then define

$$P \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_m \end{pmatrix} = \begin{pmatrix} \ell'_{11} \\ \vdots \\ \ell'_{m1} \end{pmatrix} \cdots \begin{pmatrix} \ell'_{1N} \\ \vdots \\ \ell'_{mN} \end{pmatrix};$$

so, roughly speaking, P pads its input with zeroes and views the result as a string over $\Sigma_\pm^m$. Note in particular that

$$\left[P \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_m \end{pmatrix} \right] = \begin{pmatrix} [\sigma_1] \\ \vdots \\ [\sigma_m] \end{pmatrix}.$$

I claim that for any ℓ_{ij}, any automaton (Q, q_0, δ, F), and any $q_1, q_2 \in Q$ the relation

$$\left\{ (k_{ij}) : \delta \left(q_1, P \begin{pmatrix} \ell_{11}^{k_{11}} \cdots \ell_{1n_1}^{k_{1n_1}} \\ \vdots \\ \ell_{m1}^{k_{m1}} \cdots \ell_{mn_m}^{k_{mn_m}} \end{pmatrix} \right) = q_2 \right\}$$

is definable in $(\mathbb{N},+)$. In fact this suffices to prove the lemma. Indeed, fix an automaton (Q,q_0,δ,F) for the set of representations over $\Sigma_\pm^m$ of elements of X. Then

$$\begin{pmatrix} [\ell_{11}^{k_{11}}\cdots\ell_{1n_1}^{k_{1n_1}}] \\ \vdots \\ [\ell_{m1}^{k_{m1}}\cdots\ell_{mn_m}^{k_{mn_m}}] \end{pmatrix} \in X \iff \left[P\begin{pmatrix} \ell_{11}^{k_{11}}\cdots\ell_{1n_1}^{k_{1n_1}} \\ \vdots \\ \ell_{m1}^{k_{m1}}\cdots\ell_{mn_m}^{k_{mn_m}} \end{pmatrix}\right] \in X$$

$$\iff \bigvee_{q\in F} \delta\left(q_0, P\begin{pmatrix} \ell_{11}^{k_{11}}\cdots\ell_{1n_1}^{k_{1n_1}} \\ \vdots \\ \ell_{m1}^{k_{m1}}\cdots\ell_{mn_m}^{k_{mn_m}} \end{pmatrix}\right) = q$$

is definable in $(\mathbb{N},+)$ by the claim.

We now show the claim. We apply strong induction on $(m,n_1,\ldots,n_m)$; our ordering will be given by $(m',n'_1,\ldots,n'_{m'}) \prec (m,n_1,\ldots,n_m)$ if $m'<m$ or $m'=m$ and $(n'_1,\ldots,n'_m)$ precedes $(n_1,\ldots,n_m)$ in the lexicographical order. The base case is $m=0$; this is vacuous. For the induction step, suppose that for any $m',n'_1,\ldots,n'_m$ with $(m',n'_1,\ldots,n'_{m'}) \prec (m,n_1,\ldots,n_m)$ we have for any $(\ell_{11},\ldots,\ell_{1n'_1}),\ldots,(\ell_{m'1},\ldots,\ell_{m'n'_{m'}}) \in (\Sigma_\pm)^{<\omega}$, any automaton (Q,q_0,δ,F), and any $q_1,q_2\in Q$ that the relation

$$\left\{(k_{ij}) : \delta\left(q_1, P\begin{pmatrix} \ell_{11}^{k_{11}}\cdots\ell_{1n'_1}^{k_{1n'_1}} \\ \vdots \\ \ell_{m'1}^{k_{m'1}}\cdots\ell_{m'n'_{m'}}^{k_{m'n'_{m'}}} \end{pmatrix}\right) = q_2\right\}$$

is definable in $(\mathbb{N},+)$. We show the same holds of

$$(m,n_1,\ldots,n_m).$$

Fix $(\ell_{11},\ldots,\ell_{1n_1}),\ldots,(\ell_{m1},\ldots,\ell_{mn_m}) \in (\Sigma_\pm)^{<\omega}$, an automaton (Q,q_0,δ,F), and states $q_1,q_2\in Q$.

