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OPPOSITE ALGEBRAS OF GROUPOID C^*-ALGEBRAS

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ABSTRACT

We show that every groupoid C^*-algebra is isomorphic to its opposite, and deduce that there exist C^*-algebras that are not stably isomorphic to groupoid C^*-algebras, though many of them are stably isomorphic to twisted groupoid C^*-algebras. We also prove that the opposite algebra of a section algebra of a Fell bundle over a groupoid is isomorphic to the section algebra of a natural opposite bundle.

1. Introduction

Groupoids are among the most widely used models for operator algebras. It is therefore a basic question whether a given C^*-algebra A can be realised as $C^*(G)$ or $C^*_{\mathrm{r}}(G)$ for some locally compact topological groupoid G. Many classes of C^*-algebras have groupoid models: for example, graph C^*-algebras

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and higher-rank graph C^*-algebras, C^*-algebras of actions of inverse semigroups, and C^*-algebras associated to foliations. Moreover, it follows from the main results in [5] that every UCT Kirchberg C^*-algebra (that is, every separable, simple, nuclear, purely infinite, UCT C^*-algebra) has an étale groupoid model.

We show in this paper that not every C^*-algebra has a groupoid model. We achieve this by showing that all groupoid C^*-algebras are self-opposite in the sense that they are isomorphic to their opposite C^*-algebras. Similar results for L^p-algebras of ample étale groupoids appear in [7, Corollary 6.10 and Remarks 6.8 and 6.13].

Several examples of non-self-opposite C^*-algebras are already known. The first, produced by Connes [2], is a non-self-opposite von Neumann factor. Later, examples of non-self-opposite separable C^*-algebras were found by Phillips in [11]. All of Phillips' examples are continuous-trace C^*-algebras, hence nuclear. Simple and separable non-self-opposite C^*-algebras are constructed in [13, 12]; these examples are non-nuclear, though the one in [13] is exact. It remains open whether there exists a simple, separable and nuclear non-self-opposite C^*-algebra [3]. This is related to Elliott's conjecture (see [17]) because the Elliott invariant (essentially K-groups) used in the conjecture cannot distinguish a C^*-algebra A from its opposite A^{op}.

Although our result implies the existence of C^*-algebras with no groupoid model, it is still possible that such C^*-algebras can be realised as twisted groupoid C^*-algebras. That is, they could be isomorphic to $C^*(G,\Sigma)$ or $C^*_{\mathrm{r}}(G,\Sigma)$, for some twist Σ over a groupoid G. A twist over G is essentially the same thing as a Fell line bundle L over G, and $C^*(G,\Sigma)$ and $C^*_{\mathrm{r}}(G,\Sigma)$ are then the corresponding full and reduced cross-sectional C^*-algebras $C^*(G,L)$ and $C^*_{\mathrm{r}}(G,L)$. Renault proves in [16] that every C^*-algebra A admitting a Cartan subalgebra $\mathrm{C}_0(X)\subseteq A$ is isomorphic to $C^*_{\mathrm{r}}(G,\Sigma)$ for some (second countable, locally compact Hausdorff) étale essentially principal groupoid G with $G^0=X$ and some twist Σ on G; furthermore, the pair (G,Σ) is uniquely determined by the Cartan pair $(A,\mathrm{C}_0(X))$.

Kumjian, an Huef and Sims proved in [8] that every Fell C^*-algebra (in particular, every continuous-trace C^*-algebra) is Morita equivalent to one with a diagonal subalgebra in the sense of Kumjian [9]. These diagonal subalgebras are exactly the Cartan subalgebras (in the sense of Renault) with the **unique extension property**: every pure state of the Cartan subalgebra $\mathrm{C}_0(X)$ extends uniquely to A. The corresponding twist (G,Σ) that describes $(A, \mathrm{C}_0(X))$ is over a principal, not just essentially principal, groupoid G. After stabilisation, these results imply that all continuous-trace C^*-algebras have a twisted groupoid model—including the examples of Phillips in [11] that do not admit untwisted groupoid models. The point is that the opposite algebra of $C^*(G,\Sigma)$ arises as the C^*-algebra $C^*(G,\overline{\Sigma})$ of the conjugate twist, and this corresponds to taking the negative of the associated Dixmier–Douady invariant.

We elucidate the above phenomenon by describing the opposite C^*-algebras $C^*(G,\mathcal{A})^{\mathrm{op}}$ and $C^*_{\mathrm{r}}(G,\mathcal{A})^{\mathrm{op}}$ of the cross-sectional algebras of arbitrary Fell bundles $\mathcal{A}$ over locally compact groupoids. Specifically, given a Fell bundle $\mathcal{A}$ over G, we construct an appropriate opposite bundle $\mathcal{A}^{\mathrm{o}}$ over G, and prove that

$$C^*(G,\mathcal{A})^{\mathrm{op}} \cong C^*(G,\mathcal{A}^{\mathrm{o}}).$$

This can also be described in terms of the conjugate Fell bundle $\bar{\mathcal{A}}$. In the special case of a Fell line bundle L (that is, a twist over G), this corresponds to the conjugate line bundle. When L is the trivial line bundle, $\bar{L} = L$, and $C^*_{\mathrm{r}}(G;L)$ and $C^*(G;L)$ coincide with $C^*_{\mathrm{r}}(G)$ and $C^*(G)$, so we recover our earlier result as a special case.

For a Fell bundle associated to an action α of a locally compact group G on a C^*-algebra A, our result is equivalent to the statement that the opposite C^*-algebras of the full and reduced crossed products $A \rtimes_\alpha G$ and $A \rtimes_{\alpha,\mathrm{r}} G$ are isomorphic to $A^{\mathrm{op}} \rtimes_{\alpha^{\mathrm{op}}} G$ and $A^{\mathrm{op}} \rtimes_{\alpha^{\mathrm{op}},\mathrm{r}} G$ (where α^{op} is the action of G on A^{op} determined by α upon identifying A and A^{op} as linear spaces); this was proved for full crossed products in [3].

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2. Groupoid C^*-algebras and their opposites

For background on groupoids and their C^*-algebras, we refer the reader to [15].

In this section we show that the full and reduced C^*-algebras of a locally compact, locally Hausdorff groupoid with Haar system are self-opposite. We first briefly recall how these C^*-algebras are defined.

Let G be a locally compact and locally Hausdorff groupoid with Hausdorff unit space G^0 and a (continuous) left invariant Haar system $\lambda = \{\lambda^x\}_{x\in G^0}$. Let $\mathfrak{C}_c(G,\lambda)$ be the *-algebra of compactly supported, quasi-continuous sections, that is, the linear span of continuous functions with compact support $f\colon U \to \mathbb{C}$ on open Hausdorff subsets $U \subseteq G$. These functions are extended by zero off U and hence viewed as functions $G \to \mathbb{C}$. The continuity of λ means that every such function is mapped to a continuous function $\lambda(f)\colon G^0 \to \mathbb{C}$ via

$$\lambda(f)(x) := \int_G f(g)\,\mathrm{d}\lambda^x(g).$$

By definition, λ^x is a Radon measure on G with support $G^x := r^{-1}(x)$ for all $x \in G^0$.

