POLYHARMONIC HYPERSURFACES INTO SPACE FORMS

BY

STEFANO MONTALDO*

Dipartimento di Matematica e Informatica, Università degli Studi di Cagliari
Via Ospedale 72, 09124 Cagliari, Italia
e-mail: montaldo@unica.it

AND

CEZAR ONICIUC**

Faculty of Mathematics, "Alexandru Iioan Cuza" University of Iasi
Bd. Carol I no. 11, 700506 Iasi, Romania
e-mail: oniciucc@uaic.ro

AND

ANDREA RATTO*

Dipartimento di Matematica e Informatica, Università degli Studi di Cagliari
Via Ospedale 72, 09124 Cagliari, Italia
e-mail: rattoa@unica.it

ABSTRACT

In this paper we shall consider polyharmonic hypersurfaces of order r (briefly, r-harmonic hypersurfaces), where $r \geq 3$ is an integer, into a space form $N^{m+1}(c)$ of curvature c. For this class of hypersurfaces we shall prove that, if $c \leq 0$, then any r-harmonic hypersurface must be minimal provided that the mean curvature function and the squared norm of the shape operator are constant. When the ambient space is $\mathbb{S}^{m+1}$, we shall obtain the geometric condition which characterizes the r-harmonic hypersurfaces with constant mean curvature and constant squared norm of the shape operator, and we shall establish the bounds for these two constants. In particular, we shall prove the existence of several new examples of proper r-harmonic isoparametric hypersurfaces in $\mathbb{S}^{m+1}$ for suitable values of m and r. Finally, we shall show that all these r-harmonic hypersurfaces are also ES-r-harmonic, i.e., critical points of the Eells–Sampson r-energy functional.

1. Introduction and statement of the results

Let us consider a submanifold of the Euclidean space described by the isometric immersion $\varphi : M^m \hookrightarrow \mathbb{R}^n$. We say that M^m is **polyharmonic of order r** (briefly, r-harmonic) if

$$(1.1) \qquad \Delta^r \varphi = (\Delta^r \varphi^1, \dots, \Delta^r \varphi^n) = 0,$$

where $r \geq 1$ is an integer and Δ denotes the Laplace operator associated to the induced metric on M^m. Now, let $\mathbf{H}$ be the mean curvature vector field of M^m. Since

$$\mathbf{H} = -\frac{1}{m} \Delta \varphi,$$

equation (1.1) is equivalent to

$$\Delta^{r-1} \mathbf{H} = 0.$$

In particular, minimal submanifolds are trivially r-harmonic for all $r \geq 1$. The case $r = 2$ corresponds to the so-called biharmonic submanifolds and it is the most studied in the literature (for instance, see [11, 21, 38, 39]). In particular, the Chen conjecture states that any biharmonic submanifold of $\mathbb{R}^n$ is minimal. Although there are some results which prove that the conjecture holds under suitable geometric restrictions (see [11, 12, 19, 30, 31]), the general case is still open. In a similar spirit, the Maeta conjecture (see [22, 25, 36]) states that any r-harmonic submanifold of $\mathbb{R}^n$ is minimal. Also in this case some partial results

are available. For instance, the Maeta conjecture is true for curves (see [22]), but again the general case is open.

For the purposes of this paper, it is necessary to extend these notions to the case that the ambient space is a general manifold N. To this end, the starting point is the notion of a harmonic map. **Harmonic maps** are the critical points of the **energy functional**

$$(1.2) \qquad E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 \, dV,$$

where $\varphi : M \to N$ is a smooth map between two Riemannian manifolds (M, g) and (N, h). In particular, φ is harmonic if it is a solution of the Euler–Lagrange system of equations associated to (1.2), i.e.,

$$(1.3) \qquad -d^* d\varphi = \mathrm{trace} \nabla d\varphi = 0.$$

The left member of (1.3) is a vector field along the map φ or, equivalently, a section of the pull-back bundle $\varphi^{-1} TN$: it is called **tension field** and denoted $\tau(\varphi)$. In addition, we recall that, if φ is an **isometric immersion**, then φ is a harmonic map if and only if the immersion φ defines a minimal submanifold of N (see [16, 17] for background). Let us denote ∇^M, ∇^N and ∇^φ the induced connections on the bundles TM, TN and $\varphi^{-1} TN$ respectively. The **rough Laplacian** on sections of $\varphi^{-1} TN$, denoted $\overline{\Delta}$, is defined by

$$(1.4) \qquad \overline{\Delta} = d^* d = -\sum_{i=1}^m (\nabla^\varphi_{e_i} \nabla^\varphi_{e_i} - \nabla^\varphi_{\nabla^M_{e_i} e_i}),$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field tangent to M.

Now, in order to define the notion of an r-harmonic map, we consider the following family of functionals, which represent a version of order r of the classical energy (1.2). If $r = 2s$, $s \geq 1$:

$$(1.5) \qquad \begin{aligned} E_{2s}(\varphi) &= \frac{1}{2} \int_M \langle \underbrace{(d^* d) \cdots (d^* d)}_{s \text{ times}} \varphi, \underbrace{(d^* d) \cdots (d^* d)}_{s \text{ times}} \varphi \rangle_N \, dV \\ &= \frac{1}{2} \int_M \langle \overline{\Delta}^{s-1} \tau(\varphi), \overline{\Delta}^{s-1} \tau(\varphi) \rangle_N \, dV. \end{aligned}$$

In the case that $r = 2s + 1$, $s \geq 0$:

(1.6)
$$E_{2s+1}(\varphi) = \frac{1}{2} \int_M \langle d \underbrace{(d^*d) \cdots (d^*d)}_{s \text{ times}} \varphi, d \underbrace{(d^*d) \cdots (d^*d)}_{s \text{ times}} \varphi \rangle_N$$

$$= \frac{1}{2} \int_M \sum_{j=1}^m \langle \nabla^\varphi_{e_j} \overline{\Delta}^{s-1} \tau(\varphi), \nabla^\varphi_{e_j} \overline{\Delta}^{s-1} \tau(\varphi) \rangle_N \, dV.$$

We say that a map φ is r-**harmonic** if, for all variations φ_t,

$$\frac{d}{dt} E_r(\varphi_t)|_{t=0} = 0.$$

If $r = 1$, the functional (1.6) is just the energy. In the case that $r = 2$, the functional (1.5) is called **bienergy** and its critical points are the so-called **biharmonic maps**. At present, a very ample literature on biharmonic maps is available and, again, we refer to [11, 21, 38, 39] for an introduction to this topic. More generally, the r-**energy functionals** $E_r(\varphi)$ defined in (1.5), (1.6) have been intensively studied (see [5, 6, 22, 23, 24, 32, 36, 46, 47], for instance). In particular, the Euler–Lagrange equations for $E_r(\varphi)$ were obtained by Wang [46] and Maeta [22] (see Equations (2.3) and (2.4)). We point out that an inspection of the Euler–Lagrange equations for $E_r(\varphi)$ shows that a harmonic map is always r-harmonic for any $r \geq 2$. If the target manifold N is flat, then an r-harmonic map is also $(r+1)$-harmonic for any $r \geq 1$. Differently, when the target manifold is nonflat, a non-harmonic r-harmonic map is not automatically $(r + 1)$-harmonic (similarly, an r-harmonic submanifold, i.e., an r-harmonic isometric immersion, is not necessarily $(r + 1)$-harmonic). For instance, the hypersphere $\mathbb{S}^m(R)$ of radius R into $\mathbb{S}^{m+1}(1)$ is r-harmonic if and only if either $R = 1$ (the harmonic case) or $R = 1/\sqrt{r}$, $r \geq 2$ (see [32]). Thus, when the target manifold is nonflat, we call an r-harmonic map **proper** if it is not harmonic (similarly, an r-harmonic submanifold, i.e., an r-harmonic isometric immersion, is **proper** if it is not minimal). As a general fact, when the ambient space has nonpositive sectional curvature there are several results which assert that, under suitable conditions, an r-harmonic submanifold is minimal (see [11], [22], [25] and [36], for instance).

Things drastically change when the ambient space is positively curved. Let us denote by $\mathbb{S}^{m+1}$ the sphere $\mathbb{S}^{m+1}(1)$ of radius 1. Moreover, let us indicate by A the shape operator of M^m into $\mathbb{S}^{m+1}$ and by $\mathbf{H} = f\eta$ the mean curvature vector field, where η is the unit normal vector field and f is the mean curvature

function. Throughout the whole paper, when we write that M^m is a CMC hypersurface we mean that f is a constant which will be denoted by α.

The first fundamental result in the theory of biharmonic hypersurfaces of $\mathbb{S}^{m+1}$ can be stated as follows:

THEOREM 1.1 ([8, 9, 21]): *Let M^m be a non-minimal CMC hypersurface in $\mathbb{S}^{m+1}$. Then M^m is proper biharmonic if and only if*

$$|A|^2 = m.$$

In a similar spirit, in the triharmonic case Maeta obtained a result which implies:

THEOREM 1.2 ([24]): *Let M^m be a non-minimal hypersurface in $\mathbb{S}^{m+1}$ and assume that $\nabla A = 0$. Then M^m is proper triharmonic if and only if*

$$|A|^4 - m|A|^2 - m^2\alpha^2 = 0.$$

The geometric restrictions given in Theorems 1.1 and 1.2 are of great interest in the general theory of submanifolds of the sphere. For instance, we know that the small hypersphere $\mathbb{S}^m(1/\sqrt{2})$ and certain generalised Clifford tori verify the conditions of Theorem 1.1. Similarly, $\mathbb{S}^m(1/\sqrt{3})$ and some suitable generalised Clifford tori are triharmonic, but a complete classification of proper biharmonic (triharmonic) hypersurfaces in $\mathbb{S}^{m+1}$ is a challenging open problem, even in the compact CMC case. For the sake of completeness, we mention that, when $r \geq 4$, using completely different methods (equivariant differential geometry and variational principles), we recently proved that the small hypersphere $\mathbb{S}^m(1/\sqrt{r})$ and certain suitable generalised Clifford tori are r-harmonic (see [7, 32]).