Suppose first that some $n_i=0$; say for ease of notation that $i=1$. We define a new automaton (Q,q_0,δ',F) over $\Sigma_\pm^{m-1}$ by setting

$$\delta'\left(q, \begin{pmatrix} \ell'_2 \\ \vdots \\ \ell'_m \end{pmatrix}\right) = \delta\left(q, \begin{pmatrix} 0 \\ \ell'_2 \\ \vdots \\ \ell'_m \end{pmatrix}\right)$$

for any $\ell'_2, \ldots, \ell'_m \in \Sigma_{\pm}^{m-1}$. So given an input $\sigma \in (\Sigma_{\pm}^{m-1})^*$ we get that $\delta'(q_1, \sigma) = \delta(q_1, \left(\begin{smallmatrix} 0^{|\sigma|} \\ \sigma \end{smallmatrix}\right))$ (here $0^{|\sigma|} \in \Sigma_{\pm}^*$ is a single string consisting of $|\sigma|$-many 0s). In particular we have

$$\delta\left(q_1, P\begin{pmatrix} \varepsilon \\ \ell_{21}^{k_{21}} \cdots \ell_{2n_2}^{k_{2n_2}} \\ \vdots \\ \ell_{m1}^{k_{m1}} \cdots \ell_{mn_m}^{k_{mn_m}} \end{pmatrix}\right) = q_2 \iff \delta\left(q_1, \begin{pmatrix} 0^{\max_i(k_{i1}+\cdots+k_{in_i})} \\ P\begin{pmatrix} \ell_{21}^{k_{21}} \cdots \ell_{2n_2}^{k_{2n_2}} \\ \vdots \\ \ell_{m1}^{k_{m1}} \cdots \ell_{mn_m}^{k_{mn_m}} \end{pmatrix} \end{pmatrix}\right) = q_2$$

$$\iff \delta'\left(q_1, P\begin{pmatrix} \ell_{21}^{k_{21}} \cdots \ell_{2n_2}^{k_{2n_2}} \\ \vdots \\ \ell_{m1}^{k_{m1}} \cdots \ell_{mn_m}^{k_{mn_m}} \end{pmatrix}\right) = q_2$$

and by the induction hypothesis the latter is definable in $(\mathbb{N}, +)$ (since $(m-1, n_2, \ldots, n_m) \prec (m, 0, n_2, \ldots, n_m)$).

Suppose then that no $n_i = 0$. Suppose k_{11} is minimum among the k_{i1}. Then for $q \in Q$ we get that

$$\delta\left(q, P\begin{pmatrix} \ell_{12}^{k_{12}} \cdots \ell_{1n_1}^{k_{1n_1}} \\ \ell_{21}^{k_{21}-k_{11}} \ell_{22}^{k_{22}} \cdots \ell_{2n_2}^{k_{2n_2}} \\ \vdots \\ \ell_{m1}^{k_{m1}-k_{11}} \ell_{m2}^{k_{m2}} \cdots \ell_{mn_m}^{k_{mn_m}} \end{pmatrix}\right) = q_2$$

is definable in $(\mathbb{N}, +)$ by the induction hypothesis (since we have that $(m, n_1 - 1, n_2, \ldots, n_m) \prec (m, n_1, \ldots, n_m)$). Moreover since Q is finite we get that

$$\delta\left(q_1, \begin{pmatrix} \ell_{11}^{k_{11}} \\ \vdots \\ \ell_{m1}^{k_{11}} \end{pmatrix}\right)$$

is ultimately periodic in k_{11}; hence for $q \in Q$ we get that

$$q = \delta\left(q_1, \begin{pmatrix} \ell_{11}^{k_{11}} \\ \vdots \\ \ell_{m1}^{k_{11}} \end{pmatrix}\right)$$

is definable in $(\mathbb{N},+)$. Thus

$$\delta\left(q_1, P\begin{pmatrix} \ell_{11}^{k_{11}}\cdots\ell_{1n_1}^{k_{1n_1}} \\ \vdots \\ \ell_{m1}^{k_{m1}}\cdots\ell_{mn_m}^{k_{mn_m}}\end{pmatrix}\right) = q_2$$