Throughout the paper we follow the convention established by Renault that for $g \in G$ and $f_1, f_2 \in C_c(G)$, we abuse notation a little and write

$$\int_G f_1(h) f_2(h^{-1}g)\,\mathrm{d}\lambda^{r(g)}(h)$$

rather than $\int_{G^{r(g)}} f_1(h) f_2(h^{-1}g)\,\mathrm{d}\lambda^{r(g)}(h)$—strictly speaking, the integrand makes no sense for $h \notin G^{r(g)}$, but since the set of such h has measure zero under $\lambda^{r(g)}$ it is clear what the integral means. Recall that the product and involution on $\mathfrak{C}_c(G,\lambda)$ are defined by

$$(f_1 * f_2)(g) := \int_G f_1(h) f_2(h^{-1}g)\,\mathrm{d}\lambda^{r(g)}(h) \quad \text{and} \quad f^*(g) := \overline{f(g^{-1})}.$$

Under these operations and the inductive-limit topology, $\mathfrak{C}_c(G,\lambda)$ is a topological *-algebra. The I-norm on $\mathfrak{C}_c(G,\lambda)$ is defined by

$$\|f\|_I := \max\{\|\lambda(|f|)\|_\infty, \|\lambda(|f^*|)\|_\infty\}.$$

The L^1-Banach *-algebra of G is the completion of $\mathfrak{C}_c(G,\lambda)$ with respect to $\|\cdot\|_I$; we denote it by $L^1_I(G,\lambda)$. The full C^*-algebra of G is the universal enveloping C^*-algebra of $L^1_I(G,\lambda)$; in other words, it is the C^*-completion of $\mathfrak{C}_c(G,\lambda)$ with respect to the maximum $\|\cdot\|_I$-bounded C^*-norm:

$$\|f\|_{\mathrm{u}} := \sup\{\|\pi(f)\| : \pi \text{ is an } I\text{-norm decreasing } {}^*\text{-representation of } \mathfrak{C}_c(G,\lambda)\}.$$

The **regular representations** of (G,λ) are the representations

$$\pi_x\colon \mathfrak{C}_c(G,\lambda) \to \mathbb{B}(L^2(G_x,\lambda_x)), \quad x \in G^{(0)}$$

given by $\pi_x(f)\xi(g) := (f*\xi)(g) = \int_G f(gh)\xi(h^{-1})\,\mathrm{d}\lambda^x(h)$. Here λ_x is the image of λ^x under the inversion map $G \to G$, $g \mapsto g^{-1}$; so it is a measure with support $G_x = s^{-1}(x)$. The system of measures $(\lambda_x)_{x\in G^0}$ is a right invariant Haar system on G.

The regular representations of G give rise to a $\|\cdot\|_I$-bounded C^*-norm called the **reduced C^*-norm**:

$$\|f\|_{\mathrm{r}} := \sup_{x\in G^0} \|\pi_x(f)\|.$$

The **reduced C^*-algebra** of G is the completion of $\mathfrak{C}_c(G,\lambda)$ with respect to $\|\cdot\|_{\mathrm{r}}$. It is denoted by $C^*_{\mathrm{r}}(G,\lambda)$.

Given a groupoid G, we write G^{op} for the opposite groupoid, equal to G as a topological space, but with

$$(G^{\mathrm{op}})^{(2)} = \{(h,g) : (g,h) \in G^{(2)}\}$$

and composition given by $h \cdot_{\mathrm{op}} g = gh$. We write λ^{op} for the Haar system on G^{op} defined as the image of λ under the inversion map regarded as a homeomorphism of G onto G^{op}.

Theorem 2.1: *Let (G,λ) be a locally compact, locally Hausdorff groupoid with Haar system. The inversion map $g \mapsto g^{-1}$ defines an isomorphism*

$$(G,\lambda) \cong (G^{\mathrm{op}},\lambda^{\mathrm{op}})$$

of topological groupoids with Haar systems. Given $f \in \mathfrak{C}_c(G,\lambda)$, define $f^{\mathrm{op}} : G \to \mathbb{C}$ by

$$f^{\mathrm{op}}(g) := f(g^{-1}).$$

Then $f \mapsto f^{\mathrm{op}}$ is an isomorphism $\mathfrak{C}_c(G,\lambda) \xrightarrow{\sim} \mathfrak{C}_c(G,\lambda)^{\mathrm{op}}$ of topological *-algebras. This isomorphism extends to a Banach *-algebra isomorphism

$$L^1_I(G,\lambda) \xrightarrow{\sim} L^1_I(G,\lambda)^{\mathrm{op}}$$

and to C^*-algebra isomorphisms

$$C^*(G,\lambda) \xrightarrow{\sim} C^*(G,\lambda)^{\mathrm{op}} \quad \text{and} \quad C^*_{\mathrm{r}}(G,\lambda) \xrightarrow{\sim} C^*_{\mathrm{r}}(G,\lambda)^{\mathrm{op}}.$$

Proof. The map $g \mapsto g^{-1}$ is a homeomorphism $G \to G^{\mathrm{op}}$ and satisfies

$$g^{-1} \cdot_{\mathrm{op}} h^{-1} = h^{-1}g^{-1} = (gh)^{-1},$$

so it is an isomorphism $G \xrightarrow{\sim} G^{\mathrm{op}}$ of topological groupoids. The range (resp. source) map of G^{op} is the source (resp. range) map of G, and the inversion map sends the left invariant Haar system $\lambda = (\lambda^x)_{x\in G^0}$ on G to the right invariant Haar system $(\lambda_x)_{x\in G^0}$, which is precisely λ^{op}. This yields the isomorphism $(G,\lambda) \cong (G^{\mathrm{op}},\lambda^{\mathrm{op}})$. The map

$$f \mapsto f^{\mathrm{op}}$$

is a linear involution (in particular, a bijection) which is clearly a homeomorphism with respect to the inductive-limit topology. It is also clearly isometric for the $\|\cdot\|_I$-norms on $\mathfrak{C}_c(G,\lambda)$ and $\mathfrak{C}_c(G^{\mathrm{op}},\lambda^{\mathrm{op}})$. So to prove that it is topological *-algebra isomorphism $\mathfrak{C}_c(G,\lambda)\cong\mathfrak{C}_c(G,\lambda)^{\mathrm{op}}$ and extends to isomorphisms $L^1_I(G,\lambda)\cong L^1_I(G,\lambda)^{\mathrm{op}}$ and $C^*(G,\lambda)\cong C^*(G,\lambda)^{\mathrm{op}}$, it suffices to show that $f\mapsto f^{\mathrm{op}}$ is a *-homomorphism. For $f\in\mathfrak{C}_c(G,\lambda)$,

$$(f^{\mathrm{op}})^*(g)=\overline{f^{\mathrm{op}}(g^{-1})}=\overline{f(g)}=f^*(g^{-1})=(f^*)^{\mathrm{op}}(g).$$

