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PERIODIC POINTS AND MEASURES FOR A CLASS OF SKEW-PRODUCTS

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ABSTRACT

We consider the C^1-open set $\mathcal{V}$ of partially hyperbolic diffeomorphisms on the space $\mathbb{T}^2 \times \mathbb{T}^2$ whose non-wandering set is not stable, introduced by M. Shub in [57]. Firstly, we show that the non-wandering set of each diffeormorphism in $\mathcal{V}$ is a limit of horseshoes in the sense of entropy. Afterwards, we establish the existence of a C^2-open set $\mathcal{U}$ of C^2-diffeomorphisms in $\mathcal{V}$ and of a C^2-residual subset $\mathfrak{R}$ of $\mathcal{U}$ such that any diffeomorphism in $\mathfrak{R}$ has equal topological and periodic entropies, is asymptotic per-expansive, has a sub-exponential growth rate of the periodic orbits and admits a principal strongly faithful symbolic extension with embedding. Besides, such a diffeomorphism has a unique probability measure with maximal entropy describing the distribution of periodic orbits. Under an additional assumption, we prove that the skew-products in $\mathcal{U}$ preserve a unique ergodic SRB measure, which is physical, whose basin has full Lebesgue measure and which coincides with the measure with maximal entropy.

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1. Introduction

Let $f : M \to M$ be a diffeomorphism of a manifold into itself and $\Omega(f)$ be its non-wandering set. When $\Omega(f)$ does not admit a hyperbolic structure, it may be difficult to describe completely its orbit structure. Motivated by this problem, R. Bowen suggested to look for invariant components of $\Omega(f)$ with large entropy on which the dynamics of f may be simpler to characterize. The key idea is to find closed invariant subsets, say topological horseshoes, within which the dynamics is conjugate to subshifts that may be good approximations, in some sense, of the global dynamics. For instance, this strategy might provide information on the topological entropy of a complicated dynamics by taking the least upper bound over its restrictions to those horseshoes. In this case, the system is said to be a **limit of horseshoes in the sense of entropy**. L.-S. Young studies in [61] systems that are limits of this type, including piecewise monotonic maps of the interval, the Poincaré map of the Lorenz attractor [33] and Abraham–Smale's examples [2], leaving unsolved the case of the partially hyperbolic, robustly transitive, entropy-expansive and non-Ω-stable diffeomorphisms constructed by Shub in [57]. In this work we consider precisely a class of those Shub's examples, explore the dynamical properties of their measures of maximal entropy and show that these examples are indeed limits of horseshoes.

In what follows, we will call Shub's examples to the diffeomorphisms in a C^1-open neighborhood of a skew-product F_S on $\mathbb{T}^2 \times \mathbb{T}^2$ whose construction we will detail on Subsection 4.1. The C^r-diffeomorphism F_S, $r \geqslant 1$, with base dynamics given by an Anosov diffeomorphism $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ having two fixed points p and q, was obtained in [57] through an isotopy between a linear Anosov diffeomorphism $L : \mathbb{T}^2 \to \mathbb{T}^2$ and a Derived from Anosov diffeomorphism $\mathfrak{D}$ of $\mathbb{T}^2$. The latter is generated by a smooth local bifurcation of a fixed point of L into a sink and two saddles, as described in [2]. This way, $F_S : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ is defined by $F_S(x, y) = (\Phi(x), f_x(y))$, where $f_p = L$, $f_q = \mathfrak{D}$ and there exist small values $0 < \varrho_1 < \varrho_2$ such that

$$f_x = \begin{cases} L & \text{if } x \notin B_{\varrho_2}(q) \\ \mathfrak{D} & \text{if } x \in B_{\varrho_1}(q) \end{cases}$$

where $B_\varrho(q)$ stands for the open ball centered at q with radius ϱ. The diffeomorphisms in $\mathcal{V}$ are robustly transitive, non-uniformly hyperbolic, topologically Ω-stable but not Ω-stable.

The first study of the ergodic properties of these systems was done by Newhouse and Young in [48], where it is proved that there exists a C^1-open set $\mathcal{V}$ of Shub's examples such that each $G \in \mathcal{V}$ has a unique probability measure with maximal entropy. One expects that this measure has a strong tie with other dynamical properties; in particular, it would be relevant to show that this measure describes the distribution of the periodic points of G (meaning that it is the weak*-limit of the sequence of Dirac measures supported on the sets of n-periodic points, $n \in \mathbb{N}$). We prove that this attribute, which is known to be valid within the uniformly hyperbolic setting (cf. [9]) and for Mañé's Derived from Anosov examples on $\mathbb{T}^3$ (cf. [22, Theorem 1.3]), also holds in a C^2-residual subset $\mathfrak{R}$ of $\mathcal{V}$. Both properties of G are a consequence of the existence of a semi-conjugation between G and the uniformly hyperbolic dynamics $\Phi \times L$, besides a careful analysis of the periodic fibers induced by the semi-conjugation.

The second question we address concerns the growth rate of periodic orbits with respect to the period, and whether the distribution of these orbits is detected by the measure with maximal entropy. To estimate the growth rate of periodic orbits of a diffeomorphism $f : M \to M$ one takes, for each $n \in \mathbb{N}$, the cardinal $gr_n(f)$ of the set of isolated fixed points of f^n, and verify how it changes when n goes to $+\infty$. The nature of this growth rate seems to depend mainly on the amount of hyperbolicity f exhibits and its degree of regularity. For instance, for any $r \geqslant 1$, there is a C^r-dense subset of diffeomorphisms f whose growth rate is at most exponential (cf. [5]), that is, there exists $K > 0$ such that $gr_n(f) \leqslant e^{nK}$. Besides, every Axiom A C^1-diffeomorphism f satisfies $\lim_{n\to+\infty} gr_n(f) = e^{nh_{\text{top}}(f)}$, where $h_{\text{top}}(f)$ stands for the topological entropy of f (cf. [14]). On the other hand, in the complement of the hyperbolic setting, Kaloshin proved in [40] the super-exponential growth of periodic points within Newhouse C^2-domains on surfaces. We recall that the latter are C^2-open sets of diffeomorphisms where maps with homoclinic tangencies are dense; and that the standard way to get Newhouse domains is by the generic unfolding of a homoclinic tangency of a C^2-surface diffeomorphism [47]. Kaloshin showed that, in any such a domain and for every sequence of positive integers $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$, there exists a C^2-residual subset, which depends on $\mathbf{a}$, whose elements f satisfy the condition $\limsup_{n\to+\infty} gr_n(f)/a_n = +\infty$. In particular, this indicates that the C^r-dense subset constructed in [5] is not C^r-generic when $r \geqslant 2$. A result similar to [40] in the C^1-topology and in any manifold M of dimension $\geqslant 3$ was proved by Bonatti, Díaz and Fisher [7], replacing the Newhouse domains by the open set of diffeomorphisms with a C^1-robust heterodimensional cycle. Since there are no Newhouse C^1-domains on surfaces (cf. [46]), Kaloshin's result is still an open problem in this context. It is not known (though it is not expected) whether the construction in [7] is valid for higher regularity topologies, due to the dependence on techniques which are only feasible within the C^1-topology. The C^2-diffeomorphisms of $\mathbb{T}^2 \times \mathbb{T}^2$ we consider in this work exhibit C^1-robust heterodimensional cycles, and we show that C^2-generically these examples have an asymptotic exponential growth rate of the number of periodic orbits given by the topological entropy, as happens in the hyperbolic setting. Thereby, we also convey an improved description of the symbolic dynamics of the diffeomorphisms in $\mathfrak{R}$. More precisely, we show that every diffeomorphism in the C^2-residual set $\mathfrak{R}$ has a sub-exponential growth rate of the periodic orbits in arbitrarily small scales (the so-called asymptotically per-expansiveness as defined in [17]). This result enables us to build a symbolic extension, from whose properties we conclude that C^2-generically in $\mathcal{V}$ the set of Borel invariant probability measures is homeomorphic to the space of Borel probability measures invariant by a subshift.

For topologically transitive Axiom A C^2-attractors, the work of Bowen, Ruelle and Sinai (we refer the reader to [14] and references therein) proves the existence of a unique invariant probability measure, the so-called SRB measure, that is characterized by obeying Pesin's formula [51]. From Ledrappier and L.-S. Young's work [43], the property that defines an SRB measure for a C^2-diffeomorphism is known to be equivalent to the existence of a disintegration of the measure in conditional measures on unstable manifolds which are absolutely continuous with respect to the Lebesgue measure. Moreover, the SRB measure is also the unique physical measure (cf. [14, Theorem 4.12]; a thorough essay on the existence and uniqueness of both SRB and physical measures within more general settings may be read in [62]). Regarding the C^2-diffeomorphisms in $\mathcal{V}$, the existence of an SRB measure was proved in [25]. We show that, under the additional assumption that Φ and L are both linear hyperbolic automorphisms of the 2-torus, and reducing, if necessary, the set of C^2 diffeomorphisms G we consider in the neighborhood $\mathcal{V}$ of F_S, then G is mostly contracting with a minimal strong unstable foliation, and so (cf. [6]) it has a unique ergodic SRB measure, whose basin of attraction has full Lebesgue measure (hence it is also G's unique physical measure).

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2. Main results

Denote by $\mathrm{Diff}^r(M)$, $r \geqslant 1$, the space of C^r-diffeomorphisms of a compact Riemannian manifold M in itself, endowed with the C^r-norm. Let $f \in \mathrm{Diff}^1(M)$ be the restriction of an Axiom A diffeomorphism with no cycles to a basic piece of its Smale's spectral decomposition. It is known that f satisfies the following conditions, which are strongly related to the expansiveness and specification properties that hyperbolic systems comply with:

UNIQUE MEASURE WITH MAXIMAL ENTROPY. f preserves a unique probability measure μ which satisfies $h_\mu(f) = h_{\mathrm{top}}(f)$, where $h_\mu(f)$ denotes the metric entropy of the f-invariant probability measure μ and $h_{\mathrm{top}}(f)$ stands for the topological entropy of f (cf. [11] and [59] for definitions).

EQUIDISTRIBUTION OF THE PERIODIC POINTS. The measure with maximal entropy of f is the limit in the weak* topology of the sequence of Dirac measures supported on the sets of n-periodic points, say $\mathrm{Per}_n(f) = \{x \in M : f^n(x) = x\}$, for $n \in \mathbb{N}$ (cf. [11]).

LIMIT OF HORSESHOES IN THE SENSE OF ENTROPY. Given $\varepsilon > 0$ there exists a hyperbolic f-invariant subset Λ_ε such that $f|_{\Lambda_\varepsilon}$ is conjugate to a subshift of finite type and $h_{\mathrm{top}}(f|_{\Lambda_\varepsilon}) > h_{\mathrm{top}}(f) - \varepsilon$ (cf. [61]).

EQUAL TOPOLOGICAL AND PERIODIC ENTROPIES. One has

$$\lim_{n\to+\infty} \frac{1}{n} \log \#\mathrm{Per}_n(f) = h_{\mathrm{top}}(f)$$

(cf. [14]).

SYMBOLIC EXTENSION. f is a factor of a subshift of finite type (cf. [10]). This symbolic extension of f is principal and strongly faithful with embedding, in the sense of [17].

These attributes are not valid in general outside the hyperbolic setting (cf. [7, 36]). The aim of this work is to prove them on a class of partially hyperbolic diffeomorphisms of $\mathbb{T}^2\times\mathbb{T}^2$ with a non-hyperbolic one-dimensional central direction, contained in the family of Shub's examples. In order to do so, we will need to demand more regularity of those systems and restrict to a residual subset of them.

We start remarking that, in a broad class of non-hyperbolic systems, the existence of at least one probability measure with maximal entropy is guaranteed. Indeed, this is valid for entropy-expansive diffeomorphisms (cf. [45]), and it was shown in [25] (see also [26, 27] for generalizations) that, when the central bundle is one-dimensional, then the system is entropy-expansive. So Shub's examples are endowed with a probability measure with maximal entropy.

Moreover, the construction in [48] provides a C^1-open set of Shub's examples for which the uniqueness of the measure with maximal entropy is also ensured (a generalization of this property for equilibrium states may be found in [23]). Nevertheless, without additional assumptions, this measure may not describe the distribution of the periodic points and the topological entropy may be different from the periodic one. Yet, as we will explain, Shub's examples may be obtained as the C^1-neighborhood $\mathcal{V}$ of an adequate C^∞ skew-product F_S in such a way that each diffeomorphism in $\mathcal{V}$ is a limit of horseshoes in the sense of entropy.

Besides, if we restrict to the Kupka–Smale C^2-diffeomorphisms (which we denote by $\mathcal{KS}$) then we can control the growth of the periodic orbits at arbitrarily small scales, which implies the equality of the topological and periodic entropies and the equidistribution of the periodic points, this way improving the statement of [22, Theorem 1.3].

Theorem A: *There exist a C^∞ skew-product*

$$F_S : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$$

and a C^1-open neighborhood $\mathcal{V}$ of F_S in $\mathrm{Diff}^1(\mathbb{T}^2 \times \mathbb{T}^2)$ such that every G in $\mathcal{V}$ is a limit of horseshoes in the sense of the entropy. Moreover, there is a C^2-open set $\mathcal{U} \subset \mathcal{V}$ in $\mathrm{Diff}^2(\mathbb{T}^2\times\mathbb{T}^2)$ such that every diffeomorphism G in the C^2-residual subset $\mathfrak{R} = \mathcal{U} \cap \mathcal{KS}$ satisfies the following properties:

(a) $h_{\mathrm{top}}(G) = \lim_{n\to+\infty} \frac{1}{n} \log \#\mathrm{Per}_n(G)$.

(b) *The measure with maximal entropy of G describes the distribution of periodic points.*

As previously mentioned, Shub's examples are entropy-expansive, and this is a sufficient condition for the existence of a principal symbolic extension (cf. [26]). Moreover, if we restrict to $\mathfrak{R}$, the diffeomorphisms satisfy a stronger property (namely the asymptotically per-expansiveness) and such an extension may be constructed in such a way that the corresponding semi-conjugation preserves the periodic points and induces a homeomorphism between the respective spaces of invariant probability measures.

Theorem B: *Every diffeomorphism of the C^2-residual subset $\mathfrak{R}$ has a principal strongly faithful symbolic extension with embedding.*

According to [38, Chapter 8], there exists a C^1-neighborhood $\mathcal{V}$ of F_S such that for each $G \in \mathcal{V}$ there is a homeomorphism $\Gamma_G : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ such that

$$\mathrm{Sp}(G) := \Gamma_G \circ G \circ \Gamma_G^{-1} : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$$

is a partially hyperbolic skew-product with base dynamics Φ and for which we may find a continuous surjective skew-product H_G such that

$$(H_G \circ \mathrm{Sp}(G))(x, y) = (\Phi \times L) \circ H_G(x, y) \quad \forall (x, y) \in \mathbb{T}^2 \times \mathbb{T}^2.$$

Moreover, the open neighborhoods $\mathcal{V}$ and $\mathcal{U}$ of F_S may be chosen so that, if F_S is of class C^2 and $G \in \mathcal{U}$, then $\mathrm{Sp}(G)$ satisfies the technical assumption in [39, Definition 2], and so $\mathrm{Sp}(G)$ is of class C^2 as well and belongs to $\mathcal{V}$ (cf. [39, p. 2398]). Therefore, the strong unstable foliation of $\mathrm{Sp}(G)$ is minimal (cf. Proposition 5.2), and so, if Φ is a linear hyperbolic automorphism of $\mathbb{T}^2$, then $\mathrm{Sp}(G)$ is mostly contracting (see Lemma 10.1). Thus, as stated by [6], $\mathrm{Sp}(G)$ has a unique ergodic SRB measure, whose basin of attraction has full Lebesgue measure. Consequently, this SRB measure is $\mathrm{Sp}(G)$'s unique physical measure. Under this additional assumption on Φ, the skew-product $\mathrm{Sp}(G)$ inherits from $\Phi \times L$ two further properties.