$$\iff \bigvee_{q\in Q}\left(q=\delta\left(q_1, \begin{pmatrix}\ell_{11}^{k_{11}}\\ \vdots \\ \ell_{m1}^{k_{11}}\end{pmatrix}\right)\wedge\delta\left(q, P\begin{pmatrix} \ell_{12}^{k_{12}}\cdots\ell_{1n_1}^{k_{1n_1}} \\ \ell_{21}^{k_{21}-k_{11}}\ell_{22}^{k_{22}}\cdots\ell_{2n_2}^{k_{2n_2}} \\ \vdots \\ \ell_{m1}^{k_{m1}-k_{11}}\ell_{m2}^{k_{m2}}\cdots\ell_{mn_m}^{k_{mn_m}}\end{pmatrix}\right)=q_2\right)$$

is definable in $(\mathbb{N},+)$.

Similarly we get definability in the case k_{i1} is minimum for some $i > 1$. So taking disjunctions we get that the relation

$$\left\{(k_{ij}) : \delta\left(q_1, P\begin{pmatrix} \ell_{11}^{k_{11}}\cdots\ell_{1n_1}^{k_{1n_1}} \\ \vdots \\ \ell_{m1}^{k_{m1}}\cdots\ell_{mn_m}^{k_{mn_m}}\end{pmatrix}\right) = q_2\right\}$$

is definable in $(\mathbb{N},+)$, as desired. ∎

For our proof of Theorem 6.9 it will be convenient to assume $d \geq 8$. In fact this suffices: consider for example the case $d = 4$. Assuming the theorem holds when $d = 4^2 = 16$, we get that $(\mathbb{Z},+,16^{\mathbb{N}},\times\restriction 16^{\mathbb{N}})$ is NIP. But $\times\restriction 4^{\mathbb{N}}$ is definable in $(\mathbb{Z},+,16^{\mathbb{N}},\times\restriction 16^{\mathbb{N}})$: we have $(a,b,c)\in\times\restriction 4^{\mathbb{N}}$ if and only if

$$(4^i a, 4^j b, 4^{i+j}c)\in\times\restriction 16^{\mathbb{N}} \quad \text{for some } i,j\in\{0,1\}.$$

This is because x is a power of 4 if and only if one of $x, 4x$ is a power of 16. So $(\mathbb{Z},+,4^{\mathbb{N}},\times\restriction 4^{\mathbb{N}})$ is a reduct of $(\mathbb{Z},+,16^{\mathbb{N}},\times\restriction 16^{\mathbb{N}})$, and is thus NIP. Similar arguments work for all $2 \leq d < 8$.

Proof of Theorem 6.9. We assume $d \geq 8$. We will apply an extension due to Conant and Laskowski of the result of Chernikov and Simon we used previously (Fact 6.5). Since these results only apply to subsets of the domain, our first task is to encode $d^{\mathbb{N}}$ and $\times\restriction d^{\mathbb{N}}$ as such. Let

$$B = d^{\mathbb{N}} \cup \{[7^i 6^j 4^i] : i,j\in\mathbb{N}\}.$$

The point is that from B we will be able to extract both $d^{\mathbb{N}}$ and

$$\left\{\frac{a-1}{d-1} + 2\frac{b-1}{d-1} + 4\frac{c-1}{d-1} : (a,b,c)\in\times\restriction d^{\mathbb{N}}, a\leq b\right\}.$$

These together will be enough to recover $\times\restriction d^{\mathbb{N}}$.