So $f\mapsto f^{\mathrm{op}}$ preserves involution. If $f_1,f_2\in\mathfrak{C}_c(G,\lambda)$, then

$$(f_1*f_2)^{\mathrm{op}}(g)=(f_1*f_2)(g^{-1})=\int_G f_1(h)f_2(h^{-1}g^{-1})\,\mathrm{d}\lambda^{s(g)}(h), \tag{2.2}$$

while

$$\begin{aligned}(f_2^{\mathrm{op}}*f_1^{\mathrm{op}})(g)&=\int_G f_2^{\mathrm{op}}(h)f_1^{\mathrm{op}}(h^{-1}g)\,\mathrm{d}\lambda^{r(g)}(h)\\&=\int_G f_1(g^{-1}h)f_2(h^{-1})\,\mathrm{d}\lambda^{r(g)}(h).\end{aligned} \tag{2.3}$$

Making the change of variables $h\mapsto gh$ and applying left invariance of λ shows that (2.2) and (2.3) are equal.

To prove that $C^*_{\mathrm{r}}(G,\lambda)\cong C^*_{\mathrm{r}}(G,\lambda)^{\mathrm{op}}$, observe that the map $f\mapsto f^{\mathrm{op}}$ gives an isomorphism $L^2(G_x,\lambda_x)\cong L^2(G^x,\lambda^x)=L^2(G^{\mathrm{op}}_x,\lambda^{\mathrm{op}}_x)$ which induces a unitary equivalence between the regular representations

$$\pi_x\colon\mathfrak{C}_c(G,\lambda)\to\mathbb{B}(L^2(G_x,\lambda_x)\quad\text{and}\quad\pi^{\mathrm{op}}_x\colon\mathfrak{C}_c(G^{\mathrm{op}},\lambda^{\mathrm{op}})\to\mathbb{B}(L^2(G^{\mathrm{op}}_x,\lambda^{\mathrm{op}}_x).$$

This yields the equality

$$\|f\|_{\mathrm{r}}=\|f^{\mathrm{op}}\|_{\mathrm{r}}$$

which shows that $f\mapsto f^{\mathrm{op}}$ extends to an isomorphism $C^*_{\mathrm{r}}(G,\lambda)\cong C^*_{\mathrm{r}}(G^{\mathrm{op}},\lambda^{\mathrm{op}})$. ∎

Remark 2.4: Similar arguments to those above show that the identity map on G, regarded as an anti-multiplicative homeomorphism from G to G^{op}, induces (by composition) an anti-multiplicative linear isomorphism $\mathfrak{C}_c(G,\lambda)\cong\mathfrak{C}_c(G^{\mathrm{op}},\lambda^{\mathrm{op}})$, and therefore a topological *-algebra isomorphism $\mathfrak{C}_c(G,\lambda)^{\mathrm{op}}\cong\mathfrak{C}_c(G^{\mathrm{op}},\lambda^{\mathrm{op}})$. This latter extends to isomorphisms

$$L^1_I(G,\lambda)^{\mathrm{op}}\cong L^1_I(G^{\mathrm{op}},\lambda^{\mathrm{op}}),\quad C^*(G,\lambda)^{\mathrm{op}}\cong C^*(G^{\mathrm{op}},\lambda^{\mathrm{op}})$$
$$\text{and}\quad C^*_{\mathrm{r}}(G,\lambda)^{\mathrm{op}}\cong C^*_{\mathrm{r}}(G^{\mathrm{op}},\lambda^{\mathrm{op}}).$$

Another way to prove Theorem 2.1 is to work with conjugate algebras. If A is a *-algebra, its **conjugate *-algebra** $\bar{A}$ is the conjugate vector space of A

endowed with the same algebraic operations as A. Involution, $a \mapsto a^*$ is then a linear anti-multiplicative isomorphism $A \to \bar{A}$ and therefore an isomorphism $A^{\mathrm{op}} \cong \bar{A}$. We have $\mathfrak{C}_c(G,\lambda) \cong \overline{\mathfrak{C}_c(G,\lambda)}$ via $\xi \mapsto \bar{\xi}$ and this extends to isomorphisms $L^1_I(G,\lambda) \cong \overline{L^1_I(G,\lambda)}$, $C^*(G,\lambda) \cong \overline{C^*(G,\lambda)}$ and $C^*_{\mathrm{r}}(G,\lambda) \cong \overline{C^*_{\mathrm{r}}(G,\lambda)}$.

Corollary 2.5: *There are (nuclear, separable) C^*-algebras that are not isomorphic to either $C^*(G,\lambda)$ or $C^*_{\mathrm{r}}(G,\lambda)$ for any locally compact, locally Hausdorff groupoid with Haar system.*

Proof. It is known that there are examples of nuclear and separable C^*-algebras that are not self-opposite [11, 3]. ∎

Let us say that a *-algebra A is **self-opposite** if $A \cong A^{\mathrm{op}}$. Our main result says that given a topological groupoid with Haar system (G,λ), the *-algebras $\mathfrak{C}_c(G,\lambda)$, $L^1_I(G,\lambda)$, $C^*(G,\lambda)$ and $C^*_{\mathrm{r}}(G,\lambda)$ are all self-opposite. Both the minimal and the maximal tensor product of self-opposite C^*-algebras are again self-opposite because $(A \otimes B)^{\mathrm{op}} \cong A^{\mathrm{op}} \otimes B^{\mathrm{op}}$.

Let $\mathbb{K}$ denote the C^*-algebra of compact operators on a separable, infinite dimensional Hilbert space; writing $\mathcal{R}$ for the equivalence relation $\mathbb{N}\times\mathbb{N}$ regarded as a discrete principal groupoid, we have $\mathbb{K} \cong C^*(\mathcal{R}) = C^*_{\mathrm{r}}(\mathcal{R})$. Hence the preceding paragraph shows that every self-opposite C^*-algebra is also stably self-opposite. The converse fails in general: Phillips constructs in [11] examples of (separable, continuous-trace) non-self-opposite C^*-algebras which are stably self-opposite. But Phillips also constructs examples of (separable, continuous-trace) C^*-algebras that are not stably self-opposite. This yields the following:

Corollary 2.6: *There are separable continuous-trace C^*-algebras that are not stably isomorphic to any groupoid C^*-algebra.*

Remark 2.7: By the Brown–Green–Rieffel theorem [1], Corollary 2.6 implies that there exist separable C^*-algebras that are not Morita equivalent to a separable (or even σ-unital) groupoid C^*-algebra. However, it is unclear whether these examples could be Morita equivalent to a non-σ-unital groupoid C^*-algebra.