Theorem C: *Assume that Φ and L are both linear hyperbolic automorphisms of $\mathbb{T}^2$. Then, for every $G \in \mathcal{U}$, the set $\mathbb{T}^2 \times \mathbb{T}^2$ is a partially hyperbolic attractor supporting a unique ergodic SRB probability measure whose basin has full Lebesgue measure. Thus, it is the unique physical measure of G. Moreover, for every $G \in \mathcal{U}$, one has:*

(a) *The image by $(H_G)_*$ of the SRB measure of $\mathrm{Sp}(G)$ is the Lebesgue measure of $\mathbb{T}^2 \times \mathbb{T}^2$.*
(b) *The SRB measure of $\mathrm{Sp}(G)$ is its measure with maximal entropy.*

2.1. Organization of the paper. Section 3 contains a short glossary for the reader's convenience. In Section 4 we describe the class of partially hyperbolic diffeomorphisms of $\mathbb{T}^2 \times \mathbb{T}^2$ this work comprises, state their main properties and present the construction of a C^∞ skew-product belonging to the family of Shub's examples. In Section 5 we prove some preliminary information, to be summoned later when we show the main results. The proofs of the first part and items (a) and (b) of Theorem A are given in Sections 6, 7 and 8, respectively. Theorem B is proved in Section 9 and the argument to set up Theorem C is explained in Section 10.

3. Glossary

We begin introducing the main definitions used in this work. Given a compact metric space (X, d) and a continuous map $f : X \to X$, denote by $\mathscr{P}(X)$ the set of Borel probability measures on X endowed with the weak*-topology, and by $\mathscr{P}(X, f)$ and $\mathscr{P}_e(X, f)$ its subsets of f-invariant and f-invariant ergodic elements, respectively.

3.1. Maximal entropy measures. For each μ in $\mathscr{P}(X, f)$, consider the metric entropy $h_\mu(f)$ of f with respect to μ (definition in [59, Section 4]). The Variational Principle [59, Theorem 9.10] states that the topological entropy $h_{\text{top}}(f)$ of f coincides with the supremum of the operator $\mu \mapsto h_\mu(f)$ restricted to either $\mathscr{P}(X, f)$ or $\mathscr{P}_e(X, f)$. A measure $\mu \in \mathscr{P}(X, f)$ such that

$$h_\mu(f) = h_{\text{top}}(f)$$

is called a **measure with maximal entropy** of f.

3.2. Distribution of periodic points. Assume that the cardinality $\#\text{Per}_n(f)$ of the set of the fixed points of f^n is finite for every $n \in \mathbb{N}$. We say that a probability measure $\mu \in \mathscr{P}(X, f)$ **describes the distribution of the periodic points of** f if μ is the weak* limit of the sequence of probability measures

$$n \in \mathbb{N} \mapsto \frac{1}{\#\text{Per}_n(f)} \sum_{x \in \text{Per}_n(f)} \delta_x$$

where δ_x denotes the Dirac measure supported at x.

3.3. Expansiveness. Denote by $B_\rho(x)$ the open ball in the metric d centered at x with radius ρ, and by $\overline{B_\rho(x)}$ its closure. Define, for each $n \in \mathbb{N}$, the equivalent metric

$$(x,y) \in X \times X \mapsto d_n(x,y) \stackrel{\text{def}}{=} \max_{0 \leqslant j \leqslant n-1} d(f^j(x), f^j(y)).$$

Given $\varepsilon > 0$, $n \in \mathbb{N}$ and a compact subset $Y \subset X$, a subset S of X is said to be (n,ε)-**spanning** of Y, if for every $y \in Y$ there is $a \in S$ such that

$$d_n(y,a) \leqslant \varepsilon.$$

The minimum cardinality of the (n,ε)-spanning subsets of Y is denoted by $r_n(Y,\varepsilon)$. Define

$$\overline{r}(Y,\varepsilon) \stackrel{\text{def}}{=} \limsup_{n\to+\infty} \frac{1}{n} \log r_n(Y,\varepsilon) \quad \text{and} \quad \overline{h}_{\text{top}}(f,Y) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0^+} \overline{r}(Y,\varepsilon).$$

Having fixed $\varepsilon > 0$ and $x \in X$, consider the set of points in X whose forward orbits by f are ε-close to the forward orbit of x, that is,

$$B^f_{\infty,\varepsilon}(x) \stackrel{\text{def}}{=} \bigcap_{i\in\mathbb{N}} f^{-i}(\overline{B_\varepsilon(f^i(x))}) = \{y \in X : d(f^i(x), f^i(y)) \leqslant \varepsilon, \forall i \in \mathbb{N}\}. \tag{3.1}$$

Now define

$$h^*_{\text{top}}(f,\varepsilon) \stackrel{\text{def}}{=} \sup_{x\in X} \overline{h}_{\text{top}}(f, B^f_{\infty,\varepsilon}(x)) \quad \text{and} \quad h^*_{\text{top}}(f) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0^+} h^*_{\text{top}}(f,\varepsilon). \tag{3.2}$$

When f is a homeomorphism we ought also to consider backward iterates in the previous definitions of d_n and $B^f_{\infty,\varepsilon}$; that is,

$$d_n(x,y) \stackrel{\text{def}}{=} \max_{|j|\leq n-1} d(f^j(x), f^j(y)) \quad \text{and} \quad B^f_{\infty,\varepsilon}(x) \stackrel{\text{def}}{=} \bigcap_{i\in\mathbb{Z}} f^{-i}(\overline{B_\varepsilon(f^i(x))}). \tag{3.3}$$

However, as X is compact, the new value of $h^*_{\text{top}}(f,\varepsilon)$ is equal to the one obtained in (3.2) with the definition (3.1), as proved in [12, Corollary 2.3].

The map f is said to be **entropy-expansive** if there is $\varepsilon > 0$ such that

$$h^*_{\text{top}}(f,\varepsilon) = 0,$$

and **asymptotically entropy-expansive** if

$$h^*_{\text{top}}(f) = 0.$$

Misiurewicz has shown in [45] that for asymptotically entropy-expansive maps the entropy operator $\mu \in \mathscr{P}(X,f) \to h_\mu(f)$ is upper semi-continuous, guaranteeing the existence of at least a measure with maximal entropy for f.

Given $\varepsilon > 0$, consider

$$\begin{aligned}\operatorname{Per}(f,\varepsilon) &\stackrel{\text{def}}{=} \limsup_{n\to+\infty} \frac{1}{n} \sup_{x\in X} \log \#(\operatorname{Per}_n(f) \cap B^f_{\infty,\varepsilon}(x)),\\ \operatorname{Per}^*(f) &\stackrel{\text{def}}{=} \lim_{\varepsilon\to 0^+} \operatorname{Per}(f,\varepsilon).\end{aligned} \tag{3.4}$$

Following [17], the map f is said to be **asymptotically per-expansive** if $\operatorname{Per}^*(f) = 0$. For instance, expansive or aperiodic maps are asymptotically per-expansive. An interesting connection between the entropy, the growth of the cardinality of the periodic orbits with the period and the asymptotic per-expansiveness is given in the next lemma.

Lemma 3.1 ([18, Lemma 2.2]): $\limsup_{n\to+\infty} \frac{1}{n} \log \#\operatorname{Per}_n(f) \leqslant h_{\mathrm{top}}(f) + \operatorname{Per}^*(f)$.

Thus, if f is asymptotically per-expansive then

$$\limsup_{n\to+\infty} \frac{1}{n} \log \#\operatorname{Per}_n(f) \leqslant h_{\mathrm{top}}(f),$$

an inequality that generalizes [59, Theorem 8.16].

3.4. Partial hyperbolicity. Assume that X is a compact connected Riemannian manifold and that f is a C^r-diffeomorphism, $r \geqslant 1$. An f-invariant compact set $\Lambda \subset X$ is **partially hyperbolic** if the tangent bundle on Λ admits a Df-invariant splitting

$$E^{\mathrm{s}}(f) \oplus E^{\mathrm{c}}(f) \oplus E^{\mathrm{u}}(f)$$

such that E^{s} is uniformly contracted, E^{u} is uniformly expanded and the possible contraction and expansion of Df along $E^{\mathrm{c}}(f)$ are weaker than those in the complementary bundles. More precisely, there exist constants $N \in \mathbb{N}$ and $\lambda > 1$ such that, for every $x \in \Lambda$ and every unit vector $v^* \in E^*(x,f)$, where $* = \mathrm{s}, \mathrm{c}, \mathrm{u}$, we have

(a) $\lambda \|Df^N_x(v^{\mathrm{s}})\| < \|Df^N_x(v^{\mathrm{c}})\| < \lambda^{-1} \|Df^N_x(v^{\mathrm{u}})\|$,
(b) $\|Df^N_x(v^{\mathrm{s}})\| < \lambda^{-1} < \lambda < \|Df^N_x(v^{\mathrm{u}})\|$.

We say that an f-invariant compact set $\Lambda \subset X$ is a **partially hyperbolic attracting** set if there exists an open neighborhood U of Λ such that $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(U)$, and there is a continuous Df-invariant splitting of the tangent bundle at Λ into a strong unstable sub-bundle E^{u} and a center sub-bundle E^{c} dominated by E^{u}. More precisely, $T_\Lambda X = E^{\mathrm{u}} \oplus E^{\mathrm{c}}$ and

$$\|(Df|_{E^{\mathrm{u}}})^{-1}\| < 1 \quad \text{and} \quad \|Df|_{E^{\mathrm{c}}}\| \|(Df|_{E^{\mathrm{u}}})^{-1}\| < 1.$$

Partial hyperbolicity is a C^1-robust property, and a partially hyperbolic diffeomorphism f admits stable and unstable foliations, say $W^{\mathrm{s}}(f)$ and $W^{\mathrm{u}}(f)$, which are f-invariant and tangent to $E^{\mathrm{s}}(f)$ and $E^{\mathrm{u}}(f)$. However, the center bundle $E^{\mathrm{c}}(f)$ may not have a corresponding tangent foliation (cf. [37]). For a comprehensive exposition on partial hyperbolicity, we refer the reader to [8].

Suppose that f has a partially hyperbolic attracting set. We say that f is **mostly contracting** if, from the point of view of the natural volume within the unstable leaves, the asymptotic forward behavior along the central direction is contracting; that is, given any u-dimensional disk D inside an unstable leaf of W^{u}, there exists a positive volume measure subset $A \subset D$ whose points satisfy

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|Df^n|_{E^{\mathrm{c}}(x)}\| < 0 \quad \forall x \in A.$$

We note that, by [3], the set of partially hyperbolic diffeomorphisms whose central direction is mostly contracting is open in the C^r-topology for any $r \geqslant 2$.

3.5. Symbolic extensions. A map $f : X \to X$ has a **symbolic extension** if there exists $m \in \mathbb{N}$, a closed σ-invariant subset Σ of $\{0, 1, \ldots, m\}^{\mathbb{Z}}$, and a continuous surjective map $\pi : \Sigma \to X$ such that $f \circ \pi = \pi \circ \sigma$, where σ stands for the shift map. Such a symbolic extension is **principal** if π preserves the metric entropy, that is, $h_\eta(\sigma) = h_\mu(f)$ for every f-invariant measure μ and every σ-invariant measure η such that $\mu = \pi_*(\eta)$. If, in addition, there is a Borel measurable map $\tau : X \to \Sigma$ such that

$$\pi \circ \tau = \mathrm{Identity}_X, \quad \sigma \circ \tau = \tau \circ f \quad \text{and} \quad \Sigma = \overline{\tau(X)},$$

then $(\Sigma, \sigma, \pi, \tau)$ is called a **symbolic extension with embedding**. A symbolic extension (Σ, σ, π) is said to be **strongly faithful** if the induced map $\pi_* : \mathscr{P}(\Sigma, \sigma) \to \mathscr{P}(X, f)$ is a homeomorphism and if π preserves periodic points, that is, for any $n \in \mathbb{N}$ one has

$$\pi(\mathrm{Per}_n(\sigma_{|\Sigma})) = \mathrm{Per}_n(f).$$

The existence of symbolic extensions seems to depend on hyperbolic-type properties of f and its degree of differentiability. For instance, it was proved by Boyle, D. Fiebig and U. Fiebig in [15], and independently by Downarowicz in [28], that if f is asymptotically entropy-expansive, then it has a principal symbolic extension. Years before, using Yomdin's theory [60], Buzzi established in [20] that C^∞ diffeomorphisms are asymptotically entropy-expansive; thus such systems admit principal symbolic extensions. In addition, Downarowicz and Maass proved in [29] the existence of symbolic extensions for interval C^r-maps ($r > 1$), and Burguet showed in [16] that, for C^2-diffeomorphisms on surfaces, symbolic extensions are sure to exist. On the other hand, Downarowicz and Newhouse proved in [30] that a generic area-preserving C^1-diffeomorphism of a compact surface is either Anosov or has no symbolic extension. Regarding the nonexistence of symbolic extensions for generic C^1-diffeomorphisms, we also refer the reader to [4, 24, 21].

In addition, Cowieson and L.-S. Young showed in [25] that every partially hyperbolic C^1-diffeomorphism with a one-dimensional center bundle is entropy-expansive (see generalizations in [26, 27, 19] regarding partially hyperbolic systems with either a central bundle splitting in a dominated way into one-dimensional sub-bundles or a 2-dimensional center bundle). Therefore, if f is partially hyperbolic with a one-dimensional center bundle then a principal symbolic extension exists. In particular, every Shub's example in $\mathcal{V}$ has a principal symbolic extension. We will show that, if we restrict to $\mathfrak{R}$, then the diffeomorphisms are asymptotically per-expansive and have a strongly faithful extension with embedding.

For future use, we register that, according to [17, Main Theorem], the following four conditions together are enough to guarantee that f has a principal strongly faithful symbolic extension with embedding:

(1) f is entropy-expansive.
(2) f is asymptotically per-expansive.
(3) $\mathrm{Per}(f)$ is zero-dimensional.
(4) There exists $K > 0$ such that
  (i) $h_{\mathrm{top}}(f) < \log K$;
  (ii) $\#\mathrm{Per}_n(f) \leqslant K^n$ for every $n \in \mathbb{N}$.

3.6. Hyperbolic measures. Given $x \in X$ and $v \in T_xX$, define the **upper Lyapunov exponent** of v at x by

$$\lambda^+(x,v) \stackrel{\text{def}}{=} \limsup_{n\to+\infty} \frac{1}{n} \log \|D_x f^n(v)\|.$$

The **lower Lyapunov exponent** of v at x, say $\lambda^-(x,v)$, is obtained replacing $\limsup$ by $\liminf$ in the previous definition. The function $\lambda^+ : TX \to \mathbb{R}$ can only take a finite number $\ell(x)$ of different values on each tangent space T_xX, say $\lambda_1(x) < \lambda_2(x) < \cdots < \lambda_{\ell(x)}(x)$, and associated to these there exists a filtration $L_1(x) \subset L_2(x) \subset \cdots \subset L_{\ell(x)}(x) = T_xX$ such that $\lambda^+(x,v) = \lambda_i(x)$ for every $x \in X$ and all $v \in L_i(x) \setminus L_{i-1}(x)$. Besides, the maps $(\lambda_i(x))_{1 \leqslant i \leqslant \ell(x)}$ are measurable and f-invariant; their values are called the **Lyapunov exponents** of f at x. For each $1 \leqslant i \leqslant \ell(x)$ and $x \in X$, the number

$$k_i(x) = \dim L_i(x) - \dim L_{i-1}(x)$$

is the multiplicity of the i-th exponent at x. Moreover, there exists a subset $\mathscr{O}(f) \subset X$ such that, if x belongs to $\mathscr{O}(f)$, then the limit

$$\lambda^+(x, v) = \lambda^-(x, v) = \lim_{n \to +\infty} \frac{1}{n} \log \|D_x f^n(v)\|$$

exists for every $v \neq 0$. The elements in $\mathscr{O}(f)$ are called **regular points**, and Oseledets' Theorem [49] ensures that the set of regular points $\mathscr{O}(f)$ has full μ measure for any $\mu \in \mathscr{P}(X, f)$. If, in addition, μ is ergodic, then the functions $x \to \lambda_i(x)$ and $x \to \ell(x)$ are constant at μ almost everywhere. We denote these constants by $\lambda_1(\mu) < \cdots < \lambda_\ell(\mu)$. An ergodic probability measure μ is said to be **hyperbolic** if $\lambda_i(\mu) \neq 0$ for every $i = 1, \ldots, \ell$.