Claim 6.11: *$d^{\mathbb{N}}$ and $\times{\restriction}d^{\mathbb{N}}$ are definable in $(\mathbb{Z},+,B)$.*

Proof. Note first that $d^{\mathbb{N}}$ is definable in $(\mathbb{Z},+,B)$: we have $a \in d^{\mathbb{N}}$ if and only if $a = 1$ or $0 \neq a \in B$ and $a \equiv 0 \pmod{d}$. I now claim that $(a,b,c) \in \times{\restriction}d^{\mathbb{N}}$ with $a \leq b$ if and only if $a,b,c \in d^{\mathbb{N}}$ and

$$\frac{a-1}{d-1} + 2\frac{b-1}{d-1} + 4\frac{c-1}{d-1} \in B.$$

For the left-to-right direction, note that if $(d^i, d^j, d^{i+j}) \in \times{\restriction}d^{\mathbb{N}}$ with $i \leq j$ then

$$\frac{d^i-1}{d-1} + 2\frac{d^j-1}{d-1} + 4\frac{d^{i+j}-1}{d-1} = [1^i] + [2^j] + [4^{i+j}] = [7^i 6^{j-i} 4^i] \in B.$$

For the right-to-left direction, suppose d^i, d^j, d^k satisfy

$$[1^i] + [2^j] + [4^k] = \frac{d^i-1}{d-1} + 2\frac{d^j-1}{d-1} + 4\frac{d^k-1}{d-1} \in B.$$

If $i = j = k = 0$ then $(d^i, d^j, d^k) \in \times{\restriction}d^{\mathbb{N}}$ and $d^i \leq d^j$, as desired; suppose then that at least one is non-zero. Then $[1^i] + [2^j] + [4^k] \not\equiv 0 \pmod{d}$, so $[1^i]+[2^j]+[4^k] \in B \setminus d^{\mathbb{N}}$, and is thus equal to $[7^{i'} 6^{j'} 4^{i'}] = [1^{i'}]+[2^{j'+i'}]+[4^{2i'+j'}]$ for some i', j'.

But the map $(x,y,z) \mapsto [1^x]+[2^y]+[4^z]$ is injective. Indeed, we can represent $[1^x]+[2^y]+[4^z]$ by an element of $\{1,\dots,7\}^*$; note that each element of $\{1,\dots,7\}$ can be represented uniquely as a sum of a subset of $\{1,2,4\}$. We can then recover x from the canonical representation of $[1^x]+[2^y]+[4^z]$ as the number of occurrences of $\ell \in \{1,\dots 7\}$ that use a 1 in this sum representation; we can likewise recover y,z.

So since $[1^i]+[2^j]+[4^k] = [1^{i'}]+[2^{j'+i'}]+[4^{2i'+j'}]$ we get by injectivity that $j = j'+i' \geq i' = i$ and $k = 2i'+j' = i+j$; so $(d^i, d^j, d^k) \in \times{\restriction}d^{\mathbb{N}}$ and $d^j \geq d^i$, as desired.

But $(a,b,c) \in \times{\restriction}d^{\mathbb{N}} \iff (b,a,c) \in \times{\restriction}d^{\mathbb{N}}$; so

$$(x \leq y \wedge (x,y,z) \in \times{\restriction}d^{\mathbb{N}}) \vee (y \leq x \wedge (y,x,z) \in \times{\restriction}d^{\mathbb{N}})$$

defines $\times{\restriction}d^{\mathbb{N}}$ in $(\mathbb{Z},+,B)$. ∎

So it suffices to show that $(\mathbb{Z},+,B)$ is NIP. We again check that the induced structure on B is NIP. When using Fact 6.5, we only concerned ourselves with the structure induced from the $\emptyset$-definable sets; however, to use the result of Conant and Laskowski, we will need that the structure induced by all sets definable with parameters from $\mathbb{Z}$ is NIP.

Claim 6.12: *Let $\mathcal{Z}$ be $(\mathbb{Z},+)$ expanded by names for all the constants. Then the induced structure $B_{\mathcal{Z}}$ is NIP.*