In [6], in the framework of ZFC enriched with Jensen's diamond principle (a strengthening of the continuum hypothesis), Farah and Hirshberg construct examples of non-separable approximately matricial algebras (uncountable direct limits of the CAR algebra) that are non-self-opposite, so we can also state:

Corollary 2.8: *It is consistent with ZFC that there are non-separable approximately matricial (so simple, nuclear) C^*-algebras that are not isomorphic to a groupoid C^*-algebra.*

Recall that the ordinary separable AF-algebras admit groupoid models: it is even known that they are always crossed products for a partial action of the integers, see [4].

By [8, Theorem 6.6(1)], every separable continuous-trace C^*-algebra (indeed, every Fell algebra) is Morita equivalent to a separable C^*-algebra with a diagonal subalgebra in the sense of Kumjian [9]. Kumjian shows in [9] that C^*-algebras containing diagonals are, up to isomorphism, the C^*-algebras obtained from twists on étale principal groupoids. More precisely, writing $\mathbb{T}$ for the circle group, this means a locally compact Hausdorff central groupoid extension

$$\mathbb{T} \times G^0 \hookrightarrow \Sigma \twoheadrightarrow G,$$

of a (second countable) locally compact Hausdorff groupoid G by the (trivial) group bundle $\mathbb{T} \times G^0$. To a twisted groupoid (G, Σ) one can assign a full C^*-algebra $C^*(G, \Sigma)$ and a reduced C^*-algebra $C^*_{\mathrm{r}}(G, \Sigma)$, and then every separable C^*-algebra containing a diagonal subalgebra has the form $C^*_{\mathrm{r}}(G, \Sigma)$ for some twist Σ over a principal groupoid G. Moreover, the pair (G, Σ) is unique, up to isomorphism of twisted groupoids. This follows from the more general result, proved by Renault in [16], that isomorphism classes of Cartan subalgebras correspond bijectively to isomorphism classes of twisted essentially principal étale groupoids (meaning twisted groupoids where G is not necessarily principal, but only essentially principal; see [16] for details). Using these results, we arrive at the following consequence:

Corollary 2.9: *There are separable stable continuous-trace C^*-algebras that are not isomorphic to any groupoid C^*-algebra but which are isomorphic to the reduced C^*-algebra of a twisted principal étale groupoid.*

Proof. Let A be a separable continuous-trace C^*-algebra which is not stably isomorphic to any groupoid C^*-algebra as in Corollary 2.6. Let

$$B := A \otimes \mathbb{K}$$

be the stabilisation of A. Then B is a separable stable continuous-trace C^*-algebra which is not isomorphic to any groupoid C^*-algebra. By [8, Theorem 6.(1)]

A is Morita equivalent to $C^*_{\mathrm{r}}(G,\Sigma)$, for some twisted principal étale groupoid (G,Σ). It follows from the Brown–Green–Rieffel theorem that

$$B \cong C^*_{\mathrm{r}}(G,\Sigma) \otimes \mathbb{K}.$$

To finish the proof we observe that, again writing $\mathcal{R}$ for the discrete equivalence relation $\mathbb{N}\times\mathbb{N}$, we have $C^*_{\mathrm{r}}(G,\Sigma)\otimes\mathbb{K} \cong C^*_{\mathrm{r}}(G\times\mathcal{R},\Sigma\times\mathcal{R})$. ∎

3. Section C^*-algebras of Fell bundles and their opposites

Let G be a locally compact and locally Hausdorff groupoid endowed with a continuous Haar system λ, which we fix throughout the rest of the section. In this section we generalise our previous result and describe the opposite C^*-algebras of the section C^*-algebras of Fell bundles over G. Our result generalises the observation in [3] that $(A \rtimes_\alpha G)^{\mathrm{op}} \cong A^{\mathrm{op}} \rtimes_{\alpha^{\mathrm{op}}} G$ for any action α of a locally compact group G on a C^*-algebra A.

Fell bundles over topological groupoids are defined in [10]. Only Hausdorff groupoids are considered there, but the same definition makes sense for locally Hausdorff groupoids. A Fell bundle over G consists of an upper semicontinuous Banach bundle $\mathcal{A}$ over G endowed with **multiplications** $\mathcal{A}_g\times\mathcal{A}_h\to\mathcal{A}_{gh}$, $(a,b)\mapsto a\cdot b$, for every composable pair $(g,h)\in G^2$ and **involutions** $\mathcal{A}_g\to\mathcal{A}_{g^{-1}}$, $a\mapsto a^*$, for every $g\in G$. These operations are required to be continuous (with respect to the given topology on $\mathcal{A}$) and satisfy algebraic conditions similar to those in the definition of a C^*-algebra.

We next recall, briefly, how to define the full and reduced C^*-algebras of a Fell bundle. Consider the space $\mathfrak{C}_c(G,\mathcal{A})$ of compactly supported continuous sections $\xi\colon U\to\mathcal{A}$ defined on open Hausdorff subspaces $U\subseteq G$ and extended by zero outside U and hence viewed as sections $\xi\colon G\to\mathcal{A}$. The continuity of the algebraic operations on $\mathcal{A}$ implies that for $\xi,\eta\in\mathfrak{C}_c(G,\mathcal{A})$, the formulas

$$(\xi*\eta)(g) := \int_G \xi(h)\cdot\eta(h^{-1}g)\,\mathrm{d}\lambda^{r(g)}(h) \quad\text{and}\quad \xi^*(g) := \xi(g^{-1})^*$$

define elements $\xi*\eta, \xi^*\in\mathfrak{C}_c(G,\mathcal{A})$ and so determine a convolution product $*$ and an involution * on $\mathfrak{C}_c(G,\mathcal{A})$. Under these operations, $\mathfrak{C}_c(G,\mathcal{A})$ is a *-algebra; and indeed, a topological *-algebra in the inductive-limit topology.

Since the norm function on $\mathcal{A}$ is upper semicontinuous, the function $g \mapsto \|\xi(g)\|$ from G to $[0,\infty)$ is upper semicontinuous and hence measurable. So we can define the I-norm on $\mathfrak{C}_c(G,\mathcal{A})$ by

$$\|\xi\|_I := \sup_{x\in G^{(0)}} \max\left\{ \int_{G^x} |\xi(g)|\,\mathrm{d}\lambda^x(g), \int_{G^x} |\xi^*(g)|\,\mathrm{d}\lambda^x(g)\right\}.$$

The L^1-Banach algebra of $\mathcal{A}$, denoted $L^1_I(G,\mathcal{A})$, is defined as the completion of $\mathfrak{C}_c(G,\mathcal{A})$ with respect to $\|\cdot\|_I$. The full C^*-algebra $C^*(G,\mathcal{A})$ of $\mathcal{A}$ is defined as the universal enveloping C^*-algebra of $L^1_I(G,\mathcal{A})$: the completion of $\mathfrak{C}_c(G,\mathcal{A})$ with respect to the C^*-norm

$$\|\xi\|_{\mathrm{u}} := \sup\{\|\pi(\xi)\| : \pi \text{ is an } I\text{-norm decreasing }{}^*\text{-representation of } \mathfrak{C}_c(G,\mathcal{A})\}.$$

That this is indeed a norm on $\mathfrak{C}_c(G,\mathcal{A})$, and not just a seminorm, follows from the existence of the following regular representations.