3.7. SRB measures. Let $x \in X$ be a regular point of a C^1-diffeomorphism $f : X \to X$, and consider the sum (with multiplicity) of all the positive Lyapunov exponents at x, say

$$\chi^u(x) \stackrel{\text{def}}{=} \sum_{\{i:\lambda_i(x)>0\}} k_i(x)\lambda_i(x).$$

Margulis–Ruelle inequality [56] states that the metric entropy of every $\mu \in \mathscr{P}(X, f)$ is bounded above by the space average of χ^u, that is,

$$h_\mu(f) \leqslant \int \chi^u d\mu.$$

On the other hand, by Oseledets' Theorem, if $E^u(x)$ stands for the subspace of $T_x X$ corresponding to the positive Lyapunov exponents at the regular point $x \in X$ and $J^u(x)$ denotes the Jacobian of Df restricted to the subspace $E^u(x)$, then

$$\chi^u(x) = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |J^u(f^i(x))|.$$

Thus, for every Borel f-invariant probability measure μ one has

$$h_\mu(f) \leqslant \int \log |J^u| d\mu. \tag{3.5}$$

A probability measure μ attaining the equality in (3.5) is called an **SRB measure**.

Pesin proved in [51] that if $\mu \in \mathscr{P}(X, f)$ is equivalent to the Lebesgue measure (the Riemannian volume) then μ is an SRB measure. Afterwards, Ledrappier and L.-S. Young identified all the measures satisfying Pesin's entropy formula, establishing in [43] that the equality (3.5) holds if and only if the conditional measures of μ along the (Pesin) unstable manifolds are absolutely continuous with respect to the Lebesgue measure.

3.8. Physical measures. Let μ be a Borel f-invariant probability measure on X. A point $x \in X$ is called **μ-generic** if

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \int \varphi \, d\mu \quad \forall \varphi \in C^0(X, \mathbb{R})$$

where $C^0(X, \mathbb{R})$ stands for the space of continuous maps $\varphi : X \to \mathbb{R}$ with the uniform norm. We denote by $\mathfrak{B}(\mu)$ the set of μ-generic points, also called the **basin of attraction** of μ. The measure μ is called **physical** if $\mathfrak{B}(\mu)$ has positive Lebesgue measure. Note that, if the basin of μ has full Lebesgue measure, then μ is the unique physical measure of f.

For topologically transitive Axiom A C^2-attractors, there exists a unique invariant probability measure μ which is characterized by each of the following properties, equivalent to one another (cf. [14]):

(1) Equality (3.5) holds (that is, μ is SRB).
(2) The conditional measures of μ on unstable manifolds are absolutely continuous with respect to the Lebesgue measure.
(3) Lebesgue almost every point in a neighborhood of the attractor is generic with respect to μ (that is, μ is physical).

4. The setting

In this section we describe the class of skew-products introduced in [48] (Subsection 4.1), then we detail some of its properties (Subsection 4.2) and later we rebuild this class to increase the regularity of its maps (Subsection 4.3).

4.1. Skew-products. Let Φ and L be two Anosov diffeomorphisms of the 2-torus $\mathbb{T}^2$, L being a linear automorphism. Consider a family of C^1-diffeomorphisms $(f_x)_{x\in\mathbb{T}^2}$ acting on $\mathbb{T}^2$ and take the skew-product induced by Φ and $(f_x)_{x\in\mathbb{T}^2}$, defined by

$$\begin{aligned} F : \mathbb{T}^2 \times \mathbb{T}^2 &\to \mathbb{T}^2 \times \mathbb{T}^2 \\ (x,y) &\mapsto F(x,y) = (\Phi(x), f_x(y)). \end{aligned} \tag{4.1}$$

Assume that F has the following properties (see [48, p. 612]):

($\mathbf{S}_1$) The map $x \in \mathbb{T}^2 \to f_x \in \mathrm{Diff}^1(\mathbb{T}^2)$ is continuous.

($\mathbf{S}_2$) F is homotopic to $\Phi \times L$ as a bundle map, that is, the homotopic path is made of skew-products with fixed base dynamics Φ.

($\mathbf{S}_3$) There is a one-dimensional lamination $\mathcal{F}$ of $\mathbb{T}^2 \times \mathbb{T}^2$ which is F-invariant and normally expanding.

The first property means that each leaf $\mathcal{F}(x,y)$ through (x,y) is a smoothly immersed line in $\{x\} \times \mathbb{T}^2$ such that

$$F(\mathcal{F}(x,y)) = \mathcal{F}(F(x,y));$$

the second one means that there is a continuous splitting

$$(x,y) \to E^{\mathrm{u}}(x,y) \oplus E^{\mathrm{c}}(x,y)$$

of the tangent space to $\{x\} \times \mathbb{T}^2$ such that:

- $D_y f_x(E^{\mathrm{u}}(x,y)) = E^{\mathrm{u}}(F(x,y))$ and $D_y f_x(E^{\mathrm{c}}(x,y)) = E^{\mathrm{c}}(F(x,y))$.
- $E^{\mathrm{c}}(x,y) = T_{(x,y)}\mathcal{F}(x,y)$.
- There is a Riemannian metric on $\{x\} \times \mathbb{T}^2$ with induced norm $\|.\|$ such that

$$\inf_{(x,y)\in\mathbb{T}^2\times\mathbb{T}^2} \|D_y f_x|_{E^{\mathrm{u}}(x,y)}\| > \max\{1, \sup_{(x,y)\in\mathbb{T}^2\times\mathbb{T}^2} \|D_y f_x|_{E^{\mathrm{c}}(x,y)}\|\}.$$

Thus, F is partially hyperbolic with a one-dimensional center bundle and a splitting

$$\mathrm{T}(\mathbb{T}^2 \times \mathbb{T}^2) = E^{\mathrm{ss}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{u}} \oplus E^{\mathrm{uu}} \tag{4.2}$$

where the splitting $E^{\mathrm{ss}} \oplus E^{\mathrm{uu}}$ is related to the hyperbolicity of the base Φ and the splitting $E^{\mathrm{c}} \oplus E^{\mathrm{u}}$ is related to the dynamics at the leaves of the lamination $\mathfrak{L} := \{\{x\} \times \mathbb{T}^2\}_x$. Note that F preserves the lamination $\mathfrak{L}$ thus, replacing Φ by one of its iterates if necessary, we can assume that F is normally hyperbolic to $\mathfrak{L}$.

4.2. PROPERTIES. For future use, we list here the main properties of the previous skew-products.

4.2.1. *Semi-conjugation with an Anosov diffeomorphism.* Under the previous assumptions on F, it was shown in [48, Lemmas 1 & 3] that there exists a continuous surjective skew-product $H : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ of the form

$$H(x, y) = (x, h_x(y)),$$

where $h_x : \{x\} \times \mathbb{T}^2 \to \{x\} \times \mathbb{T}^2$ is homotopic to the identity, satisfies the equality

$$h_{\Phi(x)} \circ f_x = L \circ h_x \quad \forall x \in \mathbb{T}^2 \tag{4.3}$$

and:

($\mathbf{H}_1$) For every $(x, y) \in \mathbb{T}^2 \times \mathbb{T}^2$, one has

$$(H \circ F)(x, y) = (\Phi \times L) \circ H(x, y). \tag{4.4}$$

($\mathbf{H}_2$) $h_{\mathrm{top}}(H^{-1}\{(x, y)\}) = 0 \quad \forall (x, y) \in \mathbb{T}^2 \times \mathbb{T}^2$.

The semi-conjugation H can be seen as the result of a parameterized version of a theorem due to Franks [31]. An immediate consequence of ($\mathbf{H}_2$) and Bowen's inequality [13] is the following estimate:

$$h_{\mathrm{top}}(F) = h_{\mathrm{top}}(\Phi \times L) = h_{\mathrm{top}}(\Phi) + h_{\mathrm{top}}(L). \tag{4.5}$$

4.2.2. *Unique maximal entropy measure.* Using the semi-conjugation H between F and $\Phi \times L$, Newhouse and L.-S. Young have established in [48] sufficient conditions for the existence of a unique probability measure $\mu_{\max}$ of maximal entropy for F, and proved that $H_*(\mu_{\max}) = \nu_{\max}$, where $\nu_{\max}$ stands for the probability measure with maximal entropy of $\Phi \times L$. Moreover, the pairs $(F, \mu_{\max})$ and $(\Phi \times L, \nu_{\max})$ are almost conjugate: more precisely, there exists a Φ-invariant Borel set B such that $B \times \mathbb{T}^2$ is contained in the set of injectivity points of H, say

$$\mathcal{A} \stackrel{\mathrm{def}}{=} \{(x, y) \in \mathbb{T}^2 \times \mathbb{T}^2 \colon \# H^{-1}(x, y) = 1\}$$

and satisfies:

($\mathbf{M}_1$) $\mu_{\max}(B \times \mathbb{T}^2) = \nu_{\max}(B \times \mathbb{T}^2) = 1$.

($\mathbf{M}_2$) The restrictions $F|_{B \times \mathbb{T}^2}$ and $(\Phi \times L)|_{B \times \mathbb{T}^2}$ are conjugated by $H|_{B \times \mathbb{T}^2}$.

Actually, $B \times \mathbb{T}^2$ is a subset of $\mathcal{E}$ (cf. [48, p. 624]), where

$$\mathcal{E} \stackrel{\text{def}}{=} \{(x,y) \in \mathbb{T}^2 \times \mathbb{T}^2 \colon \lambda_+^c(F)(x,y) < 0\} \subset \mathcal{A}$$

and $\lambda_+^c(F)$ stands for the upper Lyapunov exponent of F along to the one-dimension central direction $E^c(F)$. Therefore,

$$\mu_{\max}(\mathcal{E}) = \nu_{\max}(\mathcal{E}) = 1. \tag{4.6}$$

We remark that the properties ($\mathbf{M}_1$)–($\mathbf{M}_2$) are satisfied by the skew-products (which are Shub's examples) constructed in [48, p. 626].

Taking into account that $B \times \mathbb{T}^2 \subset \mathcal{A}$, we also note that the properties ($\mathbf{M}_1$) and ($\mathbf{H}_2$) of Subsection 4.2.1 allow us to apply [22, Theorem 1.5] to F, and thereby conclude that:

- ($\mathbf{M}_3$) The maximal entropy measure $\mu_{\max}$ describes the distribution of periodic classes of F.

Let us be more precise regarding this property. Consider the equivalence relation on the set $\mathbb{T}^2 \times \mathbb{T}^2$ given by

$$(x,y) \sim (x_0,y_0) \Leftrightarrow H(x,y) = H(x_0,y_0).$$

Then the elements in the class $[(x,y)]$ are the ones in $H^{-1}(\{H(x,y)\})$. The class $[(x,y)]$ is said to be n-periodic if $H(x,y)$ belongs to $\mathrm{Per}_n(\Phi \times L)$. Denote by $\widetilde{\mathrm{Per}}_n(F)$ the set of periodic classes with period n. Then $\mu_{\max}$ describes the distribution of periodic classes of F if $\mu_{\max}$ is the weak* limit of the sequence of measures

$$n \in \mathbb{N} \mapsto \zeta_n \stackrel{\text{def}}{=} \frac{1}{\#\widetilde{\mathrm{Per}}_n(F)} \sum_{[(x,y)] \in \widetilde{\mathrm{Per}}_n(F)} \delta_{[(x,y)]}$$

where $\delta_{[(x,y)]}$ is any F^n-invariant probability measure supported on the class $[(x,y)]$.

4.2.3. *Persistent properties.* According to [38, Chapter 8], there exists a C^1-neighborhood $\mathcal{V}$ of F such that for each $G \in \mathcal{V}$ there is a homeomorphism $\Gamma_G : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ so that

$$\mathrm{Sp}(G) := \Gamma_G \circ G \circ \Gamma_G^{-1} : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2 \tag{4.7}$$

is a bundle map covering Φ satisfying the conditions ($\mathbf{S}_1$)–($\mathbf{S}_3$) above. In particular, $\mathrm{Sp}(G)$ is a partial hyperbolic skew-product with splitting

$$T(\mathbb{T}^2 \times \mathbb{T}^2) = E^{ss}_{\mathrm{Sp}(G)} \oplus E^{c}_{\mathrm{Sp}(G)} \oplus E^{u}_{\mathrm{Sp}(G)} \oplus E^{uu}_{\mathrm{Sp}(G)}. \tag{4.8}$$

Therefore, for each $G \in \mathcal{V}$ there exists a continuous surjective skew-product H_G such that

$$H_G \circ \mathrm{Sp}(G)(x,y) = (\Phi \times L) \circ H_G(x,y) \quad \forall (x,y) \in \mathbb{T}^2 \times \mathbb{T}^2 \tag{4.9}$$

and H_G satisfies the conditions $(\mathbf{H}_1)$-$(\mathbf{H}_2)$. Consequently, every G in $\mathcal{V}$ has the following properties:

- $(\mathbf{P}_1)$ $h_{\mathrm{top}}(G) = h_{\mathrm{top}}(\mathrm{Sp}(G)) = h_{\mathrm{top}}(\Phi \times L) = h_{\mathrm{top}}(\Phi) + h_{\mathrm{top}}(L) > 0$.
- $(\mathbf{P}_2)$ G has a unique measure with maximal entropy.
- $(\mathbf{P}_3)$ G is semi-conjugated to $\Phi \times L$ by $h_G := H_G \circ \Gamma_G$, that is, $h_G \circ G = (\Phi \times L) \circ h_G$.

We remark that, if F is of class C^2 and satisfies the technical assumption called modified dominated splitting condition in [39, Definition 2], then we can apply [39] and conclude that, for small enough $\rho > 0$, any ρ-perturbation G of F in the C^2-topology has additional properties, such as:

- $(\mathbf{P}_4)$ There exists a continuous map $\wp_G \colon \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$\wp_G \circ G = \Phi \circ \wp_G$$

  and the homeomorphism $\Gamma_G \colon \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ has the form

$$\Gamma_G(x,y) = (\wp_G(x,y), y) \tag{4.10}$$

  (cf. [39, Theorem 1]).
- $(\mathbf{P}_5)$ The skew-product $\mathrm{Sp}(G)$ in (4.7) is of class C^2, is given by

$$(x,y) \in \mathbb{T}^2 \times \mathbb{T}^2 \mapsto \mathrm{Sp}(G)(x,y) = (\Phi(x), g_x(y)), \tag{4.11}$$

  and satisfies $d_{C^2}(f_x, g_x) \leqslant o(\rho)$ uniformly in $x \in \mathbb{T}^2$ (cf. [39, p. 2398]).

According to [39, Appendix A.2], the family $(g_x)_x$ defining the second coordinate of the skew-product $\mathrm{Sp}(G)$ satisfies

$$g_x(y) = \pi_2(G(\widetilde{\beta}_x(y), y)) = G_2(\widetilde{\beta}_x(y), y) \tag{4.12}$$

where π_2 is the natural projection on the second factor of $\mathbb{T}^2 \times \mathbb{T}^2$,

$$G(x,y) = (G_1(x,y), G_2(x,y))$$

and the pairs

$$\{(\widetilde{\beta}_x(y), y) \colon y \in \mathbb{T}^2\}$$

parameterize the leaf W_x of the G-invariant central lamination $(W_x)_{x \in \mathbb{T}^2}$ given by [38, Theorems 7.1, 7.4] (cf. [39, Subsection 3.3]). This lamination is C^2-normally hyperbolic, plaque expansive (cf. [48]) and C^2-near to the foliation $(\{x\} \times \mathbb{T}^2)_{x\in\mathbb{T}^2}$ (cf. [38, Section 6A]). In particular, as G is of class C^2, then so is $\mathrm{Sp}(G)$ and one has $d_{C^2}(f_x, g_x) \leqslant o(\rho)$ (cf. the computation in [39, p. 2398]).

4.3. Construction of Shub's examples. We now describe the construction of a Shub's example of class C^∞, say F_S, satisfying the properties $(\mathbf{S}_1)$–$(\mathbf{S}_3)$ on Subsection 4.1. Consequently, for every $1 \leqslant r \leqslant +\infty$, we can consider a C^r-open set $\mathcal{U} \subset \mathrm{Diff}^r(\mathbb{T}^2 \times \mathbb{T}^2)$ of Shub's examples containing F_S.