We are now in the right position to state the main results of this paper. We shall write $N^{m+1}(c)$ to indicate an $(m+1)$-dimensional space form of curvature c ($m \geq 1, c \in \mathbb{R}$).

THEOREM 1.3: *Let M^2 be a CMC triharmonic surface in $N^3(c)$, $c \leq 0$. Then M^2 is minimal.*

THEOREM 1.4: *Let M^m be a CMC r-harmonic hypersurface in $N^{m+1}(c)$, where $r \geq 3$ and $c \leq 0$. If $|A|^2$ is constant, then M^m is minimal.*

Remark 1.5: The conclusion of Theorem 1.4 was obtained by Maeta (see [22]) in the case that $r = 3$ and M^m is compact.

Next, we focus on the case that the ambient space is the Euclidean sphere. Our results are:

THEOREM 1.6: *Let M^2 be a CMC proper triharmonic surface in $\mathbb{S}^3$. Then M^2 is an open part of the small hypersphere $\mathbb{S}^2(1/\sqrt{3})$.*

THEOREM 1.7: *Let M^m be a compact, CMC proper triharmonic hypersurface in $\mathbb{S}^{m+1}$. Then either $|A|^2 = 2m$ and $M^m = \mathbb{S}^m(1/\sqrt{3})$, or there exists a point $p \in M^m$ such that*

$$m < |A|^2(p) < 2m.$$

The following is an immediate consequence of Theorem 1.7:

COROLLARY 1.8: *Let M^m be a compact, CMC proper triharmonic hypersurface in $\mathbb{S}^{m+1}$. If $|A|^2 \geq 2m$, then $|A|^2 = 2m$ and $M^m = \mathbb{S}^m(1/\sqrt{3})$.*

THEOREM 1.9: *Let M^m be a CMC proper r-harmonic hypersurface in $\mathbb{S}^{m+1}$, $r \geq 3$, and assume that $|A|^2$ is constant. Then:*

  (i) *$|A|^2 \in (m, m(r-1)]$, and $|A|^2 = m(r-1)$ if and only if M^m is an open part of $\mathbb{S}^m(1/\sqrt{r})$.*
  (ii) *$\alpha^2 \in (0, r-1]$, and $\alpha^2 = r-1$ if and only if M^m is an open part of $\mathbb{S}^m(1/\sqrt{r})$.*

Theorem 1.9 describes the structure of a certain family of proper r-harmonic hypersurfaces of the Euclidean sphere. The fact that this family contains several significant examples will be illustrated by means of the following result:

THEOREM 1.10: *Let M^m be a non-minimal CMC hypersurface in $\mathbb{S}^{m+1}$ and assume that $|A|^2$ is constant. Then M^m is proper r-harmonic ($r \geq 3$) if and only if*

(1.7) $$|A|^4 - m|A|^2 - (r-2)m^2\alpha^2 = 0.$$

Remark 1.11: The special case $r = 3$ in Theorem 1.10 is a slight improvement of Theorem 1.2. We also observe that the conclusions of Theorems 1.4 and 1.10 remain true for $r = 2$ even without the hypothesis $|A|^2$ equal to a constant.

In order to illustrate the main applications of Theorem 1.10, it is now necessary to recall some basic facts about isoparametric hypersurfaces of the Euclidean sphere. These hypersurfaces have a long, beautiful history which started in 1918 with the work of Laura and continued with the contributions of several

authors, amongst which we cite E. Cartan, Abresch, Ozeki-Takeuchi, Münzner, Stolz, Chi, Miyaoka and, more recently, Siffert (we refer to [1, 3, 15, 28, 29, 34, 35, 37, 41, 42, 43, 44] and references therein). A smooth function $F : \mathbb{S}^{m+1} \to \mathbb{R}$ is called **isoparametric** if both $|\nabla F|^2$ and ΔF are functions of F. Isoparametric hypersurfaces are the regular level sets of isoparametric functions. Each isoparametric hypersurface is CMC and has ℓ distinct constant principal curvatures $k_1, \ldots, k_\ell$ with constant multiplicities $m_1, \ldots, m_\ell$, $m_1 + \cdots + m_\ell = m$. Moreover, the only possible values for ℓ are $1, 2, 3, 4, 6$, and $m_{i+2} = m_i$, so that there are at most two distinct multiplicities, which we shall denote m_1, m_2. The integer ℓ is called the **degree** of the isoparametric hypersurface. Any isoparametric function F has image $F(\mathbb{S}^{m+1}) = [-1, 1]$. In order to state our results, it is convenient to describe a family of parallel isoparametric hypersurfaces as follows. As in [3], we define $M_s = F^{-1}(\cos \ell s)$, where $0 < s < \pi/\ell$. Geometrically, s represents the affine parameter such that

$$\nabla F/|\nabla F| = \nabla s,$$

and so ∇s is normal to M_s. In the cases $\ell = 1, 2$ the isoparametric hypersurfaces M_s are small hyperspheres and generalised Clifford tori respectively. These cases have been thoroughly studied, by different methods, in [7, 24, 32, 33]. Therefore, we focus on the remaining cases $\ell = 3, 4, 6$. By a suitable choice of the orientation we can assume $m_1 \leq m_2$ and we have the following explicit description of the principal curvatures of M_s, $s \in (0, \pi/\ell)$:

$$(1.8) \qquad k_i(s) = \cot\left(s + \frac{(i-1)\pi}{\ell}\right), \quad i = 1, \ldots, \ell.$$

For a fixed s, the distributions on M_s corresponding to the k_i's are integrable and the leaves are totally umbilical in $\mathbb{S}^{m+1}$, i.e., they are small spheres of dimension m_i and radius $\sin(s + (i-1)\pi/\ell)$, $i = 1, \ldots, \ell$. Moreover, these leaves are totally geodesic in M_s.

The non-existence of proper biharmonic isoparametric hypersurfaces of degree $\ell = 3, 4, 6$ was proved in [20]. Our results will extend this to the cases $r = 3, 4$, but, on the other hand, the instances described in Example 1.16 will enable us to conclude that there exist proper r-harmonic isoparametric hypersurfaces of degree $\ell = 4$ for all $r \geq 5$. Finally, we point out that the analysis of [20] shows that, when $\ell = 3, 4$ or 6, $|A|^2 > m$. This implies that, in this context, a solution of (1.7) cannot be minimal.

CASE $\ell = 3$. There exist only four examples of this type, corresponding to $m_1 = m_2 = 1, 2, 4, 8$. Our result is:

THEOREM 1.12: *Let M_s, $0 < s < \pi/3$, be a family of parallel isoparametric hypersurfaces of degree 3 in $\mathbb{S}^{m+1}$. Then*

(i) *if $2 \leq r \leq 19$, the family M_s does not contain any proper r-harmonic hypersurface;*

(ii) *if $r \geq 20$, the family M_s contains four (two geometrically distinct) proper r-harmonic hypersurfaces.*

CASE $\ell = 6$. It was proved in [1] that in this case necessarily $m_1 = m_2$ and the possible values are $m_1 = 1$ or $m_1 = 2$. Moreover, associated to each of them there is a homogeneous example. Our result is:

THEOREM 1.13: *Let M_s, $0 < s < \pi/6$, be a family of parallel isoparametric hypersurfaces of degree 6 in $\mathbb{S}^{m+1}$. Then*

(i) *if $2 \leq r \leq 109$, the family M_s does not contain any proper r-harmonic hypersurface;*

(ii) *if $r \geq 110$, the family M_s contains four proper r-harmonic hypersurfaces.*

CASE $\ell = 4$. This is the richest case. The only examples with $m_1 = m_2$ occur when $m_1 = m_2 = 1$ or $m_1 = m_2 = 2$. But we have examples with $m_1 \neq m_2$ provided that $m_1 = 4$ and $m_2 = 5$, or $1 + m_1 + m_2$ is a multiple of $2^{\xi(m_1 - 1)}$, where $\xi(n)$ denotes the number of natural numbers p such that $1 \leq p \leq n$ and $p \equiv 0, 1, 2, 4 \pmod 8$. Let C_m be the skew-symmetric Clifford algebra over $\mathbb{R}$ generated by the canonical basis $\{e_1, \ldots, e_m\}$ of $\mathbb{R}^m$ subject to the only constraint

$$e_i e_j + e_j e_i = -2\delta_{ij} \mathrm{I},$$

where I is the identity element of C_m. A more explicit description of the pairs (m_1, m_2) which can occur is the following (see [15]):

$$(m_1, m_2) = (m, k\delta_m - m - 1),$$

where δ_m denotes the dimension of an irreducible module of C_m and the positive integers m, k are such that the second entry is positive. The full classification of the corresponding isoparametric hypersurfaces is now complete and we refer the reader to the recent work of Chi [14]. Finally, for future reference, we

point out that the following pairs of multiplicities occur: $m_1 = 1, m_2 \geq 2$; $m_1 = 2, m_2 = 2k, k \geq 2$; $m_1 = 3, m_2 = 4$; $m_1 = 4, m_2 = 4k - 5, k \geq 3$. Our results are:

THEOREM 1.14: *Let M_s, $0 < s < \pi/4$, be a family of parallel isoparametric hypersurfaces of degree 4 in $\mathbb{S}^{m+1}$ and assume that $m_1 = m_2$. Then*

(i) *if $2 \leq r \leq 41$, the family M_s does not contain any proper r-harmonic hypersurface;*

(ii) *if $r \geq 42$, the family M_s contains four proper r-harmonic hypersurfaces.*

THEOREM 1.15: *Let M_s, $0 < s < \pi/4$, be a family of parallel isoparametric hypersurfaces of degree 4 in $\mathbb{S}^{m+1}$ and assume that $m_1 < m_2$. Let $b = m_2/m_1$ and define*

$$
\begin{aligned}
(1.9) \quad P_{b,r}(y) =& y^4(b^2 r + 2br + r) + y^3(-3b^2 r - 4b^2 - 4br - r + 4) \\
&+ y^2(3b^2 r + 12b^2 + 2br - 2b) + y(-b^2 r - 12b^2 + 2b) + 4b^2.
\end{aligned}
$$

Then the family M_s contains a proper r-harmonic hypersurface if and only if the fourth order polynomial $P_{b,r}(y)$ admits a root $y^* \in (0,1)$, and to any such root corresponds a proper r-harmonic hypersurface M_s with $s = (1/2)\arccos(\sqrt{y^*})$.