Proof. Let

$$D=\{(e_1,1,0,0):e_1\in\mathbb{N}\}\cup\{(0,0,e_3,e_4):e_3,e_4\in\mathbb{N}\}\subseteq\mathbb{N}^4;$$

note that D is definable in $(\mathbb{N},+)$. Consider $\Phi\colon\mathbb{N}^4\to\mathbb{Z}$ given by

$$(e_1,e_2,e_3,e_4)\mapsto[0^{e_1}1^{e_2}7^{e_3}6^{e_4}4^{e_3}];$$

note that $\Phi(D)\subseteq B$, and in fact $\Phi\colon D\to B$ is bijective. I claim that Φ defines an interpretation of $B_{\mathcal{Z}}$ in $(\mathbb{N},+)$. Recall that $(\mathbb{Z},+,0,1,\delta\mathbb{N})_{\delta>0}$ admits quantifier elimination (see, e.g., [15, Exercise 3.4.6]). So if $X\subseteq\mathbb{Z}$ is definable in $\mathcal{Z}$ then X is a Boolean combination of congruences and equalities, and hence $X\cap\mathbb{N}$ is definable in $(\mathbb{N},+)$; likewise with $-X\cap\mathbb{N}$. Thus by Fact 6.7 we get that $X\cap\mathbb{N}$ and $-X\cap\mathbb{N}$ (and hence $X\cap-\mathbb{N}$) are d-automatic. So since d-automatic sets are closed under Boolean combinations we get that X is d-automatic. One argues similarly that if $X\subseteq\mathbb{Z}^m$ is definable in $\mathcal{Z}$ then X is d-automatic. So to show that Φ defines an interpretation it suffices to show that whenever $X\subseteq\mathbb{Z}^m$ is d-automatic we have that

$$\left\{(e_{ij})\in D^m:\begin{pmatrix}[0^{e_{11}}1^{e_{12}}7^{e_{13}}6^{e_{14}}4^{e_{13}}]\\ \vdots\\ [0^{e_{m1}}1^{e_{m2}}7^{e_{m3}}6^{e_{m4}}4^{e_{m3}}]\end{pmatrix}\in X\right\}$$

is definable in $(\mathbb{N},+)$. But this follows from Lemma 6.10 (and definability of D). So Φ defines an interpretation of $B_{\mathcal{Z}}$ in $(\mathbb{N},+)$; so $B_{\mathcal{Z}}$ is NIP. ∎

Now by [8, Theorem 2.9] we get since $\mathrm{Th}(\mathbb{Z},+)$ is weakly minimal (see, e.g., [8, Proposition 3.1]) and $B_{\mathcal{Z}}$ is NIP that $(\mathbb{Z},+,B)$ is NIP. So $(\mathbb{Z},+,d^{\mathbb{N}},\times\restriction d^{\mathbb{N}})$ is NIP. ∎

Despite the similarity of methods in Theorems 6.1 and 6.9, we do not know whether $\mathrm{Th}(\mathbb{Z},+,<,d^{\mathbb{N}},\times\restriction d^{\mathbb{N}})$ is NIP. (Indeed, it is not even clear to us whether $(\mathbb{Z},+,<,d^{\mathbb{N}},\times\restriction d^{\mathbb{N}})$ is a definitional expansion of $(\mathbb{Z},+,d^{\mathbb{N}},\times\restriction d^{\mathbb{N}})$, though it seems unlikely.) One might hope to apply Fact 6.5 with $(\mathbb{Z},+,<)$ as the base NIP structure and B as the new predicate. Indeed, as in the proof of Claim 6.12 one can show that the induced structure on B is NIP by observing that the definable subsets of $(\mathbb{Z},+,<)$ are d-automatic. Checking boundedness, however, is not simply a matter of adapting the arguments of Theorem 6.1 as the quantifier elimination result of Point that applied to $d^{\mathbb{N}}$ does not seem to apply to B. Nor does the result of Conant and Laskowski yield boundedness as $(\mathbb{Z},+,<)$ is not weakly minimal. So if one wishes to use our approach to show that $\mathrm{Th}(\mathbb{Z},+,<,d^{\mathbb{N}},\times\restriction d^{\mathbb{N}})$ is NIP one needs a new way to check boundedness.

One can restate Theorem 6.9 as saying that expanding $(\mathbb{Z},+)$ by a singly generated submonoid of $(\mathbb{Z}\setminus\{0\},\times)$ yields an NIP structure. It would be natural to ask about finitely generated submonoids in general, but it seems unlikely that our automata-theoretic methods will apply as there is no obvious choice of d in general.

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