To build such a C^1-skew-product F_S and the corresponding neighborhood $\mathcal{U}$ in the C^1-topology, Shub considered a Derived from Anosov (DA) defined by splitting a saddle of the linear automorphism L into a source and two saddles by a large C^1-isotopy. Let $p \neq q$ be fixed points of L^2 and consider the map $F_S : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ defined by

$$F_S(x, y) = (L^2(x), f_x(y)),$$

where $x \in \mathbb{T}^2 \mapsto f_x \in \mathrm{Diff}^1(\mathbb{T}^2)$ is chosen so that

- $f_x = L$ for every x outside a small disc B of $\mathbb{T}^2$ such that $p \in \mathbb{T}^2 \setminus \overline{B}$;
- $f_x = \mathrm{DA}$ for every x inside a smaller disc $B' \subset B$ such that $q \in B'$;
- in between, the map $x \to f_x$ is an isotopy gluing L and DA.

Shub proceeded proving that F_S is a topologically transitive diffeomorphism, hence $\Omega(F_S) = \mathbb{T}^2 \times \mathbb{T}^2$. Moreover, by the Equivariant Fibration Theorem [57, Proposition 8.6] there exists a C^1-open neighborhood $\mathcal{V}$ of F_S such that every G in $\mathcal{V}$ is a topologically transitive partially hyperbolic diffeomorphism with a one-dimensional central direction. The results in [2] also show that no such G is structurally stable.

4.3.1. *Construction of a C^∞ skew-product F_S.* Let Φ be a C^∞ Anosov diffeomorphism having two fixed points, say $p \neq q$ and $\theta_0 \in \mathbb{T}^2$ the fixed point of L. Denote by λ_s and λ_u the eigenvalues associated to the unstable and stable eigenvectors $\mathbf{v}^u$ and $\mathbf{v}^s$ of the matrix DL (which we still denote by L if no confusion arises). Suppose that $0 < \lambda_s < 1 < \lambda_u = \lambda_s^{-1}$. Fix a small open neighborhood $W \stackrel{\text{def}}{=} W_1 \times W_2$ of (q, θ_0), within which we use coordinates $u_1\mathbf{v}^u + u_2\mathbf{v}^s$ along each fiber $\{w\} \times W_2$, where $w \in W_1$. Let $\varrho > 0$ be small enough so that the ball $B_\varrho(q, \theta_0) = B_\varrho(q) \times B_\varrho(\theta_0)$ of radius ϱ centered at (q, θ_0) is contained in W. Take a C^∞ bump function $\delta : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ defined by

$$\delta(x, y) \stackrel{\text{def}}{=} b(x)b(y),$$

where $b : \mathbb{T}^2 \to \mathbb{R}$ is a bump function satisfying $0 \leqslant b(x) \leqslant 1$ for every $x \in \mathbb{T}^2$, $b(x) = 1$ if $|x| < \varrho/2$ and $b(x) = 0$ if $|x| > \varrho$. Afterwards, consider the system of differential equations in $\mathbb{T}^2 \times \mathbb{T}^2$ given by

$$\text{(4.13)} \qquad \begin{cases} \dot{w} = \mathbf{0} & \text{in } \mathbb{T}^2, \\ (\dot{u}_1, \dot{u}_2) = (0, u_2 \delta(|w-q|, |(u_1,u_2)|)) & \text{in } \mathbb{T}^2. \end{cases}$$

Denote by φ^t the flow of the differential equation (4.13), that is,

$$\text{(4.14)} \qquad \varphi^t(w,(u_1,u_2)) = (w, \psi_w^t(u_1,u_2))$$

where

$$\psi_w^t(u_1,u_2) = (u_1, \psi_{w,2}^t(u_1,u_2)).$$

Note that $(w,u_1,u_2) \to \varphi^t(w,(u_1,u_2))$ is C^∞ and that the support of $\varphi^t - id$ is contained in W. Moreover, the derivative of the flow at (w,θ_0) in terms of the (w,u_1,u_2)-coordinates is given by

$$D_{(w,\theta_0)}\varphi^t = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & D_{\theta_0}\psi_w^t \end{pmatrix} \quad \text{where } D_{\theta_0}\psi_w^t = \begin{pmatrix} 1 & 0 \\ 0 & e^{tb(|w-q|)} \end{pmatrix}$$

while the bold numbers $\mathbf{0}$ and $\mathbf{1}$ stand for the null 2×2 matrix and the 2×2 identity matrix, respectively. Fix now $T > 0$ such that $1 < \lambda_s e^T < \lambda_u$ and define $F_S : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ by

$$\text{(4.15)} \qquad F_S \stackrel{\text{def}}{=} \varphi^T \circ (\Phi \times L).$$

Note that $f_x(\theta_0) = \theta_0$ for all $x \in \mathbb{T}^2$ and that, by the choice of T, the fixed point θ_0 is a source of f_q at the fiber $\{q\} \times \mathbb{T}^2$ (see Figure 1).

Furthermore, for each $x \in \mathbb{T}^2$ we have $f_x = \psi_{\Phi(x)}^T \circ L$, so the map

$$x \in \mathbb{T}^2 \to f_x \in \mathrm{Diff}^\infty(\mathbb{T}^2)$$

is of class C^∞ (property $(\mathbf{S}_1)$). Besides, for every $t \in [0,T]$, the map $\varphi^t \circ (\Phi \times L)$ is a skew-product with fixed base Φ, so F_S is homotopic to $\Phi \times L$ as bundle map (property $(\mathbf{S}_2)$). Finally, for every t, the flow φ^t preserves the stable foliation $\mathcal{F} := W_L^s$ of L. Using the arguments of [55, p. 300], it is not difficult to verify the existence of a DF_S-invariant expanding fiber bundle $E^{\mathrm{u}}(x,\cdot)$ of the tangent space to $\{x\} \times \mathbb{T}^2$, whose integration provides a foliation W^u transverse to $\mathcal{F}$. Summarizing,

$$\text{(4.16)} \quad T_{\{x\}\times\mathbb{T}^2} = E^{\mathrm{u}}(x,\cdot) \oplus E^{\mathrm{c}}(x,\cdot), \quad E^{\mathrm{c}}(x,y) = T_{(x,y)}\mathcal{F}(x,y), \quad \mathcal{F} = W_L^s.$$

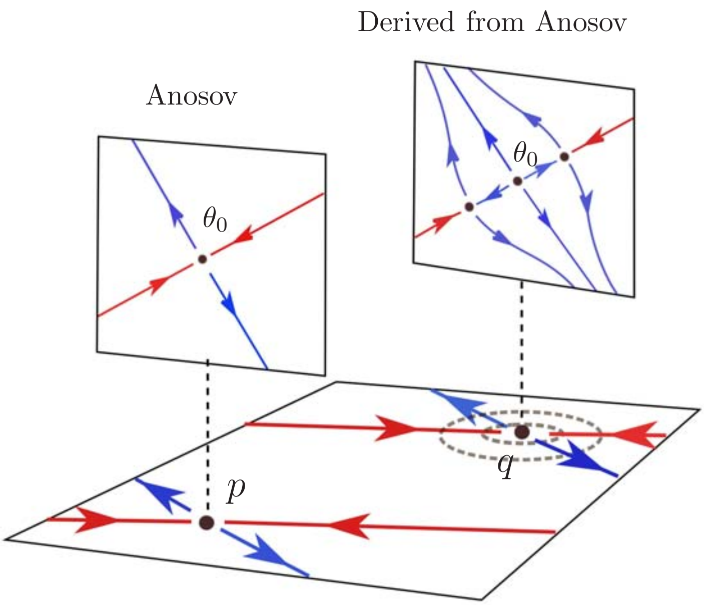

Figure 1. Homotopic deformation from $\Phi \times L$ to F_S.

The inequality $(\mathbf{S}_3)$ relating the norms $\|D_y f_x|_{E^{\mathrm{u}}(x,y)}\|$ and $\|D_y f_x|_{E^{\mathrm{c}}(x,y)}\|$ follows from the choice of T, which also ensures that F_S satisfies the modified dominated splitting of [39, Definition 2] we referred to on Subsection 4.2.3, namely

$$\max\{\max\{\kappa_1, \kappa_2\} + \|D_x(f_x)^{\pm}\|_{C^0}, \|D_y(f_x)^{\pm}\|_{C^0}\} < \min\{\kappa_1^{-1}, \kappa_2^{-1}\} \tag{4.17}$$

where $0 < \kappa_1 < 1$, $0 < \kappa_2 < 1$, $\|D\Phi|_{E^{\mathrm{s}}}\| \leqslant \kappa_1$ and $\|D\Phi^{-1}|_{E^{\mathrm{u}}}\| \leqslant \kappa_2$.

5. Periodic points and minimal foliations

In this section we collect some additional information that will be used in the proofs of our main results. Throughout this section $\mathcal{U} \subset \mathrm{Diff}^2(\mathbb{T}^2 \times \mathbb{T}^2)$ denotes a small C^2-neighborhood of the C^∞ diffeomorphism F_S defined by (4.15), whose elements comply with the results we have mentioned from Shub's [57], Newhouse–Young's [48], Ilyashenko–Negut's [39] and Andersson's [3] articles.

5.1. Hyperbolic periodic points. Recall that to each $G \in \mathcal{U}$ we can associate a C^2-skew-product $\mathrm{Sp}(G)$ satisfying the properties of Section 4, and a homeomorphism Γ_G of $\mathbb{T}^2 \times \mathbb{T}^2$ such that $\mathrm{Sp}(G) \circ \Gamma_G = \Gamma_G \circ G$ (cf. (4.7)).

Proposition 5.1: *Let $n \in \mathbb{N}$. If (x, y) is a hyperbolic fixed point of G^n, then $\Gamma_G(x, y)$ is a hyperbolic fixed periodic point of $\mathrm{Sp}(G)^n$.*

Proof. Recall from (4.11) that $\mathrm{Sp}(G)(x, y) = (\Phi(x), g_x(y))$, where the family of maps $(g_x)_{x\in\mathbb{T}^2}$ in $\mathrm{Diff}^2(\mathbb{T}^2)$ is the one presented in (4.12). For every $n \in \mathbb{N}$, write

$$G^n(x, y) = (G_1^n(x, y), G_2^n(x, y)) \quad \text{and} \quad \mathrm{Sp}(G)^n(x, y) = (\Phi^n(x), g_x^n(y))$$

where $g_x^n \colon \mathbb{T}^2 \to \mathbb{T}^2$ stands for the composition

$$g_x^n(y) \stackrel{\text{def}}{=} g_{\Phi^{n-1}(x)} \circ g_{\Phi^{n-2}(x)} \circ \cdots \circ g_x(y).$$

Since $\mathrm{Sp}(G)^n \circ \Gamma_G = \Gamma_G \circ G^n$ for every $n \in \mathbb{N}$ and $\Gamma_G(x, y) = (\wp_G(x, y), y)$ (see (4.10)) one has for all $(x, y) \in \mathbb{T}^2 \times \mathbb{T}^2$

$$\begin{aligned} g^n_{\wp_G(x,y)}(y) &= g_{\Phi^{n-1}(\wp_G(x,y))} \circ g_{\Phi^{n-2}(\wp_G(x,y))} \circ \cdots \circ g_{\wp_G(x,y)}(y) \\ &= \pi_2 \circ \mathrm{Sp}(G)^n(\Gamma_G(x, y)) \\ &= \pi_2 \circ \Gamma_G(G^n(x, y)) \\ &= G_2^n(x, y). \end{aligned} \tag{5.1}$$

Suppose now that the $G^n(x_0, y_0) = (x_0, y_0)$ and that (x_0, y_0) is hyperbolic. Then $\Gamma_G(x_0, y_0)$ is a fixed point of $\mathrm{Sp}(G)^n$ and

$$D_y g^n_{\wp_G(x_0,y_0)}(E^c_{\mathrm{Sp}(G)}(\Gamma_G(x_0, y_0))) = E^c_{\mathrm{Sp}(G)}(\Gamma_G(x_0, y_0))$$

where $E^c_{\mathrm{Sp}(G)}$ of $\mathrm{Sp}(G)$ is the fiber central bundle mentioned in (4.8). The equations (5.1) imply that $E^c_{\mathrm{Sp}(G)}(\Gamma_G(x_0, y_0))$ is also invariant by $D_y G_2^n(x_0, y_0)$. Thus, the hyperbolicity of (x_0, y_0) with respect to G^n ensures that the restriction of $D_y g^n_{\wp_G(x_0,y_0)}$ to the bundle $E^c_{\mathrm{Sp}(G)}(\Gamma_G(x_0, y_0))$ is different from the identity. Therefore, $\Gamma_G(x_0, y_0)$ is a fixed hyperbolic point of $\mathrm{Sp}(G)^n$. This ends the proof of the proposition. ∎

5.2. Minimal foliations. Consider $r \geqslant 1$ and $f \in \mathrm{Diff}^r(M)$. An f-invariant foliation $\mathcal{F}_f$ of M is called **minimal** if its leaves are dense on the manifold M. If, in addition, the foliation has a continuation $\mathcal{F}_g$ for every diffeomorphism g which is C^r-close to f and this continuation is minimal, then $\mathcal{F}_f$ is said to be **C^r-robustly minimal**.

Let $F_S \in \mathrm{Diff}^r(\mathbb{T}^2 \times \mathbb{T}^2)$ be the map defined in (4.15), $r \geqslant 1$, and consider its unstable foliation $W^u(F_S)$, tangent to $E^{\mathrm{u}} \oplus E^{\mathrm{uu}}$. Note that, for every G which is C^r-near F_S, there is a continuation $E^{\mathrm{u}}_G \oplus E^{\mathrm{uu}}_G$ of those bundles, and so G has an unstable foliation $W^u(G)$ as well.

Proposition 5.2: *The strong unstable foliation $W^u(F_S)$ is C^r-robustly minimal.*

Proof. The proposition follows directly from [53, Section 5]. ∎

6. Proof of Theorem A: First part

Throughout this section, $\mathcal{V} \subset \mathrm{Diff}^1(\mathbb{T}^2 \times \mathbb{T}^2)$ denotes a small C^1-neighborhood of the diffeomorphism F_S defined by (4.15). To prove the first part of Theorem A we need to recall an auxiliary result which extends to C^1-diffeomorphisms the classic Katok's theorem [41, Corollary 4.3] on the existence of horseshoes in the presence of hyperbolic measures.

Lemma 6.1 (Theorem 1-(iv) in [32]): *Let f be a C^1-diffeomorphism of a smooth Riemannian manifold M and μ a hyperbolic ergodic f-invariant Borel probability measure with positive entropy $h_\mu(f) > 0$. Suppose that the support of μ admits a dominated splitting. Then, for every $\varepsilon > 0$, there exists a basic set $\Lambda_\varepsilon \subset M$ such that*

$$|h_{\mathrm{top}}(f|_{\Lambda_\varepsilon}) - h_{\mathrm{top}}(f)| < \varepsilon.$$

We are left to prove that, when G is in $\mathcal{V}$, then the assumptions of Lemma 6.1 are valid for the corresponding skew-product $\mathrm{Sp}(G)$.

Proposition 6.2: *For every diffeomorphism $G \in \mathcal{V}$, there exists an $\mathrm{Sp}(G)$-invariant Borel probability measure $\mu_{\mathrm{Sp}(G)}$ which is hyperbolic and maximizes the entropy.*

Assume for the moment this proposition, and let us complete the proof of the first part of Theorem A.

Proof. If $\mu_{\mathrm{Sp}(G)}$ is the measure as in Proposition 6.2, we know that (see item $(\mathbf{P}_1)$ in Subsection 4.2)

$$h_{\mu_{\mathrm{Sp}(G)}}(\mathrm{Sp}(G)) = h_{\mathrm{top}}(\mathrm{Sp}(G)) = h_{\mathrm{top}}(G) = h_{\mathrm{top}}(F_S) = h_{\mathrm{top}}(\Phi) + h_{\mathrm{top}}(L) > 0.$$

So we can apply Lemma 6.1 and get, for every $\varepsilon > 0$, a set $\Lambda_\varepsilon \subset \mathbb{T}^2 \times \mathbb{T}^2$ which is invariant by $\mathrm{Sp}(G)$ and satisfies

$$h_{\mathrm{top}}(\mathrm{Sp}(G)|_{\Lambda_\varepsilon}) > h_{\mathrm{top}}(\mathrm{Sp}(G)) - \varepsilon.$$

Since G and $\mathrm{Sp}(G)$ are conjugate by Γ_G, then $\Delta_{G,\varepsilon} := \Gamma_G^{-1}(\Lambda_\varepsilon)$ is a G-invariant set such that

$$h_{\mathrm{top}}(G|_{\Delta_\varepsilon}) = h_{\mathrm{top}}(\mathrm{Sp}(G)|_{\Lambda_\varepsilon}) > h_{\mathrm{top}}(\mathrm{Sp}(G)) - \varepsilon = h_{\mathrm{top}}(G) - \varepsilon.$$

So G is a limit of horseshoes in the sense of entropy. ∎

Let us now show the pending result.