Moreover, there exist two natural numbers $r^{**} \geq r^* \geq 5$ such that

(i) *if $2 \leq r < r^*$, the family M_s does not contain any proper r-harmonic hypersurface;*

(ii) *if $r \geq r^*$, the family M_s contains at least one proper r-harmonic hypersurface;*

(iii) *if $r \geq r^{**}$, the family M_s contains four proper r-harmonic hypersurfaces.*

Example 1.16: We observe that $P_{b,r}(0) = 4b^2$ and $P_{b,r}(1) = 4$. It follows that, in most examples, there is an even (possibly zero) number of roots in $(0,1)$. For instance, performing a numerical analysis with the software Mathematica, we were able to check that, when $m_1 = 1$ and $m_2 = 10000$, there are at least two solutions for all $r \geq r^* = 5$, and $r^{**} = 312919$. This shows that, in general, the estimate $r^* \geq 5$ in Theorem 1.15 is sharp. In subsection 3.1 we shall provide some upper bound for r^* and r^{**}.

1.1. THE DEFINITION OF EELLS AND SAMPSON. The notion of r-harmonicity which we have used in this paper represents, from both the geometric and the analytic point of view, a convenient approach to the study of higher order versions of the classical energy functional. On the other hand, we point out that

the first idea of studying higher order energies was formulated in a different way. More precisely, in 1965 Eells and Sampson (see [18]) proposed the following functionals which we denote $E_r^{ES}(\varphi)$ to commemorate these two outstanding mathematicians:

$$(1.10) \qquad E_r^{ES}(\varphi) = \frac{1}{2} \int_M |(d^* + d)^r (\varphi)|^2 dV.$$

Again, harmonic maps are trivially absolute minima for the functionals (1.10). Therefore, we say that φ is a **proper ES-r-harmonic map** if it is a not harmonic critical point of $E_r^{ES}(\varphi)$. The study of (1.10) was suggested again in [16], but so far not much is known about this subject. The functionals $E_r^{ES}(\varphi)$ and $E_r(\varphi)$ coincide when $r = 2, 3$ and the difference between them appears when we consider the case $r = 4$, where we have

$$E_4^{ES}(\varphi) = \frac{1}{2} \int_M |d^2(\tau(\varphi))|^2 \, dV + E_4(\varphi).$$

As observed in [16], $d^2(\tau(\varphi))$ is a 2-form with values in $\varphi^{-1}TN$ which can be described as follows:

$$(1.11) \qquad d^2(\tau(\varphi))(X, Y) = R^N(d\varphi(X), d\varphi(Y))\tau(\varphi).$$

In general, aside from the special cases where $\dim M = 1$ or the target N is flat, the curvature term $d^2(\tau(\varphi))$ does not vanish and gives rise to difficulties which increase as r does. In particular, by contrast to the case of $E_r(\varphi)$, the explicit derivation of the Euler–Lagrange equation for the Eells–Sampson functionals $E_r^{ES}(\varphi)$ seems, in general, a very complicated task. These problems are explained in detail in the recent paper [7], where the Euler–Lagrange equation of the functional $E_4^{ES}(\varphi)$ was computed.

Nevertheless, in the context of the present paper, we can establish a significant relationship between r-harmonicity and ES-r-harmonicity. That is achieved proving the following rather general result, where, in a standard way, $\overline{\Delta}^0$ denotes the identity operator:

THEOREM 1.17: *Let $\varphi : (M^m, g) \to N^n(c)$ be any smooth map from a Riemannian manifold (M^m, g) into a space form. Let $r \geq 4$ and assume that*

$$(1.12) \qquad \overline{\Delta}^i \tau(\varphi)(x) \perp d\varphi(T_x M) \quad \forall x \in M, \quad i = 0, \ldots, r - 4.$$

Then φ is r-harmonic if and only if it is ES-r-harmonic.

The following corollary of the previous theorem is very important in our context:

COROLLARY 1.18: *Let $\varphi : M^m \to N^{m+1}(c)$ be a CMC hypersurface with $|A|^2$ equal to a constant. Then φ is r-harmonic if and only if it is ES-r-harmonic.*

We point out that, as a consequence of Corollary 1.18, the conclusion of Theorem 1.10 holds if, in its statement, the word r-harmonic is replaced by ES-r-harmonic. The same occurs for our results on isoparametric hypersurfaces and, in particular, all the proper r-harmonic hypersurfaces obtained in Theorems 1.12, 1.13, 1.14 and 1.15 are also proper ES-r-harmonic.

2. Proof of the first general results

In this section we prove Theorems 1.3–1.10. As a preliminary step, in the following lemma we state without proof some standard facts which we shall use in this section.

LEMMA 2.1: *Let $\varphi : M^m \to N^{m+1}(c)$ be a hypersurface. Let A denote the shape operator and $f = (1/m)\operatorname{trace} A$ the mean curvature function. Then*

    (a) $(\nabla A)(\cdot, \cdot)$ *is symmetric;*
    (b) $\langle (\nabla A)(\cdot, \cdot), \cdot \rangle$ *is totally symmetric;*
    (c) $\operatorname{trace}(\nabla A)(\cdot, \cdot) = m \operatorname{grad} f$.

Next, we perform our first computation:

LEMMA 2.2: *Let $\varphi : M^m \to N^{m+1}(c)$ be a hypersurface and denote by η the unit normal vector field. Then*

$$(2.1) \qquad \overline{\Delta}\mathbf{H} = (\Delta f + f|A|^2)\eta + 2A(\operatorname{grad} f) + mf \operatorname{grad} f.$$

Proof. We work with a geodesic frame field $\{X_i\}_{i=1}^m$ around an arbitrarily fixed point $p \in M^m$. Also, we simplify the notation writing ∇ for ∇^M. Since $\mathbf{H} = f\eta$, around p we have

$$\nabla^{\varphi}_{X_i}\mathbf{H} = \nabla^{\perp}_{X_i}\mathbf{H} - A_{\mathbf{H}}(X_i) = (X_i f)\eta - fA(X_i).$$

Then, denoting by B the second fundamental form, at p we have

$$\begin{aligned}\nabla^{\varphi}_{X_i}\nabla^{\varphi}_{X_i}\mathbf{H} =&(X_iX_if)\eta - (X_if)A(X_i) - (X_if)A(X_i)\\ &- f(\nabla_{X_i}A(X_i) + B(X_i, A(X_i)))\\ =&(X_iX_if)\eta - 2(X_if)A(X_i) - f(\nabla A)(X_i, X_i) - f|A(X_i)|^2\eta.\end{aligned}$$

Now, taking the sum over i and using Lemma 2.1, we obtain (2.1) (note that the sign convention for Δ and $\overline{\Delta}$ is as in (1.4)).   ∎

Next, we assume that the mean curvature function f is constant and we obtain

LEMMA 2.3: *Let $\varphi : M^m \to N^{m+1}(c)$ be a hypersurface and assume that its mean curvature function f is equal to a constant α. Then*

$$(2.2)\qquad\qquad \overline{\Delta}^2\mathbf{H} = \alpha(\Delta|A|^2 + |A|^4)\eta + 2\alpha A(\text{grad}\,|A|^2).$$

Proof. Since f is constant, according to Lemma 2.2 we have $\overline{\Delta}\mathbf{H} = \alpha|A|^2\eta$ and so

$$\overline{\Delta}^2\mathbf{H} = \alpha\overline{\Delta}(|A|^2\eta).$$

Now, around p

$$\nabla^{\varphi}_{X_i}(|A|^2\eta) = (X_i|A|^2)\eta - |A|^2A(X_i).$$

At p

$$\begin{aligned}\nabla^{\varphi}_{X_i}\nabla^{\varphi}_{X_i}(|A|^2\eta) =&(X_iX_i|A|^2)\eta - (X_i|A|^2)A(X_i)\\ &- (X_i|A|^2)A(X_i) - |A|^2(\nabla_{X_i}A(X_i) + B(X_i, A(X_i))).\end{aligned}$$