For each $x \in G^{(0)}$, let $L^2(G_x,\mathcal{A})$ be the right Hilbert $\mathcal{A}_x$-module completion of the space $\mathfrak{C}_c(G_x,\mathcal{A})$ of quasi-continuous sections $G_x \to \mathcal{A}$ with respect to the norm induced by the $\mathcal{A}_x$-valued inner product

$$\langle\xi|\eta\rangle_{\mathcal{A}_x} := \int_G \xi(h)^*\eta(h)\,\mathrm{d}\lambda_x(h) = \int_G \xi(h^{-1})^*\eta(h^{-1})\,\mathrm{d}\lambda^x(h).$$

Then for each $x\in G^{(0)}$, the regular representation $\pi_x : \mathfrak{C}_c(G,\mathcal{A}) \to \mathbb{B}(L^2(G_x,\mathcal{A}))$ is defined by

$$\big(\pi_x(\xi)\eta\big)(g) := \int_G \xi(gh)\eta(h^{-1})\,\mathrm{d}\lambda^x(h) = \int_G \xi(gh^{-1})\eta(h)\,\mathrm{d}\lambda_x(h),$$

for all $\xi \in \mathfrak{C}_c(G,\mathcal{A})$, $\eta \in \mathfrak{C}_c(G_x,\mathcal{A})$ and $g \in G_x$. The reduced C^*-norm on $\mathfrak{C}_c(G,\mathcal{A})$ is defined by

$$\|\xi\|_{\mathrm{r}} := \sup_{x\in G^0} \|\pi_x(\xi)\|.$$

This is, indeed, a norm: if $\pi_x(\xi) = 0$ then $(\xi * \eta)(g) = 0$ for all $\eta \in \mathfrak{C}_c(G_x,\mathcal{A})$ and $g \in G_x$; so

$$\xi * \xi^*(x) = \int_G \xi(h)\xi(h)^*\,\mathrm{d}\lambda^x(h) = 0 \quad \text{for all } x \in G^{(0)},$$

forcing $\xi|_{G^x} = 0$ for all x. A standard computation shows that $\|\xi\|_{\mathrm{r}} \le \|\xi\|_I$. Therefore $\|\cdot\|_{\mathrm{u}}$ is also a C^*-norm and $\|\cdot\|_{\mathrm{r}} \le \|\cdot\|_{\mathrm{u}}$. The completion of $\mathfrak{C}_c(G,\mathcal{A})$ with respect to $\|\cdot\|_{\mathrm{r}}$ is the **reduced section C^*-algebra** of $\mathcal{A}$, and is denoted by $C^*_{\mathrm{r}}(G,\mathcal{A})$.

Our goal is to describe the opposite C^*-algebras $C^*(G,\mathcal{A})^{\mathrm{op}}$ and $C^*_{\mathrm{r}}(G,\mathcal{A})^{\mathrm{op}}$. We show that $C^*(G,\mathcal{A})^{\mathrm{op}} \cong C^*(G,\mathcal{A}^{\mathrm{o}})$, for an appropriate opposite Fell bundle $\mathcal{A}^{\mathrm{o}}$ over G associated to $\mathcal{A}$. It is more natural to first define an opposite Fell bundle $\mathcal{A}^{\mathrm{op}}$ over the opposite groupoid G^{op} and then later use the canonical anti-isomorphism $G \cong G^{\mathrm{op}}$ induced by the inversion map to obtain the desired Fell bundle $\mathcal{A}^{\mathrm{o}}$ over G.

The opposite Fell bundle $\mathcal{A}^{\mathrm{op}}$ over G^{op} is defined as follows. As a Banach bundle, $\mathcal{A}^{\mathrm{op}}$ does not differ from $\mathcal{A}$. In particular, the fibres are equal, $\mathcal{A}^{\mathrm{op}}_g = \mathcal{A}_g$ for all $g \in G$, and also the topology on $\mathcal{A}^{\mathrm{op}}$ is equal to that on $\mathcal{A}$. Moreover, $\mathcal{A}^{\mathrm{op}}$ is also endowed with the same involution as $\mathcal{A}$, which makes sense because G and G^{op} carry the same inversion map. The only thing that changes in $\mathcal{A}^{\mathrm{op}}$ is the multiplication: given $g,h \in G^{\mathrm{op}}$ the condition $s^{\mathrm{op}}(g) = r^{\mathrm{op}}(h)$ means $r(g) = s(h)$, so we can use the multiplication map $\mu_{h,g}\colon \mathcal{A}_h \times \mathcal{A}_g \to \mathcal{A}_{hg}$ and define the multiplication maps

$$\mu^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}}_g \times \mathcal{A}^{\mathrm{op}}_h = \mathcal{A}_g \times \mathcal{A}_h \to \mathcal{A}^{\mathrm{op}}_{g\cdot_{\mathrm{op}} h} = \mathcal{A}_{hg} \quad \text{by } \mu^{\mathrm{op}}(a,b) := \mu(b,a).$$

In other words, $a \cdot_{\mathrm{op}} b := b \cdot a$ if we use $\cdot$ and $\cdot_{\mathrm{op}}$ to denote the multiplications on $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$, respectively. It is straightforward to see that $\mathcal{A}^{\mathrm{op}}$ is indeed a Fell bundle over G^{op}. Now we use the anti-isomorphism $G^{\mathrm{op}} \cong G$ induced by the inversion map $g \mapsto g^{-1}$ to form the pullback Fell bundle of $\mathcal{A}^{\mathrm{op}}$. In other words, $\mathcal{A}^{\mathrm{o}}$ is a Fell bundle over G with fibres $\mathcal{A}^{\mathrm{o}}_g = \mathcal{A}_{g^{-1}}$ and the topology induced by the sections $\xi^{\mathrm{o}}(g) := \xi(g^{-1})$ for $\xi\colon U \to \mathcal{A}$ a continuous section defined on a Hausdorff open subset $U \subseteq G$. The involution map $\mathcal{A}^{\mathrm{o}}_g \to \mathcal{A}^{\mathrm{o}}_{g^{-1}}$ is the involution map $\mathcal{A}_{g^{-1}} \to \mathcal{A}_g$ from $\mathcal{A}$ and the multiplication map $\mathcal{A}^{\mathrm{o}}_g \times \mathcal{A}^{\mathrm{o}}_h \to \mathcal{A}^{\mathrm{o}}_{gh}$ is given by $(a,b) \mapsto b \cdot a$ for all $b \in \mathcal{A}^{\mathrm{o}}_g = \mathcal{A}_{g^{-1}}$, $b \in \mathcal{A}^{\mathrm{o}}_h = \mathcal{A}_{h^{-1}}$ and $g,h \in G$ with $s(g) = r(h)$.