Proof of Proposition 6.2. Recall that, for every $G \in \mathcal{V}$, the non-wandering set of the skew-product $\mathrm{Sp}(G)$, equal to $\mathbb{T}^2 \times \mathbb{T}^2$, is partially hyperbolic with a splitting of the form (see (4.8))

$$T(\mathbb{T}^2 \times \mathbb{T}^2) = E^{ss}_{\mathrm{Sp}(G)} \oplus E^{c}_{\mathrm{Sp}(G)} \oplus E^{u}_{\mathrm{Sp}(G)} \oplus E^{uu}_{\mathrm{Sp}(G)}. \tag{6.1}$$

In particular, the support of every $\mathrm{Sp}(G)$-invariant probability measure admits a dominated splitting. Moreover, since $\dim(E^*_{\mathrm{Sp}(G)}) = 1$, $* = ss, c, u, uu$, the splitting (6.1) is the (unique) **finest** dominated splitting of $T(\mathbb{T}^2 \times \mathbb{T}^2)$, that is, the bundles of any other dominated splitting of $T(\mathbb{T}^2 \times \mathbb{T}^2)$ can be obtained by the union of the bundles $E^*_{\mathrm{Sp}(G)}$, $* = s, c, u, uu$. Therefore, for every ergodic $\mathrm{Sp}(G)$-invariant measure μ the corresponding Oseledets' splitting coincides with (4.8) at μ almost every point in $\mathbb{T}^2 \times \mathbb{T}^2$ (for more details see [1, Subsection 2.4]). Thus, there exist four Lyapunov exponents for μ, namely

$$\lambda^{ss}(\mu) < \lambda^{c}(\mu) < \lambda^{u}(\mu) < \lambda^{uu}(\mu) \tag{6.2}$$

satisfying $\lambda^{ss}(\mu) < 0 < \lambda^{u}(\mu)$.

We will now check that the (unique ergodic) measure with maximal entropy $\mu_{\max}$ of F_S provided in [48] is hyperbolic, that is, $\lambda^c(\mu_{\max}) \neq 0$. For that, consider the set $\mathscr{O}(F_S)$ of regular points of F_S given by Oseledets's Theorem [49]. Both $\mathscr{O}(F_S) \cap (B \times \mathbb{T}^2)$ and $\mathscr{O}(F_S) \cap \mathcal{E}$ have full $\mu_{\max}$ measure. Thus, the points (x, y) in this intersections satisfy

$$\lambda^c_+(F_S)(x, y) = \lambda^c(\mu_{\max}) < 0. \tag{6.3}$$

We must also show that, if $G \in \mathcal{V}$, then the diffeomorphism $\mathrm{Sp}(G)$, which has a unique (ergodic) measure with maximal entropy as well, satisfies a property similar to (6.3). Consider the measure $\mu_{\mathrm{Sp}(G)}$ of maximal entropy for $\mathrm{Sp}(G)$. From [48, Theorem 1-(3)], the push-forward $(\pi_1)_*(\mu_{\mathrm{Sp}(G)})$, where $\pi_1 : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ is the natural projection on the first factor, is the measure with maximal entropy ν of Φ. To verify that $\lambda_+^c(\mathrm{Sp}(G)) < 0$, we will show that there exist a set $B_{\mathrm{Sp}(G)} \subset \mathbb{T}^2$ such that $\nu(B_{\mathrm{Sp}(G)}) = 1$ (so $\mu_{\mathrm{Sp}(G)}(B_{\mathrm{Sp}(G)} \times \mathbb{T}^2) = 1$) and $\lambda_+^c(\mathrm{Sp}(G))(x, y) < 0$ at every point $(x, y) \in B_{\mathrm{Sp}(G)} \times \mathbb{T}^2$.

Recall from [48] that the set B for F_S is obtained applying Birkhoff's Ergodic Theorem to Φ, ν and the map $\varsigma : \mathbb{T}^2 \to \mathbb{R}$ defined by

$$\varsigma(x) = \sup_{y \in \mathbb{T}^2} \| D_y f_x(y)|_{E^c(x,y)} \|$$

which satisfies

$$\int \log \varsigma \, d\nu < 0 \tag{6.4}$$

and

$$\lambda^c(F_G)(x, y) \leqslant \widetilde{\log} \varsigma(x) \quad \text{for every } (x, y) \in \mathbb{T}^2 \times \mathbb{T}^2,$$

where $\widetilde{\log} \varsigma(x)$ denotes the limit given by Birkhoff's Ergodic Theorem of the sequence

$$\left(\frac{1}{n} \sum_{i=0}^{n-1} \log \varsigma(\Phi^i(x)) \right)_{n \in \mathbb{N}}.$$

Since, for every $G \in \mathcal{V}$, the homeomorphism Γ_G is C^0 arbitrarily near the identity, then the function ς_G for $\mathrm{Sp}(G)$, defined as done with ς, satisfies an inequality similar to (6.4) (see [48, p. 627]), which means that the set $B_{\mathrm{Sp}(G)}$ is obtained by an analogous application of Birkhoff's Ergodic Theorem. ∎

7. Proof of Theorem A: Second part

In the remainder of the paper, $\mathcal{U} \subset \mathrm{Diff}^2(\mathbb{T}^2 \times \mathbb{T}^2)$ denotes a small C^2-neighborhood of the diffeomorphism F_S in (4.15). Let $\mathcal{KS} \subset \mathrm{Diff}^2(\mathbb{T}^2 \times \mathbb{T}^2)$ be the C^2-residual subset of Kupka–Smale diffeomorphisms. We write

$$\mathfrak{R} \overset{\text{def}}{=} \mathcal{U} \cap \mathcal{KS}.$$

This section is committed to prove the following proposition, which implies Theorem A-(a):

PROPOSITION 7.1: *For every* $G \in \mathfrak{R}$ *and* $n \in \mathbb{N}$, *one has*

$$\#\mathrm{Per}_n(\Phi \times L) \leqslant \#\mathrm{Per}_n(G) \leqslant 3\#\mathrm{Per}_n(\Phi \times L). \tag{7.1}$$

Consequently,

$$\lim_{n\to+\infty} \frac{1}{n} \log \#\mathrm{Per}_n(G) = h_{\mathrm{top}}(G).$$

The key idea to show this proposition consists in finding upper and lower estimates for the cardinals

$$(x, y) \in \mathrm{Per}_n(\Phi \times L) \text{ and } n \in \mathbb{N} \mapsto \#(H_G^{-1}(x, y) \cap \mathrm{Per}_n(\mathrm{Sp}(G)))$$

where H_G is the semi-conjugation between $\mathrm{Sp}(G)$ and $\Phi \times L$ given in (4.9). Thus, recalling that, for every $G \in \mathcal{U}$ (see property $(\mathbf{P}_1)$ in Subsection 4.2 and [14]), one has

$$h_{\mathrm{top}}(G) = h_{\mathrm{top}}(\mathrm{Sp}(G)) = h_{\mathrm{top}}(\Phi \times L) = \lim_{n\to+\infty} \frac{1}{n} \log \#\mathrm{Per}_n(\Phi \times L)$$

those estimates allow us to get equation (7.1) for $\mathrm{Sp}(G)$, hence for G by conjugation.

This task has two parts: the first one is to show that, if $(x, y) \in \mathrm{Per}_n(\Phi \times L)$, then the set $H_G^{-1}(x, y)$ is an interval (Subsection 7.1); the second one consists of an analysis of the dynamic of $\mathrm{Sp}(G)$ on the periodic leaf $(\{x\} \times \mathbb{T}^2)_{x\in\mathrm{Per}(\Phi)}$ (Subsection 7.2).

7.1. Connectedness of the induced classes.

In [48, Lemma 3], it is shown that, given $G \in \mathcal{V}$, for each $(x, y) \in \mathbb{T}^2 \times \mathbb{T}^2$ the set $H_G^{-1}(x, y)$, called a **class induced by** H, is contained in an interval inside a single center leaf of $\mathrm{Sp}(G)$, whose length is bounded by a constant independent of (x, y). In what follows we need a stronger assertion, though: that these induced classes are connected subsets of those intervals whenever $G \in \mathcal{U}$.

We start studying the connectedness of the induced classes of F_S. Recall that $\mathcal{W}^u$ is the foliation tangent to the expanding bundle E^u in (4.16), $\mathcal{F}$ is the central foliation of F_S in (4.16) and that there exists a semi-conjugation $H : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ between F_S and $\Phi \times L$ in (4.4). The goal of this subsection is to show that, for all $(x, y) \in \mathbb{T}^2 \times \mathbb{T}^2$, $H^{-1}(x, y)$ is a one-dimensional compact connected subset (an interval) of a single leaf of $\mathcal{F}$.

Consider a foliation $\mathcal{W}$ of a simply connected compact Riemannian manifold M and lift it to the universal cover $\widetilde{M}$, obtaining a foliation we denote by $\widetilde{\mathcal{W}}$. For points x, y on the same leaf $\widetilde{W}$ of $\widetilde{\mathcal{W}}$, one can define a distance $\mathcal{D}_{\widetilde{W}}(x, y)$ as the length of the shortest path inside the leaf $\widetilde{W}$ linking x and y. We say that the lifted foliation $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ is *quasi-isometric* if there is a constant $C > 1$ such that for any $x, y \in \widetilde{M}$ lying on the same leaf of $\widetilde{\mathcal{W}}$ we have

$$\mathcal{D}_{\widetilde{W}}(x, y) < C\mathcal{D}(x, y) + C$$

where $\mathcal{D}$ denotes the metric on $\widetilde{M}$. The next assertion is inspired by [34].

Claim 7.2: *For every $x \in \mathbb{T}^2$, both $\widetilde{\mathcal{W}}^u(x, \cdot)$ and $\widetilde{\mathcal{F}}(x, \cdot)$ are quasi-isometric.*

Proof. Since we wish to estimate the intrinsic distance between two points of the same leaf of either $\widetilde{\mathcal{W}}^u$ or $\widetilde{\mathcal{F}}$, which is contained in some fiber $\{\tilde{x}\} \times \mathbb{R}^2$ with $\tilde{x} \in \mathbb{R}^2$, it is sufficient to consider the lifts of $\mathcal{W}^u$ and $\mathcal{F}$, which we still denote by $\widetilde{\mathcal{W}}^u$ and $\widetilde{\mathcal{F}}$, to the universal cover $\mathbb{T}^2 \times \mathbb{R}^2$ of $\mathbb{T}^2 \times \mathbb{T}^2$ with respect to the second factor.

Firstly, we observe that from [52, Lemma 4.A.5] we know that, for each $x \in \mathbb{T}^2$, the foliations $\widetilde{\mathcal{W}}^u(x, \cdot)$ and $\widetilde{\mathcal{F}}(x, \cdot)$ inside $\{x\} \times \mathbb{R}^2$ have a global product structure. Therefore, $\widetilde{\mathcal{W}}^u(x, \cdot)$ and $\widetilde{\mathcal{F}}(x, \cdot)$ are quasi-isometric due to [52, Proposition 4.3.9]. Indeed, this result informs that, for every $x \in \mathbb{T}^2$, there exist $C_{1,x}, C_{2,x} > 1$ such that, for every $\tilde{y}, \tilde{z}$ in $\mathbb{R}^2$, one has

$$\mathcal{D}_{\widetilde{W}^u}((x, \tilde{y}), (x, \tilde{z})) < C_{1,x}\|\tilde{y} - \tilde{z}\| + C_{1,x}$$

and

$$\mathcal{D}_{\widetilde{\mathcal{F}}}((x, \tilde{y}), (x, \tilde{z})) < C_{2,x}\|\tilde{y} - \tilde{z}\| + C_{2,x}. \qquad \blacksquare$$

The next result is a parameterized version of [58, Proposition 3.1].

Lemma 7.3: *For every $(x, y) \in \mathbb{T}^2 \times \mathbb{T}^2$, the set $H^{-1}(x, y)$ is a one-dimensional compact connected subset of a single center leaf of F_S.*

Proof. The equality (4.4) can be expressed in $\mathbb{T}^2 \times \mathbb{R}^2$ by lifting (4.3) to $\{x\} \times \mathbb{R}^2$, which provides the equality $\widetilde{H} \circ \widetilde{F}_S = (\Phi \times \widetilde{L}) \circ \widetilde{H}$, where $\widetilde{H}(x, \tilde{y}) = (x, \widetilde{h}_x(\tilde{y}))$ is a proper map at a bounded distance from the identity. The former property of $\widetilde{H}$ implies that $\widetilde{h}_x^{-1}(\tilde{y})$ is a compact subset of $\mathbb{R}^2$ for every $(x, \tilde{y}) \in \mathbb{T}^2 \times \mathbb{R}^2$. The latter leads to the following estimate: for every $x \in \mathbb{T}^2$ and $\tilde{y}, \tilde{z} \in \mathbb{R}^2$,

$$(7.2) \qquad \widetilde{h}_x(\tilde{y}) = \widetilde{h}_x(\tilde{z}) \Leftrightarrow \exists\, K > 0 : \|\widetilde{F}_S^n(x, \tilde{y}) - \widetilde{F}_S^n(x, \tilde{z})\| < K \quad \forall n \in \mathbb{Z}.$$

Besides, if $\widetilde{W}^s_{\Phi\times L}$ stands for the lifts of the weak stable foliation of $\Phi \times L$ to $\mathbb{T}^2 \times \mathbb{R}^2$, then (cf. [48, Lemma 2])

$$\widetilde{h}_x(\widetilde{\mathcal{F}}(x,\tilde{y})) = \widetilde{W}^s_{\Phi\times L}(\widetilde{H}(x,\tilde{y})).$$

We are left to verify that $\widetilde{h}_x^{-1}(\tilde{y})$ is a connected set. This is a consequence of the following adaptation of [58, Lemma 3.2].