Taking the sum over i and using Lemma 2.1 we find

$$\overline{\Delta}(|A|^2\eta) = (\Delta|A|^2)\eta + 2A(\text{grad}\,|A|^2) + m|A|^2\,\text{grad}\,f + |A|^4\eta$$

and, since f is constant, the proof ends immediately.   ∎

Next, we recall the formulas which describe the r-tension field of a general map $\varphi : M \to N$ between two Riemannian manifolds (see [22]):

$$\begin{aligned}\tau_{2s}(\varphi) =&\overline{\Delta}^{2s-1}\tau(\varphi) - R^N(\overline{\Delta}^{2s-2}\tau(\varphi), d\varphi(e_i))d\varphi(e_i)\\ (2.3)\qquad &- \sum_{\ell=1}^{s-1}\{R^N(\nabla^{\varphi}_{e_i}\overline{\Delta}^{s+\ell-2}\tau(\varphi), \overline{\Delta}^{s-\ell-1}\tau(\varphi))d\varphi(e_i)\\ &- R^N(\overline{\Delta}^{s+\ell-2}\tau(\varphi), \nabla^{\varphi}_{e_i}\overline{\Delta}^{s-\ell-1}\tau(\varphi))d\varphi(e_i)\},\end{aligned}$$

where $\overline{\Delta}^{-1} = 0$ and $\{e_i\}_{i=1}^m$ is a local orthonormal frame field tangent to M (the sum over i is not written but understood). Similarly,

$$
\begin{aligned}
\tau_{2s+1}(\varphi) =& \overline{\Delta}^{2s}\tau(\varphi) - R^N(\overline{\Delta}^{2s-1}\tau(\varphi), d\varphi(e_i))d\varphi(e_i) \\
& - \sum_{\ell=1}^{s-1}\{R^N(\nabla_{e_i}^\varphi \overline{\Delta}^{s+\ell-1}\tau(\varphi), \overline{\Delta}^{s-\ell-1}\tau(\varphi))d\varphi(e_i) \\
& \quad - R^N(\overline{\Delta}^{s+\ell-1}\tau(\varphi), \nabla_{e_i}^\varphi \overline{\Delta}^{s-\ell-1}\tau(\varphi))d\varphi(e_i)\} \\
& - R^N(\nabla_{e_i}^\varphi \overline{\Delta}^{s-1}\tau(\varphi), \overline{\Delta}^{s-1}\tau(\varphi))d\varphi(e_i).
\end{aligned}
$$

(2.4)

We shall also need the expression for the sectional curvature tensor field in the special case that N is a space form:

$$(2.5) \qquad R^{N(c)}(X,Y)Z = c(\langle Y, Z\rangle X - \langle X, Z\rangle Y) \quad \forall X, Y, Z \in C(TN(c)).$$

We are now in the right position to prove our first theorem.

Proof of Theorem 1.3. The 3-tension field is described by (2.4) with $s = 1$. In the first part of the proof, for future reference, we do not make any assumption on the dimension m and the curvature c. We observe that $\tau(\varphi) = m\mathbf{H}$ and use Lemma 2.2, Lemma 2.3 and (2.5). We have

$$
\begin{aligned}
\sum_{i=1}^m & R^{N(c)}(\overline{\Delta}\tau(\varphi), d\varphi(X_i))d\varphi(X_i) \\
& = cm\sum_{i=1}^m \{\langle d\varphi(X_i), d\varphi(X_i)\rangle \overline{\Delta}\mathbf{H} - \langle d\varphi(X_i), \overline{\Delta}\mathbf{H}\rangle d\varphi(X_i)\} \\
& = cm\{m\alpha|A|^2\eta - 0\} = cm^2\alpha|A|^2\eta.
\end{aligned}
$$

(2.6)

Similarly, we compute

$$(2.7) \qquad \sum_{i=1}^m R^{N(c)}(\nabla_{X_i}^\varphi \tau(\varphi), \tau(\varphi))d\varphi(X_i) = cm^3\alpha^3\eta.$$

Inserting (2.2), (2.6) and (2.7) into (2.4) we obtain the explicit expression of the 3-tension field:

$$\tau_3(\varphi) = m\alpha[\Delta|A|^2 + |A|^4 - mc|A|^2 - m^2c\alpha^2]\eta + 2m\alpha A(\text{grad}\,|A|^2).$$

Therefore, we conclude that M^m is a triharmonic hypersurface in $N^{m+1}(c)$ if and only if either it is minimal or

(2.8)
$$\begin{cases} \text{(i)} \quad \Delta|A|^2 + |A|^4 - mc|A|^2 - m^2 c\alpha^2 = 0, \\ \text{(ii)} \quad A(\operatorname{grad}|A|^2) = 0. \end{cases}$$

From now on, we assume that M^m is not minimal and we use the hypothesis $c \leq 0$. If $\operatorname{grad}|A|^2 = 0$ on M^m, then $\Delta|A|^2 = 0$. Therefore, from (2.8)(i) we deduce $|A|^4 = 0$ which implies that $\alpha = 0$, i.e., M^m is minimal (a contradiction). Assume now that there exists a point $p_0 \in M^m$ such that $\operatorname{grad}|A|^2(p_0) \neq 0$. Then $\operatorname{grad}|A|^2 \neq 0$ in a neighbourhood U of p_0. We deduce from (2.8)(ii) that 0 is an eigenvalue of A on U. Now we use the assumption that $m = 2$. Since M^2 is CMC, it is immediate to conclude that the two principal curvatures are constant on U. But then also $|A|^2$ is constant on U: this contradiction ends the proof. ∎

Now, it is convenient to proceed to the

Proof of Theorem 1.6. From the proof of Theorem 1.3 we know that $\operatorname{grad}(|A|^2) = 0$, i.e., $|A|^2$ is a constant on M^2. It follows that, since M^2 is CMC,

$$k_1 + k_2 = \text{constant} \quad \text{and} \quad k_1^2 + k_2^2 = \text{constant}$$

and consequently M^2 is an isoparametric surface in $\mathbb{S}^3$. Then M^2 is an open part of either a small hypersphere $\mathbb{S}^2(R)$ of $\mathbb{S}^3$, $0 < R \leq 1$, or a Clifford torus $\mathbb{S}^1(R_1) \times \mathbb{S}^1(R_2)$, $R_1^2 + R_2^2 = 1$. But, since M^2 is proper triharmonic, we know from [22, 24] that the only possibility is that M^2 is an open part of $\mathbb{S}^2(1/\sqrt{3})$. ∎

Proof of Theorem 1.7. The Cauchy inequality tells us that, for any hypersurface, we have

(2.9)
$$|A|^2 = \sum_{i=1}^{m} k_i^2 \geq \frac{(\sum_{i=1}^{m} k_i)^2}{m} = mf^2.$$

Applying (2.9) to (2.8)(i) (with $c = 1$) we immediately deduce

$$\Delta|A|^2 \leq |A|^2(2m - |A|^2)$$

and, integrating on the compact manifold M^m,

$$0 \leq \int_{M^m} |A|^2(2m - |A|^2) \, dV.$$

Thus, either $|A|^2 = 2m$ or there exists a point $p_1 \in M^m$ such that $|A|^2(p_1) < 2m$. On the other hand, (2.8)(i) (with $c = 1$) implies

$$\Delta|A|^2 > |A|^2(m - |A|^2)$$

on M^m. Therefore integration yields

$$0 > \int_{M^m} |A|^2(m - |A|^2)\, dV.$$

Thus there exists a point $p_2 \in M^m$ such that $|A|^2(p_2) > m$. Finally, as M^m is connected, by an obvious continuity argument we deduce that, if $|A|^2 = 2m$ does not hold on M^m, then there exists a point $p \in M^m$ such that $m < |A|^2(p) < 2m$.

It only remains to prove that, if $|A|^2 = 2m$ on M^m, then $M^m = \mathbb{S}^m(1/\sqrt{3})$. Indeed, if $|A|^2 = 2m$ on M^m, then (2.8)(i) (with $c = 1$) implies $|A|^2 = m\alpha^2$. But this is equivalent to saying that we have equality in (2.9), which means that M^m is umbilical. Then $M^m = \mathbb{S}^m(R)$, $0 < R \leq 1$ and, as M^m is proper triharmonic, we know that the only possibility is $R = 1/\sqrt{3}$. ∎

Theorems 1.4 and 1.10 are an immediate consequence of the following general result:

THEOREM 2.4: *Let M^m be a non-minimal CMC hypersurface in $N^{m+1}(c)$ and assume that $|A|^2$ is constant. Then M^m is proper r-harmonic ($r \geq 3$) if and only if*

$$|A|^4 - mc|A|^2 - (r - 2)m^2 c\alpha^2 = 0.$$

Proof. As $|A|^2$ is constant, it follows from Lemma 2.3 that

$$(2.10) \qquad \overline{\Delta}^2 \mathbf{H} = \alpha|A|^4 \eta.$$

Now, we show that

$$(2.11) \qquad \overline{\Delta}\eta = |A|^2 \eta.$$

Indeed, $\nabla^\varphi_{X_i} \eta = -A(X_i)$ and so

$$\nabla^\varphi_{X_i} \nabla^\varphi_{X_i} \eta = -(\nabla A)(X_i, X_i) - |A(X_i)|^2 \eta.$$

Summing over i and using Lemma 2.1 we obtain

$$\overline{\Delta}\eta = m\operatorname{grad} f + |A|^2 \eta = |A|^2 \eta.$$

Next, putting together (2.1), (2.10) and (2.11), we easily deduce that

$$(2.12) \qquad \overline{\Delta}^p \mathbf{H} = \alpha|A|^{2p} \eta \quad \forall p \in \mathbb{N}^*.$$