Theorem 3.1: *Let $\mathcal{A}$ be a Fell bundle over a locally compact, locally Hausdorff groupoid with Haar system (G,λ), and consider the Fell bundle $\mathcal{A}^{\mathrm{o}}$ over (G,λ) described above. The map $\xi \mapsto \xi^{\mathrm{o}}$ defined by*

$$\xi^{\mathrm{o}}(g) := \xi(g^{-1})$$

gives an isomorphism of topological *-algebras $\mathfrak{C}_c(G,\mathcal{A})^{\mathrm{op}} \xrightarrow{\sim} \mathfrak{C}_c(G,\mathcal{A}^{\mathrm{o}})$. Moreover, this isomorphism extends to an isomorphism of Banach *-algebras $L^1_I(G,\mathcal{A})^{\mathrm{op}} \xrightarrow{\sim} L^1_I(G,\mathcal{A}^{\mathrm{o}})$ and C^*-algebras $C^*(G,\mathcal{A})^{\mathrm{op}} \xrightarrow{\sim} C^*(G,\mathcal{A}^{\mathrm{o}})$ and $C^*_{\mathrm{r}}(G,\mathcal{A})^{\mathrm{op}} \xrightarrow{\sim} C^*_{\mathrm{r}}(G,\mathcal{A}^{\mathrm{o}})$.

Proof. We prove the equivalent assertion that $\mathfrak{C}_c(G,\mathcal{A})^{\mathrm{op}} \cong \mathfrak{C}_c(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}})$ via the canonical linear isomorphism $\mathfrak{C}_c(G,\mathcal{A}) \ni \xi \mapsto \xi^{\mathrm{op}} := \xi \in \mathfrak{C}_c(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}})$, and that this isomorphism extends to isomorphisms

$$L^1_I(G,\mathcal{A})^{\mathrm{op}} \xrightarrow{\sim} L^1_I(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}}), \quad C^*(G,\mathcal{A})^{\mathrm{op}} \xrightarrow{\sim} C^*(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}}), \quad \text{and}$$
$$C^*_{\mathrm{r}}(G,\mathcal{A})^{\mathrm{op}} \xrightarrow{\sim} C^*_{\mathrm{r}}(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}}).$$

Since the topologies on $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ are the same, the map $\xi \mapsto \xi^{\mathrm{op}}$ is clearly a linear bijection $\mathfrak{C}_c(G,\mathcal{A}) \to \mathfrak{C}_c(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}})$ which is a homeomorphism with respect to the inductive-limit topologies. Also, this map preserves the involution; that is, $(\xi^{\mathrm{op}})^* = \xi^*$ on $\mathfrak{C}_c(G,\mathcal{A})$ (which is the same as the involution on $\mathfrak{C}_c(G,\mathcal{A})^{\mathrm{op}}$), and on $\mathfrak{C}_c(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}})$ because the involutions on $\mathcal{A}$ and on $\mathcal{A}^{\mathrm{op}}$ are the same. It remains to check that the map is a homomorphism $\mathfrak{C}_c(G,\mathcal{A})^{\mathrm{op}} \to \mathfrak{C}_c(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}})$. But, remembering that the left Haar system λ^{op} on G^{op} is the right Haar system $(\lambda_x)_{x\in G^0}$ on G, we get

$$\begin{aligned}
\xi^{\mathrm{op}} * \eta^{\mathrm{op}}(g) &= \int_{G^{\mathrm{op}}} \xi(h) \cdot_{\mathrm{op}} \eta(h^{-1}g)\,\mathrm{d}(\lambda^{\mathrm{op}})^{r^{\mathrm{op}}(g)}(h)\\
&= \int_G \eta(gh^{-1})\xi(h)\,\mathrm{d}\lambda_{s(g)}(h)\\
&= \int_G \eta(gh)\xi(h^{-1})\,\mathrm{d}\lambda^{s(g)}(h) = (\eta * \xi)(g)
\end{aligned}$$

for all $\xi,\eta \in \mathfrak{C}_c(G,\mathcal{A})$ and $g \in G$. This shows that the identity map is an anti-homomorphism $\mathfrak{C}_c(G,\mathcal{A}) \to \mathfrak{C}_c(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}})$, that is, a homomorphism $\mathfrak{C}_c(G,\mathcal{A})^{\mathrm{op}} \to \mathfrak{C}_c(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}})$, as desired. A similar computation shows that $\|\xi^{\mathrm{op}}\|_I = \|\xi\|_I$ and that therefore the identity map extends to an isomorphism $L^1_I(G,\mathcal{A})^{\mathrm{op}} \xrightarrow{\sim} L^1_I(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}})$ and hence also to the corresponding universal enveloping C^*-algebras $C^*(G,\mathcal{A})^{\mathrm{op}} \xrightarrow{\sim} C^*(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}})$.

Finally, to check that $C^*_{\mathrm{r}}(G,\mathcal{A})^{\mathrm{op}} \cong C^*_{\mathrm{r}}(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}})$, fix $x \in G^0$. The regular representation π^{op}_x defines a representation of $C^*(G^{\mathrm{op}},\mathcal{A}^{\mathrm{op}})$ by adjointable operators on the right-Hilbert $\mathcal{A}^{\mathrm{op}}$-module $L^2(G^{\mathrm{op}}_x,\mathcal{A}^{\mathrm{op}})$. Recall that a left-Hilbert module over a C^*-algebra C is a left C-module $\mathcal{E}$ with an inner product ${}_C\langle\cdot|\cdot\rangle$ in which ${}_C\langle c\cdot\xi|\eta\rangle = c\,{}_C\langle\xi|\eta\rangle$ for $c \in C$ and $\xi,\eta \in \mathcal{E}$. Any right Hilbert module $\mathcal{E}$ over the opposite B^{op} of a C^*-algebra B determines a left Hilbert B-module $\mathcal{E}$ with left B-action $b\cdot\xi := \xi\cdot b$ and left B-valued inner product ${}_B\langle\xi|\eta\rangle := \langle\eta|\xi\rangle_{B^{\mathrm{op}}}$. This process preserves the C^*-algebras of adjointable operators, meaning that the identity map on $\mathcal{E}$ yields an isomorphism $\mathbb{B}(\mathcal{E}_{B^{\mathrm{op}}}) \cong \mathbb{B}({}_B\mathcal{E})$. Applying this to the right Hilbert $\mathcal{A}_x^{\mathrm{op}}$-module $L^2(G_x^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}})$ we get a left Hilbert $\mathcal{A}_x$-module with left $\mathcal{A}_x$-action given by $a \cdot \xi = \xi \cdot_{\mathrm{op}} a$ for all $\xi \in L^2(G_x^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}})$; the right hand side denotes the right $\mathcal{A}_x^{\mathrm{op}}$-action on $L^2(G_x^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}})$, so it is given by $(a \cdot_{\mathrm{op}} \xi)(g) = \xi(g) \cdot_{\mathrm{op}} a(s^{\mathrm{op}}(g)) = a(r(g)) \cdot \xi(g)$. The left $\mathcal{A}_x$-valued inner product on $L^2(G^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}})$ is given by