Claim 7.4: *If $\widetilde{h}_x(\tilde{y}) = \widetilde{h}_x(\tilde{z})$, then $(x,\tilde{z}) \in \widetilde{\mathcal{F}}(x,\tilde{y})$.*

Proof. Suppose that $(x,\tilde{z}) \notin \widetilde{\mathcal{F}}(x,\tilde{y})$. Let $(x,\tilde{w}) = \widetilde{W}^u(x,\tilde{z}) \cap \widetilde{\mathcal{F}}(x,\tilde{y})$. Note that such a point $(x,\tilde{w})$ exists and is unique (cf. [35, Proposition 2.4]). Consider

$$D_c = \mathcal{D}_{\widetilde{\mathcal{F}}}((x,\tilde{y}),(x,\tilde{w})) \quad \text{and} \quad D_u = \mathcal{D}_{\widetilde{\mathcal{W}}^u}((x,\tilde{z}),(x,\tilde{w})).$$

Recall now the choice of $T > 0$ in the definition of F_S (see (4.15)) and the eigenvalues $0<\lambda_s<1<\lambda_u=\lambda_s^{-1}$ of L, and take $\gamma_1 := \lambda_s$ and $1<\gamma_2 := e^T\lambda_s < \lambda_u$. So, by definition $0 < \gamma_1 < \gamma_2^{-1} < 1$. Using γ_1 and γ_2, one can find constants $0 < \widetilde{\gamma}_1 < \widetilde{\gamma}_2^{-1} < 1$ such that

$$\|\widetilde{F}^n_S(x,\tilde{y}) - \widetilde{F}^n_S(x,\tilde{w})\| \leqslant \widetilde{\gamma}_2^n D_c \quad \text{and} \quad \mathcal{D}_{\widetilde{\mathcal{W}}^u}(\widetilde{F}^n_S(x,\tilde{z}),\widetilde{F}^n_S(x,\tilde{w})) \geqslant \widetilde{\gamma}_1^{-n} D_u.$$

Since $\widetilde{W}^u(x,\cdot)$ is quasi-isometric (Claim 7.2), we also have

$$\|\widetilde{F}^n_S(x,\tilde{z}) - \widetilde{F}^n_S(x,\tilde{w})\| \geqslant \frac{1}{K}(\widetilde{\gamma}_1^{-n} D_u - K).$$

Therefore,

$$\|\widetilde{F}^n_S(x,\tilde{y}) - \widetilde{F}^n_S(x,\tilde{z})\| > \frac{1}{K}(\widetilde{\gamma}_1^{-n} D_u - K) - \widetilde{\gamma}_2^n D_c.$$

The last quantity goes to infinity as $n \to +\infty$, which implies, by (7.2), that $\widetilde{h}_x(\tilde{y}) \neq \widetilde{h}_x(\tilde{z})$. This finishes the proof of the claim. ∎

Claim 7.5: *For every $x \in \mathbb{T}^2$ and $\tilde{y} \in \mathbb{R}^2$, the pre-image $\widetilde{h}_x^{-1}(\tilde{y})$ is connected.*

Proof. Fix $x \in \mathbb{T}^2$. We will see that, given $\tilde{z}$ and $\tilde{w}$ in $\widetilde{h}_x^{-1}(\tilde{y})$, then the arc in the center manifold joining $\tilde{z}$ and $\tilde{w}$ is contained in $\widetilde{h}_x^{-1}(\tilde{y})$. From (7.2), we know that there exists $K > 0$ such that $\|\widetilde{F}^n_S(x,\tilde{z}) - \widetilde{F}^n_S(x,\tilde{w})\| < K$ for every $n \in \mathbb{Z}$. Let $\tilde{\vartheta}$ be a point in that arc. Bringing forth Claim 7.2, we conclude that, for every $n \in \mathbb{Z}$,

$$\begin{aligned}\|\widetilde{F}^n_S(x,\tilde{z}) - \widetilde{F}^n_S(x,\tilde{\vartheta})\| &\leqslant \mathcal{D}_{\widetilde{\mathcal{F}_S}}(\widetilde{F}^n(x,\tilde{z}),\widetilde{F}^n_S(x,\tilde{\vartheta})) \\ &\leqslant \mathcal{D}_{\widetilde{\mathcal{F}}}(\widetilde{F}^n_S(x,\tilde{z}),\widetilde{F}^n_S(x,\tilde{y})) \leqslant C_{2,x}K + C_{2,x}.\end{aligned}$$

Therefore, by (7.2), we know that $\tilde{\vartheta}$ belongs to $\widetilde{h}_x^{-1}(\tilde{y})$. By projecting, the same property is valid for the map h_x. This ends the proof of Claim 7.5 and of Lemma 7.3. ∎

Now we turn to a more general $G \in \mathcal{V}$ and its skew-product $\mathrm{Sp}(G)$ with the semi-conjugation H_G with $\Phi \times L$. Since F_S abides by the stronger estimates of the partial hyperbolicity (called **absolute partial hyperbolicity** in [58, Proposition 3.1]) demanded from the values γ_1 and γ_2 that were used to prove Claim 7.4, and the absolute partial hyperbolicity is a C^1-open condition, then $\mathrm{Sp}(G)$ satisfies this property as well. Moreover, the proof of the quasi-isometric nature of the foliations asserted in Claim 7.2 also works for $\mathrm{Sp}(G)$. Consequently, a statement analogous to the one of Lemma 7.3 is true for $\mathrm{Sp}(G)$, that is, for every $(x,y) \in \mathbb{T}^2 \times \mathbb{T}^2$, the set $H_G^{-1}(x,y)$, which is contained in an interval inside a single center leaf of $\mathrm{Sp}(G)$ (cf. [48, Lemma 3]), is connected.

7.2. The dynamics on the periodic fiber $\{x\} \times \mathbb{T}^2$. Consider $G \in \mathfrak{R}$ and its corresponding skew-product $\mathrm{Sp}(G)$ (see (4.11)) defined by

$$(x,y) \in \mathbb{T}^2 \times \mathbb{T}^2 \mapsto \mathrm{Sp}(G)(x,y) = (\Phi(x), g_x(y)).$$

For every $n \in \mathbb{N}$, write

$$\mathrm{Sp}(G)^n(x,y) = (\Phi^n(x), g_x^n(y))$$

where $g_x^n \colon \mathbb{T}^2 \to \mathbb{T}^2$ stands for the map

$$y \in \mathbb{T}^2 \mapsto g_x^n(y) \stackrel{\text{def}}{=} g_{\Phi^{n-1}(x)} \circ g_{\Phi^{n-2}(x)} \circ \cdots \circ g_x(y).$$

Recall that there is a skew-product H_G, say $H_G(x,y) = (x, h_x^G(y))$ (see (4.4)), which semi-conjugates $\mathrm{Sp}(G)$ and $\Phi \times L$ (cf. (4.9)).

Proposition 7.6: *Let $n \in \mathbb{N}$ and $x_0 \in \mathrm{Per}_n(\Phi)$. Then either $g_{x_0}^n$ is Anosov (conjugated to L^n) or a Derived from Anosov (obtained from L^n).*

Proof. Firstly note that $g_{x_0}^n$ and L^n are semi-conjugated. Indeed, as $x_0 \in \mathrm{Per}_n(\Phi)$ then $h_{\Phi^n(x_0)}^G = h_{x_0}^G$ (see (4.3)) and so, for every $y \in \mathbb{T}^2$, one has

$$\begin{aligned} h_{x_0}^G \circ g_{x_0}^n(y) &= h_{\Phi^n(x_0)}^G \circ g_{\Phi^{n-1}(x_0)} \circ g_{x_0}^{n-1}(y) \\ &= L \circ h_{\Phi^{n-1}(x_0)}^G \circ g_{x_0}^{n-1}(y) \\ &\ \ \vdots \\ &= L^n \circ h_{x_0}^G(y). \end{aligned}$$

Thus, if for every $y \in \mathbb{T}^2$ the interval $(H_G)^{-1}(x_0, y) = \{x_0\} \times (h^G_{x_0})^{-1}(y)$ reduces to a point, then $y \mapsto H_G(x_0, y)$ is a conjugation between $g^n_{x_0}$ and L^n, hence $g^n_{x_0}$ is an Anosov diffeomorphism. The remaining case is dealt with on the next lemma.

Lemma 7.7: *Consider $n \in \mathbb{N}$ and $x_0 \in \mathrm{Per}_n(\Phi)$. If for some $y \in \mathbb{T}^2$ the interval $H_G^{-1}(x_0, y)$ is non-degenerate, then the diffeomorphism $g^n_{x_0}$ is a Derived from Anosov obtained from L^n.*

Proof. To check that $g^n_{x_0}$ satisfies the standard properties of a Derived from Anosov we will follow the reference [55, p. 300].

Claim 7.8: *θ_0 is a source of $g^n_{x_0}$.*

Proof. Since, by construction, when any expansion exists within E^c the greatest expansion is attained at θ_0, we have

$$\|Dg^n_x|_{E^c(x,\theta_0)}\| \geqslant \|Dg^n_x|_{E^c(x,y)}\| \quad \forall (x, y) \in \mathbb{T}^2 \times \mathbb{T}^2 \ \forall n \in \mathbb{N}.$$

On the other hand, if $H_G^{-1}(x_0, y)$ is a non-degenerated interval then

$$\lambda^c_+(x_0, y) \geqslant 0$$

(recall that $\mathcal{E} \subset \mathcal{A}$, see Subsection 4.2). As (x_0, θ_0) is a fixed point of $\mathrm{Sp}(G)^n$, the Lyapunov exponent $\lambda^c(\mathrm{Sp}(G)^n)(x_0, \theta_0)$ is well defined and satisfies

$$\begin{aligned} \lambda^c(\mathrm{Sp}(G)^n)(x_0, \theta_0) &= n \limsup_{k \to +\infty} \frac{1}{nk} \log \|Dg^{nk}_{x_0}|_{E^c(x_0,\theta_0)}\| \\ &= n\lambda^c_+(\mathrm{Sp}(G))(x_0, \theta_0) \\ &\geqslant n\lambda^c_+(\mathrm{Sp}(G))(x_0, y) \geqslant 0. \end{aligned}$$

Thus, $\|Dg^n_{x_0}|_{E^c(x_0,\theta_0)}\| \geqslant 1$. Yet, as $G \in \mathfrak{R}$ then, due to Proposition 5.1, one must have $\|Dg^n_{x_0}|_{E^c(x_0,\theta_0)}\| > 1$, and so θ_0 is indeed a source of $g^n_{x_0}$. ∎

Claim 7.9: *The map $g^n_{x_0}$ has three fixed points in $W^s(\theta_0, L^n)$, namely θ_0 and two new saddle points θ_1 and θ_2, one in each connected component of $W^s(\theta_0, L^n) \setminus \{\theta_0\}$.*

Proof. Recall, from the construction of F_S in Subsection 4.3, the definition and use of the neighborhood $W = W_1 \times W_2$ of (q, θ_0) and the ball

$$B_\varrho(q, \theta_0) = B_\varrho(q) \times B_\varrho(\theta_0)$$

contained in W. Since $H_G^{-1}(x_0, y)$ is a non-degenerate interval, there exists $0 \leqslant i \leqslant n$ such that $\Phi^i(x_0) \in B_\varrho(q)$. By construction, outside the open set W_2 the slope of the graph of the restriction of the map

$$g^n_{\Phi^i(x_0)} \colon \mathbb{T}^2 \to \mathbb{T}^2$$

to $W^s(\theta_0, L^n)$ is smaller than one: due to the dynamics within the stable manifold $W^s(\theta_0, L^n)$, the one-dimensional map $g^n_{\Phi^i(x_0)}$ is a contraction, so its derivative has absolute value smaller than one. Therefore, there exist two fixed points by $g^n_{\Phi^i(x_0)}$, say θ_1^i and θ_2^i, on each side of θ_0, which belong to $W_2 \cap W^s(\theta_0, L^n)$. The points θ_1 and θ_2 we were looking for are obtained intersecting the orbits of θ_1^i and θ_2^i with the fiber $\{x_0\} \times \mathbb{T}^2$. Figure 2 illustrates this information. ∎

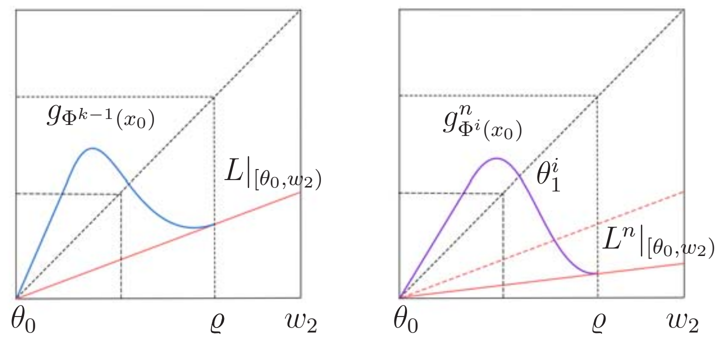

Figure 2. The maps $g_{\Phi^{k-1}(x_0)}$ when $\Phi^k(x_0) \in B_\varrho(q)$ (left) and the fixed point θ_1^i of $g^n_{\Phi^i(x_0)}$ (right).

Since G is Kupka–Smale, both (x_0, θ_1^i) and (x_0, θ_2^i) are hyperbolic periodic points of $\mathrm{Sp}(G)$. Furthermore, the fixed points θ_1^i and θ_2^i of $g^n_{\Phi^i(x_0)}$ are the unique saddles inside W_2 which are fixed by $g^n_{\Phi^i(x_0)}$. Indeed, denoting by

$$[\theta_0, w_2] \subset \{\Phi^i(x_0)\} \times \mathbb{T}^2$$

the closure of the connected component of $(W^s(\theta_0, L^n) \setminus \{\theta_0\}) \cap W_2$ containing the saddle θ_1^i (the corresponding notation for θ_2^i is $[-w_2, \theta_0]$) and identifying all the fibers $\{\Phi^j(x_0)\} \times \mathbb{T}^2$ with $\mathbb{T}^2$, we deduce that each one-dimensional map

$$g_{\Phi^i(x_0)} \colon [\theta_0, w_2] \to [\theta_0, w_2], \quad i \in \{0, 1, \ldots, n-1\}$$

satisfies

- $g_{\Phi^i(x_0)}(\theta_0) = \theta_0$;
- $g_{x_0}(w_2) = g_{\Phi^j(x_0)}(w_2)$, for every $j \in \{0, \ldots, n-1\}$;
- there is $i \in \{0, 1, \ldots, n-1\}$ such that the restriction $g_{\Phi^i(x_0)}|_{(\theta_0, w_2)}$ has a unique (saddle) fixed point (different from θ_0).

Observe that these properties are due to the fact that the map $g_{\Phi^i(x_0)}$ is uniformly C^2-close to $f_{\Phi^i(x_0)}$, the unstable and center foliations of $\mathrm{Sp}(G)$ and F_S are also C^2-close (cf. [38, Section 6]), the Derived from Anosov is a structurally stable diffeomorphism (cf. [54]) and the maps $(f_{\Phi^i(x_0)})_{i\in\{0,1,\ldots,n-1\}}$ have these properties by construction of F_S.

Similarly, for every $i = 0, 1, \ldots, n-1$, the map

$$g_{\Phi^i(x_0)} \colon [-w_2, \theta_0] \to [-w_2, \theta_0]$$

is C^2-close to the corresponding $f_{\Phi^i(x_0)}$, which ensures the existence of a unique saddle θ_2^i inside $(-w_2, \theta_0)$ which is fixed by $g^n_{\Phi^i(x_0)}$. Consequently, apart from θ_0, the points θ_1^i and θ_2^i are the unique fixed points of $g^n_{\Phi^i(x_0)}$ in $\{\Phi^i(x_0)\} \times \mathbb{T}^2$.

Claim 7.10: *The non-wandering set of $g^n_{x_0}$ is given by $\Omega(g^n_{x_0}) = \{\theta_0\} \cup \Lambda^n_{x_0}$, where $\Lambda^n_{x_0}$ is a hyperbolic attractor of topological dimension one.*

Proof. Note that, regarding the splitting $E^u(L) \oplus E^s(L)$ of the tangent space $T\mathbb{T}^2$, the derivative of each $g_{\Phi^i(x_0)}$ is determined by a matrix $Dg_{\Phi^i(x_0)} = (a_{ij})$, which is lower triangular since $a_{11} = \lambda_u$ and $a_{12} = 0$ for the whole family $(g_x)_{x\in\mathbb{T}^2}$. Thus,

$$Dg^n_{x_0}(y) = \begin{pmatrix} (\lambda_u)^n & 0 \\ b_{21}(y) & b_{22}(y) \end{pmatrix} \tag{7.3}$$

with $0 < b_{22} < 1$ at the saddle fixed points θ_1 and θ_2. Moreover, we can assume that both $b_{22}(\theta_1)$ and $b_{22}(\theta_2)$ are smaller than or equal to λ_s^n. Let $V \subset \mathbb{T}^2$ be a neighborhood of θ_0 not containing θ_1 and θ_2, and such that

(i) $b_{22} > 1$ for $w \in V$ (that is, $g^n_{x_0}$ is an expansion along E^c in V);
(ii) $0 < b_{22} < 1$ for $w \notin g^n_{x_0}(V)$ (that is, $g^n_{x_0}$ is a contraction along E^c outside $g^n_{x_0}(V)$);
(iii) $g^n_{x_0}(V) \supset V$.