Now we are in a good position to perform the explicit calculation of the r-tension field $\tau_r(\varphi)$ described in (2.3), (2.4). We begin with $\tau_{2s}(\varphi)$, $s \geq 2$. Using (2.12), (2.5) and computing we obtain (as in (2.3), we omit to write the sum over i)

$$
\begin{aligned}
\frac{1}{m}\tau_{2s}(\varphi) =\,& \alpha|A|^{4s-2}\eta \\
& - c\{\langle d\varphi(X_i), d\varphi(X_i)\rangle \alpha|A|^{4s-4}\eta - \langle d\varphi(X_i), \alpha|A|^{4s-4}\eta\rangle d\varphi(X_i)\} \\
& - cm\sum_{\ell=1}^{s-1}\{\langle d\varphi(X_i), \alpha|A|^{2s-2\ell-2}\eta\rangle(-\alpha|A|^{2s+2\ell-4}A(X_i)) \\
& \qquad\qquad - \langle d\varphi(X_i), -\alpha|A|^{2s+2\ell-4}A(X_i)\rangle\alpha|A|^{2s-2\ell-2}\eta\} \\
& + cm\sum_{\ell=1}^{s-1}\{\langle d\varphi(X_i), -\alpha|A|^{2s-2\ell-2}A(X_i)\rangle\alpha|A|^{2s+2\ell-4}\eta \\
& \qquad\qquad - \langle d\varphi(X_i), \alpha|A|^{2s+2\ell-4}\eta\rangle(-\alpha|A|^{2s-2\ell-2}A(X_i))\} \\
=\,& \alpha|A|^{4s-2}\eta - cm\alpha|A|^{4s-4}\eta \\
& - cm\left\{\sum_{\ell=1}^{s-1}[m\alpha^3|A|^{4s-6}\eta] + \sum_{\ell=1}^{s-1}[m\alpha^3|A|^{4s-6}\eta]\right\} \\
=\,& \alpha|A|^{4s-6}\{|A|^4 - mc|A|^2 - (2s-2)m^2c\alpha^2\}\eta.
\end{aligned}
$$

This completes the proof in the case $r = 2s$. The case $r = 2s+1$ is similar and so we omit the details. ∎

Proof of Theorem 1.9. (i) Since (1.7) holds, we use the Cauchy inequality (2.9) and deduce

$$
|A|^4 = m|A|^2 + (r-2)m^2\alpha^2 \leq m|A|^2 + (r-2)m|A|^2,
$$

and, consequently,

$$
|A|^2 \leq m(r-1)
$$

and $|A|^2 = m(r-1)$ if and only if M^m is umbilical, i.e., M^m is an open part of some small hypersphere $\mathbb{S}^m(R)$, $0 < R \leq 1$. But, since M^m is proper r-harmonic, we know that the only possibility is $R = 1/\sqrt{r}$ (see [7, 32]). On the other hand, as $\alpha \neq 0$,

$$
|A|^4 = m|A|^2 + (r-2)m^2\alpha^2 \Rightarrow |A|^2 > m
$$

and so the proof of statement (i) is complete.

(ii) Again, we start with (1.7) and deduce

$$|A|^2 = \frac{m}{2}(1 + \sqrt{1 + 4(r-2)\alpha^2}).$$

But $|A|^2 \geq m\alpha^2$ implies

(2.13) $1 + \sqrt{1 + 4(r-2)\alpha^2} \geq 2\alpha^2.$

We observe that when $\alpha^2 = r - 1$ we have equality in (2.13). From this it is easy to conclude that

$$1 + \sqrt{1 + 4(r-2)\alpha^2} \geq 2\alpha^2 \quad \Leftrightarrow \quad \alpha^2 \in (0, r-1].$$

Finally, when $\alpha^2 = r - 1$, we have

$$|A|^2 = \frac{m}{2}\Big(1 + \sqrt{1 + 4(r-2)(r-1)}\Big) = m(r-1) = m\alpha^2,$$

so M^m is umbilical and the conclusion is as in part (i).   ■

3. Proof of the results on r-harmonic isoparametric hypersurfaces

Proof of Theorem 1.12. We know that $m_1 = m_2 = m_3$ and using (1.8) with $\ell = 3$ we can compute

$$|A|^2 = m_1\left(\sum_{i=1}^{3} k_i^2\right) \quad \text{and} \quad \alpha^2 = (1/9)\left(\sum_{i=1}^{3} k_i\right)^2.$$

Then, using standard trigonometric identities, we find that equation (1.7) becomes

(3.1)
$$m_1^2\Big[(2-r)\Big(\tan\Big(\frac{\pi}{6}-s\Big) - \tan\Big(s+\frac{\pi}{6}\Big) + \cot s\Big)^2$$
$$+ \Big(\tan^2\Big(\frac{\pi}{6}-s\Big) + \tan^2\Big(s+\frac{\pi}{6}\Big) + \cot^2 s\Big)^2$$
$$- 3\Big(\tan^2\Big(\frac{\pi}{6}-s\Big) + \tan^2\Big(s+\frac{\pi}{6}\Big) + \cot^2 s\Big)\Big] = 0.$$

Next, using the standard addition and subtraction formulas for trigonometric functions, we compute and find that (3.1) is equivalent to

(3.2) $$\frac{9m_1^2 \csc^4 s[r\cos(12s) - r + 28\cos(6s) + 44]}{8(2\cos(2s)+1)^4} = 0.$$

Now, let $x = \cos(6s)$. Then (3.2) is equivalent to

$$rx^2 + 14x + 22 - r = 0,$$

whose solutions are

(3.3) $x_1 = \dfrac{-\sqrt{r^2 - 22r + 49} - 7}{r}$ and $x_2 = \dfrac{\sqrt{r^2 - 22r + 49} - 7}{r}$.

Now, a routine inspection of (3.3) shows that there is no acceptable solution if $r \leq 19$. By contrast, when $r \geq 20$, we have

$$-(1/2) \geq x_1 > -1 \quad \text{and} \quad -(1/5) \leq x_2 < 1.$$

From this the conclusion of the proof is immediate.

Finally, we observe that the two hypersurfaces M_s, $M_{s'}$ associated to x_1 are congruent and their distance from the minimal hypersurface $M_{\pi/6}$ tends to zero as r grows to $+\infty$. Indeed, since $s' = \pi/\ell - s = \pi/3 - s$, it is immediate to deduce that, if we write $M_s = F^{-1}(a)$, where $a = \cos 3s$, then $M_{s'} = F^{-1}(-a)$. Now, since F is the restriction to $\mathbb{S}^{m+1}$ of a homogeneous polynomial of degree 3 (odd), then M_s and $M_{s'}$ are congruent via the antipodal map. Similarly, the two hypersurfaces associated to x_2 are congruent and each of them approaches one of the focal varieties as r increases to $+\infty$, so the proof is completed.   ∎

Proof of Theorem 1.13. The proof follows exactly the scheme of the proof of Theorem 1.12. Here we have $m_1 = m_2 = \cdots = m_6$ and instead of (3.1) we find

$$m_1^2 \Big[(2-r)\Big(-\tan(\tfrac{\pi}{6} - s) + \tan s + \tan\Big(s + \tfrac{\pi}{6}\Big)$$
$$+ \cot\Big(\tfrac{\pi}{6} - s\Big) - \cot s - \cot\Big(s + \tfrac{\pi}{6}\Big)\Big)^2$$
$$- 6\Big(\tan^2\Big(\tfrac{\pi}{6} - s\Big) + \tan^2 s + \tan^2\Big(s + \tfrac{\pi}{6}\Big)$$
$$+ \cot^2\Big(\tfrac{1}{6}(\pi - 6s)\Big) + \cot^2 s + \cot^2\Big(\tfrac{\pi}{6} - s\Big)\Big)$$
$$\times \Big(\tan^2\Big(\tfrac{\pi}{6} - s\Big) + \tan^2 s + \tan^2\Big(s + \tfrac{\pi}{6}\Big)$$
$$+ \cot^2\Big(\tfrac{\pi}{6} - s\Big) + \cot^2 s + \cot^2\Big(s + \tfrac{\pi}{6}\Big)\Big)^2\Big] = 0$$

which turns out to be equivalent to

(3.4) $\dfrac{9m_1^2 \csc^4 s \sec^4 s[r\cos(24s) - r + 64\cos(12s) + 224]}{32(2\cos(4s) + 1)^4} = 0.$

Now, let $x = \cos(12s)$. Then (3.4) is equivalent to

$$rx^2 + 32x + 112 - r = 0,$$

whose solutions are

$$(3.5) \quad x_1 = \frac{-\sqrt{r^2 - 112r + 256} - 16}{r} \quad \text{and} \quad x_2 = \frac{\sqrt{r^2 - 112r + 256} - 16}{r}.$$

Now, a routine inspection of (3.5) shows that there is no acceptable solution if $r \leq 109$. By contrast, when $r \geq 110$, we have

$$-(1/5) \geq x_1 > -1 \quad \text{and} \quad -(1/11) \leq x_2 < 1.$$

Now the conclusion of the proof is as in Theorem 1.12. The only difference, in this case, is that F is the restriction to $\mathbb{S}^{m+1}$ of a homogeneous polynomial of even degree, and so we cannot conclude that the two solutions associated to x_1 or x_2 are congruent. In particular, we point out that Miyaoka showed that, when $\ell = 6$ and $m_1 = m_2 = 1$, the two focal varieties $F^{-1}(1)$ and $F^{-1}(-1)$ are not congruent (see [27]).   ∎

Proof of Theorem 1.14. Again, we use (1.8). Without making any assumption on m_1, m_2 we find that (1.7) becomes

$$(3.6) \quad \begin{aligned} &-4(r - 2)(m_1 \cot(2s) - m_2 \tan(2s))^2 \\ &+ \left(m_1 \tan^2 s + m_1 \cot^2 s + m_2 \tan^2 \left(s + \frac{\pi}{4}\right) + m_2 \cot^2 \left(s + \frac{\pi}{4}\right)\right)^2 \\ &- 2(m_1 + m_1) \\ &\times \left(m_1 \tan^2 s + m_1 \cot^2 s + m_2 \tan^2 \left(s + \frac{\pi}{4}\right) + m_2 \cot^2 \left(s + \frac{\pi}{4}\right)\right) = 0. \end{aligned}$$