$$ {}_{\mathcal{A}_x}\langle \xi | \eta \rangle = \langle \eta | \xi \rangle_{\mathcal{A}_x^{\mathrm{op}}} = \int_{G^{\mathrm{op}}} \eta(h)^* \cdot_{\mathrm{op}} \xi(h) \, \mathrm{d}\lambda_x^{\mathrm{op}}(h) = \int_G \xi(h)\eta(h)^* \, \mathrm{d}\lambda^x(h) $$

for all $\xi, \eta \in \mathfrak{C}_c(G_x^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}}) = \mathfrak{C}_c(G^x, \mathcal{A})$. Therefore the left Hilbert $\mathcal{A}_x$-module obtained from the right Hilbert $\mathcal{A}_x^{\mathrm{op}}$-module $L^2(G_x^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}})$ in this way equals the left Hilbert $\mathcal{A}_x$-module $L^2(G^x, \mathcal{A})$ defined as the completion of $\mathfrak{C}_c(G^x, \mathcal{A})$ with respect to the norm associated to the left $\mathcal{A}_x$-valued inner product given by the above formula and the left $\mathcal{A}_x$-action also defined above. Therefore we may view π_x^{op} as a representation of $C^*(G^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}})$ on

$$ \mathbb{B}({}_{\mathcal{A}_x} L^2(G^x, \mathcal{A})) \cong \mathbb{B}(L^2(G_x^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}})_{\mathcal{A}_x^{\mathrm{op}}}). $$

Under the isomorphism $C^*(G^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}}) \cong C^*(G, \mathcal{A})^{\mathrm{op}}$, this corresponds to the canonical representation $\tilde{\pi}_x$ of $C^*(G, \mathcal{A})^{\mathrm{op}}$ on ${}_{\mathcal{A}_x} L^2(G^x, \mathcal{A})$ via the formula

$$ \tilde{\pi}_x(\xi)\eta(g) := (\eta * \xi)(g) = \int_G \eta(h)\xi(h^{-1}g) \, \mathrm{d}\lambda^x(h) $$

for $\xi \in \mathfrak{C}_c(G, \mathcal{A})$, $\eta \in \mathfrak{C}_c(G^x, \mathcal{A})$ and $g \in G^x$. Straightforward computations show that the above formula defines a representation

$$ \tilde{\pi}_x \colon C^*(G, \mathcal{A})^{\mathrm{op}} \to \mathbb{B}({}_{\mathcal{A}_x} L^2(G^x, \mathcal{A})) $$

of the opposite C^*-algebra $C^*(G, \mathcal{A})^{\mathrm{op}}$.

Given a left Hilbert B-module $\mathcal{E}$, let $\tilde{\mathcal{E}}$ denote the dual right Hilbert B-module of $\mathcal{E}$: as a vector space $\tilde{\mathcal{E}} = \{\tilde{\xi} : \xi \in \mathcal{E}\}$ is the conjugate of $\mathcal{E}$ and the right B-action and right B-valued inner product are defined by

$$ \tilde{\xi} \cdot b := (b^* \cdot \xi)^\sim \quad \text{and} \quad \langle \xi | \eta \rangle_B := {}_B\langle \xi | \eta \rangle. $$

Then each representation $\pi \colon A^{\mathrm{op}} \to \mathbb{B}({}_B\mathcal{E})$ of an opposite C^*-algebra A^{op} on the C^*-algebra of adjointable operators $\mathbb{B}({}_B\mathcal{E})$ of a left Hilbert B-module $\mathcal{E}$ induces a representation

$$ \pi^{\mathrm{op}} \colon A \to \mathbb{B}({}_B\mathcal{E})^{\mathrm{op}} \cong \mathbb{B}(\tilde{\mathcal{E}}_B). $$

The isomorphism $\mathbb{B}({}_B\mathcal{E})^{\mathrm{op}} \cong \mathbb{B}(\tilde{\mathcal{E}}_B)$ we used above is induced by the involution; that is, it sends an operator $T \in \mathbb{B}({}_B\mathcal{E})$ to $\tilde{T} \in \mathbb{B}(\tilde{\mathcal{E}}_B)$ defined by

$$ \tilde{T}(\tilde{\xi}) := (T^*(\xi))^\sim. $$

For $\xi \in \mathfrak{C}_c(G^x, \mathcal{A})$, the formula

$$\xi^*(g) := \xi(g^{-1})^*$$

determines an element $\xi^* \in \mathfrak{C}_c(G_x, \mathcal{A})$. The map $\xi \mapsto \xi^*$ induces an isomorphism $(L^2(G^x, \mathcal{A}))\tilde{}_{\mathcal{A}_x} \cong L^2(G_x, \mathcal{A})_{\mathcal{A}_x}$ from the dual Hilbert $\mathcal{A}_x$-module of ${}_{\mathcal{A}_x}L^2(G^x, \mathcal{A})$ to the right Hilbert $\mathcal{A}_x$-module $L^2(G_x, \mathcal{A})$ that carries the regular representation $\pi_x \colon C^*(G, \mathcal{A}) \to \mathbb{B}(L^2(G_x, \mathcal{A})_{\mathcal{A}_x})$. This isomorphism intertwines the representations

$$\pi_x \colon C^*(G, \mathcal{A}) \to \mathbb{B}(L^2(G_x, \mathcal{A})_{\mathcal{A}_x})$$

and

$$\tilde{\pi}_x^{\mathrm{op}} \colon C^*(G, \mathcal{A}) \to \mathbb{B}({}_{\mathcal{A}_x}L^2(G^x, \mathcal{A}))^{\mathrm{op}} \cong \mathbb{B}((L^2(G^x, \mathcal{A}))\tilde{}_{\mathcal{A}_x}).$$

We conclude that

$$\|\pi_x^{\mathrm{op}}(\xi)\| = \|\tilde{\pi}_x(\xi^{\mathrm{o}})\| = \|\tilde{\pi}_x^{\mathrm{op}}(\xi^{\mathrm{o}})\| = \|\pi_x(\xi^{\mathrm{o}})\|.$$

Since $x \in G^0$ was arbitrary, we get the equality $\|\xi^{\mathrm{o}}\|_{\mathrm{r}} = \|\xi\|_{\mathrm{r}}$ and therefore the desired isomorphism $C^*_{\mathrm{r}}(G^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}}) \cong C^*_{\mathrm{r}}(G, \mathcal{A})^{\mathrm{op}}$. ∎

Remark 3.2: As in the case of groupoid C^*-algebras, we can rephrase the preceding result in terms of conjugate bundles as well. Let $\mathcal{A}$ be a Fell bundle over a groupoid G. For $g \in G$, let $\overline{\mathcal{A}}_g$ be the conjugate vector space of $\mathcal{A}_g$; that is, $\overline{\mathcal{A}}_g$ is a copy $\{\overline{a} : a \in \mathcal{A}_g\}$ of $\mathcal{A}_g$ as an abelian group under addition, but with scalar multiplication given by $\lambda\overline{a} = \overline{\overline{\lambda}a}$. Via the map $a \mapsto \bar{a}$, the operations on the Fell bundle $\mathcal{A}$ induce operations on $\overline{A} := \bigsqcup_{g\in G} \overline{A}_g$: $\overline{a}\overline{b} = \overline{ab}$, and $\overline{a}^* = \overline{a^*}$. Under these operations, $\overline{\mathcal{A}}$ is a Fell bundle over G, called the **conjugate bundle** of $\mathcal{A}$.