We observe that such a neighborhood V exists (cf. Exercise 7.36 of [55]) and $V \subset W^u(\theta_0, g^n_{x_0})$. So it is a local unstable manifold of θ_0 and

$$W^u(\theta_0, g^n_{x_0}) = \bigcup_{i \geqslant 1} g^{in}_{x_0}(V).$$

Let $N = \mathbb{T}^2 \setminus V$. Then N is a trapping region because $g^n_{x_0}(V) \supset V$. Set

$$\Lambda^n_{x_0} \stackrel{\text{def}}{=} \bigcap_{i \geqslant 1} g^{in}_{x_0}(N).$$

This is an attracting set and $\Lambda^n_{x_0} = \mathbb{T}^2 \setminus W^u(\theta_0, g^n_{x_0})$. Thus,

$$\Omega(g^n_{x_0}) = \{\theta_0\} \cup \Lambda^n_{x_0}.$$

We are left to show that $\Lambda^n_{x_0}$ is hyperbolic. Due to (7.3), $E^s(L) = E^c(\mathrm{Sp}(G))$ is an invariant bundle and every vector in this bundle is contracted by $D_z g^n_{x_0}$ for $z \in \Lambda^n_{x_0}$. This is precisely the stable bundle on $\Lambda^n_{x_0}$. Let $C > 0$ be a global upper bound of $|b_{21}|$. Consider $\alpha = C[(\lambda_u)^n - (\lambda_s)^n]^{-1}$ and take the cones

$$\mathcal{C} \stackrel{\text{def}}{=} \{(v_1, v_2) \in E^u(L) \oplus E^s(L) : |v_2| < \alpha |v_1|\}.$$

Then it can be checked, using the lower triangular nature of the derivative of g_x, that these cones are invariant and

$$E^u(g^n_{x_0}, z) = \bigcap_{j=1}^{\infty} D_{g^{-jn}_{x_0}(z)} g^{jn}_{x_0}(\mathcal{C}(g^{-jn}_{x_0}(z)))$$

is an invariant bundle on which the derivative is an expansion for every point $z \in \Lambda^n_{x_0}$. This provides the unstable bundle on $\Lambda^n_{x_0}$, hence assigning a hyperbolic splitting at the points of this set. This ends the proof of the claim. ∎

As mentioned, Claims 7.9 and 7.8 complete the proof of Lemma 7.7. ∎

Lemma 7.7 was the missing part to solve the remaining case, so the proof of Proposition 7.6 is finished. ∎

7.3. Proof of Proposition 7.1. We start establishing the first of the inequalities in (7.1).

Lemma 7.11: *For every $n \in \mathbb{N}$ and every $(x, y) \in \mathrm{Per}_n(\Phi \times L)$, one has*

$$1 \leqslant \#(H_G^{-1}(x, y) \cap \mathrm{Per}_n(\mathrm{Sp}(G))).$$

Consequently,

$$\#\mathrm{Per}_n(\Phi \times L) \leqslant \#\mathrm{Per}_n(\mathrm{Sp}(G)) \quad \forall n \in \mathbb{N}.$$

Proof. By Lemma 7.3, for every $(x,y) \in \mathrm{Per}_n(\Phi \times L)$ the map

$$\mathrm{Sp}(G)^n : H_G^{-1}(x,y) \to H_G^{-1}(x,y)$$

is a homeomorphism of a closed (possibly degenerate) interval. Brouwer's Fixed Point Theorem guarantees the existence of a fixed point of $\mathrm{Sp}(G)^n|_{H_G^{-1}(x,y)}$, for every $(x,y) \in \mathrm{Per}_n(\Phi \times L)$. Hence the desired inequality. ∎

The remaining inequality in (7.1) is a consequence of the following lemma.

Lemma 7.12: *For every $n \in \mathbb{N}$ and every $(x,y) \in \mathrm{Per}_n(\Phi \times L)$, one has*

$$\#(H_G^{-1}(x,y) \cap \mathrm{Per}_n(\mathrm{Sp}(G))) \leqslant 3.$$

Consequently,

$$\#\mathrm{Per}_n(\mathrm{Sp}(G)) \leqslant 3\#\mathrm{Per}_n(\Phi \times L) \quad \forall n \in \mathbb{N}.$$

Proof. From Proposition 7.6, we already know that, given $x \in \mathrm{Per}_n(\Phi)$, either g_x^n is Anosov or a Derived from Anosov. In the former case, the interval $H_G^{-1}(x,y)$ is a point. In the latter, the interval $H_G^{-1}(x,\theta_0)$ associated to the fixed point (x,θ_0) has exactly three fixed points by g_x^n. Yet, we must also estimate the cardinality of $H_G^{-1}(x,y) \cap \mathrm{Per}_n(G)$ when y is different from θ_0.

Claim 7.13: *Take $(x,y) \in \mathrm{Per}_n(\Phi \times L)$ and assume that g_x^n is a Derived from Anosov. If $y \neq \theta_0$, then $H_G^{-1}(x,y)$ is a point.*

Proof. Suppose, on the contrary, that $H_G^{-1}(x,y)$ is a non-degenerated interval. Then the map $\mathrm{Sp}(G)^n : H_G^{-1}(x,y) \to H_G^{-1}(x,y)$ is a Morse-Smale diffeomorphism of this interval (recall that $G \in \mathfrak{R}$). Since g_x^n is a preserving orientation map, the boundary points of the interval $H_G^{-1}(x,y)$, say (x,a_1) and (x,a_2), are necessarily fixed by $\mathrm{Sp}(G)^n$. This implies, using the fact that $H_G^{-1}(x,\theta_0) \cap H_G^{-1}(x,y) = \emptyset$, that

$$\{(x,a_1),(x,a_2)\} \subset \{x\} \times \Omega(g_x^n) \setminus \{(x,\theta_0)\} = \{x\} \times \Lambda_x^n$$

and therefore (x,a_1) and (x,a_2) are two sinks of $\mathrm{Sp}(G)^n|_{H_G^{-1}(x,y)}$. This forces the existence of a third point

$$(x,a_3) \in H_G^{-1}(x,y) \setminus \{(x,a_1),(x,a_2)\}$$

such that $\mathrm{Sp}(G)^n(x,a_3) = (x,a_3)$ and (x,a_3) is a source of $\mathrm{Sp}(G)^n|_{H_G^{-1}(x,y)}$. But (x,a_3) also belongs to $\{x\} \times \Omega(g_x^n) \setminus \{(x,\theta_0)\} = \{x\} \times \Lambda_x^n$, so this conclusion contradicts Claim 7.10. ∎

This completes the proof of Lemma 7.12. ∎

Finally, we observe that, for every G in $\mathfrak{R}$ and $n \in \mathbb{N}$, one has

$$\begin{aligned}\operatorname{Per}_n(\operatorname{Sp}(G)) &= H_G^{-1}(\operatorname{Per}_n(\Phi \times L)) \cap \operatorname{Per}_n(\operatorname{Sp}(G)) \\ &= \bigcup_{(x,y)\in \operatorname{Per}_n(\Phi\times L)} H_G^{-1}(x,y) \cap \operatorname{Per}_n(\operatorname{Sp}(G)).\end{aligned} \tag{7.4}$$

Thus, $\#\operatorname{Per}_n(\Phi \times L) \leqslant \#\operatorname{Per}_n(SP(G)) \leqslant 3\#\operatorname{Per}_n(\Phi \times L)$ for every $n \in \mathbb{N}$, as claimed. This ends the proof of the proposition.

Remark 7.14: As $G \in \mathfrak{R}$ is conjugate to $\operatorname{Sp}(G)$, there exists $K > 0$ such that

$$\frac{1}{n} \log \#\operatorname{Per}_n(G) \leqslant \log K \quad \forall n \in \mathbb{N}$$

since this is true for $\Phi \times L$, hence for $\operatorname{Sp}(G)$ by Lemma 7.12.

8. Proof of Theorem A: Third part

Let $G \in \mathfrak{R}$ and consider its unique measure $\mu_{\max}(G)$ of maximal entropy (cf. [48]). We now prove that $\mu_{\max}(G)$ is the weak* limit of the following sequence of probability measures on $\mathbb{T}^2 \times \mathbb{T}^2$

$$n \in \mathbb{N} \mapsto \mu_n(G) \stackrel{\text{def}}{=} \frac{1}{\#\operatorname{Per}_n(G)} \sum_{(x,y)\in \operatorname{Per}_n(G)} \delta_{(x,y)} \in \mathscr{P}(\mathbb{T}^2 \times \mathbb{T}^2, G).$$

Firstly, observe that the map Γ_G introduced in (4.10) satisfies

$$(\Gamma_G)_*\delta_{(x,y)} = \delta_{\Gamma_G(x,y)} \quad \text{for all } (x,y),$$

so

$$(\Gamma_G)_*(\mu_{\max}(G)) = \mu_{\max}(\operatorname{Sp}(G)) \ \text{ and } \ (\Gamma_G)_*(\mu_n(G)) = \mu_n(\operatorname{Sp}(G)) \ \ \forall n \in \mathbb{N}. \tag{8.1}$$

To simplify the notation, in what follows we write $\mu_{\max}$ instead of $\mu_{\max}(\operatorname{Sp}(G))$ and μ_n instead of $\mu_n(\operatorname{Sp}(G))$. Consider the sequence of probabilities $(\nu_n)_{n\in\mathbb{N}}$ on $\mathbb{T}^2 \times \mathbb{T}^2$ defined by

$$n \in \mathbb{N} \mapsto \nu_n \stackrel{\text{def}}{=} \frac{1}{\#\operatorname{Per}_n(\Phi \times L)} \sum_{(x,y)\in \operatorname{Per}_n(\Phi\times L)} \delta_{(x,y)}.$$

It is known that this sequence of measures converges in the weak* topology to the measure $\nu_{\max}$ of maximal entropy of $\Phi \times L$.

Proposition 8.1: *The sequence $((H_G)_*(\mu_n))_{n\in\mathbb{N}}$ converges to $\nu_{\max}$ in the weak* topology.*

To prove this assertion it is enough to show that the weak* limit of any convergent subsequence of $((H_G)_*(\mu_n))_n$ is equal to $\nu_{\max}$. This is a consequence of the following two statements.

Lemma 8.2: *Let $f : X \to X$ be a continuous map defined on a compact metric space (X, d). Consider two sequences of f-invariant Borel probability measures $(\eta_k)_{k\in\mathbb{N}}$ and $(\zeta_k)_{k\in\mathbb{N}}$ on X satisfying*

$$\exists C > 1 : C^{-1}\zeta_k \leqslant \eta_k \leqslant C\zeta_k \quad \forall k \in \mathbb{N}. \tag{8.2}$$

Assume that $(\zeta_k)_{k\in\mathbb{N}}$ and $(\eta_k)_{k\in\mathbb{N}}$ converge in the weak* topology to probability measures ζ and η respectively. Then $C^{-1}\zeta \leqslant \eta \leqslant C\zeta$. In particular, ζ and η are equivalent.

Lemma 8.3: *If η and ζ are f-invariant probability measures on X such that η is ergodic and ζ is absolutely continuous with respect to η, then $\zeta = \eta$.*

Let us postpone for the moment the proofs of these lemmas and complete the argument to show Proposition 8.1.

Proof of Proposition 8.1. Using equation (7.4) and the fact that for every $(x, y) \in \mathbb{T}^2 \times \mathbb{T}^2$ we have

$$(H_G)_*\delta_{(x,y)} = \delta_{H_G(x,y)},$$

we deduce that the $(\Phi \times L)$-invariant probability measure $(H_G)_*(\mu_n)$ satisfies

$$\begin{aligned}(H_G)_*(\mu_n) =& \frac{1}{\#\mathrm{Per}_n(\mathrm{Sp}(G))} \sum_{(x,y)\in\mathrm{Per}_n(\Phi\times L)} \#(H_G^{-1}(x,y) \cap \mathrm{Per}_n(\mathrm{Sp}(G)))\delta_{(x,y)} \\ =& \Big(\frac{\#\mathrm{Per}_n(\Phi \times L)}{\#\mathrm{Per}_n(\mathrm{Sp}(G))}\Big) \frac{1}{\#\mathrm{Per}_n(\Phi \times L)} \\ & \times \sum_{(x,y)\in\mathrm{Per}_n(\Phi\times L)} \#(H_G^{-1}(x,y) \cap \mathrm{Per}_n(\mathrm{Sp}(G)))\delta_{(x,y)}.\end{aligned}$$

Besides, after Lemmas 7.12 and 7.11 we know that

$$\forall n \in \mathbb{N} \quad 1 \leqslant \#(H_G^{-1}(x,y) \cap \mathrm{Per}_n(\mathrm{Sp}(G))) \leqslant 3 \quad \text{and} \quad \frac{1}{3} \leqslant \frac{\#\mathrm{Per}_n(\Phi \times L)}{\#\mathrm{Per}_n(\mathrm{Sp}(G))} \leqslant 1.$$

Thus,

$$\forall n \in \mathbb{N}\ \forall \text{ Borel set } A \subset \mathbb{T}^2 \times \mathbb{T}^2 \quad \frac{1}{3}\nu_n(A) \leqslant (H_G)_*(\mu_n)(A) \leqslant 3\nu_n(A).$$

Let $\eta_k := (H_G)_*(\mu_{n_k})$ be a subsequence converging to a probability measure ν_0 in the weak* topology. Since $\zeta_k := \nu_{n_k}$ converges to $\nu_{\max}$, it follows from Lemma 8.2 that ν_0 and $\nu_{\max}$ are equivalent measures. On the other hand, as $\nu_{\max}$ is ergodic, Lemma 8.3 implies that $\nu_0 = \nu_{\max}$. ∎

We now return to the proof of the two pending lemmas.

Proof of Lemma 8.2. By symmetry of the inequality (8.2) it is enough to check that for every open set U of $\mathbb{T}^2 \times \mathbb{T}^2$ we have $\eta(U) \leqslant C\zeta(U)$. Indeed, due the regularity of the measures ζ and η, from the previous inequality we get, for every Borel set A in $\mathbb{T}^2 \times \mathbb{T}^2$,

$$\begin{aligned}\eta(A) &= \inf\{\eta(G) : G \text{ is open and } A \subset G\}\\ &\leqslant C \inf\{\zeta(G) : G \text{ is open and } A \subset G\} = C\zeta(A).\end{aligned}$$

So, $\zeta(A) = 0$ implies $\eta(A) = 0$. Now, consider the sequence of closed sets in $\mathbb{T}^2 \times \mathbb{T}^2$ defined by

$$k \in \mathbb{N} \mapsto \Delta_k = \Big\{x \in X : d(x, X \setminus U) \geqslant \frac{1}{k}\Big\}.$$

From Uryshon's Lemma there exists a continuous function $\xi_k : X \to [0,1]$ such that

$$\mathbb{1}_{\Delta_k} \leqslant \xi_k \leqslant \mathbb{1}_U \quad \forall k \in \mathbb{N}.$$

We may assume that, letting k go to $+\infty$, the sequence $(\xi_k)_k$ converges to $\mathbb{1}_U$ in a monotonic and increasing way. Thus,

$$\begin{aligned}\eta(U) &= \sup_k \int \xi_k d\eta && \text{(by the Monotone Convergence Theorem)}\\ &= \sup_k \lim_n \int \xi_k d\eta_n && \text{(by the weak* convergence of } (\eta_n)_{n\in\mathbb{N}})\\ &\leqslant C \sup_k \lim_n \int \xi_k d\zeta_n && \text{(by equation (8.2))}\\ &= C \sup_k \int \xi_k d\zeta && \text{(by the weak* convergence of } (\zeta_n)_{n\in\mathbb{N}})\\ &= C\zeta(U) && \text{(by the Monotone Convergence Theorem).}\end{aligned}$$

∎

Proof of Lemma 8.3. Consider a Borel set $A \subset X$. By Birkhoff's Ergodic Theorem we have

$$\phi_A(x) := \lim_{n\to+\infty} \frac{1}{n}\{0 \leqslant j \leqslant n-1 : f^j(x) \in A\} = \mu(A)$$

for μ almost every $x \in X$, and $\nu(A) = \int \phi_A(x)d\nu(x)$. Since $\nu \ll \mu$, we also get $\phi_A(x) = \mu(A)$ for ν almost every x. So, $\int \phi_A(x)d\nu(x) = \mu(A)$. Hence $\nu(A) = \mu(A)$. ∎

Corollary 8.4: *The sequence $(\mu_n(G))_{n\in\mathbb{N}}$ converges to $\mu_{\max}(G)$ in the weak* topology.*

Proof. Taking into account both the continuity of $\eta \to (\Gamma_G)_*^{-1}(\eta)$ and the equation (8.1), it is enough to show that the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to $\mu_{\max}$ in the weak* topology. Consider a subsequence $(\mu_{n_k})_k$ converging to a probability measure μ_0. We will verify that $h_{\mu_0}(\mathrm{Sp}(G)) = h_{\mathrm{top}}(\mathrm{Sp}(G))$, and so, by the uniqueness of the measure with maximal entropy of $\mathrm{Sp}(G)$, we deduce that $\mu_0 = \mu_{\max}$.