Now we use the assumption $m_1 = m_2$ and the proof follows the patterns of Theorem 1.12. In particular, when $m_1 = m_2$, (3.6) becomes

$$(3.7) \quad 2m_1^2 \csc^4(4s)[r \cos(16s) - r + 40 \cos(8s) + 88] = 0.$$

Next, let $x = \cos(8s)$. Then (3.7) is equivalent to

$$(3.8) \quad rx^2 + 20x + 44 - r = 0,$$

whose solutions are

$$(3.9) \quad x_1 = \frac{-\sqrt{r^2 - 44r + 100} - 10}{r} \quad \text{and} \quad x_2 = \frac{\sqrt{r^2 - 44r + 100} - 10}{r}.$$

As above, inspection of (3.9) shows that there is no acceptable solution if $r \leq 41$. By contrast, when $r \geq 42$, we have $-(1/3) \geq x_1 > -1$ and $-(1/7) \leq x_2 < 1$ and so the proof ends as in Theorem 1.13.   ∎

Proof of Theorem 1.15. We simplify (3.6) without the assumption $m_1 = m_2$. We obtain

(3.10)
$$4r(m_1 + m_2)^2 + 4m_2 \sec^2(2s)[2m_1 - m_2(r + 4) + 4m_2 \sec^2(2s)]$$
$$+ \frac{1}{8} m_1 \csc^4 s \sec^4 s [\cos(4s)(m_1(r + 4) - 2m_2) - m_1(r - 4) + 2m_2] = 0.$$

Next, we set $y = \cos^2(2s)$. Then a straightforward computation shows that, in terms of y, (3.10) becomes, up to the multiplicative quantity $4m_1^2/y$, $P_{b,r}(y) = 0$, where $P_{b,r}(y)$ is the polynomial defined in (1.9). This completes the proof of the first part of the theorem.

Next, we define

$$T_r(s) = |A|^4(s) - m|A|^2(s) - (r - 2)m^2\alpha^2(s),$$

where $|A|^2(s)$ and $\alpha^2(s)$ are computed using the principal curvatures given in (1.8) with $\ell = 4$ and $m = 2(m_1 + m_2)$. We observe that, if $r_1 > r_2$, then $T_{r_1}(s) \leq T_{r_2}(s)$ on $(0, \pi/4)$. For any fixed value of r, we have

(3.11)
$$\lim_{s \to 0^+} T_r(s) = \lim_{s \to (\pi/4)^-} T_r(s) = +\infty.$$

Moreover, for future use we point out that, if we restrict r to any arbitrary set $(0, K]$ $(K > 0)$, there always exists a sufficiently small $\varepsilon > 0$ such that

(3.12)
$$T_r(s) \geq 1 \quad \forall s \in (0, \varepsilon] \cup \left[\frac{\pi}{4} - \varepsilon, \frac{\pi}{4}\right) \text{ and } \forall r \in (0, K].$$

Next, we set $\mathcal{R}^* = \inf X$, where X is the set defined as follows:

$$X = \{r \in (0, +\infty): \min\{T_r(s), 0 < s < \pi/4\} < 0\}.$$

For future reference, we point out that inspection of (3.10) shows that $\mathcal{R}^*$ depends on $b = m_1/m_2$. We can choose $s_0 \in (0, \pi/4)$ such that $\alpha^2(s_0) \neq 0$. It follows that $T_r(s_0) < 0$ provided that r is sufficiently large. This fact, together with (3.11), shows that the open set X is not empty. In fact, X is an open half-line. On the other hand, we know from [20] that $|A|^4(s) - m|A|^2(s) > 0$ on $(0, \pi/4)$, and from this it is immediate to conclude that $\mathcal{R}^* \geq 2$. We note that $\mathcal{R}^* \notin X$, i.e., $T_{\mathcal{R}^*}(s) \geq 0$ on $(0, \pi/4)$, and $T_r(s) > 0$ on $(0, \pi/4)$ for all $r < \mathcal{R}^*$.

Next, we shall prove that $\mathcal{R}^* > 4$. To this purpose, first we observe that, if $0 < r \leq 4$, $P_{0,r}(y)$ is positive on $(0, 1)$. Then it suffices to show that, if $0 < r \leq 4$, for any $y \in (0, 1)$ we have

(3.13)
$$P_{b,r}(y) > P_{0,r}(y) \quad \forall b > 0.$$

The property (3.13) is an immediate consequence of the fact that, for all $b \geq 0$,

$$\frac{dP_{b,r}(y)}{db} = 2(1-y)^3(4-ry)b + 2y(1-y)(1+yr(1-y)) > 0$$

provided that $0 < r \leq 4$ and $0 < y < 1$. Thus $P_{b,r}(y) > 0$ for any $b > 0$, $y \in (0,1)$ and $0 < r \leq 4$, so the equation $T_4(s) = 0$ does not admit any solution in $(0, \pi/4)$ and consequently $T_4(s) > 0$ on this interval. The last inequality implies $\mathcal{R}^* \geq 4$.

Then it remains to prove that $\mathcal{R}^* > 4$. We assume $\mathcal{R}^* = 4$ and derive a contradiction. Using (3.12) with $K = 5$ we deduce that, since $\mathcal{R}^* = 4 = \inf X$, for any $n \in \mathbb{N}^*$ there exists $s_n \in (\varepsilon, \pi/4 - \varepsilon)$ such that $T_{4+1/n}(s_n) < 0$. Now we can extract a subsequence of $\{s_n\}$ which converges to some $s_1 \in [\varepsilon, \pi/4 - \varepsilon]$ and passing to the limit we get $T_4(s_1) \leq 0$. But this represents a contradiction, as $T_4(s)$ is positive on $(0, \pi/4)$. By a similar continuity argument, using again (3.12) we also deduce that there exists an interior point $\hat{s}$ such that $T_{\mathcal{R}^*}(\hat{s}) = 0$.

Finally, if $\mathcal{R}^*$ is an integer we set $r^* = \mathcal{R}^*$, otherwise $r^* = \lfloor \mathcal{R}^* \rfloor + 1$, where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$. By construction, this definition of r^* ensures that r^* has the properties stated in (i), (ii) of the theorem.

By way of summary, it only remains to prove statement (iii). We observe that $\alpha(s)$ is a strictly decreasing function on $(0, \pi/4)$ and

$$\lim_{s \to 0^+} \alpha(s) = +\infty, \quad \lim_{s \to (\pi/4)^-} \alpha(s) = -\infty.$$

Therefore, there exists a unique value $s^* \in (0, \pi/4)$ such that $\alpha(s^*) = 0$ (note that $s^* = (1/2)\arccos(\sqrt{m_2/(m_1+m_2)})$ and M_{s^*} is the minimal isoparametric hypersurface of the family). It follows that $T_r(s^*)$ is a positive real number which does not depend on r. From this, arguing similarly to the construction of r^* separately on $(0, s^*)$ and $(s^*, \pi/4)$, it is easy to conclude that there exists r^{**} with the property stated in (iii) and so the proof is completed. Of course, two solutions are always in the interval $(0, s^*)$, while the other two must belong to $(s^*, \pi/4)$.   ∎

3.1. UPPER ESTIMATES FOR r^* AND r^{**}. The arguments of the proof of Theorem 1.15 apply to all the families of isoparametric hypersurfaces. In particular, in the situations of Theorems 1.12, 1.13, 1.14 we were able to determine exactly $r^* = r^{**} = 20, 110, 42$ respectively. By contrast, in the case of isoparametric hypersurfaces of degree 4 and different multiplicities, it seems to be a difficult task

to determine the explicit values of $r^* = r^*(b)$ and $r^{**} = r^{**}(b)$, $b = (m_2/m_1) > 1$. The aim of this subsection is to explain the type of difficulties which arise and also provide some upper bounds for both r^* and r^{**}.

We rewrite the polynomial $P_{b,r}(y)$ in (1.9) as follows:

$$P_{b,r}(y) = Q_b(y) - rR_b(y),$$

where

$$Q_b(y) = 2[2b^2(1-y)^3 + by(1-y) + 2y^3],$$
$$R_b(y) = (1-y)y[b(y-1) + y]^2.$$

We note that $Q_b(y)$ is positive on $(0,1)$, while $R_b(y)$ is positive on $(0, y_0)$ and $(y_0, 1)$, where $y_0 = b/(1+b) = m_2/(m_1 + m_2)$ corresponds to the minimal hypersurface M_{s_0}, where

$$s_0 = \frac{1}{2}\arccos(\sqrt{y_0}).$$

Moreover,

$$R_b(y_0) = 0 \quad \text{and} \quad Q_b(y_0) = P_{b,r}(y_0) = \frac{6b^2}{(1+b)^2} > 0.$$

Now, since we want to determine the smallest value of r such that $P_{b,r}(y)$ admits a root in $(0,1)$, it is natural to define

$$\mathcal{R}(y) = \frac{Q_b(y)}{R_b(y)}, \quad y \in (0,1), y \neq y_0,$$

and set

$$\mathcal{R}_1 = \inf\{\mathcal{R}(y) \colon y \in (0, y_0)\},$$
$$\mathcal{R}_2 = \inf\{\mathcal{R}(y) \colon y \in (y_0, 1)\}.$$

The connection with the proof of Theorem 1.15 is the following:

$$\mathcal{R}^* = \min\{\mathcal{R}_1, \mathcal{R}_2\}; \quad r^{**} = \lfloor \mathcal{R}^{**} \rfloor + 1, \quad \text{where } \mathcal{R}^{**} = \max\{\mathcal{R}_1, \mathcal{R}_2\}.$$