Let $\mathcal{A}^{\mathrm{o}}$ be the opposite bundle of $\mathcal{A}$ defined above; so $\mathcal{A}^{\mathrm{o}}_g = \mathcal{A}_{g^{-1}}$, and write $\cdot_{\mathrm{o}}$ for the multiplication in this bundle. Then the maps $\mathcal{A}^{\mathrm{o}}_g \ni a \mapsto \overline{a^*} \in \overline{\mathcal{A}}_g$ are linear isometries because the maps $a \mapsto a^*$ and $a \mapsto \bar{a}$ are both conjugate linear. We have $\overline{(a \cdot_{\mathrm{o}} b)^*} = \overline{(ba)^*} = \overline{a^*b^*} = \overline{a^*}\,\overline{b^*}$ and $(\overline{a^*})^* = \overline{a} = \overline{(a^*)^*}$, so $a \mapsto \overline{a^*}$ determines an isomorphism $\mathcal{A}^{\mathrm{o}} \cong \overline{\mathcal{A}}$ of Fell bundles over G. Thus Theorem 3.1 shows that there is a topological-*-algebra isomorphism $\xi \mapsto \overline{\xi}$ from $\mathfrak{C}_c(G, \mathcal{A})$ to $\mathfrak{C}_c(G, \overline{\mathcal{A}})$ given by $\overline{\xi}(g) := \overline{\xi(g^{-1})^*}$ that extends to isomorphisms

$$L^1_I(G, \mathcal{A}^{\mathrm{o}}) \cong L^1_I(G, \overline{\mathcal{A}}), \quad C^*(G, \mathcal{A}^{\mathrm{o}}) \cong C^*(G, \overline{\mathcal{A}}), \quad \text{and} \quad C^*_{\mathrm{r}}(G, \mathcal{A}^{\mathrm{o}}) \cong C^*_{\mathrm{r}}(G, \overline{\mathcal{A}}).$$

Remark 3.3: Remark 3.2 is closely related to the idea behind Phillips' construction in [11] of non-self-opposite continuous-trace C^*-algebras A; the observation underlying his construction is that the Dixmier–Douady class of the opposite algebra A^{op} is the inverse of the Dixmier–Douady class of A. To see how this relates to our results, fix a compact Hausdorff space X, and let $\mathcal{S}$ denote the sheaf of germs of continuous $\mathbb{T}$-valued functions on X. The Raeburn–Taylor construction [14] shows that (after identifying $\check{H}^3(X,\mathbb{Z})$ with $H^2(X,\mathcal{S})$) any class $\delta \in H^2(X,\mathcal{S})$ is the Dixmier–Douady invariant of a twisted groupoid C^*-algebra $C^*(G,\sigma)$ associated to a continuous 2-cocycle σ on a principal étale groupoid G with unit space $G^{(0)} = \bigsqcup_{i,j} U_{ij}$ for some precompact open cover $\{U_i\}$ of X. The cocycle σ determines, and is determined up to cohomology by, the Fell line-bundle L_σ over G given by $L_\sigma = G \times \mathbb{T}$ with twisted multiplication $(g,w)(h,z) = (gh, c(g,h)wz)$ and the obvious involution. Remark 3.2 shows that $C^*(G,\sigma)^{\mathrm{op}}$ is given by the conjugate bundle $\overline{L_\sigma}$, so the corresponding class in $H^2(X,\mathcal{S})$ is determined by the pointwise conjugate of the class δ; that is, $\delta(C^*(G,\sigma)) = \delta(C^*(G,\sigma)^{\mathrm{op}})^{-1}$.

References

[1] L. G. Brown, P. Green and M. A. Rieffel, *Stable isomorphism and strong Morita equivalence of C^*-algebras*, Pacific Journal of Mathematics **71** (1977), 349–363.

[2] A. Connes, *A factor not anti-isomorphic to itself*, Annals Mathematics **101** (1975), 536–554.

[3] M. Dadarlat, I. Hirshberg aqnd N. C. Phillips, *Simple nuclear C*-algebras not equivariantly isomorphic to their opposites*, Journal of Noncommutative Geometry **12** (2018), 1227–1253.

[4] R. Exel, *Approximately finite C^*-algebras and partial automorphisms*, Mathematica Scandinavica **77** (1995), 281–288.

[5] R. Exel and E. Pardo *Self-similar graphs, a unified treatment of Katsura and Nekrashevych* C**-algebras*, Advances in Mathematics **306** (2017), 1046–1129.

[6] I. Farah and I. Hirshberg, *Simple nuclear C*-algebras not isomorphic to their opposites*, Proceedings of the National Academy of Sciences of the United States of America **114** (2017), 6244–6249.

[7] E. Gardella and M. Lupini, *Representations of étale groupoids in L^p-spaces*, Advances in Mathematics**318** (2017), 233–278.

[8] A. an Huef, A. Kumjian and A. Sims, *A Dixmier–Douady theorem for Fell algebras*, Journal of Functional Analysis **260** (2011), 1543–1581.

[9] A. Kumjian, *On C^*-diagonals*, Canadian Journal of Mathematics **38** (1986), 969–1008.

[10] A. Kumjian, *Fell bundles over groupoids*, Proceedings of the American Mathematical Society **126** (1998), 1115–1125.

[11] N. C. Phillips, *Continuous-trace C*-algebras not isomorphic to their opposite algebras*, International Journal of Mathematics **12** (2001), 263–275.

[12] N. C. Phillips, *A simple separable C^*-algebra not isomorphic to its opposite algebra*, Proceedings of the American Mathematical Society **132** (2004), 2997–3005.

[13] N. C. Phillips and M. G. Viola, *A simple separable exact C*-algebra not anti-isomorphic to itself*, Mathematische Annalen **355** (2013), 783–799.

[14] I. Raeburn and J. L. Taylor, *Continuous trace C^*-algebras with given Dixmier–Douady class*, Journal of the Australian Mathematical Society . Series A **38** (1985), 394–407.

[15] J. Renault, *A Groupoid Approach to C^*-algebras*, Lecture Notes in Mathematics, Vol. 793, Springer, Berlin, 1980.

[16] J. Renault, *Cartan subalgebras in C^*-algebras*, Irish Mathematical Society Bulletin **61** (2008), 29–63.

[17] M. Rørdam and E. Størmer, *Classification of nuclear, simple C^*-algebras*, in *Classification of Nuclear C^*-algebras. Entropy in Operator Algebras*, Encyclopaedia of Mathematical Sciences, Vol. 126, Springer, Berlin, 2002, pp. 1–145.