Using item $(\mathbf{H}_2)$ in Subsection 4.2 and Ledrappier–Walters' formula, if follows that

$$h_\eta(\mathrm{Sp}(G)) = h_{(H_G)_*(\eta)}(\Phi \times L), \quad \forall \eta \in \mathscr{P}(\mathbb{T}^2 \times \mathbb{T}^2, \mathrm{Sp}(G)). \tag{8.3}$$

Besides, from Proposition 8.1 and the continuity of $\eta \to (H_G)_*(\eta)$, we deduce that $(H_G)_*(\mu_0) = \nu_{\max}$. Then, using (8.3) and property $(\mathbf{P}_1)$ in Subsection 4.2, we obtain

$$\begin{aligned} h_{\mu_0}(\mathrm{Sp}(G)) &= h_{(H_G)_*(\mu_0)}(\Phi \times L) = h_{\nu_{\max}}(\Phi \times L) \\ &= h_{\mathrm{top}}(\Phi \times L) = h_{\mathrm{top}}(\mathrm{Sp}(G)). \end{aligned}$$ ∎

9. Proof of Theorem B

We start recalling that the set of periodic points of $G \in \mathfrak{R}$ is countable, and so zero-dimensional. Besides, G has the small boundary property (cf. [17, Subsection 2.1] or [44], where it was proved that on a finite-dimensional manifold any dynamical system whose set of periodic points is countable have this property). Moreover, as already mentioned, the central direction of G is one-dimensional, thus G is entropy-expansive. After summoning Remark 7.14 and property $(\mathbf{P}_1)$ in Subsection 4.2, to show the existence of a principal strongly faithful symbolic extension with embedding for G we are left to control the growth rate of the periodic points with the period at arbitrarily small scales.

Lemma 9.1: *Every diffeomorphism G in $\mathfrak{R}$ is asymptotically per-expansive.*

Proof. Since the conjugation Γ_G (see (4.10)) between G and $Sp(G)$ is C^0 close to the identity, it is enough to show that $\mathrm{Per}^*(\mathrm{Sp}(G)) = 0$ (recall that the definition of $\mathrm{Per}^*(f)$ was given in (3.4)). For that, take a small $\varepsilon > 0$ and $(x_0, y_0) \in \mathbb{T}^2 \times \mathbb{T}^2$, and consider the set (see (3.3))

$$B^{\mathrm{Sp}(G)}_{\infty,\varepsilon}(x_0, y_0) := \{(x, y) \in \mathbb{T}^2 \times \mathbb{T}^2 : d(\mathrm{Sp}(G)^i(x, y), \mathrm{Sp}(G)^i(x_0, y_0)) \leqslant \varepsilon, \forall i \in \mathbb{Z}\}.$$

We claim that

$$\forall \varepsilon > 0\ \forall n \in \mathbb{N}\ \forall (x_0, y_0) \in \mathbb{T}^2 \times \mathbb{T}^2 \quad \#(\mathrm{Per}_n(\mathrm{Sp}(G)) \cap B^{\mathrm{Sp}(G)}_{\infty,\varepsilon}(x_0, y_0)) \leqslant 3.$$

Firstly, note that the central foliation $\mathcal{F}_{\mathrm{Sp}(G)}$ of $\mathrm{Sp}(G)$ (see $(\mathbf{S}_3)$ in Subsection (4.1)) is plaque expansive (cf. [38, p. 116] and [48, p. 626]), that is, there exists $\varepsilon_0 > 0$ such that if (x, y) belongs to $B^{\mathrm{Sp}(G)}_{\infty,\varepsilon_0}(x_0, y_0)$, then both points (x_0, y_0) and (x, y) lie on the same leaf of $\mathcal{F}_{\mathrm{Sp}(G)}$ (in particular $x_0 = x$), which is sent by the semi-conjugation H_G into a stable leaf. On the other hand, if

$$\mathrm{Per}_n(\mathrm{Sp}(G)) \cap B^{\mathrm{Sp}(G)}_{\infty,\varepsilon}(x_0, y_0) \neq \emptyset,$$

then x_0 is periodic and so, by Proposition 7.6, $g^n_{x_0}$ is Anosov or a Derived from Anosov. In the former case,

$$B^{\mathrm{Sp}(G)}_{\infty,\varepsilon}(x_0, y_0) \subset B^{\mathrm{Sp}(G)^n}_{\infty,\varepsilon}(x_0, y_0) = \{(x_0, y_0)\}.$$

In the latter case, the intersection cannot have more than three periodic points: otherwise, if we assume the existence of at least four elements of $\mathrm{Per}_n(\mathrm{Sp}(G))$ in $B^{\mathrm{Sp}(G)}_{\infty,\varepsilon}(x_0, y_0)$, then we may find two hyperbolic point (x_0, y_1) and (x_0, y_2) in $\mathrm{Per}_n(\mathrm{Sp}(G)) \cap B^{\mathrm{Sp}(G)}_{\infty,\varepsilon}(x_0, y_0)$ such that

$$H_G(x_0, y_1) \neq H_G(x_0, y_2)$$

are both in $\mathrm{Per}_n(\Phi \times L)$ and belong to the same stable leaf of $\Phi \times L$. This contradicts the known dynamics within stable leaves. ∎

To end the proof of Theorem B we just make a straightforward application of the Main Theorem of [17], which we have quoted at the end of Subsection 3.5.

10. Proof of Theorem C

Firstly, we note that, by [25], every C^2 diffeomorphism $G \in \mathcal{V}$ has at least one SRB measure on $\mathbb{T}^2 \times \mathbb{T}^2$, which is a partially hyperbolic global attractor for G, with splitting $\mathbb{E}^c_G = E^{ss}_G \oplus E^c_G$ and $\mathbb{E}^u_G = E^u_G \oplus E^{uu}_G$. Besides, under the additional assumption that Φ is a linear hyperbolic automorphism of the 2-torus, one has:

Lemma 10.1: *If Φ is a linear hyperbolic automorphism of the 2-torus, then every skew-product $F \in \mathcal{V}$, satisfying the properties ($\boldsymbol{S}_1$)-($\boldsymbol{S}_3$) in Subsection 4.1, is mostly contracting along the central direction with respect to the splitting $\mathbb{E}^{\mathrm{c}}_F = E^{\mathrm{ss}}_F \oplus E^{\mathrm{c}}_F$ and $\mathbb{E}^{\mathrm{u}}_F = E^{\mathrm{u}}_F \oplus E^{\mathrm{uu}}_F$.*

Proof. If Φ is a linear hyperbolic automorphism, then the measure $\nu_{\max}$ with maximal entropy of $\Phi \times L$ is the Lebesgue measure on $\mathbb{T}^2 \times \mathbb{T}^2$; we denote it by Leb. Thus, for this special type of Φ, the property ($\mathbf{M}_1$) and the equation (4.6) for $F \in \mathcal{V}$ (see Subsection 4.2.2) inform that

$$\mathrm{Leb}(B \times \mathbb{T}^2) = \mathrm{Leb}(\mathcal{E}) = 1.$$

Besides, by [43], the Lebesgue measure on $\mathbb{T}^2$ (which is the measure with maximal entropy of Φ and its SRB measure, and we denote by m) disintegrates into marginal measures $(m_x)_{x\in\mathbb{T}^2}$ that are absolutely continuous with respect to the Lebesgue measures $(\mathrm{Leb}_{W^u_\Phi(x)})_{x\in\mathbb{T}^2}$ restricted to the Φ-invariant unstable manifolds $(W^u_\Phi(x))_{x\in\mathbb{T}^2}$, and are supported on the sets of a partition subordinated to the unstable foliation W^u_Φ (that is, the atom of such a partition containing x is a subset of $W^u_\Phi(x)$ at m almost every point x). Moreover, for every Borel set D of $\mathbb{T}^2$, the map

$$x \in \mathbb{T}^2 \mapsto m_x(D)$$

is m-measurable and

$$m(D) = \int m_x(D)dm(x).$$

Let $B_\delta(p)$ be a small ball centered at the point p with radius δ (which we may take as the product of small local Φ-invariant manifolds at p). Then, as $m(B) = 1$,

$$0 < m(B_\delta(p)) = m(B_\delta(p) \cap B) = \int m_x(B_\delta(p) \cap B)dm(x).$$

Therefore, there exists $x_0 \in B_\delta(p)$ such that $m_{x_0}(B_\delta(p) \cap B) > 0$. Thus,

$$m_{x_0}(B_\delta(p) \cap W^u_{\Phi,\delta}(x_0)) > 0,$$

which implies, due to the absolute continuity, that

$$\mathrm{Leb}_{W^u_\Phi(x_0)}(B \cap W^u_{\Phi,\delta}(x_0)) > 0.$$

This way, we ensure that the u-dimensional disk $D^u := W^u_{\Phi,\delta}(x_0) \times W^u(\theta_0)$ has a subset with positive volume where the central exponent is negative. So, by Proposition 5.2 and (cf. [6]), F is mostly contracting. ∎

To complete the proof of the first part of Theorem C we observe that, as the strong unstable foliation of F_S is robustly minimal and G is C^1-close to F_S, then the strong unstable foliation of G is also minimal. On the other hand, as F_S is mostly contracting and this property is C^2-robust (cf. [3]), then $G \in \mathfrak{U}$ is mostly contracting as well. Therefore, we may apply [6, Theorem B] to G, and this way conclude that it has a unique ergodic SRB measure whose basin has full Lebesgue measure (hence, it is G's unique physical measure).

Now we move on to the items (a) and (b) of Theorem C. Suppose that Φ is a linear hyperbolic automorphism of $\mathbb{T}^2$ and consider $G \in \mathfrak{U}$. Let ν_{SRB} be the SRB measure and $\nu_{\max}$ be the probability measure with maximal entropy of $\Phi \times L$. Under the assumption that both Φ and L are linear automorphisms of $\mathbb{T}^2$, the measures ν_{SRB} and $\nu_{\max}$ are the same, and coincide with the Lebesgue measure in $\mathbb{T}^2 \times \mathbb{T}^2$ (cf. [59, Theorem 8.15]), which we abbreviate into Leb. Denote by μ_{SRB} and $\mu_{\max}$ the SRB measure and the measure with maximal entropy of $\mathrm{Sp}(G)$, respectively.

10.1. The SRB measure of $\mathrm{Sp}(G)$. Given $G \in \mathfrak{U}$ and $(x, y) \in \mathbb{T}^2 \times \mathbb{T}^2$, let $\widetilde{J}^u_{\mathrm{Sp}(G)}(x, y)$ be the Jacobian of $D_{(x,y)} \mathrm{Sp}(G)$ restricted to the unstable bundle $E^u(x, y) \oplus E^{uu}(x, y)$ of $\mathrm{Sp}(G)$. Analogously, define $\widetilde{J}^u_{\Phi\times L}(x, y)$. Note that $\widetilde{J}^u_{\Phi\times L}(x, y)$ coincides with $J^u_{\Phi\times L}(x, y)$, where $J^u_{\Phi\times L} = J^u$ is as in Subsection 3.7. By the ergodicity of μ_{SRB}, the corresponding Oseledets' splitting coincides with the partially hyperbolic splitting (6.1) of $\mathrm{Sp}(G)$ at μ_{SRB} almost every point in $\mathbb{T}^2 \times \mathbb{T}^2$ (cf. the proof of Proposition 6.2). Thus,

$$\widetilde{J}^u_{\mathrm{Sp}(G)}(x, y) = J^u_{\mathrm{Sp}(G)}(x, y)$$

for μ_{SRB} almost every point (x, y) in $\mathbb{T}^2 \times \mathbb{T}^2$.

PROPOSITION 10.2: *Assume that at μ_{SRB} almost every (x,y) in $\mathbb{T}^2 \times \mathbb{T}^2$ we have*

$$|J^u_{\Phi\times L} \circ H_G(x,y)| \leqslant |J^u_{\mathrm{Sp}(G)}(x,y)|. \tag{10.1}$$

Then $(H_G)_*(\mu_{\mathrm{SRB}})$ is the SRB measure of $\Phi \times L$.

Proof. Set $\nu = (H_G)_*(\mu_{\mathrm{SRB}})$. After Margulis–Ruelle inequality (3.5), we are left to verify that

$$\int \log|J^u_{\Phi\times L}| d\nu \leqslant h_\nu(\Phi \times L).$$

Firstly, we note that

$$h_{\mu_{\mathrm{SRB}}}(\mathrm{Sp}(G)) = h_\nu(\Phi \times L).$$

Indeed, property $(\mathbf{H}_2)$ on Subsection 4.2.1 and Ledrappier–Walters' formula [42, (1.2)] yield

$$h_{\mu_{\mathrm{SRB}}}(\mathrm{Sp}(G)) \leqslant h_\nu(\Phi \times L)$$

which, together with the existence of the semi-conjugation H_G and the well-known fact [59, Theorem 4.11] that $h_{\mu_{\mathrm{SRB}}}(\mathrm{Sp}(G)) \geqslant h_\nu(\Phi \times L)$, implies the equality. Thus, using (10.1) one gets

$$\begin{aligned}\int \log|J^u_{\Phi\times L}| d\nu &= \int \log|J^u_{\Phi\times L} \circ H_G| d\mu_{\mathrm{SRB}} \\ &\leqslant \int \log|J^u_{\mathrm{Sp}(G)}| d\mu_{\mathrm{SRB}} \\ &= h_{\mu_{\mathrm{SRB}}}(\mathrm{Sp}(G)) = h_\nu(\Phi \times L). \qquad \blacksquare\end{aligned}$$

Let $\beta_1 > 1$ and $\beta_2 > 1$ be the expanding eigenvalues of Φ and L, respectively, with $\beta_1 \geqslant \beta_2$. By Pesin's formula, the topological entropy of $\Phi \times L$ is given by

$$h_{\mathrm{top}}(\Phi \times L) = \log\beta_1 + \log\beta_2.$$

Indeed, on the corresponding regular sets, the positive Lyapunov exponents $\lambda^{uu} > \lambda^u > 0$ of Leb are given by (cf. [55])

$$\lambda^{uu}(\mathrm{Leb}) = \log\beta_1 \quad \text{and} \quad \lambda^u(\mathrm{Leb}) = \log\beta_2$$

and, as the mapping $(x,y) \mapsto J^u_{\Phi\times L}(x,y)$ is constant and equal to $\beta_1\beta_2$,

$$h_{\mathrm{Leb}}(\Phi \times L) = \int \log J^u_{\Phi\times L} d\mathrm{Leb} = \log\beta_1 + \log\beta_2 = h_{\mathrm{top}}(\Phi \times L).$$

To complete the proof of Theorem C (a), we summon the fact that, by construction of the Shub's examples, for every $(x, y) \in \mathbb{T}^2 \times \mathbb{T}^2$ one has

$$J^u_{\mathrm{Sp}(G)}(x, y) \geqslant \beta_1 \beta_2$$

(see property ($\mathbf{S}_3$) on Subsection 4.1). So, $J^u_{\mathrm{Sp}(G)}$ and $J^u_{\Phi\times L}$ satisfy the assumption (10.1) of Proposition 10.2. Therefore, $(H_G)_*(\mu_{\mathrm{SRB}}) = \mathrm{Leb}$.

To show Theorem C (b), we use the previous item (a), the property ($\mathbf{P}_1$) and (8.3) to deduce that

$$h_{\mu_{\mathrm{SRB}}}(\mathrm{Sp}(G)) = h_{(H_G)_*(\mu_{\mathrm{SRB}})}(\Phi\times L) = h_{\mathrm{Leb}}(\Phi\times L) = h_{\mathrm{top}}(\Phi\times L) = h_{\mathrm{top}}(\mathrm{Sp}(G))$$

and thereby conclude that $h_{\mu_{\mathrm{SRB}}}(\mathrm{Sp}(G)) = h_{\mathrm{top}}(\mathrm{Sp}(G))$, as claimed.

Remark 10.3: Under the assumption that Φ is a linear hyperbolic automorphism of $\mathbb{T}^2$, we have concluded that G also has a unique ergodic SRB measure, which is its unique physical measure. In addition, if $m_{\max}$ denotes the measure with maximal entropy of G, then clearly $(h_G)_*(m_{\max}) = (H_G)_*(\mu_{\max}) = \mathrm{Leb}$, where h_G is the semi-conjugation between G and $\Phi \times L$ defined in ($\mathbf{P}_3$). Yet, we do not know if the SRB measure of G coincides with $m_{\max}$. Anyway, Theorem C indicates that the change from G to the conjugate skew-product $\mathrm{Sp}(G)$ triggers the synchronization of the canonical measures (see [50] for a related topic).

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