The function $\mathcal{R}(y)$ is positive and tends to $+\infty$ when $y \to 0^+$, $y \to y_0^\pm$ and $y \to 1^-$. Therefore $\mathcal{R}(y)$ admits a minimum point y_1 on $(0, y_0)$ and a minimum point y_2 on $(y_0, 1)$, so that $\mathcal{R}_i = \mathcal{R}(y_i)$, $i = 1, 2$. Unfortunately, it is difficult to compute the exact values of y_1 and y_2. A numerical analysis shows that, when b is not too big, y_1 is on the interval $(0, y_0/2)$, while y_1 belongs to $(y_0/2, y_0)$ for large values of b. Similarly, y_2 increases as b does. Therefore,

a first reasonable upper estimate for $\mathcal{R}_1$ can be obtained by considering the medium point. We have

$$\mathcal{R}_1 \le \mathcal{R}(y_0/2) = \frac{8(b^2 + 6b + 10)}{b(b+2)}.$$

Similarly, an upper estimate for $\mathcal{R}_2$ is given by

$$\mathcal{R}_2 \le \mathcal{R}((1+y_0)/2) = \frac{(8 + 48b + 80b^2)}{(1+2b)}.$$

Next, we observe that

$$\mathcal{R}((1+y_0)/2) > \mathcal{R}(y_0/2) \quad \forall b > 1.$$

The conclusion of this analysis is the following. Let $r^* = r^*(b)$, $r^{**} = r^{**}(b)$ $(b > 1)$ be the integers defined in Theorem 1.15. Then

(3.14)

$$\text{(i)} \quad r^*(b) \le 1 + \frac{8(b^2 + 6b + 10)}{b(b+2)},$$

$$\text{(ii)} \quad r^{**}(b) \le 1 + \frac{(8 + 48b + 80b^2)}{(1+2b)}.$$

For instance, when b is close to 1, (3.14) tells us that $r^*(b) \le 46$ and $r^{**}(b) \le 46$ (we know from Theorem 1.14 that $r^* = r^{**} = 42$ when $b = 1$). When $b = 10000$, (3.14) gives $r^* \le 9$ and $r^{**} \le 400005$ (we pointed out in Example 1.16 that, when $b = 10000$, $r^* = 5$ and $r^{**} = 312919$). Another geometrically significant case which occurs is $(m_1, m_2) = (7, 8)$. In this case, (3.14) with $b = 8/7$ yields $r^* \le 41$ and $r^{**} \le 51$. A numerical analysis shows that, when $b = 8/7$, the exact values are $r^* = 38$ and $r^{**} = 47$.

3.2. REMARKS. (i) By a suitable choice of the orientation we assumed, for convenience, $m_1 \le m_2$ so that $b = m_2/m_1 \ge 1$. We point out that, if one decides to carry out the analysis for all $b > 0$, then there are no significant differences because of the following fact, which is true for all $b \ne 0$, $y, r \in \mathbb{R}$ and can be verified by a direct computation:

$$b^2 P_{1/b,r}(1 - y) = P_{b,r}(y).$$

(ii) In order to recover the correspondance between $P_{1,r}(y)$ and the second order equation (3.8), one has to remember that $y = \cos^2(2s)$, while $x = \cos(8s)$ and consequently perform the changes of variable

$$y = \frac{1}{4}(2 \pm \sqrt{2}\sqrt{x+1}).$$

Unfortunately, these changes of variable do not yield any simplification if $b \ne 1$.

(iii) We think that it could be interesting to find an isoparametric family with one or three distinct proper solutions for some r (necessarily, the degree of such a family must be 4 and $m_1 \neq m_2$). A necessary condition to have just one solution is that the real number $\mathcal{R}^*$ defined in the proof of Theorem 1.15 is an integer.

(iv) Using (1.7) and (1.8) with $\ell = 1, 2$, together with the technique of proof of the present paper, it is possible to recover the characterisation of proper r-harmonic small hyperspheres and generalised Clifford tori obtained by variational methods in [32].

4. ES-r-harmonicity and r-harmonicity

Proof of Theorem 1.17. It is convenient to separate two cases: r even and r odd. First, let us assume that $r = 2s \geq 4$. In this case, since if $p \neq p'$ the subspace of p-forms is orthogonal to that of p'-forms, we have

$$(4.1) \qquad E_{2s}^{ES}(\varphi) = \frac{1}{2} \int_M \sum_{k=0}^{s-1} \langle \omega_{2k}, \omega_{2k} \rangle \, dV,$$

where ω_{2k} are $\varphi^{-1}TN$-valued $2k$-differential forms on M. Each ω_{2k} is the sum of $2k$-differential forms obtained by applying to τ a string of length $2s-2$ made of d's and d^*'s. Since d^* is the null operator on 0-forms, if we read from right to left the string of a nonzero form, the number of d's that we encounter in any truncated string is always not smaller than the number of d^*'s, otherwise the form automatically vanishes. In each of these strings the number n_d of occurrences of d minus the number n_{d^*} of occurrences of d^* is equal to $2k$. To help the reader, let us give an explicit example of a $2k$-form which can occur when $r = 26, 2k = 8$:

$$(4.2) \qquad \alpha_8 = dd^*d^6(d^*)^3d^5(d^*d)^4\tau \quad (n_d = 16, n_{d^*} = 8).$$

Definition 4.1: Let α_{2k} be any $2k$-form obtained applying to τ any string made of d's and d^*'s. We say that α_{2k} is **good** if its string starts on the right with a term of the type $d^2\overline{\Delta}^i\tau$, for some i such that $r - 4 \geq i \geq 0$.

For instance, the 8-form in (4.2) is good with $i = 4$. The following lemma is important for our purposes.

LEMMA 4.2: *Let α_{2k} be a good form in the sense of Definition 4.1. Then, if the map φ satisfies the hypotheses of Theorem 1.17, α_{2k} vanishes.*

Proof of Lemma 4.2. As in (1.11), we have

$$d^2(\overline{\Delta}^i \tau)(X,Y) = R^{N(c)}(d\varphi(X), d\varphi(Y))\overline{\Delta}^i \tau$$

and so, using (2.5), the conclusion follows immediately from (1.12). ∎

Next, for $k \geq 1$, let ω_{2k} be the $2k$-form which appears in (4.1). Then ω_{2k} is the sum of good forms. Indeed, this is a simple consequence of the fact that $n_d - n_{d^*} = 2k \geq 2$. Therefore, since $\langle , \rangle$ is bilinear, $\langle \omega_{2k}, \omega_{2k} \rangle$ is the sum of addends of the type $\langle \alpha_{2k}, \beta_{2k} \rangle$, where α_{2k}, β_{2k} are good forms. Now, as in [16], we consider a variation $\{\varphi_t\}_t$ as a map $\Phi : \mathbb{R} \times M \to N$ and denote by ∇^Φ the corresponding covariant derivative. Then we can study the first variation and we have

$$
\begin{aligned}
(4.3) \quad & \frac{d}{dt} \int_M \langle (\alpha_{2k})_t, (\beta_{2k})_t \rangle \, dV|_{t=0} \\
& = \int_M \{ \langle \nabla^\Phi_{\frac{\partial}{\partial t}} (\alpha_{2k})_t|_{t=0}, (\beta_{2k})_0 \rangle + \langle (\alpha_{2k})_0, \nabla^\Phi_{\frac{\partial}{\partial t}} (\beta_{2k})_t|_{t=0} \rangle \} dV \\
& = 0,
\end{aligned}
$$

where the $\varphi_t^{-1} TN$-valued $2k$-form $(\alpha_{2k})_t$ is identified with a section in

$$\Lambda^{2k} T^* M \otimes \Phi^{-1} TN,$$

and the last equality is a consequence of the fact that $(\alpha_{2k})_0$ and $(\beta_{2k})_0$ are good forms and so they vanish by Lemma 4.2.

By way of summary, if φ is a map which verifies the hypotheses of Theorem 1.17, the first variation associated to the forms ω_{2k} ($k \geq 1$) in (4.1) automatically vanishes. It remains to understand the contribution of ω_0. To this purpose, we observe that

$$\omega_0 = \overline{\Delta}^{s-1} \tau + \text{good forms of degree zero} .$$

It follows that

$$\langle \omega_0, \omega_0 \rangle = \langle \overline{\Delta}^{s-1} \tau, \overline{\Delta}^{s-1} \tau \rangle + \text{addends of type 1} + \text{addends of type 2},$$

where the terms of type 1 and 2 are, respectively, of the form $\langle \alpha_0, \beta_0 \rangle$ and $\langle \overline{\Delta}^{s-1} \tau, \alpha_0 \rangle$, where α_0, β_0 are good forms of degree 0. Arguing as in (4.3), we conclude that the first variation of the terms of type 1 vanishes automatically.

As for type 2, let us first assume that M is compact. Then we can use the fact that d^* is the adjoint operator of d, i.e.,

$$(4.4) \qquad \int_M \langle d\alpha, \beta \rangle \, dV = \int_M \langle \alpha, d^*\beta \rangle \, dV$$

whenever α, β are, respectively, a p-form and a $(p+1)$-form. Therefore, before taking the derivative with respect to t, we can transform, by means of suitable iterated applications of (4.4), any term of type 2 into a term of the type $\langle \alpha_{2k}, \beta_{2k} \rangle$, where α_{2k}, β_{2k} are good forms ($k \geq 1$) (it is easy to check that, in this process, the maximum value which we can obtain for the exponent i of Definition 4.1 is equal to $r - 4$).

The case that M is not compact can be handled by using compactly supported variations. More precisely, we consider all compact subsets $D \subset M$ with smooth boundary and define

$$E_r^{ES}(\varphi; D) = \frac{1}{2} \int_D |(d^* + d)^r(\varphi)|^2 \, dV.$$

For each subset D we consider all smooth variations of φ which have their support in D, i.e., $\Phi = \{\varphi_t\}_t$ such that $\varphi_t = \varphi$ on $M \backslash D$ for any t. Denoting by V its variation vector field, we observe that obviously V and its covariant derivatives vanish on $M \backslash D$ and, by continuity, on $\overline{M \backslash D}$. Now, the only difference with respect to the compact case appears when we study the first variation of the terms of type 2. However, we can again proceed as above because (4.4) remains true if one integrates on D instead of M. Indeed, as $\nabla^{\Phi}_{\partial/\partial t}(\cdot)|_{t=0}$ and d, or d^*, commute (on the whole M), we have

$$\frac{d}{dt} \int_D \langle d\alpha_t, \beta_t \rangle \, dV|_{t=0} = \int_D \{ \langle \nabla^{\Phi}_{\frac{\partial}{\partial t}}(d\alpha_t)|_{t=0}, \beta_0 \rangle + \langle d\alpha_0, \nabla^{\Phi}_{\frac{\partial}{\partial t}} \beta_t|_{t=0} \rangle \} \, dV$$

$$= \int_D \{ \langle d(\nabla^{\Phi}_{\frac{\partial}{\partial t}} \alpha_t|_{t=0}), \beta_0 \rangle + \langle d\alpha_0, \nabla^{\Phi}_{\frac{\partial}{\partial t}} \beta_t|_{t=0} \rangle \} \, dV.$$

We note that the terms like $\nabla^{\Phi}_{\partial/\partial t} \alpha_t|_{t=0}$ can always be expressed as functions of V and its covariant derivatives. For example,

$$\nabla^{\Phi}_{\frac{\partial}{\partial t}} \tau(\varphi_t)|_{t=0} = -\overline{\Delta} V - \operatorname{trace} R^N(d\varphi(\cdot), V) d\varphi(\cdot).$$

Then it is well-known that, in general, the difference

$$\langle d\sigma, \rho \rangle - \langle \sigma, d^*\rho \rangle$$

is the divergence of a vector field defined on M. Since the components of this vector field are given in terms of V and its covariant derivatives, its divergence

vanishes on the boundary ∂D. Therefore

$$\frac{d}{dt} \int_D \langle d\alpha_t, \beta_t \rangle \, dV|_{t=0} = \int_D \{ \langle \nabla^{\Phi}_{\frac{\partial}{\partial t}} \alpha_t|_{t=0}, d^* \beta_0 \rangle + \langle \alpha_0, d^* (\nabla^{\Phi}_{\frac{\partial}{\partial t}} \beta_t|_{t=0}) \rangle \} \, dV$$

$$= \frac{d}{dt} \int_D \langle \alpha_t, d^* \beta_t \rangle \, dV|_{t=0}$$

and now the proof continues as in the compact case.

In summary, under the hypotheses of Theorem 1.17 the only relevant term for the computation of the first variation of the functionals $E_{2s}^{ES}(\varphi)$ is

$$\langle \overline{\Delta}^{s-1} \tau, \overline{\Delta}^{s-1} \tau \rangle = |\overline{\Delta}^{s-1} \tau|^2,$$

i.e.,

$$\frac{d}{dt} E_{2s}^{ES}(\varphi_t)|_{t=0} = \frac{1}{2} \frac{d}{dt} \int_M \sum_{k=0}^{s-1} \langle (\omega_{2k})_t, (\omega_{2k})_t \rangle \, dV|_{t=0}$$

$$= \frac{1}{2} \frac{d}{dt} \int_M \langle (\omega_0)_t, (\omega_0)_t \rangle \, dV|_{t=0}$$

$$= \frac{1}{2} \frac{d}{dt} \int_M \langle \overline{\Delta}^{s-1} \tau(\varphi_t), \overline{\Delta}^{s-1} \tau(\varphi_t) \rangle \, dV|_{t=0}$$

$$= \frac{d}{dt} E_{2s}(\varphi_t)|_{t=0}.$$

So the critical points of $E_{2s}^{ES}(\varphi)$ coincide with those of $E_{2s}(\varphi)$ and the proof of the case $r = 2s$ is ended.

In the case that $r = 2s + 1 \geq 5$ we have

$$E_{2s+1}^{ES}(\varphi) = \frac{1}{2} \int_M \sum_{k=0}^{s-1} \langle \omega_{2k+1}, \omega_{2k+1} \rangle \, dV.$$

Now the proof is very similar to that of the case $r = 2s$ and so we omit further details. ∎

Proof of Corollary 1.18. Since $\tau = m\mathbf{H}$, we know from (2.12) that

$$\overline{\Delta}^i \tau = (m\alpha |A|^{2i}) \eta \quad \forall i \geq 0.$$

Thus (1.12) holds and so the conclusion follows immediately from Theorem 1.17. ∎

5. Further studies

(i) We point out that, as in the case of Theorem 1.10, the conclusion of The-
orem 2.4 also holds if, in its statement, the word r-harmonic is replaced by
ES-r-harmonic.

(ii) In the proof of Theorem 1.17, the geometric assumption (1.12) has been
used, together with the explicit form of the curvature tensor field of a space
form, to deduce that

(5.1) $$d^2 \overline{\Delta}^i \tau(\varphi) = 0, \quad i = 1, \ldots, r - 4.$$

Now, let $\varphi : (M, g) \to (N, h)$ be any smooth map between two Riemannian
manifolds. We point out that, as a consequence of the proof of Theorem 1.17,
if φ satisfies (5.1), then φ is r-harmonic if and only if it is ES-r-harmonic.

(iii) In this paper we have worked with the hypothesis that the mean curvature
function f and $|A|^2$ are both constant. In general, if M^m is any hypersurface in
a space form of curvature c, then these two quantities are related to the scalar
curvature S of M^m by the following formula:

$$S = c(m - 1)m + m^2 f^2 - |A|^2.$$

It follows that if two amongst f, $|A|^2$ and S are constant, then so is the third.
In the literature, the study of hypersurfaces in $\mathbb{S}^{m+1}$ with constant scalar and
mean curvatures has attracted the attention of several geometers. For instance,
Chang proved the following result:

THEOREM 5.1 ([10]): *A compact hypersurface M^m of constant scalar curvature
and constant mean curvature in S^{m+1} is isoparametric provided that it has 3
distinct principal curvatures everywhere.*

To our knowledge, in the literature there are no examples of hypersurfaces M^m
in $\mathbb{S}^{m+1}$ which have constant f and S but are not isoparametric. Moreover, we
know that, when $m = 3$, any compact CMC hypersurface of constant scalar
curvature in $\mathbb{S}^4$ is isoparametric (see [2, 13]). Therefore, putting together the
results of [7, 24, 32], Theorem 1.12 and Corollary 1.18, we have a complete
classification of the compact, r-harmonic (ES-r-harmonic) hypersurfaces in $\mathbb{S}^4$
with constant f and $|A|^2$. These hypersurfaces are precisely:

- The small hyperspheres $\mathbb{S}^3(1/\sqrt{r})$.
- The generalised Clifford tori $\mathbb{S}^1(R_1) \times \mathbb{S}^2(R_2)$ obtained in Theorem 1.2
  of [32].
- The isoparametric hypersurfaces obtained in Theorem 1.12 with $m_1 = 1$.

On the other hand, we point out that recently Tang and Yan proved that any hypersurface M^m in $\mathbb{S}^{m+1}$ which has constant f and S is isoparametric provided that some extra conditions on the principal curvatures are verified (see [45]).

(iv) In conclusion, the known results and conjectures concerning biharmonic hypersurfaces in a sphere (see [4, 38]) suggest to us to formulate the following:

COMJECTURE 1: *Any proper r-harmonic (ES-r-harmonic) hypersurface in $\mathbb{S}^{m+1}$ is CMC and has $|A|^2$ equal to a constant.*

Or, even stronger:

COMJECTURE 2: *Any proper r-harmonic (ES-r-harmonic) hypersurface in $\mathbb{S}^{m+1}$ is isoparametric.*

As an additional comment to the above conjectures, we point out that results similar to Theorem 5.1 hold without the assumption M compact or complete. Indeed, it is not difficult to show that if a CMC hypersurface in a space form $N^{m+1}(c)$ has $|A|^2$ constant and at most two distinct principal curvatures at each point, then it is isoparametric of degree either 1 or 2. When the hypersurface has three distinct principal curvatures we can prove the following local version of Theorem 5.1:

PROPOSITION 5.2: *Let M^m be a CMC hypersurface in $\mathbb{S}^{m+1}$ with $|A|^2$ constant. Assume that, at each point, M has three distinct principal curvatures, each of multiplicity at least two. Then M is isoparametric.*

Proof. As M has a fixed number of distinct principal curvatures at each point, their multiplicities are constant on M and the k_i's are smooth functions. Moreover, the corresponding distributions T_i are integrable and, since $\dim T_i \geq 2$, $X_i k_i = 0$ for any vector $X_i \in T_i$, $i = 1, 2, 3$ (see [40]). Then on M we have

$$\begin{cases} m_1 k_1 + m_2 k_2 + m_3 k_3 = m\alpha, \\ m_1 k_1^2 + m_2 k_2^2 + m_3 k_3^2 = a^2, \end{cases}$$

where $a = |A| > 0$. Taking the derivative with respect to any $X_1 \in T_1$ we get

$$\begin{cases} m_2(X_1 k_2) + m_3(X_1 k_3) = 0, \\ m_2 k_2(X_1 k_2) + m_3 k_3(X_1 k_3) = 0. \end{cases}$$

Now, since

$$\begin{vmatrix} m_2 & m_3 \\ m_2 k_2 & m_3 k_3 \end{vmatrix} = m_2 m_3 (k_3 - k_2) \neq 0,$$

we deduce that $X_1 k_2 = X_1 k_3 = 0$. In the same way, we obtain $X_2 k_1 = X_2 k_3 = 0$ and $X_3 k_1 = X_3 k_2 = 0$. Therefore, k_1, k_2 and k_3 are constant functions on M and so M is isoparametric. ∎

Remark 5.3: If we assume that M^m is complete, then, by a result in [26], Proposition 5.2 holds without the hypothesis that $|A|^2$ is constant.

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